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FORTRAN OPTICAL LENS  
DESIGN PROGRAM

Volume I

Mathematical Specifications

Paul J. Firnett  
Leslie A. Wilson

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INFORMATICS INC.  
2000 Riverside Drive  
Los Angeles, California 90039

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## INTRODUCTION

### 1. PURPOSE

FOLDP, which is an acronym standing for FORTRAN Optical Lens Design Program, is a digital computer program written in 7094 FORTRAN IV source language which employs the principles of geometric optics in order to perform automatic designing of optical systems. It is modeled after an earlier program<sup>1</sup> written by C.A. Lehman of the Los Alamos Scientific Laboratory.

The performance of an arbitrary optical system is measured by a merit function, which is essentially the sum of the squares of a set of weighted aberrations. Designing is a linearized least squares process for iteratively reducing the magnitude of the merit function (aberrations) by automatically adjusting system parameters.

### 2. SCOPE

The complete description of FOLDP comprises 3 volumes where:

- a) Volume I is a mathematical specification including geometry, ray specification and tracing, system characteristics, computation and display of spot diagrams, sensitivity analysis, designing, and cross sectional plotting.
- b) Volume II is a user's manual which describes, by means of examples, the input, output and display functions of the program.

- c) Volume III relates to the program logic and includes a description of the structure, flow charts and the program listing.

FOLDP is designed to sequentially execute a set of user selected options. In general, each option (currently there are thirteen) requires certain specific inputs and performs some major computational or display function. The following table contains a list of the options which are currently available. They are described in detail, in Volume 2.

Option Number	Function
0	Terminate processing
1	Input on octal cards (Restart)
2	Design computation
3	Spot diagram computations
4	Twinray diagnostic maximum object
5	Twinray diagnostic zero object
6	Punch octal cards
7	Plot spot diagram
8	Sensitivity computations
9	Print geometry data
10	Print all input data
11	Cross section plot
12	MTF data card punch

FOLDP can handle systems containing up to 100 surfaces (including aspherics), 7 object points, 6 colors and 200 rays.

3. REFERENCES

- 1) Lehman, C. A., LASL Lens Design Program for the IBM 7090, LA-2837, Los Alamos Scientific Laboratory, Los Alamos, N. M., February 1963.
- 2) Schmidt, L. F., Casad, T. A., Automatic Design of Optical Systems by means of an IBM Digital Computer Employing the IBJOB Monitor, Jet Propulsion Laboratory, California Institute of Technology, February 1, 1966.
- 3) Brixner, B., Automatic Lens Design for Nonexperts, Applied Optics, Vol. 2 No. 12, December 1963.
- 4) Firnett, P. J., Lens Designing Technique with 1962 LASL Code, Mesa Scientific Corporation, September 1965.
- 5) Brewer, S. H., Gold, B. J., PAGOS - Program for the Analysis of General Optical Systems, Report No. TOR-1001(2574-02)-1, Aerospace Corp., 1967.
- 6) Steinberg, B., PAGOS Conversion CDC 6600/IBM 7094, TR-68-700-77-1, Informatics Inc., May 1968.
- 7) O'Brien, K. B., Automatic Optical Design of Desired Image Distributions Using Orthogonal Constraints, J. Opt. Soc. Am., Volume 54, Number 10, 1964.



## MATHEMATICAL SPECIFICATIONS

## Section 1

## GEOMETRY

The basic frame of reference is a rectangular cartesian coordinate system oriented so that the  $z$  axis coincides with the optic axis and is positive to the right.

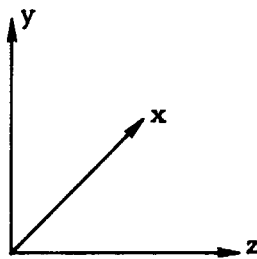


Figure 1. Coordinate System

An arbitrary lens system comprises the following:

- 1) An object which is the source of the rays to be traced through the system.
- 2) A set of  $N$  surfaces (including aspherics) denoted by  $\sigma_i$  where  $1 \leq i \leq N$ , with a material of specified index of refraction  $n_i$  between each consecutive pair.
- 3) An entrance pupil plane  $\sigma_0$  of radius  $\rho_0$  which is used to specify a pattern of rays to be traced through the lens system.
- 4) An image plane (or surface) to which the rays are traced.

The arrangement of the components is illustrated in Figure 2, where:  $S$  is the signed axial distance from the object to  $\sigma_1$ ,  $S_i$  is the signed axial distance from  $\sigma_{i-1}$  to  $\sigma_i$  for  $1 \leq i \leq N$ , and  $S^*$  is the signed axial distance from  $\sigma_N$  to the image plane.

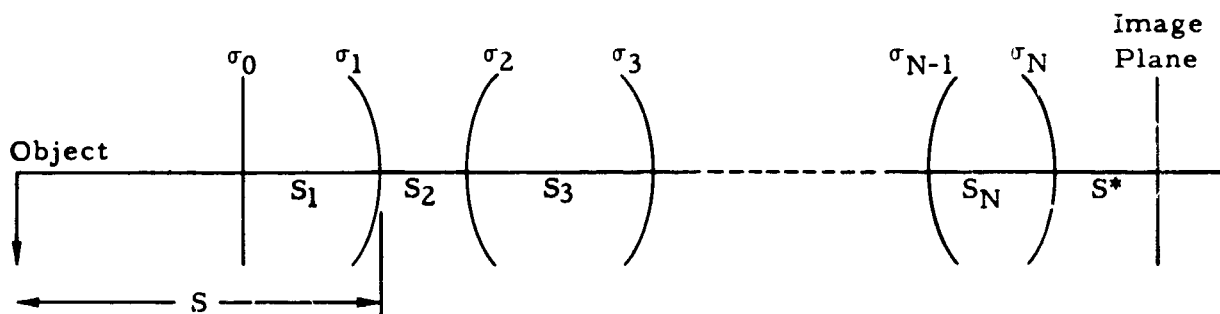


Figure 2. Lens System

### 1.1 OBJECT POINTS

Let  $C'$  be a cartesian system located at the object. An object point is normally defined by specifying its height  $h$  ( $h \geq 0$ ) which the program interprets as a point on the negative  $y'$  axis with coordinates  $(0, -h, 0)$ , as illustrated in Figure 3. This "on axis" object point is the source of a set of rays to be traced through the lens system.

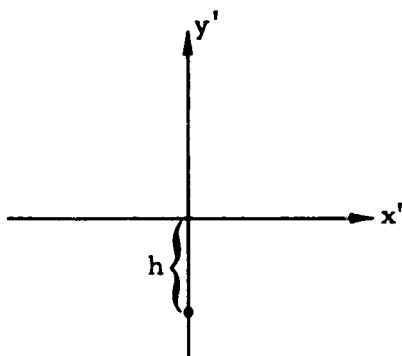


Figure 3. On Axis Object

In certain non-symmetric lens systems, e. g. , one which contains a cylindrical element it is desirable to have the rays originate from an object point which is not necessarily located on the y axis. This is accomplished by specifying, in addition to  $h$ , an azimuth angle  $\theta$  which is measured clockwise from the negative y-axis so that  $(h, \theta)$  are, in effect, polar coordinates of the object point.

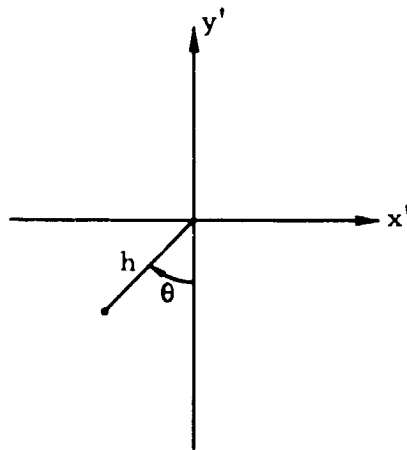


Figure 4. Off Axis Object

The rectangular coordinates of the object point are:

$$x' = -h \sin \theta \quad (1.001)$$

$$y' = -h \cos \theta \quad (1.002)$$

$$z' = 0 \quad (1.003)$$

The case  $\theta = 0$  corresponds to an on axis object. The program has the capability of handling  $N_H$  object points ( $1 \leq N_H \leq 7$ ) distributed along the radius vector  $\theta$  at equal increments of  $h$ . These  $N_H$  points are specified by the object heights  $h_j$  where:

$$h_j = h_o + (j - 1)\Delta h \quad 1 \leq j \leq N_H \quad (1.004)$$

The rectangular coordinates of the  $j$ th object point are  $(-h_j \sin \theta, -h_j \cos \theta, 0)$  where  $h_o$ ,  $\Delta h$ ,  $N_H$ , and  $\theta$  are input to the program.

## 1.2 LENS SURFACES

Each of the lens surfaces ( $\sigma_i$ ,  $1 \leq i \leq N$ ) is parameterized in a cartesian coordinate system  $C^{(i)}$  such that the vertex of the surface is at the origin of  $C^{(i)}$  and is described by an equation of the form:

$$F_i(x, y, z) = 0. \quad (1.005)$$

In principle, there are only two restrictions on  $F_i(x, y, z)$ :

- 1) A procedure must exist for determining the point of intersection,  $(x_1, y_1, z_1)$ , of an arbitrary straight line with  $F_i(x, y, z)$ ,

and,

- 2) The gradient of  $F_i$ ,  $\nabla F_i$ , must exist at  $(x_1, y_1, z_1)$ .

In practice the program is designed to consider 3 types of surfaces - plane, conic, and polynomial.

### 1.2.1 Plane

A plane is described by the equation:

$$F_i(x, y, z) = z = 0. \quad (1.006)$$

1.2.2 Conic

A conic of revolution about the optic (z) axis having its surface vertex at (0, 0, 0) is described by the equation:

$$x^2 + y^2 + (z - R)^2 = R^2 + bz^2 \quad (1.007)$$

where R is the radius of curvature at (0, 0, 0) and b is an eccentricity dependent parameter,  $b = b(\epsilon)$ , whose value determines the type of conic as follows:

$b = 0$	Sphere
$b = 1$	Paraboloid
$0 < b < 1$	Oblate Ellipsoid
$-1 < b < 0$	Prolate Ellipsoid
$b > 1$	Hyperboloid

If  $R > 0$  the center of curvature is to the right of the origin and the surface is convex. If  $R < 0$  the center of curvature is to the left of the origin and the surface is concave.

The conic of revolution defined by equation (1.007) is produced by rotating the plane conic,

$$y^2 + (z - R)^2 = R^2 + bz^2, \quad (1.008)$$

about the z-axis.

1.2.2.1 Circle (b = 0). Setting  $b = 0$  in equation (1.008) yields:

$$y^2 + (z - R)^2 = R^2, \quad (1.009)$$

which is a circle of radius  $R$  in the  $y$ - $z$  plane with center at  $(R, 0)$ , as illustrated in Figure 5.

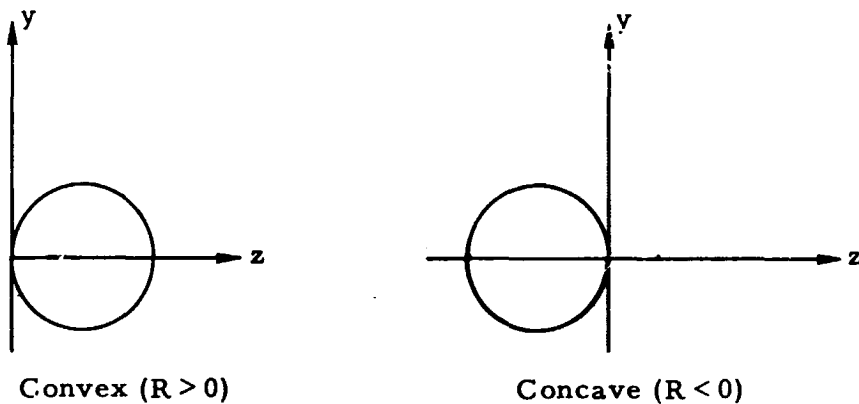


Figure 5. Circle ( $b = 0$ )

Since  $\epsilon = 0$  for a circle it follows that:

$$b = \epsilon \quad (1.010)$$

1.2.2.2 Parabola (b = 1). Setting  $b = 1$  in equation (1.008) yields:

$$y^2 = 2Rz, \quad (1.011)$$

which is the standard equation of a parabola in the  $y$ - $z$  plane. (See Figure 6.)

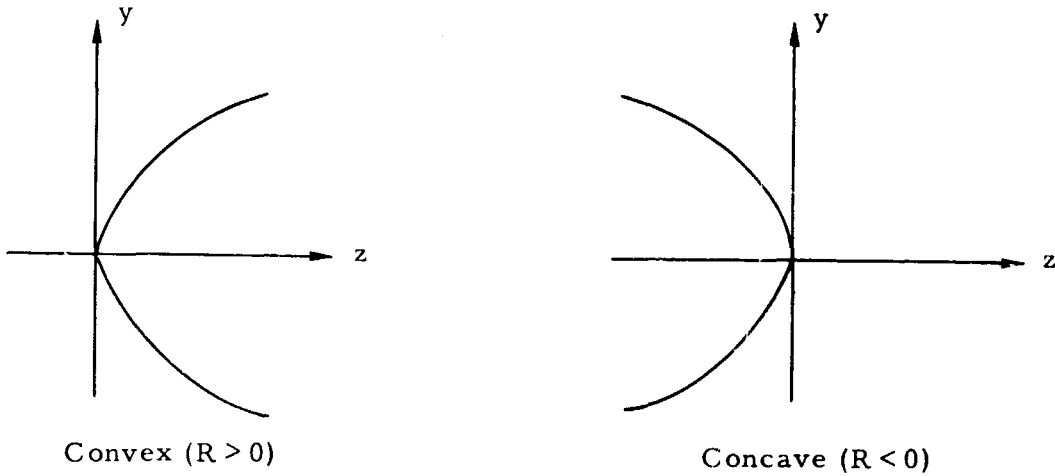


Figure 6. Parabola ( $b = 1$ )

Since  $\epsilon = 1$  for a parabola, then:

$$b = \epsilon \quad (1.012)$$

1.2.2.3 Ellipse ( $0 < |b| < 1$ ). The standard equation of an ellipse in the  $y$ - $z$  plane whose vertex is at the origin is:

$$\frac{y^2}{a_1^2} + \frac{(z - a_2)^2}{a_2^2} = 1 \quad (1.013)$$

where  $a_1 > 0$  and  $a_2 \neq 0$ . Comparing (1.013) to (1.008), it follows that:

$$R = \frac{a_1^2}{a_2^2} \quad (1.014)$$

$$b = 1 - \frac{a_1^2}{a_2^2} \quad (1.015)$$

or,

$$a_2 = \frac{R}{1-b} \quad (1.016)$$

$$a_1^2 = \frac{R^2}{1-b} \quad (1.017)$$

a)  $|a_2| > a_1$  Oblate Ellipse

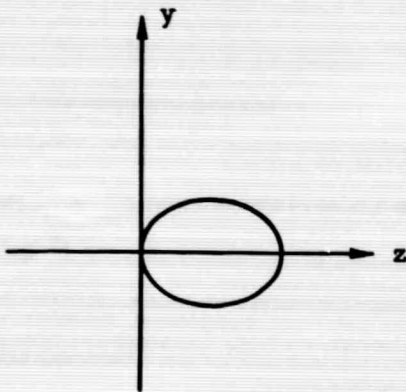
Eccentricity is given by:

$$\epsilon^2 = 1 - \frac{a_1^2}{a_2^2} \quad (1.018)$$

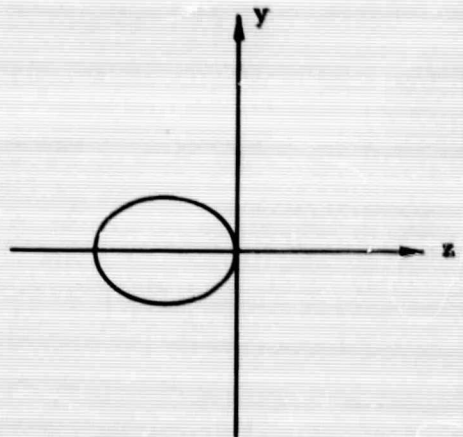
so that

$$b = \epsilon^2 \quad (1.019)$$

where  $0 < b < 1$ . The oblate ellipse is illustrated in Figure 7.



Convex ( $R > 0$ )



Concave ( $R < 0$ )

Figure 7. Oblate Ellipse ( $0 < b < 1$ )



b)  $|a_2| < a_1$  Prolate Ellipse

Eccentricity is given by:

$$\epsilon^2 = 1 - \frac{a_2^2}{a_1^2} \quad (1.020)$$

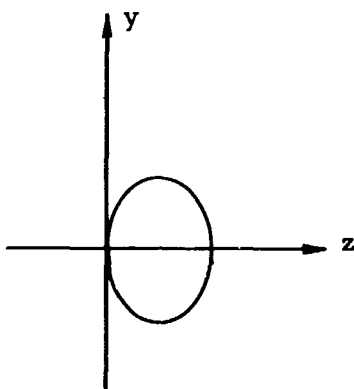
where  $0 < \epsilon < 1$ . It follows from Equation (1.015) that:

$$b = \frac{-\epsilon^2}{1 - \epsilon^2} \quad (1.021)$$

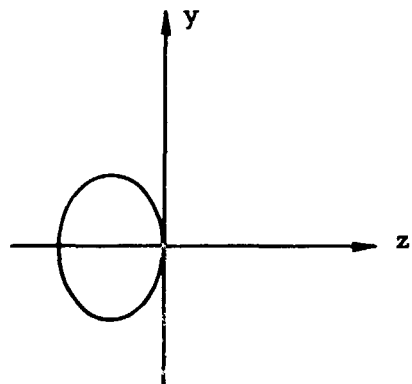
and

$$\epsilon^2 = \frac{-b}{1 - b} \quad (1.022)$$

The prolate ellipse, which is specified by  $-1 < b < 0$ , is illustrated in Figure 8.



Convex ( $R > 0$ )



Concave ( $R < 0$ )

Figure 8. Prolate Ellipse ( $-1 < b < 0$ )

1.2.2.4 Hyperbola ( $b > 1$ ). The standard equation of a hyperbola in the y-z plane whose vertex is at the origin is:

$$\frac{(z - a_2)^2}{a_2^2} - \frac{y^2}{a_1^2} = 1 \quad (1.023)$$

where  $a_1 > 0$  and  $a_2 \neq 0$ . Comparing (1.023) and (1.008):

$$R = \frac{-a_1^2}{a_2^2} \quad (1.024)$$

$$b = 1 + \frac{a_1^2}{2a_2^2} \quad (1.025)$$

or,

$$a_2 = \frac{-R}{b-1} \quad (1.026)$$

$$a_1^2 = \frac{R^2}{b-1} \quad (1.027)$$

The eccentricity  $\epsilon$  ( $\epsilon > 1$ ) is given by:

$$\epsilon^2 = 1 + \frac{a_1^2}{2a_2^2} \quad (1.028)$$

so that

$$b = \epsilon^2. \quad (1.029)$$

The two types of hyperbolas are illustrated in Figure 9.

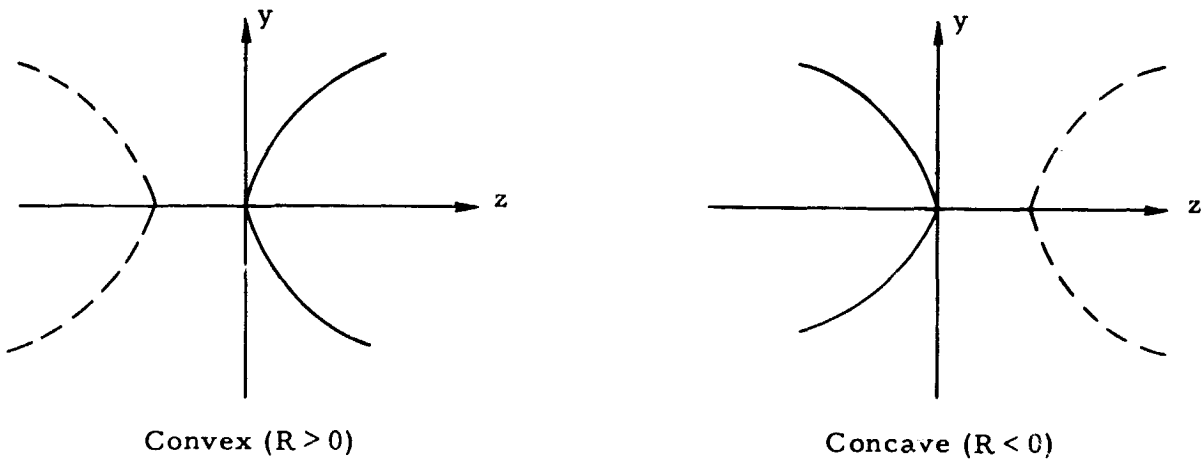


Figure 9. Hyperbola ( $b > 1$ )

In addition to the conic of revolution, defined by Equation (1.007), the program will accept the two types of conic cylinders described below.

The horizontal conic cylinder (axis of cylinder parallel to x axis) is defined by:

$$y^2 + (z - R)^2 = R^2 + bz^2 \quad (1.030)$$

The vertical conic cylinder (axis of cylinder parallel to y axis) is defined by:

$$x^2 + (z - R)^2 = R^2 + bz^2 \quad (1.031)$$

### 1.2.3 Nonstandard Conic

Equation (1.007) describes a conic of revolution in standard position in the sense that the vertex is at the origin and the axis of

revolution is the z (optic) axis. A nonstandard conic of revolution is one whose vertex is located at a specified point  $(x_T, y_T, 0)$  in the x-y plane (translated) or one whose axis of revolution points in a specified direction  $\vec{S} = (u, v, w)$  where  $\vec{S}$  is a unit vector (tilted). To specify a tilted conic, one must supply u and v such that  $u^2 + v^2 \leq 1$  from which w is computed:

$$w = + \left\{ 1 - (u^2 + v^2) \right\}^{1/2} \quad (1.032)$$

The triplet  $(u, v, w)$  may be interpreted as the components of a unit vector  $\vec{S}$  which lies along the axis of revolution or the direction cosines of the line collinear with this axis. A conic of revolution may be translated and tilted by the appropriate choice of  $(x_T, y_T)$  and  $(u, v)$ .

In a coordinate system whose origin is at the point  $(x_T, y_T, 0)$  and whose z axis points along  $\vec{S}$ , the conic of revolution is described by Equation (1.007). As will be seen, the nonstandard conic requires special treatment during ray tracing.

#### 1.2.4 Polynomial Aspherics

The aspherics are parameterized in such a manner that the polynomial terms represent the result of deforming a standard conic.

The first step is to solve Equation (1.007) for z to obtain:

$$z = \frac{\phi}{R + \epsilon \left\{ R^2 - (1-b)\phi \right\}^{1/2}} \quad (1.033)$$

where

$$\phi = x^2 + y^2 \quad (1.034)$$

$$\epsilon = \begin{cases} 1. & R > 0 \\ -1. & R < 0 \end{cases} \quad (1.035)$$

The aspheric of revolution is obtained by adding deformation terms:

$$z = \frac{\phi}{R + \epsilon \left\{ R^2 - (1-b)\phi \right\}^{1/2}} + \sum_{i=2}^5 a_i \phi^i \quad (1.036)$$

where  $a_i$  are the deformation coefficients.

A plane may be deformed by applying the usual convention that  $R = 0$  in which case the equation becomes:

$$z = \sum_{i=2}^5 a_i \phi^i. \quad (1.036-1)$$

Equation (1.036) may also be used to represent horizontal and vertical aspheric cylinders by defining  $\phi$  such that:

$$\phi = \begin{cases} y^2 & \text{Horizontal Cylinder} \\ x^2 & \text{Vertical Cylinder.} \end{cases} \quad (1.036-2)$$

No provision is made for translating or tilting aspheric elements.

### 1.3 IMAGE SURFACE

Associated with each of the  $N_H$  object points considered by the program, there is a principal image plane (perpendicular to the z axis) to which the specified pattern of rays is traced. The image plane is positioned by specifying its axial distance  $S_j^*$  ( $1 \leq j \leq N_H$ ) from the last lens surface ( $\sigma_N$ ) as follows:

$$S_j^* = \begin{cases} S^* + d_j & \text{IMODE} = 1 \\ S^* + B_F + d_j & \text{IMODE} = 0 \end{cases} \quad (1.037)$$

IMODE is an input flag.  $B_F$  is the computed back-focal distance for the first color.\*  $S^*$  is an input which, in the case IMODE = 1, can be interpreted as the axial distance from  $\sigma_N$  to the image surface.

The  $d_j$  are a set of  $N_H$  (input) distance increments which can be employed to specify a curved image surface. A plane is always specified by setting  $d_j = 0$ . for  $1 \leq j \leq N_H$ .

Let  $(\bar{x}_j, \bar{y}_j)$  be the centroid of a set of points on an image plane whose position is given by Equation (1.037). This set of points is produced by tracing a specified pattern of rays from the object point  $(h_j, \theta)$  to the image plane.

The polar coordinates of the centroid  $(\bar{R}_j, \bar{\theta}_j)$  are:

$$\bar{R}_j = \left\{ \bar{x}_j^2 + \bar{y}_j^2 \right\}^{1/2} \quad (1.038)$$

$$\bar{\theta}_j = \text{Tan}^{-1} \left\{ \frac{\bar{y}_j}{\bar{x}_j} \right\}. \quad (1.039)$$

\*See Section 4.2

The image vector is a radius vector on the image plane which is rotated  $180^\circ$  clockwise from the object as illustrated in Figure 10.

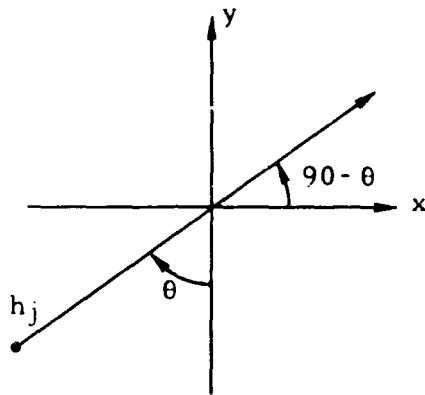


Figure 10. Image Vector

By definition, image height  $h_j'$  is the projection of  $\bar{R}_j$  onto the image vector. Thus:

$$h_j' = \bar{R}_j \cos \theta_j' \quad (1.040)$$

where  $\theta_j' = \bar{\theta}_j - (90 - \theta)$  is the angle between  $\bar{R}_j$  and the image vector (Figure 11.)

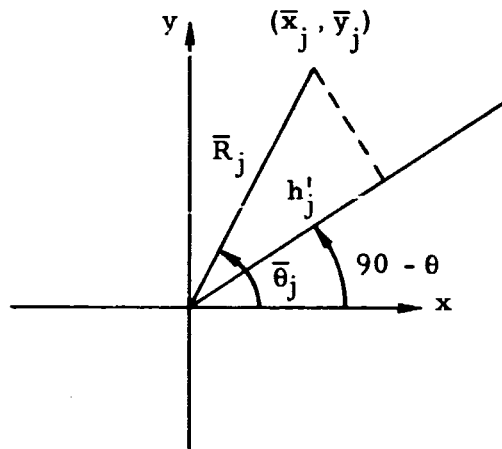


Figure 11. Image Height

Since  $\cos \theta'_j = \cos \bar{\theta}_j \sin \theta + \sin \bar{\theta}_j \cos \theta$  then Equation (1.040) becomes

$$h'_j = \bar{x}_j \sin \theta + \bar{y}_j \cos \theta \quad (1.041)$$



Section 2

RAY SPECIFICATION

Each ray to be traced through the lens system is initially specified by its coordinates,  $(\hat{x}_0, \hat{y}_0, 0)$ , on a unit radius entrance pupil. The actual coordinates  $(x_0, y_0, 0)$  are:

$$\begin{aligned}x_0 &= \rho_0 \hat{x}_0, \\y_0 &= \rho_0 \hat{y}_0,\end{aligned}\tag{2.001}$$

and  $z_0 = 0.$

where  $\rho_0$  is the radius of the entrance pupil. The ray is thus described by the equation:

$$\frac{x - x_0}{Q_x} = \frac{y - y_0}{Q_y} = \frac{z - z_0}{Q_z} .\tag{2.002}$$

The initial direction of the ray,  $\vec{Q} = (Q_x, Q_y, Q_z)$ , is determined by the conditions:

1. The ray originates at a specified object point  $(h, \theta)$ .
2.  $\vec{Q}$  is a unit vector.

Let  $S'$  be the (signed) axial distance from the object to entrance pupil.

$$S' = S - S_1 \quad (2.003)$$

From section 1.1, the rectangular coordinates  $(x_1, y_1, z_1)$  of the object point are:

$$\begin{aligned} x_1 &= -h \sin \theta \\ y_1 &= -h \cos \theta \\ z_1 &= -S' \end{aligned} \quad (2.004)$$

Condition 1. requires:

$$\frac{x_1 - x_0}{Q_x} = \frac{y_1 - y_0}{Q_y} = \frac{z_1}{Q_z}$$

or,

$$Q_x = Q_z \left( \frac{x_0 - x_1}{S'} \right)$$

$$Q_y = Q_z \left( \frac{y_0 - y_1}{S'} \right)$$

Condition 2. requires:

$$Q_z^2 \left\{ 1 + \left( \frac{x_0 - x_1}{S'} \right)^2 + \left( \frac{y_0 - y_1}{S'} \right)^2 \right\} = 1$$

Define:

$$\Delta = \left\{ (S')^2 + (x_0 + h \sin \theta)^2 + (y_0 + h \cos \theta)^2 \right\}^{1/2} \quad (2.005)$$

Then:

$$Q_x = \frac{x_0 + h \sin \theta}{\Delta} \quad (2.006)$$

$$Q_y = \frac{y_0 + h \cos \theta}{\Delta} \quad (2.007)$$

$$Q_z = \frac{S'}{\Delta} \quad (2.008)$$

The specified ray is illustrated in Figure 12. The object point  $(h, \theta)$  determines the ray position since the ray originates at the object; whereas, the entrance pupil point  $(x_0, y_0, 0)$  determines the ray direction since the ray is along a line joining  $(h, \theta)$  and  $(x_0, y_0, 0)$ .

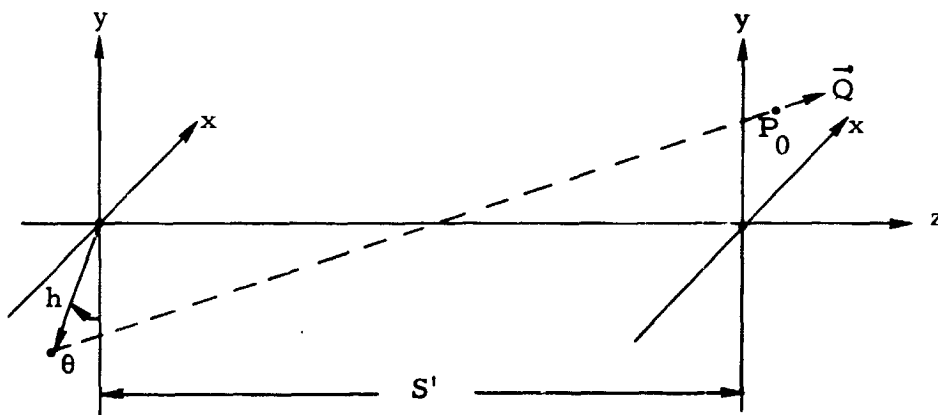


Figure 12. Ray Specification

The specification of a pattern of rays (on the unit radius entrance pupil) to be traced from each object point  $(h_j, \theta)$  where  $1 \leq j \leq N_H$  and for each color  $i$  where  $1 \leq i \leq N_c$  requires the following input:

$N_R$  = Total number of rays to be traced.

$\Delta y$  = Vertical spacing increment of rays in entrance pupil.

$N_{SL}$  = Number of vertical slices into which entrance pupil is divided such that  $1 \leq N_{SL} \leq 26$ .

The entrance pupil is divided up into  $N_{SL}$  vertical slices where  $(x_K, y_K)$  is the coordinate of the first ray in the  $K$ th slice. Succeeding rays in this slice have coordinates  $(x_K, y_K + \Delta y)$ ,  $(x_K, y_K + 2\Delta y)$ , ...,  $(x_K, y_K + [m_K - 1]\Delta y)$ , where  $m_K$  is the total number in the  $K$ th slice. This information is contained in the  $3 \times N_{SL}$  matrix  $l$  where:

$$l = \begin{bmatrix} x_1 & x_2 & \dots & x_K & \dots & x_{N_{SL}} \\ y_1 & y_2 & & y_K & & y_{N_{SL}} \\ m_1 & m_2 & \dots & m_K & \dots & m_{N_{SL}} \end{bmatrix} \quad (2.009)$$

Rays are generated starting with the first point of slice 1, continuing to the last point of slice 1, and then proceeding to the next slice. The above data must be selected so that  $x^2 + y^2 \leq 1$  for every  $(x, y)$  in the pattern. An additional constraint is that

$$\sum_{K=1}^{N_{SL}} m_K \geq N_R \quad (2.010)$$

If the inequality applies in equation (2.010) only the first  $N_R$  rays of the total pattern will be generated.

An additional degree of flexibility in pattern specification is provided by the input trigger LMODE which must be set to 0, 1, or 2. If  $(x', y')$  are the coordinates of an arbitrary ray in the pattern described above:

1. LMODE=0 instructs the program to also trace  $(-x', y')$  and to compute an average  $x$  at the image plane.
2. LMODE=1 instructs the program to trace only the specified rays and to assume average  $x$  is zero at the image plane.
3. LMODE=2 instructs the program to trace only the specified rays and to compute average  $x$  at the image plane.

Section 3

RAY TRACING

The technique of tracing a ray from the object point  $(h, \theta)$  through the lens system to a point on  $\sigma_N$  is a three step iterative procedure whereby the coordinates  $P_i = (x_i, y_i, z_i)$  of the ray on  $\sigma_i$  and its direction  $\vec{Q}_i = (Q_{xi}, Q_{yi}, Q_{zi})$  following refraction\* at  $\sigma_i$  are determined from  $P_{i-1}$  and  $\vec{Q}_{i-1}$ . The starting conditions  $P_0$  and  $\vec{Q}_0$  are assumed to be known (see Section 2). The three steps in the iteration are:

- a) Determine  $P_i = (x_i, y_i, z_i)$  as the simultaneous solution of:

$$\frac{x - x_{i-1}}{Q_{x,i-1}} = \frac{y - y_{i-1}}{Q_{y,i-1}} = \frac{z - (z_{i-1} - S_i)}{Q_{z,i-1}} \quad (3.001)$$

and the equation of  $\sigma_i$ :

$$F_i(x, y, z) = 0.$$

- b) Determine the unit vector  $\vec{N}_i$ , which is normal to  $\sigma_i$  at  $(x_i, y_i, z_i)$ , as follows:

$$\vec{N}_i = \frac{\nabla F_i}{|\nabla F_i|} \quad (3.002)$$

where  $\nabla F_i$  is evaluated at  $P_i$ .

\*Refraction is used in the general sense so as to include reflection as a special case.

- c) Determine, by means of Snell's law, the direction of the refracted ray  $\bar{Q}_i$ .

### 3.1 POINT OF INTERSECTION AND NORMAL VECTOR

If during the computation of  $P_i$  it is determined that the ray missed  $\sigma_i$  then the ray trace should be terminated. This is the case if  $Q_{z,i-1} = 0$  or  $Q_{z,i-1} \neq 0$  but there is no real solution for  $P_i$ .

Another condition which can occur is vignetting when there is an aperture located at  $\sigma_i$ . Vignetting is not considered to be of sufficient importance to abort the ray trace. If it occurs a tally of the first surface at which it happened should be made and the trace continued.

The aperture at  $\sigma_i$  may be circular or rectangular according to the values assigned to  $A_{1i}$  and  $A_{2i}$  as follows:

- 1)  $A_{1i} = A_{2i} = 0$  no aperture. The program will check for vignetting at  $\sigma_i$ .
- 2)  $A_{1i} > 0$   $A_{2i} = 0$  circular aperture. The ray vignets if  $x_i^2 + y_i^2 > A_{1i}^2$ .
- 3)  $A_{1i} < 0$   $A_{2i} = 0$  circular aperture with central opacity. The ray vignets if  $x_i^2 + y_i^2 < A_{1i}^2$ .
- 4)  $A_{1i} > 0$   $A_{2i} > 0$  rectangular aperture. The ray vignets if  $|x_i| > A_{1i}$  or  $|y_i| > A_{2i}$ .

- 5)  $A_{1i} < 0$   $A_{2i} > 0$  rectangular aperture with central opacity. The ray vignets if  $|x_i| < A_{1i}$  and  $|y_i| < A_{2i}$ .
- b)  $A_{2i} < 0$  is not permitted.

The determination of  $P_i$  and  $\vec{\nabla}F_i$  depends upon the shape of  $\sigma_i$ ; hence, the different types of surfaces are considered separately.

### 3.1.1 Standard Conic

Let  $P_i^* = (x_i^*, y_i^*, 0)$  be the point where the ray intersects the x-y (vertex) plane. From Equation (3.001), with  $z = 0$ :

$$\begin{aligned} x_i^* &= x_{i-1} + \lambda_1 Q_{x,i-1} \\ y_i^* &= y_{i-1} + \lambda_1 Q_{y,i-1} \\ z_i^* &= 0. \end{aligned} \tag{3.002}$$

where

$$\lambda_1 = \frac{-(z_{i-1} - S_i)}{Q_{z,i-1}} \tag{3.003}$$

If  $R = 0$ ,  $\sigma_i$  is the plane  $z = 0$  and:

$$\begin{aligned} \frac{\partial F_i}{\partial x} &= 0 \\ \frac{\partial F_i}{\partial y} &= 0 \\ \frac{\partial F_i}{\partial z} &= 1 \end{aligned} \tag{3.004}$$



Suppose  $R \neq 0$ , then the ray equation may be written:

$$\begin{aligned} x &= \lambda_2 Q_{x, i-1} + x_i^* \\ y &= \lambda_2 Q_{y, i-1} + y_i^* \\ z &= \lambda_2 Q_{z, i-1} \end{aligned} \quad (3.005)$$

where

$$\lambda_2^2 = (x - x_i^*)^2 + (y - y_i^*)^2 + z^2. \quad (3.006)$$

Clearly,  $\lambda_2$  is the distance along the ray from  $P_i^*$  to any other point  $P = (x, y, z)$  on the ray.

If  $\sigma_i$  is a conic of revolution then:

$$F_i \equiv x^2 + y^2 + (1-b)z^2 - 2Rz = 0 \quad (3.007)$$

Using Equations (3.005) to eliminate  $x, y,$  and  $z$  in Equation (3.007) yields the equation:

$$-\alpha \lambda_2^2 + 2\beta \lambda_2 + \gamma = 0 \quad (3.008)$$

where

$$\alpha = b Q_{z, i-1}^2 - 1 \quad (3.009)$$

$$\beta = x_i^* Q_{x, i-1} + y_i^* Q_{y, i-1} - R Q_{z, i-1} \quad (3.010)$$

$$\gamma = (x_i^*)^2 + (y_i^*)^2 \quad (3.011)$$

In order to circumvent the difficulties which arise when  $b=1$ , Equation (3.008) is solved for  $1/\lambda_2$  and the result inverted to get  $\lambda_2$ . Let

$$\Delta = \beta^2 + \alpha\gamma. \quad (3.012)$$

If  $\Delta < 0$  there is no real solution for  $\lambda_2$  which implies that the ray missed  $\sigma_i$ .

If  $\Delta = 0$ , then:

$$\lambda_2 = \frac{-\gamma}{\beta} \quad (3.013)$$

If  $\Delta > 0$ , then:

$$\lambda_2 = \frac{-\gamma}{(\beta \pm \Delta^{1/2})} \quad (3.014)$$

From the definition of  $\lambda_2$  it is clear that the smallest of the two values expressed in Equation (3.014) is the desired root; hence,  $\Delta^{1/2}$  should be given the sign of  $\beta$ .

Using the computed value of  $\lambda_2$ ,  $P_i = (x_i, y_i, z_i)$  is given by Equation (3.005).  $\vec{\nabla} F_i$  comes directly from Equation (3.007) since:

$$\begin{aligned} \frac{\partial F_i}{\partial x} &= x \\ \frac{\partial F_i}{\partial y} &= y \\ \frac{\partial F_i}{\partial z} &= (1-b)z - R \end{aligned} \quad (3.015)$$

the partials being evaluated at  $(x_i, y_i, z_i)$ .

If  $\sigma_i$  is a horizontal cylinder

$$F_i \equiv y^2 + (1-b)z^2 - 2Rz = 0 \quad (3.016)$$

$P_i$  is determined in exactly the same manner except that:

$$\alpha = (b-1)Q_{z,i-1}^2 - Q_{y,i-1}^2 \quad (3.017)$$

$$\beta = y_i^* Q_{y,i-1} - RQ_{z,i-1} \quad (3.018)$$

$$\gamma = (y_i^*)^2 \quad (3.019)$$

$\vec{\nabla} F_i$  for a horizontal cylinder is given by:

$$\frac{\partial F_i}{\partial x} = 0 .$$

$$\frac{\partial F_i}{\partial y} = y \quad (3.020)$$

$$\frac{\partial F_i}{\partial z} = (1-b)z - R$$

evaluated at  $P_i = (x_i, y_i, z_i)$ .

If  $\sigma_i$  is a vertical cylinder

$$F_i \equiv x^2 + (1-b)z^2 - 2Rz = 0 . \quad (3.021)$$

To determine  $P_i$  the following definitions apply:

$$\alpha = (b-1) Q_{z,i-1}^2 - Q_{x,i-1}^2 \quad (3.022)$$

$$\beta = x_i^* Q_{x,i-1} - R Q_{z,i-1} \quad (3.023)$$

$$\gamma = (x_i^*)^2 \quad (3.024)$$

$\nabla F_i$  for a vertical cylinder is, from Equation (3.021):

$$\frac{\partial F_i}{\partial x} = x$$

$$\frac{\partial F_i}{\partial y} = 0 \quad (3.025)$$

$$\frac{\partial F_i}{\partial z} = (1-b)z - R$$

evaluated at  $P_i = (x_i, y_i, z_i)$ . The program must check for vignetting at  $\sigma_i$  before proceeding with refraction.

3.1.2 Nonstandard Conic

The determination of  $P_i$  and  $\vec{\nabla}F_i$  when  $\sigma_i$  is a nonstandard conic of revolution involves several coordinate transformations. Consider the following definitions of the terminology.

C	Cartesian coordinate system with origin on the optic axis and z axis collinear with it.
$\vec{X}$	Column vector with components (x y z).
C'	Cartesian coordinate system with origin at $(x_T, y_T, 0)$ and $z'$ parallel to z.
$\vec{X}'$	Column vector with components $(x' y' z')$ .
C''	Cartesian coordinate system with same origin as C' but $z''$ in the direction $\vec{S} = (u, v, w)$ .
$\vec{X}''$	Column vector with components $(x'' y'' z'')$ .
$P_0$	Point of intersection of ray on $\sigma_{i-1}$ .
$\vec{Q}$	Direction of ray incident on $\sigma_i$ . ( $\vec{Q} \equiv \vec{Q}_{i-1}$ )
$P_1$	Point where ray intersects $x''$ - $y''$ plane.
$P_i$	Point where ray intersects $\sigma_i$ .
$\vec{X}_0, \vec{X}'_0, \vec{X}''_0$	Coordinates of $P_0$ .
$\vec{X}_1, \vec{X}'_1, \vec{X}''_1$	Coordinates of $P_1$ .
$\vec{Q}'$	Direction of ray with C' components $(Q'_x, Q'_y, Q'_z)$ .
$\vec{Q}''$	Direction of ray with C'' components $(Q''_x, Q''_y, Q''_z)$ .
$\vec{X}_T$	Column vector with components $(x_T y_T 0)$ .
$\vec{X}_i, \vec{X}'_i, \vec{X}''_i$	Coordinates of $P_i$ .
$\vec{S}$	Vector with components (u v w)

The transformation from  $C$  to  $C'$  is a translation defined by the vector equation

$$\vec{X}' = \vec{X} - \vec{X}_T, \quad (3.026)$$

and the inverse

$$\vec{X} = \vec{X}' + \vec{X}_T. \quad (3.027)$$

The transformation from  $C'$  to  $C''$  is a rotation about a line perpendicular to the plane defined by  $z'$  and  $z''$ . Thus:

$$\vec{X}'' = A\vec{X}' \quad (3.028)$$

where  $A$  is the orthogonal matrix given by:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ u & v & w \end{pmatrix}. \quad (3.029)$$

Since  $A$  is orthogonal the transformation preserves inner products (and consequently distance) and the inverse is given by

$$\vec{X}' = A^t \vec{X}'' \quad (3.030)$$

where "t" represents transpose.

The equation of the ray (in C) is

$$\bar{X} = \lambda \bar{Q} + \bar{X}_0 \quad (3.031)$$

where  $\lambda^2 = (\bar{X} - \bar{X}_0) \cdot (\bar{X} - \bar{X}_0) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$  is the distance along the ray from  $P_0$  to any other point P.  $\bar{X}_0$  is given by:

$$\bar{X}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} x_{i-1} \\ y_{i-1} \\ z_{i-1} - S_i \end{pmatrix} \quad (3.032)$$

Transforming Equation (3.031) to  $C'$  yields:

$$\bar{X}' = \lambda \bar{Q} + \bar{X}'_0 \quad (3.033)$$

where  $\bar{X}'_0 = \bar{X}_0 - \bar{X}_T$ . If  $\sigma_i$  is not tilted ( $u = v = 0$ ), then the conic is in standard position in system  $C'$ . In this situation the method described in Section 3.1.1 may be applied to determine  $P_i$  and  $\vec{\nabla}F_i$  starting with  $\bar{X}'_0$  in place of  $(x_{i-1}, y_{i-1}, z_{i-1})$ ,  $\bar{Q}$  in place of  $\bar{Q}_{i-1}$  (which are identical), and  $\lambda_1 = -z'_0 / Q_z$ . The procedure produces  $\bar{X}'_i$  (coordinates of  $P_i$  in  $C'$ ) and  $\vec{\nabla}F_i$ . After checking for vignetting using  $(x'_i, y'_i)$ ,  $\bar{X}'_i$  is given by Equation (3.027).

If  $\sigma_i$  is tilted ( $|u| + |v| \neq 0$ ) the situation is significantly more complicated the first step being to determine  $P_1$ . Transforming the ray Equation (3.033) into  $C''$  by employing Equation (3.028):

$$\bar{X}'' = \lambda \bar{Q}'' + \bar{X}''_0 \quad (3.034)$$

where  $\bar{Q}'' = A\bar{Q}' = A\bar{Q}$ .

$P_1$  is the point where  $z'' = 0$ . From Equation (3.034):

$$z'' = \lambda Q_z'' + z_0'' \quad (3.035)$$

with

$$z_0'' = ux_0' + vy_0' + wz_0' \quad (3.036)$$

$$Q_z'' = uQ_x + vQ_y + wQ_z. \quad (3.037)$$

Let  $\lambda_1$  be that value of  $\lambda$  for which  $z'' = 0$ ; hence:

$$\lambda_1 = -\frac{z_0''}{Q_z''}. \quad (3.038)$$

It follows immediately that

$$\bar{X}_1'' = \lambda_1 \bar{Q}'' + \bar{X}_0'' \quad (3.039)$$

which can be transformed into  $C'$  giving

$$\bar{X}_1' = \lambda_1 \bar{Q} + \bar{X}_0'. \quad (3.040)$$

If  $R = 0$  ( $\sigma_i$  is a plane) it follows that  $P_i \equiv P_1$ ; therefore,  $\bar{X}_i' = \bar{X}_1'$  and  $\bar{\nabla}F_i = \bar{S}$ .

If  $R \neq 0$ ,  $\bar{X}_i'$  must be computed as the simultaneous solution of

$$\bar{X}' = \lambda \bar{Q} + \bar{X}_1' \quad (3.041)$$

and the equation of  $\sigma_i$ . In  $C''$  the equation of  $\sigma_i$  may be written:

$$(x'')^2 + (y'')^2 + (z'')^2 - 2Rz'' - b(z'')^2 = 0 \quad (3.042)$$



Since the transformation from  $C'$  to  $C''$  preserves distance and  $z'' = ux' + vy' + wz'$  from Equation (3.028), Equation (3.042) transforms into:

$$(x')^2 + (y')^2 + (z')^2 - 2R(ux' + vy' + wz') - b(ux' + vy' + wz')^2 = 0 \quad (3.043)$$

Equation (3.041) may be used to eliminate  $x'$ ,  $y'$ , and  $z'$  from Equation (3.043) since:

$$\begin{aligned} (x')^2 + (y')^2 + (z')^2 &= \bar{X}' \cdot \bar{X}' \\ &= \lambda^2 + 2\bar{X}'_1 \cdot \bar{Q}\lambda + \bar{X}'_1 \cdot \bar{X}'_1 \end{aligned}$$

and

$$\begin{aligned} ux' + vy' + wz' &= \lambda(uQ_x + vQ_y + wQ_z) + (ux'_1 + vy'_1 + wz'_1) \\ &= \lambda Q''_z + Z''_1 \\ &= \lambda Q''_z \end{aligned}$$

The resultant equation is:

$$-\alpha\lambda^2 + 2\beta\lambda + \gamma = 0 \quad (3.044)$$

where

$$\alpha = bQ''_z - 1 \quad (3.045)$$

$$\beta = x'_1 Q_x + y'_1 Q_y + z'_1 Q_z - RQ''_z \quad (3.046)$$

$$\gamma = (x'_1)^2 + (y'_1)^2 + (z'_1)^2 \quad (3.047)$$

Equation (3.044) is solved in exactly the same manner as Equation (3.008) of Section 3.1.1 to yield the real root  $\lambda_2$  which when substituted into Equation (3.041) gives the desired solution  $\bar{X}_i'$ .  $\bar{X}_i$  is given by Equation (3.027). Since  $\partial F_i / \partial x \equiv \partial F_i / \partial x'$ ,  $\partial F_i / \partial y = \partial F_i / \partial y'$ , and  $\partial F_i / \partial z = \partial F_i / \partial z'$  then  $\bar{\nabla} F_i$  comes directly from Equation (3.043).

Let

$$K = -R - bz_i'' \quad (3.048)$$

where

$$z_i'' = ux_i' + vy_i' + wz_i' \quad (3.049)$$

Then, at the point  $P_i$ , it follows from Equation (3.043) that:

$$\bar{\nabla} F_i = \bar{X}_i' + K\bar{S} \quad (3.050)$$

In checking for vignetting at  $\sigma_i$  the aperture parameters  $A_{1i}$  and  $A_{2i}$  describe an aperture in  $C''$  (not  $C$ ); hence, one must use  $(x_i'', y_i'')$  instead of  $(x_i, y_i)$ .

If the aperture is circular ( $A_{1i} \neq 0$  and  $A_{2i} = 0$ ), the ray vignets if  $(x_i'')^2 + (y_i'')^2 > A_{1i}^2$  ( $A_{1i} > 0$ ) or  $(x_i'')^2 + (y_i'')^2 < A_{1i}^2$  ( $A_{1i} < 0$ ) where:

$$(x_i'')^2 + (y_i'')^2 = (x_i')^2 + (y_i')^2 + (z_i')^2 - (z_i'')^2 \quad (3.051)$$

If the aperture is rectangular ( $A_{1i} \neq 0$  and  $A_{2i} > 0$ ), the ray vignets if:

$$|x_i''| > A_{1i} \text{ or } |y_i''| > A_{2i} \quad (A_{1i} > 0),$$

or

$$|x_i''| < |A_{1i}| \text{ and } |y_i''| < A_{2i} \quad (A_{1i} < 0).$$

where

$$\bar{X}_i'' = A \bar{X}_i' \quad (3.052)$$

In a document entitled "Least Squares Fitting of a Paraboloid to a Discrete Set of Data Points," C. L. Lawson of JPL shows that the transformation matrix  $A$  is given by:

$$A = \begin{pmatrix} 1 - g_1 u^2 & -g_1 uv & -u \\ -g_1 uv & 1 - g_1 v^2 & -v \\ u & v & w \end{pmatrix} \quad (3.053)$$

where

$$g_1 = \frac{1 - w}{u^2 + v^2}, \quad (3.054)$$

making it possible to compute  $(x_i'', y_i'')$ .

3.1.3 Aspherics

A method of successive approximations is used to compute  $P_i$  when  $\sigma_i$  is an aspheric element. The procedure begins with an estimate  $P_0$  from which a sequence of improved estimates  $P_1, P_2, \dots, P_K$  is generated such that  $P_K$  converges to  $P_i$  as  $K$  increases.

The equation of  $\sigma_i$  is

$$z = \begin{cases} \frac{\phi}{R + \epsilon \{R^2 - (1-b)\phi\}^{1/2}} + \sum_{i=2}^5 a_i \phi^i & R \neq 0 \\ \sum_{i=2}^5 a_i \phi^i & R = 0 \end{cases} \quad (3.055)$$

with

$$\epsilon = \begin{cases} 1. & R > 0 \\ -1. & R < 0 \end{cases} \quad (3.056)$$

$$\phi = \begin{cases} x^2 + y^2 & \text{Surface of Revolution} \\ y^2 & \text{Horizontal Cylinder} \\ x^2 & \text{Vertical Cylinder} \end{cases} \quad (3.057)$$

The initial estimate  $P_0$  is the point where the ray intersects the undeformed conic, that is, the solution with  $a_i = 0$ . With the deformation coefficients all zero,  $\sigma_i$  is a standard conic; hence, the procedure described in Section 3.1.1 is employed to compute  $P_0 = (x_0, y_0, z_0)$ .

Let  $\vec{P}_K = (x_K, y_K, z_K)$  be a vector from the origin to  $(x_K, y_K, z_K)$  which represents the Kth estimate. The first step in generating  $\vec{P}_{K+1}$  is to project  $\vec{P}_K$  onto the aspheric, i. e., to the point  $\vec{P}'_K = (x_K, y_K, z'_K)$  where  $z'_K$  is given by Equation (3.055) with  $(x, y) = (x_K, y_K)$ .  $\vec{P}_{K+1}$  is then taken as the point  $(x_{K+1}, y_{K+1}, z_{K+1})$  where the ray intersects the plane  $z = z'_K$ . Since  $(x_K, y_K, z_K)$  always lies on the ray, the equation of the ray is:

$$\vec{P} = \vec{P}_K + \theta \vec{Q}. \quad (3.058)$$

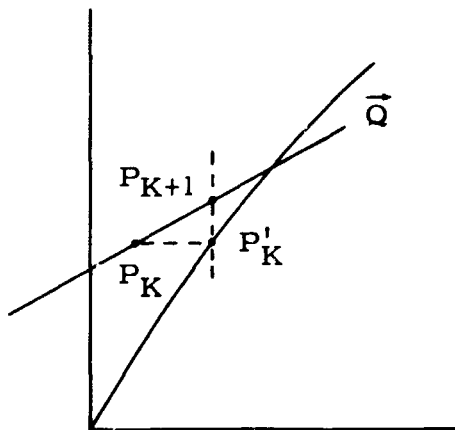


Figure I-3-1. Aspheric Iteration

Let  $\theta_K$  be that value of  $\theta$  such that  $\bar{P}_{K+1} = \bar{P}_K + \theta_K \bar{Q}$ . Since  $z_{K+1} = z'_K$  it follows that:

$$\theta_K = \frac{z'_K - z_K}{Q_z} \quad (3.059)$$

Since  $z_K$  lies on the ray and  $z'_K$  lies on the aspheric, if, during the iteration,  $z'_K = z_K$  it follows immediately that  $P_K$  is the required solution. Having determined  $\theta_K$ ,  $\bar{P}_{K+1}$  is given by Equation (3.058). The iteration continues until  $|\theta_K| \leq 10^{-7}$  which is then followed by 3 additional iterations.

There are two conditions under which the ray is considered to be a miss:

- 1) If for some  $K$  the term  $R^2 - (1-b)\phi_K$  which appears in Equation (3.055) is negative.
- 2) The sequence  $|\theta_K|$   $K = 0, 1, 2, \dots$  is assumed to be decreasing, i. e.,  $|\theta_K| > |\theta_{K+1}|$ . If this is not true for some  $K$  the ray is treated as a miss.

From Equation (3.055) the equation of  $\sigma_i$  may be written:

$$F_i(x, y, z) = z - G[\phi(x, y)] = 0. \quad (3.060)$$

It follows that:

$$\frac{\partial F_i}{\partial x} = - \frac{\partial G}{\partial \phi} \frac{\partial \phi}{\partial x} \quad (3.061)$$

$$\frac{\partial F_i}{\partial y} = - \frac{\partial G}{\partial \phi} \frac{\partial \phi}{\partial y} \quad (3.062)$$

$$\frac{\partial F_i}{\partial z} = 1. \quad (3.063)$$

From Equation (3.057):

$$\frac{\partial \phi}{\partial x} = \begin{cases} 2x & \text{Surface of Revolution} \\ 0. & \text{Horizontal Cylinder} \\ 2x & \text{Vertical Cylinder} \end{cases} \quad (3.064)$$

$$\frac{\partial \phi}{\partial y} = \begin{cases} 2y & \text{Surface of Revolution} \\ 2y & \text{Horizontal Cylinder} \\ 0. & \text{Vertical Cylinder} \end{cases} \quad (3.065)$$

If  $R \neq 0$  then  $G(\phi)$  is given by:

$$G(\phi) = \frac{\phi}{R + \epsilon \{R^2 - (1-b)\phi\}^{1/2}} + \sum_{i=2}^5 a_i \phi^i \quad (3.066)$$

Therefore:

$$\frac{\partial G}{\partial \phi} = \frac{1}{2\epsilon \{R^2 - (1-b)\phi\}^{1/2}} + \sum_{i=2}^5 i a_i \phi^{i-1} \quad (3.067)$$

If  $R = 0$  then:

$$G(\phi) = \sum_{i=2}^5 a_i \phi^i \quad (3.068)$$

from which:

$$\frac{\partial G}{\partial \phi} = \sum_{i=2}^5 i a_i \phi^{i-1} \quad (3.069)$$



## 3.2 REFRACTION

Snell's law makes it possible to determine  $\vec{Q}_i$  from  $\vec{Q}_{i-1}$ ,  $\vec{N}_i$ ,  $n_i$ , and  $n_{i+1}$  where  $n_i$  is the index of refraction of the material between  $\sigma_{i-1}$  and  $\sigma_i$ .

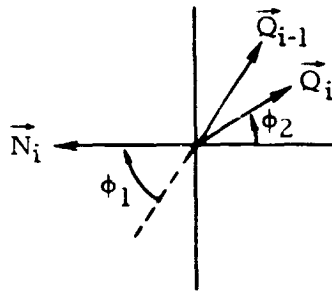


Figure 13. Refraction

In Figure 13,  $\phi_1$  is called the angle of incidence,  $\phi_2$  the angle of refraction, and the plane defined  $\vec{N}_i$  and  $\vec{Q}_{i-1}$  is called the plane of incidence. It should be noted that there is no loss in generality in assuming that  $\vec{N}_i$  points to the left instead of to the right.

The law of refraction states that the refracted ray ( $\vec{Q}_i$ ) lies in the plane of incidence and is directed such that:

$$n_i \sin \phi_1 = n_{i+1} \sin \phi_2. \quad (3.089)$$

Since  $|\vec{Q}_{i-1} \times \vec{N}_i| = \sin(\pi - \phi_1) = \sin \phi_1$ , and  $|\vec{Q}_i \times \vec{N}_i| = \sin(\pi - \phi_2) = \sin \phi_2$ , equation (3.089) becomes:

$$n_i (\vec{Q}_{i-1} \times \vec{N}_i) = n_{i+1} (\vec{Q}_i \times \vec{N}_i). \quad (3.090)$$

Define  $\nu$  such that

$$\nu = \frac{n_i}{n_{i+1}}, \quad (3.091)$$

and re-write (3.090) as follows:

$$(\vec{Q}_i - \nu \vec{Q}_{i-1}) \times \vec{N}_i = 0. \quad (3.092)$$

Since  $\vec{Q}_i$  lies in the plane of incidence, equation (3.092) implies that there is some scalar  $\mu$  such that:

$$\vec{Q}_i = \nu \vec{Q}_{i-1} + \mu \vec{N}_i. \quad (3.093)$$

Insisting that  $\vec{Q}_i \cdot \vec{Q}_i = 1$  produces the following equation in  $\mu$ .

$$\mu^2 + 2\nu (\vec{Q}_{i-1} \cdot \vec{N}_i) \mu + \nu^2 - 1 = 0. \quad (3.094)$$

Let

$$\Delta = \nu^2 - \frac{(\nu^2 - 1)}{(\vec{Q}_{i-1} \cdot \vec{N}_i)^2} \quad (3.095)$$

Equation (3.094) has (in general) two solutions given by:

$$\mu = (\vec{Q}_{i-1} \cdot \vec{N}_i) \left(-\nu \pm \Delta^{\frac{1}{2}}\right). \quad (3.096)$$

It will be shown in Section 3.2.1 that  $\Delta < 0$  implies internal reflection.

The solutions  $\vec{Q}_i$  and  $\vec{Q}'_i$  corresponding to the two values of  $\mu$  are illustrated in Figure 14.

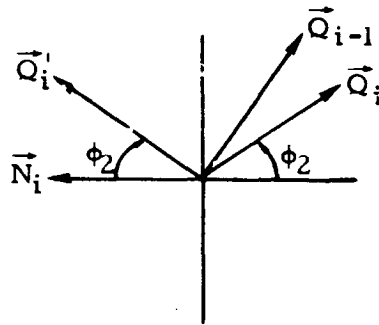


Figure 14.  $\vec{Q}_i$  and  $\vec{Q}'_i$

The correct solution is determined by the requirement that  $\vec{Q}_i \cdot \vec{N}_i$  has the same algebraic sign as  $\vec{Q}_{i-1} \cdot \vec{N}_i$ .

$$\begin{aligned} \vec{Q}_i \cdot \vec{N}_i &= \nu(\vec{Q}_{i-1} \cdot \vec{N}_i) + \mu \\ &= (\vec{Q}_{i-1} \cdot \vec{N}_i)(\pm \Delta^{\frac{1}{2}}). \end{aligned}$$

It is immediately clear that  $\Delta^{\frac{1}{2}}$  must have a positive sign, that is,

$$\mu = (\vec{Q}_{i-1} \cdot \vec{N}_i)(-\nu + \Delta^{\frac{1}{2}}). \tag{3.097}$$

$\vec{Q}_i$  is then given by equation (3.093) using the value of  $\mu$  from (3.097).

3.2.1 Internal Reflection

It follows from equation (3.095) that  $\Delta < 0$  only if  $\nu > 1$ . This situation occurs when  $n_i > n_{i+1}$ , that is, when the ray is travelling from an optically dense to an optically rare medium, and, as a consequence,

$$\phi_2 > \phi_1.$$

That value of  $\phi_1$  for which  $\phi_2 = 90^\circ$  is called the critical angle  $\phi_{1c}$ . Substituting  $\phi_2 = 90^\circ$  into equation (3.090):

$$\sin \phi_{1c} = \frac{1}{\nu} \quad (3.098)$$

and 
$$\cos \phi_{1c} = \frac{\sqrt{\nu^2 - 1}}{\nu}. \quad (3.099)$$

At values of  $\phi_1$  greater than  $\phi_{1c}$  the ray is not refracted but, instead, is internally reflected at  $\sigma_i$ .

It is a simple matter to show that  $\phi_1 > \phi_{1c}$  implies  $\Delta < 0$ . Since  $\cos \phi_1 < \cos \phi_{1c}$  and  $\cos \phi_1 = \bar{Q}_{i-1} \cdot \bar{N}_i$  then:

$$(\bar{Q}_{i-1} \cdot \bar{N}_i)^2 < \cos^2 \phi_{1c},$$

$$(\bar{Q}_{i-1} \cdot \bar{N}_i)^2 < \frac{\nu^2 - 1}{\nu^2},$$

$$\nu^2 - \frac{(\nu^2 - 1)}{(\bar{Q}_{i-1} \cdot \bar{N}_i)^2} < 0,$$

or  $\Delta < 0$ .

If this condition occurs when tracing a ray the process should be terminated. It also follows that  $\Delta = 0$  if and only if  $\phi_1 = \phi_{1c}$ .

### 3.2.2 Reflection

Suppose  $\sigma_1$  is a mirror so that the ray is reflected. In Figure 15,  $\phi_1$  is the angle of incidence and  $\phi_2$  is the angle of reflection. The law of reflection states that the reflected ray ( $\vec{Q}_i$ ) lies in the plane of incidence and is directed such that:

$$\phi_2 = \phi_1. \quad (3.100)$$

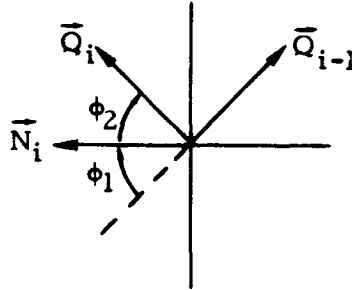


Figure 15. Reflection

Since  $\vec{Q}_{i-1} \cdot \vec{N}_i = \cos(\pi - \phi_1) = -\cos \phi_1$ , the following relations must hold for reflection:

$$\vec{Q}_i \cdot \vec{N}_i = -(\vec{Q}_{i-1} \cdot \vec{N}_i) = \cos \phi_1 \quad (3.101)$$

$$\vec{Q}_i \cdot \vec{Q}_{i-1} = -\cos 2\phi_1. \quad (3.102)$$

Consider the direction  $\vec{Q}_i^*$  of a refracted ray which results by setting  $n_{i+1} = -n_i$ . It follows that  $\nu = -1$ ,  $\Delta = 1$ ,  $\mu = 2 (\vec{Q}_{i-1} \cdot \vec{N}_i)$ , and

$$\vec{Q}_i^* = -\vec{Q}_{i-1} + 2 (\vec{Q}_{i-1} \cdot \vec{N}_i) \vec{N}_i. \quad (3.102)$$

The orientation of  $\vec{Q}_i^*$  with respect to  $\vec{Q}_{i-1}$  and  $\vec{N}_i$  is given by:

$$\vec{Q}_i^* \cdot \vec{N}_i = \vec{Q}_{i-1} \cdot \vec{N}_i = \cos (\pi - \phi_1) \quad (3.103)$$

$$\vec{Q}_i^* \cdot \vec{Q}_{i-1} = -1 + 2 (\vec{Q}_{i-1} \cdot \vec{N}_i)^2 = \cos 2\phi_1 \quad (3.104)$$

and is illustrated in Figure 16.

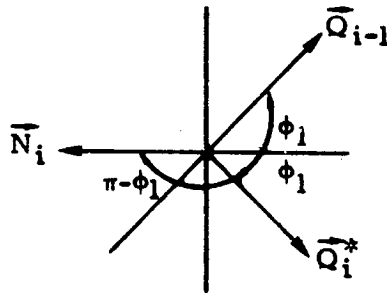


Figure 16.  $\vec{Q}_i^*$

Since  $-\vec{Q}_i^* \cdot \vec{N}_i = \cos \phi_1$  and  $-\vec{Q}_i^* \cdot \vec{Q}_{i-1} = -\cos 2\phi_1$  it follows that

$$\vec{Q}_i = -\vec{Q}_i^*. \quad (3.105)$$

Consequently, reflection is a special case of refraction which is specified by setting  $n_{i+1} = -n_i$ . The resultant direction is  $-\vec{Q}_i$  where  $\vec{Q}_i$  is the direction of the refracted ray.

## SECTION 4

### SYSTEM CHARACTERISTICS

The procedures outlined below for computing such things as focal length were abstracted from Lehman's program', and no attempt will be made to reconcile the technique with the usual definition of the parameter being computed.

#### 4.1 FOCAL POINT AND FOCAL LENGTH

A focal point  $F_p$  and a focal length  $F_L$  are determined for each of the  $N_c$  colors by tracing a paraxial ray with entrance pupil coordinates  $(\hat{x}_0, \hat{y}_0) = (0, \Omega_2)$  from the object point  $(h, \theta) = (\rho_0 \Omega_2, \pi)$  where  $\Omega_2$  is input.

From Section 2., the initial position  $P_0$  and the initial direction  $\vec{Q}_0$  of the ray are:

$$P_0 = (0, \rho_0 \Omega_2, 0),$$

and 
$$\vec{Q}_0 = (0, 0, 1).$$

Let this ray be traced to a point  $P_1 = (x_1, y_1, z_1)$  on  $\sigma_N$  and let

$\vec{Q}_1 = (Q_{x1}, Q_{y1}, Q_{z1})$  be the direction of the ray following refraction at  $\sigma_N$ .

If, during the ray trace, the ray misses a surface or is internally reflected at a surface the program should print an appropriate message containing the surface number and then terminate the run. If the ray vignets at some surface it will be ignored.

If  $Q_{y1} \neq 0$  the ray intersects the  $x - z$  plane at a point  $(x_2, 0, z_2)$  where:

$$x_2 = -\frac{Q_{x1}}{Q_{y1}} y_1 + x_1 \quad (4.001)$$

$$z_2 = -\frac{Q_{z1}}{Q_{y1}} y_1 + z_1. \quad (4.002)$$

The focal point is, by definition,

$$F_p = \begin{cases} z_2 & Q_{y1} \neq 0 \text{ and } z_2 \neq 0 \\ \infty & Q_{y1} = 0 \text{ or } z_2 = 0 \end{cases} \quad (4.003)$$

and is illustrated in Figure 17.

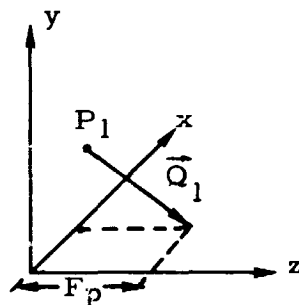


Figure 17. Focal Point



The focal length is given by:

$$F_L = \begin{cases} z_3 & Q_{y1} \neq 0 \quad n_{N+1} > 0 \quad z_3 \neq 0 \\ -z_3 & Q_{y1} \neq 0 \quad n_{N+1} < 0 \quad z_3 \neq 0 \\ \infty & Q_{y1} = 0 \quad \text{or} \quad z_3 = 0 \end{cases} \quad (4.004)$$

where  $n_{N+1}$  is the index of refraction of the material following  $\sigma_N$ , and

$$z_3 = -\frac{Q_{z1}}{Q_{y1}} \rho_0 \Omega_2. \quad (4.005)$$

If L is a line through the point  $(0, \rho_0 \Omega_2, 0)$  with direction  $\vec{Q}_1$ , then  $z_3$  is the  $z$  coordinate of the point where L intersects the  $x - z$  plane (see Figure 18).

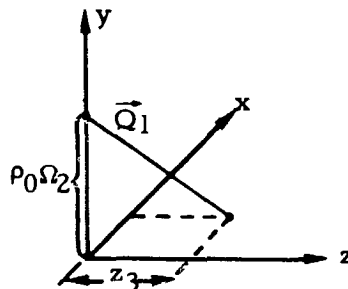


Figure 18. Focal Length

The sign change in (4.004) when  $n_{N+1} < 0$  is required to maintain compatibility with the focal length sign convention for a reflective element.

#### 4.2 BACK FOCUS AND $f/$ NUMBER

Back focus  $B_F$  and  $f$  number  $f/$  are determined for each of the  $N_c$  colors by tracing a meridional ray with entrance pupil coordinates  $(\hat{x}_0, \hat{y}_0) = (0, \Omega_1)$  from the object point  $(h, \theta) = (0, 0)$  where  $\Omega_1$  is input.

From Section 2. the initial position  $P_0$  and the initial direction  $\vec{Q}_0$  of the ray are:

$$P_0 = (0, \rho_0 \Omega_1, 0),$$

and 
$$\vec{Q}_0 = (0, Q_{y0}, Q_{z0})$$

where 
$$Q_{y0} = \frac{\rho_0 \Omega_1}{\Delta}$$

$$Q_{z0} = \frac{S'}{\Delta}$$

$$\Delta = \left[ (S')^2 + (\rho_0 \Omega_1)^2 \right]^{\frac{1}{2}}$$

Let  $P_1 = (x_1, y_1, z_1)$  be the point where the ray intersects  $\sigma_N$  and  $\vec{Q}_1 = (Q_{x1}, Q_{y1}, Q_{z1})$  its direction following refraction. If the ray misses a surface or is internally reflected the program should print an appropriate message including the surface number and terminate the run. Vignetting is ignored.

If  $Q_{y1} \neq 0$  the ray intersects the  $x - z$  plane at a point  $P_2 = (x_2, 0, z_2)$  where  $x_2$  and  $z_2$  are given by equations (4.001 and 4.002).

Back focus is, by definition,

$$B_F = \begin{cases} z_2 & Q_{y1} \neq 0 \text{ and } z_2 \neq 0 \\ \infty & Q_{y1} = 0 \text{ or } z_2 = 0 \end{cases} \quad (4.006)$$

Let  $L$  be a line through the point  $(0, \Omega_1, 0)$  with direction  $\bar{Q}_1$ . If  $z_3$  is the coordinate of the point where  $L$  intersects the  $x - z$  plane then:

$$z_3 = -\frac{Q_{z1}}{Q_{y1}} \Omega_1. \quad (4.007)$$

$f$  number is, by definition,

$$f/ = \begin{cases} \frac{1}{2} |z_3| & Q_{y1} \neq 0 \text{ and } z_3 \neq 0 \\ \infty & Q_{y1} = 0 \text{ or } z_3 = 0 \end{cases} \quad (4.008)$$

### 4.3 EXIT PUPIL LOCATION

Exit pupil location  $E_p$  is determined for every different combination of color and non-zero object height. Let  $(i, h_j)$  represent some combination of color ( $1 \leq i \leq N_c$ ) and object height  $h_j$  where  $1 \leq j \leq N_H$  and  $h_j > 0$ .  $E_p$  is computed by tracing a ray with entrance pupil coordinates  $(\hat{x}_0, \hat{y}_0) = (0, 0)$  from the object point  $(h, \theta) = (h_j, 0)$ .

From Section 2. the initial position  $P_0$  and the initial direction  $\vec{Q}_0$  are:

$$P_0 = (0, 0, 0)$$

and 
$$\vec{Q}_0 = (0, Q_{y0}, Q_{z0}),$$

where 
$$Q_{y0} = \frac{h_j}{\Delta}$$

$$Q_{z0} = \frac{S'}{\Delta}$$

$$\Delta = \left[ (S')^2 + (h_j)^2 \right]^{\frac{1}{2}}$$

Let  $P_1 = (x_1, y_1, z_1)$  be the point of intersection on  $\sigma_N$  and  $\vec{Q}_1 = (Q_{x1}, Q_{y1}, Q_{z1})$  be the direction of the ray following refraction at  $\sigma_N$ . If the ray misses a surface, is internally reflected, or  $Q_{y1} = 0$  then  $E_p = \infty$ . Vignetting is ignored.

If  $Q_{y1} \neq 0$  the ray intersects the  $x - z$  plane at a point  $P_2 = (x_2, 0, z_2)$  where  $x_2$  and  $z_2$  are given by equations (4.001) and (4.002).

Exit pupil location is then given by:

$$E_p = \begin{cases} z_2 & Q_{y1} \neq 0 \text{ and } z_2 \neq 0 \\ \infty & Q_{y1} = 0 \text{ or } z_2 = 0. \end{cases} \quad (4.009)$$

Section 5

TWINRAY DIAGNOSTICS

The following definitions are pertinent to the discussion:

$C_i$  - Cartesian coordinate system whose origin is at the vertex of  $\sigma_i$ . ( $0 \leq i \leq N$ )

$S_i$  - Directed distance from vertex of  $\sigma_{i-1}$  to vertex of  $\sigma_i$ . ( $1 \leq i \leq N$ )

$n_i$  - Index of refraction between  $\sigma_{i-1}$  and  $\sigma_i$ . ( $1 \leq i \leq N + 1$ )

$P_i^k$  - Point where ray  $k$  intersects  $\sigma_i$  with coordinates  $(x_i^k, y_i^k, z_i^k)$  relative to  $C_i$ .

$\vec{Q}_i^k$  - Unit vector with components  $(Q_{xi}^k, Q_{yi}^k, Q_{zi}^k)$  which is along ray  $k$  as it leaves  $\sigma_i$ .

$\tau_i^k$  - Thickness associated with surfaces  $\sigma_i$  and  $\sigma_{i+1}$ . ( $0 \leq i \leq N$ ).

5.1 ZERO OBJECT HEIGHT

Consider two rays,  $R_1$  and  $R_2$ , both of which originate at the object point  $(h, \theta) = (0., 0.)$  where  $R_1$  has entrance pupil coordinates  $(\hat{x}_0, \hat{y}_0) = (0., \Omega_2)$  and  $R_2$  has entrance pupil coordinates  $(0., 1.)$ . It follows from Section 2 that

$$P_o^1 = (0., \rho_o, \Omega_2, 0.)$$

and

$$P_o^2 = (0., \rho_o, 0.).$$

Let  $R_k$  be traced through the lens system to produce  $\{(P_i^k, \vec{\Omega}_i^k); i=0, 1, \dots, N_k\}$ . If  $R_k$  misses or is internally reflected at surface  $M_k$  where  $1 \leq M_k \leq N$  then an appropriate comment should be printed including  $k$  and  $M_k$ , and  $N_k = M_k - 1$ . If  $R_k$  vignets at surface  $M_k$  a comment should be printed and  $N_k = N$ . If  $R_k$  reaches  $\sigma_n$  without vignetting then  $N_k = N$ .

Compute  $\{\lambda_i^k; i=0, 1, \dots, N_k\}$  where:

$$\lambda_i^k = \begin{cases} z_{i+1}^k - (z_i^k - S_{i+1}) & 0 \leq i \leq N_k - 1 \\ -z_{N_k}^k & i = N_k. \end{cases} \quad (5.001)$$

It follows from Figure 19 that  $\lambda_i^k$  is the distance measured parallel to the optic axis between  $P_i^k$  and  $P_{i+1}^k$ .

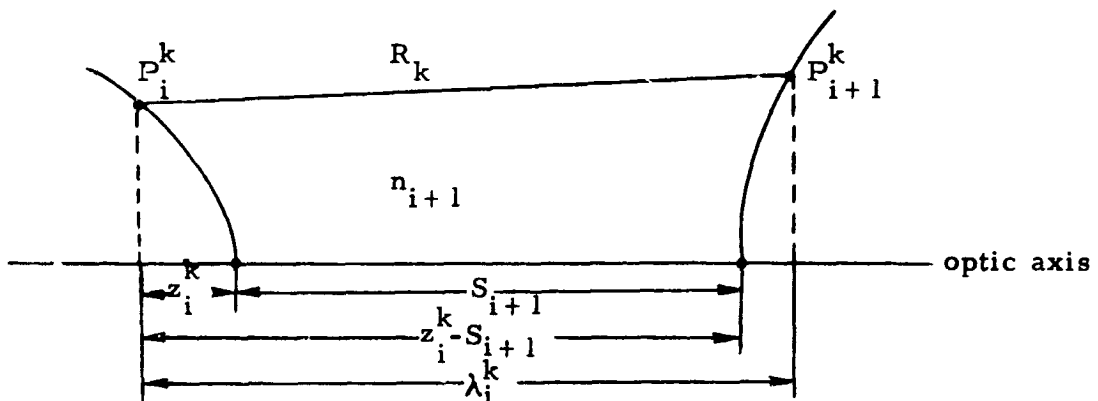


Figure 19 Feathering Condition

Compute thickness set  $\{\tau_i^k; i=0, 1, \dots, N_k\}$  where:

$$\tau_i^k = \frac{n_{i+1}}{|n_{i+1}|} \lambda_i^k \quad (5.002)$$

The data  $\{(P_i^k, \vec{Q}_i^k, \tau_i^k); 0 \leq i \leq N_k; k=1, 2\}$  should be printed with appropriate headings.

## 5.2 MAXIMUM OBJECT HEIGHT

Consider two rim rays,  $R_1$  and  $R_2$ , both of which originate at the object point  $(h, \theta) = (h_{\max}, 0.)$  where  $h_{\max} = h_o + (N_H - 1)\Delta h$ .  $R_1$  has entrance pupil coordinates  $(\hat{x}_o, \hat{y}_o) = (0., -1.)$  and  $R_2$  has entrance pupil coordinates  $(\hat{x}_o, \hat{y}_o) = (0., 1.)$ . From Section 2, it follows that:

$$P_o^1 = (0., -\rho_o, 0.)$$

and  $P_o^2 = (0., \rho_o, 0.)$

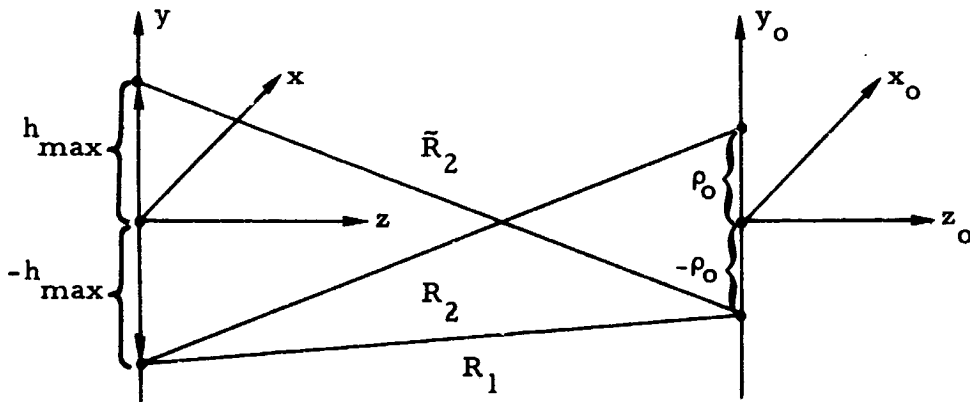


Figure 20. Rim Rays

Ray  $\tilde{R}_2$  is referred to as the conjugate of  $R_2$ . The position and direction of  $\tilde{R}_2$  at  $\sigma_i$  is designated by  $\tilde{P}_i^2 = (\tilde{x}_i^2, \tilde{y}_i^2, \tilde{z}_i^2)$  and  $\tilde{Q}_i^2 = (\tilde{Q}_{xi}^2, \tilde{Q}_{yi}^2, \tilde{Q}_{zi}^2)$  where  $0 \leq i \leq N$ . The relationship between  $R_2$  and  $\tilde{R}_2$  is initially given by:

$$\tilde{x}_o^2 = x_o^2 = 0.$$

$$\tilde{y}_o^2 = -y_o^2$$

$$\tilde{z}_o^2 = z_o^2$$

$$\tilde{Q}_{xo}^2 = Q_{xo}^2 = 0.$$

$$\tilde{Q}_{yo}^2 = -Q_{yo}^2$$

$$\tilde{Q}_{zo}^2 = Q_{zo}^2$$

Trace  $R_1$  and  $R_2$  through the lens system and compute thickness (see Section 5.2) to yield:

$$\{(P_i^1, \vec{Q}_i^1, \tau_i^1); i = 0, 1, \dots, N_1\}$$

and

$$\{(P_i^2, \vec{Q}_i^2, \tau_i^2); i = 0, 1, \dots, N_2\}.$$

The next step is to compute  $\{(A_{Li}, A_{Ri}); i = 0, 1, \dots, N\}$  where  $A_{Li}$  is aperture location and  $A_{Ri}$  is aperture radius associated with  $\sigma_i$ .



Let  $M = \text{Min}(N_1, N_2)$ . The following computations are performed for  $0 \leq i \leq M$ . If  $Q_{zi}^1 = 0$ . or  $Q_{zi}^2 = 0$ . set

$$A_{Li} = 0.$$

and

$$A_{Ri} = -1.$$

If  $Q_{zi}^1 \neq 0$  and  $Q_{zi}^2 \neq 0$  compute:

$$m_i^1 = \frac{Q_{yi}^1}{Q_{zi}^1} \quad (5.003)$$

$$m_i^2 = \frac{Q_{yi}^2}{Q_{zi}^2} \quad (5.004)$$

If  $m_i^1 + m_i^2 = 0$  set  $A_{Li} = 0$  . and  $A_{Ri} = -1$ . If  $m_i^1 + m_i^2 \neq 0$  then:

$$A_{Li} = \frac{m_i^2 z_i^2 - y_i^2 + m_i^1 z_i^1 - y_i^1}{m_i^1 + m_i^2} \quad (5.005)$$

and

$$A_{Ri} = |m_i^1 (A_{Li} - z_i^1) + y_i^1| \quad (5.006)$$

If  $M < N$  then for  $M < i \leq N$  set  $A_{Li} = 0$ . and  $A_{Ri} = -1$ .

In an attempt to interpret  $(A_{Li}, A_{Ri})$  geometrically suppose that the system contains only standard conics and aspherics. The normal

vector at any point  $(0, y, z)$  in the  $y$ - $z$  plane at any of the  $\sigma_i$  has no  $x$  component. It follows that any ray starting out in the  $y$ - $z$  plane is confined to that plane as it travels through the lens system, hence:

$$\begin{aligned} P_i^1 &= (0., y_i^1, z_i^1) & \vec{Q}_i^1 &= (0., Q_{yi}^1, Q_{zi}^1) \\ P_i^2 &= (0., y_i^2, z_i^2) & \vec{Q}_i^2 &= (0., Q_{yi}^2, Q_{zi}^2) \end{aligned}$$

for  $i = 0, 1, \dots, N$ .

A second consequence of the assumed symmetry of the system in conjunction with the law of refraction is that the initial relation between  $\tilde{R}_2$  and  $R_2$  is preserved at every surface, that is:

$$\begin{aligned} \tilde{P}_i^2 &= (\tilde{x}_i^2, \tilde{y}_i^2, \tilde{z}_i^2) \equiv (0., -y_i^2, z_i^2) \\ \vec{\tilde{Q}}_i^2 &= (\tilde{Q}_{xi}^2, \tilde{Q}_{yi}^2, \tilde{Q}_{zi}^2) = (0., -Q_{yi}^2, Q_{zi}^2) \end{aligned} \tag{5.007}$$

Let  $L_1$  be the line through  $P_i^1$  with direction  $\vec{Q}_i^1$  and  $\tilde{L}_2$  be the line through  $\tilde{P}_i^2$  with direction  $\vec{\tilde{Q}}_i^2$ .  $L_1$  and  $\tilde{L}_2$  intersect at a point,  $(y_I, z_I)$ , which is the simultaneous solution of:

$$y = m_i^1 (z - z_i^1) + y_i^1 \tag{L_1} \tag{5.008}$$

and

$$y = -m_i^2 (z - z_i^2) - y_i^2. \tag{\tilde{L}_2} \tag{5.009}$$

Solving for  $(y_I, z_I)$ :

$$z_I = \frac{m_i^2 z_i^2 - y_i^2 + m_i^1 z_i^1 - y_i^1}{m_i^1 + m_i^2} \quad (5.010)$$

$$y_I = m_i^1 (z_1 - z_i^1) + y_i^1 \quad (5.011)$$

It follows that:

$$A_{Li} = z_I$$

and  $A_{Ri} = |y_I|$

Therefore,  $(A_{Li}, A_{Ri})$  are in essence the coordinates of the point of intersection of  $L_1$  and  $\tilde{L}_2$ , the situation being illustrated in Figure 21.

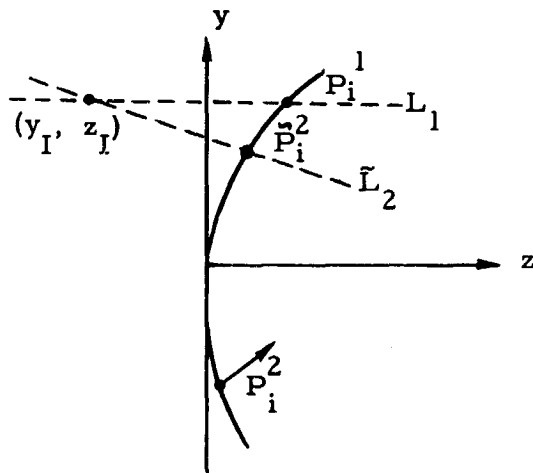


Figure 21. Aperture Location and Radius

If this system contains a translated or tilted element it is not clear how  $(A_{Li}, A_{Ri})$  should be interpreted.

The data  $\{(P_i^1, \vec{Q}_i^1, \tau_i^1); i = 0, 1, \dots, N_1\}$ ,  
 $\{(P_i^2, \vec{Q}_i^2, \tau_i^2); i = 0, 1, \dots, N_2\}$  and  $\{(A_{Li}, A_{Ri}); i = 0, 1, \dots, N\}$   
should be printed with appropriate headings. If  $A_{Ri} = -1$  do not print  $(A_{Li}, A_{Ri})$ .

Section 6

SPOT PATTERN STATISTICS

Let  $i$  and  $(h_1, \theta)$  be an arbitrary combination of color and object point where  $1 \leq i \leq N_C$  and  $1 \leq j \leq N_H$ . The ray pattern specified in Section 2 is traced through the lens system to the final surface  $\sigma_N$ . The total number of rays,  $M_R$ , that are traced is

$$M_R = \begin{cases} N_R & \text{LMODE} \neq 0 \\ 2N_R & \text{LMODE} = 0. \end{cases} \quad (6.001)$$

From this pattern of  $M_R$  rays let:

$N_M$  = Number of rays which missed a surface

$N_{RF}$  = Number of internally reflected rays

$N_V$  = Number of rays which vignetted

$\bar{N}_R$  = Number of successfully traced rays

If vignetting is counted as a failure then:

$$\bar{N}_R = M_R - (N_M + N_{RF} + N_V). \quad (6.002)$$

Let  $P_i^0 = (x_i^0, y_i^0, z_i^0)$  be the coordinates of the  $i$ th ray on  $\sigma_N$  and  $\vec{Q}_i = (Q_{xi}, Q_{yi}, Q_{zi})$  be its direction following refraction at  $\sigma_N$  where  $1 \leq i \leq N_R$ . It is assumed that  $Q_{zi} \neq 0$ , for when it is, the ray is counted as an internally reflected ray.

Let  $\lambda$  represent the signed axial distance from the vertex of  $\sigma_N$  to some arbitrary image plane which is perpendicular to the optic axis. The point where the  $i$ th ray intersects this plane has coordinates  $(x_i, y_i)$  given by

$$x_i = \frac{Q_{xi}}{Q_{zi}} (\lambda - z_i^o) + x_i^o \quad (6.003)$$

$$y_i = \frac{Q_{yi}}{Q_{zi}} (\lambda - z_i^o) + y_i^o \quad (6.004)$$

Let:

$$m_{xi} = \frac{Q_{xi}}{Q_{zi}} \quad (6.005)$$

$$k_{xi} = x_i^o - m_{xi} z_i^o \quad (6.006)$$

$$m_{yi} = \frac{Q_{yi}}{Q_{zi}} \quad (6.007)$$

$$k_{yi} = y_i^o - m_{yi} z_i^o \quad (6.008)$$

Then:

$$x_i(\lambda) = m_{xi} \lambda + k_{xi} \quad (6.009)$$

$$y_i(\lambda) = m_{yi} \lambda + k_{yi} \quad (6.010)$$

The pattern of points  $\{(x_i, y_i); i = 1, 2, \dots, \bar{N}_R\}$  describes a monochromatic spot on the image whose center is, by definition, the point  $(\bar{x}, \bar{y})$  given by:

$$\bar{x}(\lambda) = \begin{cases} \frac{1}{\bar{N}_R} \sum_{i=1}^{\bar{N}_R} x_i(\lambda) & \text{LMODE} \neq 1 \\ 0. & \text{LMODE} = 1 \end{cases} \quad (6.011)$$

$$\bar{y}(\lambda) = \frac{1}{\bar{N}_R} \sum_{i=1}^{\bar{N}_R} y_i(\lambda) \quad (6.012)$$

Let:

$$\bar{m}_x = \begin{cases} \frac{1}{\bar{N}_R} \sum_{i=1}^{\bar{N}_R} m_{xi} & \text{LMODE} \neq 1 \\ 0. & \text{LMODE} = 1 \end{cases} \quad (6.013)$$

$$\bar{k}_x = \begin{cases} \frac{1}{\bar{N}_R} \sum_{i=1}^{\bar{N}_R} k_{xi} & \text{LMODE} \neq 1 \\ 0. & \text{LMODE} = 1 \end{cases} \quad (6.014)$$

$$\bar{m}_y = \frac{1}{\bar{N}_R} \sum_{i=1}^{\bar{N}_R} m_{yi} \quad (6.015)$$

$$\bar{k}_y = \frac{1}{\bar{N}_R} \sum_{i=1}^{\bar{N}_R} k_{yi} \quad (6.016)$$

Then:

$$\bar{x}(\lambda) = \bar{m}_x \lambda + \bar{k}_x \quad (6.017)$$

$$\bar{y}(\lambda) = \bar{m}_y \lambda + \bar{k}_y \quad (6.018)$$

The monochromatic spot may also be described in a coordinate system whose origin is at the center  $(\bar{x}, \bar{y})$  giving rise to the point set  $\{(\tilde{x}_i, \tilde{y}_i); i = 1, 2, \dots, \bar{N}_R\}$  where:

$$\tilde{x}_i(\lambda) = x_i(\lambda) - \bar{x}(\lambda) \quad (6.019)$$

$$\tilde{y}_i(\lambda) = y_i(\lambda) - \bar{y}(\lambda) \quad (6.020)$$

Let:

$$\tilde{m}_{xi} = m_{xi} - \bar{m}_x \quad (6.021)$$

$$\tilde{k}_{xi} = k_{xi} - \bar{k}_x \quad (6.022)$$

$$\tilde{m}_{yi} = m_{yi} - \bar{m}_y \quad (6.023)$$

$$\tilde{k}_{yi} = k_{yi} - \bar{k}_y \quad (6.024)$$



Then:

$$\tilde{x}_i(\lambda) = \tilde{m}_{xi} \lambda + \tilde{k}_{xi} \quad (6.025)$$

$$\tilde{y}_i(\lambda) = \tilde{m}_{yi} \lambda + \tilde{k}_{yi} \quad (6.026)$$

Let  $x^*(\lambda)$ ,  $y^*(\lambda)$ , and  $R^*(\lambda)$  represent RMS x, RMS y, and spot size, respectively. By definition:

$$x^*(\lambda) = \left\{ \frac{1}{\bar{N}_R} \sum_{i=1}^{\bar{N}_R} \tilde{x}_i^2(\lambda) \right\}^{\frac{1}{2}} \quad (6.027)$$

$$y^*(\lambda) = \left\{ \frac{1}{\bar{N}_R} \sum_{i=1}^{\bar{N}_R} \tilde{y}_i^2(\lambda) \right\}^{\frac{1}{2}} \quad (6.028)$$

$$R^*(\lambda) = \left\{ [x^*(\lambda)]^2 + [y^*(\lambda)]^2 \right\}^{\frac{1}{2}} \quad (6.029)$$

Let:

$$A_x = \frac{1}{\bar{N}_R} \sum_{i=1}^{\bar{N}_R} \tilde{m}_{xi}^2 \quad (6.030)$$

$$B_x = \frac{1}{\bar{N}_R} \sum_{i=1}^{\bar{N}_R} \tilde{m}_{xi} \tilde{k}_{xi} \quad (6.031)$$

$$C_x = \frac{1}{\bar{N}_R} \sum_{i=1}^{\bar{N}_R} \tilde{k}_{xi}^2 \quad (6.032)$$

$$A_y = \frac{1}{\bar{N}_R} \sum_{i=1}^{\bar{N}_R} \tilde{m}_{yi}^2 \quad (6.033)$$

$$B_y = \frac{1}{\bar{N}_R} \sum_{i=1}^{\bar{N}_R} \tilde{m}_{yi} \tilde{k}_{yi} \quad (6.034)$$

$$C_y = \frac{1}{\bar{N}_R} \sum_{i=1}^{\bar{N}_R} \tilde{k}_{yi}^2 \quad (6.035)$$

$$A = A_x + A_y \quad (6.036)$$

$$B = B_x + B_y \quad (6.037)$$

$$C = C_x + C_y \quad (6.038)$$

Then:

$$x^*(\lambda) = \left\{ A_x \lambda^2 + 2B_x \lambda + C_x \right\}^{\frac{1}{2}} \quad (6.039)$$

$$y^*(\lambda) = \left\{ A_y \lambda^2 + 2B_y \lambda + C_y \right\}^{\frac{1}{2}} \quad (6.040)$$

$$R^*(\lambda) = \left\{ A \lambda^2 + 2B \lambda + C \right\}^{\frac{1}{2}} \quad (6.041)$$

Let  $\lambda_x$  denote that value of  $\lambda$  for which  $x^*(\lambda)$  is a minimum and define  $\lambda_y$  and  $\lambda_R$  analogously with respect to  $y^*(\lambda)$  and  $R^*(\lambda)$ .

It follows directly from equations (6.039), (6.040), and (6.041) that:

$$\lambda_x = - \frac{B_x}{A_x} \quad (6.042)$$

$$x^* (\lambda_x) = \left\{ \frac{A_x C_x - B_x^2}{A_x} \right\}^{\frac{1}{2}} \quad (6.043)$$

$$\lambda_y = - \frac{B_y}{A_y} \quad (6.044)$$

$$y^* (\lambda_y) = \left\{ \frac{A_y C_y - B_y^2}{A_y} \right\}^{\frac{1}{2}} \quad (6.045)$$

$$\lambda_R = - \frac{B}{A} \quad (6.046)$$

$$R^* (\lambda_R) = \left\{ \frac{AC - B^2}{A} \right\}^{\frac{1}{2}} \quad (6.047)$$

Let  $h^i (\lambda)$  denote the image height associated with the point set  $\{(x_i, y_i); i = 1, 2, \dots, \bar{N}_R\}$ . It follows from equation (1.041) of Section 1.3 that:

$$h^i (\lambda) = \bar{x} (\lambda) \text{Sin } \theta + \bar{y} (\lambda) \text{Cos } \theta \quad (6.048)$$

where  $\theta$  is the azimuth of the object.

The principal plane associated with the object point  $(h_j, \theta)$  is positioned at  $\lambda = S_j^*$  where

$$S_j^* = \begin{cases} S^* + d_j & \text{IMODE} = 1 \\ S^* + B_{F1} + d_j & \text{IMODE} = 0 \end{cases} \quad (6.049)$$

$S^*$  and  $d_j$  are input and  $B_{F1}$  is the computed back focus for color 1.

There are 4 segments of the program which utilize data from tracing the ray pattern:

- 1) Optics Diagnostics
- 2) Design Computations
- 3) Spot Diagram Point Plotting
- 4) Spot Size Sensitivity

Except during design computations, a vignetted ray is always counted as a failure.

For designing and plotting the following parameters are required:

- 1)  $N_M, N_{RF}, N_V, \bar{N}_R$
- 2)  $\bar{x}(S_j^*), \bar{y}(S_j^*)$
- 3)  $\{\tilde{x}_i(S_j^*), \tilde{y}_i(S_j^*); i = 1, 2, \dots, \bar{N}_R\}$

For optics diagnostics the required data is:

- 1)  $N_M, N_{RF}, N_V, \bar{N}_R$

2)  $\bar{m}_x, \bar{k}_x, \bar{m}_y, \bar{k}_y$

3)  $A_x, B_x, C_x, A_y, B_y, C_y, A, B, C$

For sensitivity analysis the same information is required as for designing plus  $x^*(S_j^*)$ ,  $y^*(S_j^*)$ , and  $R^*(S_j^*)$ .

## SECTION 7

### SENSITIVITY COMPUTATIONS

The purpose of the sensitivity analysis is to determine the effect on rms  $x$ , spot size etc. when some specified parameter is perturbed by a given percentage.

Let  $T = \{t_i; i = 1, 2, \dots, Mp\}$  denote a set of parameters such that  $1 \leq Mp \leq 30$ , the members of  $T$  being identified by means of input symbols. Parameters are such things as element separation, radius of curvature, index of refraction etc. Let  $P = \{p_i; i = 1, 2, \dots, Mp\}$  denote a set of (input) percentage values in one-to-one correspondence with the  $t_i$ . Let  $T^0 = \{t_i^0; i = 1, 2, \dots, Mp\}$  and  $T^1 = \{t_i^1; i = 1, 2, \dots, Mp\}$  denote sets such that  $t_i^0$  is the nominal value of  $t_i$  and  $t_i^1$  the perturbed value where:

$$t_i^1 = \begin{cases} .01 p_i & t_i^0 = 0 \\ t_i^0 (1 + .01 p_i) & t_i^0 \neq 0 \end{cases} \quad (7.001)$$

Let  $\Delta T = \{\Delta t_i; i = 1, 2, \dots, Mp\}$  be a parameter increment set such that:

$$\Delta t_i = t_i^1 - t_i^0. \quad (7.002)$$

For some arbitrary combination of object point  $(h_j, \theta)$  and color  $i$  let  $V$  denote the set of statistical data  $\{\bar{x}, \bar{y}, x^*, y^*, R^*\}$ , the elements being defined in Section 6. If  $v_k$  is some element of  $V$  it follows that  $v_k = v_k(t_1, t_2, \dots, t_{Mp})$ . Define  $v_k^0$  to be the nominal value of  $v_k$  and  $v_k^i$  the value when  $t_i$  is changed from  $t_i^0$  to  $t_i^1$ , that is,

$$v_k^0 \equiv v_k(t_1^0, t_2^0, \dots, t_i^0, \dots, t_{Mp}^0) \quad (7.003)$$

$$v_k^i \equiv v_k(t_1^0, t_2^0, \dots, t_i^1, \dots, t_{Mp}^0) \quad (7.004)$$

The absolute change  $\Delta v_k^i$  and the percentage change (sensitivity)  $\sigma_k^i$  are given by:

$$\Delta v_k^i = v_k^i - v_k^0 \quad (7.005)$$

$$\sigma_k^i = \begin{cases} 0. & v_k^0 = 0 \\ (100 \Delta v_k^i) / v_k^0 & v_k^0 \neq 0 \end{cases} \quad (7.006)$$

It should be noted that  $v_k^0$  depends only on object point and color and hence should not be re-computed for the several values of parameter number  $i$ .

This is accomplished by generating a 3 dimensional matrix of nominal statistical data, prior to the parameter variation, in such a way that layer  $j$  contains the information for all colors and the  $j$ th object point. For fixed  $j$  let  $\{\bar{x}_i, \bar{y}_i, x_i^*, y_i^*, R_i^*\}$  denote the statistical data for color  $i$ . Layer  $j$  of the matrix contains the following nominal data.

$$6 \times 9 \times 7 \begin{bmatrix} \bar{x}_1 & \bar{y}_1 & x_1^* & y_1^* & R_1^* & \bar{N}_{R_1} & N_{M_1} & N_{RF_1} & N_{V_1} \\ \bar{x}_2 & \bar{y}_2 & x_2^* & y_2^* & R_2^* & \bar{N}_{R_2} & N_{M_2} & N_{RF_2} & N_{V_2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \bar{x}_{N_c} & \bar{y}_{N_c} & x_{N_c}^* & y_{N_c}^* & R_{N_c}^* & \bar{N}_{RN_c} & N_{MN_c} & N_{RFN_c} & N_{VN_c} \end{bmatrix}$$

All of the data in this layer is computed on the principal image plane whose position  $S_j^*$  is defined by equation (6.049).

Consider some arbitrary parameter  $t_k$  and arbitrary object point  $(h_j, \theta)$ . Suppose that  $t_k$  has been perturbed, i.e.,  $t_k = t_k^1$ . The perturbed statistical data for the  $i$ th color is, from Section 6.

$$\bar{x}_i(\lambda) = \bar{m}_{xi} \lambda + \bar{k}_{xi} \quad (7.007)$$

$$\bar{y}_i(\lambda) = \bar{m}_{yi} \lambda + \bar{k}_{yi} \quad (7.008)$$

$$x_i(\lambda) = \{A_{xi} \lambda^2 + 2B_{xi} \lambda + C_{xi}\}^{1/2} \quad (7.009)$$

$$y_i(\lambda) = \{A_{yi} \lambda^2 + 2B_{yi} \lambda + C_{yi}\}^{1/2} \quad (7.010)$$

$$R_i(\lambda) = \{A_i \lambda^2 + 2B_i \lambda + C_i\}^{1/2} \quad (7.011)$$

where  $\lambda$  represents the position of the image plane upon which the statistical variables are to be evaluated. Let  $G$  be the  $N_c \times 17$  matrix whose  $i$ th row consists of the following data:

$$\bar{m}_{xi}, \bar{k}_{xi}, \bar{m}_{yi}, \bar{k}_{yi}, A_{xi}, B_{xi}, C_{xi}, A_{yi}, B_{yi}, C_{yi}, \\ A_i, B_i, C_i, \bar{N}_{R_i}, N_{M_i}, N_{RF_i}, N_{Vi}$$

Ordinarily the perturbed statistical data and corresponding sensitivities are evaluated only on the principal plane of the perturbed system. However, on option (REFOCS>0), the computations are also performed on a "refocused" plane whose position  $\lambda_F$  is computed as follows:



a) REFOCS = 1

The refocused plane is that plane upon which rms  $x$  for all colors taken together is a minimum. Define an overall rms  $x$  as follows:

$$[x^*(\lambda)]^2 = \sum_{i=1}^{N_c} [x_i^*(\lambda)]^2 \quad (7.012)$$

Making use of equation (7.009),

$$x^*(\lambda) = \{A_x \lambda^2 + 2B_x \lambda + C_x\}^{1/2} \quad (7.013)$$

where:

$$A_x = \sum_{i=1}^{N_c} A_{xi} \quad (7.014)$$

$$B_x = \sum_{i=1}^{N_c} B_{xi} \quad (7.015)$$

$$C_x = \sum_{i=1}^{N_c} C_{xi} \quad (7.016)$$

It is assumed that  $A_{xi} = B_{xi} = C_{xi} = 0$  if  $\bar{N}_{Ri} = 0$ .

Equating the derivative of (7.013) to zero and solving for  $\lambda$  gives:

$$\lambda_F = -\frac{B_x}{A_x} \quad (7.017)$$

b) REFOCS = 2

The refocused plane is that plane upon which rms  $y$  for all colors taken together is a minimum. It follows immediately that:

$$\lambda_F = - \frac{B_y}{A_y} \quad (7.018)$$

$$B_y = \sum_{i=1}^{N_c} B_{yi} \quad (7.019)$$

$$A_y = \sum_{i=1}^{N_c} A_{yi} \quad (7.020)$$

c) REFOCS = 3

The refocused plane is that plane upon which spot size for all colors taken together is a minimum.

$$\lambda_F = - \frac{B}{A} \quad (7.021)$$

$$B = \sum_{i=1}^{N_c} B_i \quad (7.022)$$

$$A = \sum_{i=1}^{N_c} A_i \quad (7.023)$$

## Section 8

### LENS PERFORMANCE OPTIMIZATION

By lens designing is meant a technique for automatically altering certain members of some specified set of lens system parameters in such a way that the performance of the system is improved.

A computerized procedure for lens designing introduces two requirements. The first requirement is to define a merit function  $\varphi$  whose magnitude is a measure of the performance. The definition of  $\varphi$  must be such that any process which reduces the magnitude of  $\varphi$  results in improved performance. The second requirement is then to devise a numerical process for iteratively decreasing the magnitude of  $\varphi$ .

#### 8.1 MERIT FUNCTION

The basic criterion employed in the definition of  $\varphi$  is that a set of monochromatic rays originating at a single object point should converge to the same image point after passing through the lens system. If  $\{(x_i, y_i); i = 1, 2, \dots, N\}$  are the image coordinates produced from tracing  $N$  rays and  $(\bar{x}, \bar{y})$  is the mean, then the deviation from the mean for the  $i$ th point has two components,  $x_i - \bar{x}$  and  $y_i - \bar{y}$ . The complete set of weighted deviations become independent components of an error vector  $\vec{E}$  whose length is taken as the merit function.

$$\varphi = \vec{E} \cdot \vec{E} \quad (8.001)$$

Because there are  $N_H$  different object points and  $N_C$  colors to be considered simultaneously as well as such things as chromatic aberration and magnification, the structure of the weighted error vector will be developed in detail.

Definitions

$N_R$	Number of specified rays in pattern.
$M_R$	Number of rays that are traced.
$N_H$	Number of object points.
$N_C$	Number of colors.
$\vec{E}$	Error Vector ( $e_1 e_2 \dots e_M$ ).
$M$	Order of $\vec{E}$
$f^{(R)}$	Required $f/$ number.
$F_L^{(R)}$	Required focal length.
$W^*$	Weight applied to focal length deviation.
$E_p^{(R)}$	Required exit pupil position.
$h_E$	Object height used to compute $E_p$ .
$W^{**}$	Weight applied to exit pupil deviation.
$W_{sj}$	Weights which reflect the importance of the different object points ( $1 \leq j \leq N_H$ )

Definitions

$W_{ci}$	Weights which reflect the importance of the different colors. ( $1 \leq i \leq N_c$ )
$W_x$	Weight on x-direction.
$W_y$	Weight on y-direction.
$W_{Aj}$	Weights on lateral chromatic aberration which reflect the importance of the different object points. ( $1 \leq j \leq N_H$ )
$h'_{Ej}$	Expected image height corresponding to jth object point. ( $1 \leq j \leq N_H$ )
$W_{Ij}$	Weights on image height deviations for the $N_H$ different object points.

If there are  $N_s > 0$  substitution sets to be considered in addition to the nominal system then  $\vec{E}$  actually comprises  $N_s + 1$  error vectors. Let  $\vec{E}_0$  and  $M_0$  be the error vector and its order corresponding to the nominal system and let  $\vec{E}_i$  and  $M_i$  be the error vector and its order corresponding to the  $i$ th substitution set where  $1 \leq i \leq N_s$ . It follows that:

$$\vec{E} = (\vec{E}_0 \quad \vec{E}_1 \quad \vec{E}_2 \quad \dots \quad \vec{E}_{N_s}) \quad (8.002)$$

$$M = \sum_{i=0}^{N_s} M_i \quad (8.003)$$

It is important that the program determine M before initiating design in order to ascertain if the problem requires more computer storage than that which is available. Each of the  $\vec{E}_i$  is structured in the same way; hence, the discussion which follows applies to any arbitrary one of the  $\vec{E}_i$ . In practice  $\vec{E}$  is produced by generating each of the  $\vec{E}_i$  in succession starting with  $\vec{E}_0$ . It will be convenient to drop the subscript on  $\vec{E}_i$  and  $M_i$  so that  $\vec{E}$  and M in the following context actually refer to the appropriate subset of the final error vector.

If  $f^{(R)}/ \neq 0$  the entrance pupil radius,  $\rho_0$ , is re-computed in an attempt to force the f/ number of the lens (for color 1) to be the same as  $f^{(R)}/$ . Although this will not affect the structure of  $\vec{E}$ , it is certainly a part of the design process in the sense of improving performance.

Let  $\rho_0^{(0)}$  represent the current value of  $\rho_0$ . For  $i=1, 2, 3$ , first compute the current f/ \* number of the lens, designated  $f^{(i)}/$ . Then replace  $\rho_0$  with  $\rho_0^{(i)}$  where:

$$\rho_0^{(i)} = \left| \rho_0^{(i-1)} \frac{f^{(i)}/}{f^{(R)}/} \right| \quad (8.003)$$

and repeat until  $\rho_0^{(3)}$  has replaced  $\rho_0$ .

If  $W^* \neq 0$  the first component of  $\vec{E}$  is a weighted focal length deviation which is computed by tracing a paraxial ray for color 1 from object point  $(h, \theta) = (\rho_0 \Omega_2, \pi)$  with entrance pupil coordinates

---

\*See Section 4.2

$(\hat{x}_0, \hat{y}_0) = (0., \Omega_2)$ . Let  $(x_1, y_1, z_1)$  and  $(Q_{x1}, Q_{y1}, Q_{z1},)$  be the coordinates and direction of this ray at  $\sigma_N$ . If the ray misses or reflects or  $Q_{z1} = 0$ , a diagnostic is printed and the run is terminated. If  $Q_{z1} \neq 0$  compute:

$$\frac{1}{F_L} = \begin{cases} -\frac{Q_{y1}}{Q_{z1}} \cdot \frac{1}{\rho_0 \Omega_2} & n_{N+1} > 0 \\ \frac{Q_{y1}}{Q_{z1}} \cdot \frac{1}{\rho_0 \Omega_2} & n_{N+1} < 0 \end{cases} \quad (8.004)$$

where  $n_{N+1}$  is the index of refraction for color 1 of the material following  $\sigma_N$ . The deviation  $e_1$  is then:

$$e_1 = \begin{cases} \left( \frac{1}{F_L} - \frac{1}{F_L^{(R)}} \right) W^* & F_L^{(R)} \neq 0 \\ \frac{W^*}{F_L} & F_L^{(R)} = 0 \end{cases} \quad (8.005)$$

If  $W^* \neq 0$  the next component of  $\vec{E}$  (that is,  $e_2$  if  $W^* \neq 0$  or  $e_1$  if  $W^* = 0$ ) is a weighted exit pupil position deviation which is computed by tracing a ray for color 1 from the object point  $(h, \theta) = (h_E, 0.)$  with entrance pupil coordinates  $(\hat{x}_0, \hat{y}_0) = (0., 0.)$ . Let  $(x_1, y_1, z_1)$  and  $(Q_{x1}, Q_{y1}, Q_{z1})$  designate the position and direction of this ray on  $\sigma_N$ . If the ray misses or reflects or  $Q_{z1} = 0$ , a diagnostic is printed and the design process is terminated. If  $Q_{z1} \neq 0$  the component, say  $e_2$ , is given by:

$$e_2 = W^{**} \left\{ y_1 + \frac{Q}{Q_{z1}} y_1 (E_P^{(R)} - z_1) \right\}. \quad (8.006)$$

As a point of review the number of rays traced is

$$M_R = \begin{cases} N_R & \text{LMODE} \neq 0 \\ 2N_R & \text{LMODE} = 0 \end{cases} \quad (8.007)$$

where  $1 \leq M_R \leq 200$ .

It will be convenient to refer to the remaining components of  $\vec{E}$  as constituting a set  $\Psi$  containing  $M'$  elements. It follows that:

$$M = \begin{cases} M' & W^* = W^{**} = 0 \\ M' + 1 & W^* \neq 0 \quad W^{**} = 0 \text{ or } W^* = 0 \quad W^{**} \neq 0 \\ M' + 2 & W^* \neq 0 \quad W^{**} \neq 0 \end{cases}$$

The set  $\Psi$  is the logical sum (union) of  $N_H$  disjoint subsets  $\psi_j$  corresponding to the  $N_H$  different object points.

$$\Psi = \bigcup_{j=1}^{N_H} \psi_j \quad (8.009)$$

Each of the  $\psi_j$  contains  $N_\psi$  elements where:

$$N_\psi \begin{cases} = & 2M_R N_c + N_c + 1 & \text{LMODE} = 1 \\ & 2M_R N_c + 2N_c + 1 & \text{LMODE} \neq 1 \end{cases} \quad (8.010)$$

Since there are  $N_H$  mutually disjoint subsets then

$$M' = N_H N_\psi \quad (8.011)$$



Since  $1 \leq M_R \leq 200$ ,  $1 \leq N_c \leq 6$ , and  $1 \leq N_H \leq 7$  then:

$$4 \leq N \leq 2413,$$

$$4 \leq M' \leq 16891,$$

and  $4 \leq M \leq 16893.$

To define the elements of  $\psi_j$ , for arbitrary  $1 \leq j \leq N_H$ , let  $(x_{ijk}, y_{ijk})$  be the coordinates of the point of intersection of the  $k$ th ray (color  $i$ ) and the principal image plane associated with  $j$ . The pattern of points  $\left\{ (x_{ijk}, y_{ijk}); k = 1, 2, \dots, M_R \right\}$  produced by tracing the  $M_R$  specified monochromatic (fixed  $i$ ) rays defines a "spot" on the image plane whose "center" is the point  $(\bar{x}_{ij}, \bar{y}_{ij})$  given by

$$\bar{x}_{ij} = \frac{1}{M_R} \sum_{k=1}^{M_R} x_{ijk} \quad (8.012)$$

$$\bar{y}_{ij} = \frac{1}{M_R} \sum_{k=1}^{M_R} y_{ijk} \quad (8.013)$$

It should be noted that if any of the rays miss or reflect a diagnostic comment should be printed and the generation of  $\bar{E}$  aborted.

If the coordinate system is translated to  $(\bar{x}_{ij}, \bar{y}_{ij})$ , the center of the spot, the pattern of points becomes  $\left\{ (\tilde{x}_{ijk}, \tilde{y}_{ijk}); k = 1, 2, \dots, M_R \right\}$  where:

$$\tilde{x}_{ijk} = x_{ijk} - \bar{x}_{ij} \quad (8.014)$$

$$\tilde{y}_{ijk} = y_{ijk} - \bar{y}_{ij} \quad (8.015)$$

Each of these  $2 M_R$  numbers, after weighting, becomes an independent component of the error vector  $\vec{E} = (e_{x_{ijk}}, e_{y_{ijk}})$ , which adds  $2 M_R$  elements to  $\psi_j$ .

$$e_{x_{ijk}} = W_{ijk}^x \tilde{x}_{ijk} \quad (8.015.1)$$

$$e_{y_{ijk}} = W_{ijk}^y \tilde{y}_{ijk} \quad (8.015.2)$$

Nominally, the weights are given by:

$$W_{ijk}^x = \frac{W_{sj} W_{ci} W_x}{\sqrt{M_R}} = W_{ij}^x \quad (8.016)$$

and

$$W_{ijk}^y = \frac{W_{sj} W_{ci} W_y}{\sqrt{M_R}} = W_{ij}^y \quad (8.017)$$

where  $W_{sj}$ ,  $W_{ci}$ ,  $W_x$ , and  $W_y$  are input parameters which remain constant during the design.

The first  $2M_R N_c$  elements of  $\psi_j$  are then:

$$\begin{array}{lll}
 \text{Color 1:} & e_{x1jk}, e_{y1jk} & 1 \leq k \leq M_R \\
 \text{Color 2:} & e_{x2jk}, e_{y2jk} & 1 \leq k \leq M_R \\
 \vdots & \vdots & \vdots \\
 \text{Color } i: & e_{xijk}, e_{yijk} & 1 \leq k \leq M_R \\
 \vdots & \vdots & \vdots \\
 \text{Color } N_c: & e_{xN_cjk}, e_{yN_cjk} & 1 \leq k \leq M_R
 \end{array}$$

This scheme permits weighting the components of a monochromatic spot according to both object point number ( $W_{sj}$ ) and color number ( $W_{ci}$ ).

Considering, for the moment, only the  $N_H N_c$  monochromatic spots, the merit function  $\varphi = \vec{E} \cdot \vec{E}$  may be written:

$$\varphi = \sum_{i,j} W_{sj}^2 W_{ci}^2 \left\{ \frac{W_x^2}{M_R} \sum_{k=1}^{M_R} \tilde{x}_{ijk}^2 + \frac{W_y^2}{M_R} \sum_{k=1}^{M_R} \tilde{y}_{ijk}^2 \right\} \quad (8.017.1)$$

For any given spot (fixed  $i$  and  $j$ ), all  $x$  directional error items ( $\tilde{x}_{ijk}$ ) are weighted the same ( $W_x$ ), and all  $y$  directional error items ( $\tilde{y}_{ijk}$ ) are weighted the same ( $W_y$ ), that is, the weights are independent of  $k$ . The directional weights,  $W_x$  and  $W_y$ , allow a system to be designed in which the spot is not necessarily circular, e. g., with  $W_x = 1$ . and  $W_y = 0$ . the spot is permitted to smear in an unconstrained fashion in the  $y$  direction to produce a vertical line image.

In the usual case  $W_x = W_y = 1$ , so that:

$$\phi = \sum_{i,j} W_{sj}^2 W_{ci}^2 \left\{ \frac{1}{M_R} \sum_{k=1}^{M_R} (\tilde{x}_{ijk}^2 + \tilde{y}_{ijk}^2) \right\}, \quad (8.017.2)$$

the quantity in brackets being the square of the spot size,  $R_{ij}^*$ . With  $\tilde{r}_{ijk}^2 = \tilde{x}_{ijk}^2 + \tilde{y}_{ijk}^2$ , it is seen that spot size is actually the rms radius of the points describing the spot.

O'Brien<sup>7</sup> suggests a Gaussian weighting procedure for systems in which contrast characteristics (MTF) are important. For the common case of  $W_x = W_y = 1$ , the deviations,  $\tilde{x}_{ijk}$  and  $\tilde{y}_{ijk}$ , are weighted by

$$W_{ijk} = \frac{W_{sj} W_{ci}}{\sqrt{M_R}} e^{-\frac{\tilde{r}_{ijk}^2}{2\sigma^2}} \quad (8.017.3)$$

with  $\sigma$ , an input parameter, representing the radius about the center within which 68% of the energy should lie.

For the general case equations (8.016) and (8.017) become:

$$W_{ijk}^x = \frac{W_{sj} W_{ci} W_x}{\sqrt{M_R}} e^{-\frac{\tilde{r}_{ijk}^2}{2\sigma^2}} \quad (8.017.4)$$

and

$$W_{ijk}^y = \frac{W_{sj} W_{ci} W_y}{\sqrt{M_R}} e^{-\frac{\tau_{ijk}^2}{2\sigma^2}} \quad (8.017.5)$$

where

$$\tau_{ijk}^2 = W_x^2 \tilde{x}_{ijk}^2 + W_y^2 \tilde{y}_{ijk}^2 \quad (8.017.6)$$

The major disadvantage of using a Gaussian weighting procedure is the increased computation time which results from having to compute the exponential term for each and every image point. The case  $\sigma = 0$ , will be used to specify the absence of a Gaussian.

Consider the "spot" described by the point set

$\{(\bar{x}_{ij}, \bar{y}_{ij}); i = 1, 2, \dots, N_c\}$  which are the "centers" of the  $N_c$  monochromatic spots. The center of this spot has coordinates  $(\bar{x}_j, \bar{y}_j)$  given by:

$$\bar{x}_j = \frac{1}{N_c} \sum_{i=1}^{N_c} \bar{x}_{ij} \quad (8.018)$$

$$\bar{y}_j = \frac{1}{N_c} \sum_{i=1}^{N_c} \bar{y}_{ij} \quad (8.019)$$

Translating the origin to  $(\bar{x}_j, \bar{y}_j)$  the point set becomes

$\{(x_{ij}, y_{ij}); i = 1, 2, \dots, N_c\}$  where:

$$\tilde{x}_{ij} = \bar{x}_{ij} - \bar{x}_j \quad (8.020)$$

$$\tilde{y}_{ij} = \bar{y}_{ij} - \bar{y}_j \quad (8.021)$$

Treating  $\{(\tilde{x}_{ij}, \tilde{y}_{ij}); i = 1, 2, \dots, N_c\}$  as a set of  $2N_c$  independent deviations, the inclusion of these (after weighting) into  $\psi_j$  corresponds to reducing lateral chromatic aberration since reducing the length of  $\vec{E}$  forces the centers of the  $N_c$  monochromatic spots to a single point.

Let

$$\bar{W}_{Aj} = \frac{W_{Aj}}{\sqrt{N_c}} \quad (8.022)$$

The next  $2N_c$  elements of  $\psi_j$  are:

$$\bar{W}_{Aj} \tilde{y}_{ij} \quad i = 1, 2, \dots, N_c$$

and 
$$\bar{W}_{Aj} \tilde{x}_{ij} \quad i = 1, 2, \dots, N_c.$$

If LMCDE=1 the program assumes  $\bar{x}_{ij} = 0$  for all  $i$  (and  $j$ ). It follows that  $\bar{x}_j = 0$  and  $\tilde{x}_{ij} = 0$  and, in this instance, the elements  $\bar{W}_{Aj} \tilde{x}_{ij}$  for  $1 \leq i \leq N_c$  are not included in  $\psi_j$ .

If equations (8.012) and (8.013) are substituted into equations (8.018) and (8.019), respectively, there results the following equations, which illustrate that  $(\bar{x}_j, \bar{y}_j)$  is the average  $x$  and  $y$  for all rays traced from the object point  $(h_j, \theta)$ .

$$\bar{x}_j = \frac{1}{M_R N_c} \sum_{k=1}^{M_R} \sum_{i=1}^{N_c} x_{ijk} \quad (8.023)$$

$$\bar{y}_j = \frac{1}{M_R N_c} \sum_{k=1}^{M_R} \sum_{i=1}^{N_c} y_{ijk} \quad (8.024)$$

If  $(\bar{x}_j, \bar{y}_j)$  is interpreted as the image point of the object point  $(h_j, \theta)$  then the image height  $h'_j$  is, from Section 1.3.,

$$h'_j = \bar{x}_j \sin \theta + \bar{y}_j \cos \theta \quad (8.025)$$

The final element of  $\psi_j$  is a weighted deviation of  $h'_j$  from an (input) expected image height,  $h'_{Ej}$ , that is,  $W_{Ij}(h'_j - h'_{Ej})$ .

### 8.1.1 Image Plane Computation

As has been mentioned, there is a principal image plane associated with the object point  $(h_j, \theta)$  whose position is given by equation (1.037) where  $S^*$  (or  $S^* + B_F$ ) is the location of the vertex of the image surface and  $d_j$  is the sagitta which positions the plane.

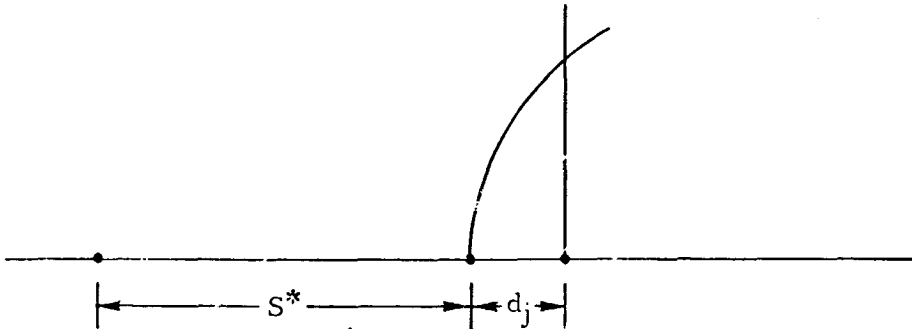


Figure 21.1 Curved Image Surface

Suppose that the image surface is a standard conic of revolution with the equation

$$x^2 + y^2 + (z - R)^2 = R^2 + bz^2 \quad (8.025.1)$$

The user selects  $d_j$  so that the expected image height,  $h'_{Ej}$ , is a point lying on the image surface, that is,  $d_j$  must satisfy,

$$(b-1) d_j^2 + 2 R d_j - (h'_{Ej})^2 = 0, \quad (8.025.2)$$

since  $h'_{Ej}$  is a radial distance. This equation has, in general, two solutions the appropriate one depending upon the type of conic ( $b$ ) and the sign of the curvature ( $R$ ). As an example for a spherical image surface with positive  $R$ ,

$$d_j = R - \sqrt{R^2 - (h'_{Ej})^2} \quad (8.025.3)$$

the other solution,  $R + \sqrt{R^2 - (h'_{Ej})^2}$ , being rejected on physical grounds. Lack of a real solution indicates an incompatibility between the image surface and the expected image height.

Since the computed image height,  $h'_j$ , on the plane  $z = d_j$  is usually different from  $h'_{Ej}$  the image is not actually in focus. This is especially bad when constructing  $\vec{E}$  since the spot is examined on a poorly positioned image plane. It would be more appropriate to first compute a new value of  $d_j$  by insisting that  $h'_j$ , as given by equation (8.025), satisfy equation (8.025.2)



With  $(\bar{x}_{ij}, \bar{y}_{ij})$  as the mean of all rays with color index  $i$  from the object point  $(h_j, \theta)$ , it follows from Section 6 that:

$$\bar{x}_{ij} = \bar{m}_{xij} \lambda + \bar{k}_{xij} \quad (8.025.4)$$

$$\bar{y}_{ij} = \bar{m}_{yij} \lambda + \bar{k}_{yij} \quad (8.025.5)$$

where  $\lambda$  is the position of a plane relative to the last optical surface, that is,  $\lambda = S^* + z$  where  $z$  is measured from the vertex of the image surface. Equations (8.018) and (8.019) become:

$$\bar{x}_j = \bar{m}_{xj} \lambda + \bar{k}_{xj} \quad (8.025.6)$$

$$\bar{y}_j = \bar{m}_{yj} \lambda + \bar{k}_{yj} \quad (8.025.7)$$

with

$$\bar{m}_{xj} = \begin{cases} 0. & \text{LMODE}=1 \\ \frac{1}{N_c} \sum_{i=1}^{N_c} \bar{m}_{xij} & \text{LMODE} \neq 1 \end{cases} \quad (8.025.8)$$

$$\bar{k}_{xj} = \begin{cases} 0. & \text{LMODE}=1 \\ \frac{1}{N_c} \sum_{i=1}^{N_c} \bar{k}_{xij} & \text{LMODE} \neq 1 \end{cases} \quad (8.025.9)$$

$$\bar{m}_{yj} = \frac{1}{N_c} \sum_{i=1}^{N_c} \bar{m}_{yij} \quad (8.025.10)$$

$$\bar{k}_{yj} = \frac{1}{N_c} \sum_{i=1}^{N_c} \bar{k}_{yij} \quad (8.025.11)$$

Equation (8.025) which defines  $h_j^1$  in terms of  $(\bar{x}_j, \bar{y}_j)$  is written:

$$h_j^1 = a_j (S^* + d_j) + b_j \quad (8.025.12)$$

where

$$a_j = \bar{m}_{xj} \sin \theta + \bar{m}_{yj} \cos \theta \quad (8.025.13)$$

and

$$b_j = \bar{k}_{xj} \sin \theta + \bar{k}_{yj} \cos \theta \quad (8.025.14)$$

In order for  $h_j^1$  to lie on the image surface,  $d_j$  must satisfy:

$$[a_j (S^* + d_j) + b_j]^2 = (b-1) d_j^2 + 2 R d_j \quad (8.025.15)$$

or

$$-\alpha d_j^2 + 2\beta d_j + \gamma = 0, \quad (8.025.16)$$

where

$$\alpha = (b-1) - a_j^2 \quad (8.025.17)$$

$$\beta = a_j (b_j + a_j S^*) - R \quad (8.025.18)$$

and 
$$\gamma = (b_j + a_j S^*)^2. \quad (8.025.19)$$

Let 
$$\Delta = \beta^2 + \alpha\gamma. \quad (8.025.20)$$

If  $\Delta < 0$ , there is no real solution for  $d_j$  and the program should retain the old value. If  $\Delta = 0$  then:

$$d_j = -\frac{\gamma}{\beta} \quad (8.025.21)$$

If  $\Delta > 0$  then:

$$d_j = \frac{-\gamma}{\beta \pm \Delta^{1/2}} \quad (8.025.22)$$

where  $\Delta^{1/2}$  is given the sign of  $\beta$ .

One of the major disadvantages of computing  $d_j$  is the necessity of tracing all rays of all colors before calculating the desired plane position. In order to avoid the costly re-tracing of these rays, for each color the following data must be preserved:  $\tilde{m}_{xi}$ ,  $\tilde{k}_{yi}$ ,  $\tilde{m}_{yi}$ , and  $\tilde{k}_{yi}$  where  $1 \leq i \leq N_R$ , a total of  $4 \cdot 200 \cdot 6 = 480$  items in the maximum case.

The automatic computation of the sagitta ( $d_j$ ) during design would be a program option controlled by means of a trigger, say ATRGGR, where ATRGGR = 0 instructs the program to always use the input  $d_j$  and ATRGGR = 1 instructs the program to compute  $d_j$  using the equation of the image surface and the computed image height. Initially, the user must supply  $b$  and  $R$  for the image surface and a starting set of  $d_j$  based upon desired image heights.

The program then overstores the input  $d_j$  with new values computed according to the procedure described above. If  $h_j'$  is such that there is no real solution for  $d_j$ , the program will retain the old value rather than abort the design.

Since the program overstores  $d_j$  with new values it follows that when using this option with substitution sets the parameters CIMPL(1), CIMPL(2)\*, . . . must be included as substitution parameters; otherwise, at the conclusion of generating  $\vec{E}$ , the  $d_j$  will have been replaced with values corresponding to the final substitution set instead of the nominal system.

As will be seen a design cycle begins with the initial vector  $\vec{E}_0 = \vec{E}(\vec{U}_0)$  and proceeds to compute the "derivative" vectors  $\vec{I}_1, \vec{I}_2, \dots, \vec{I}_n$  as given by equation (8.042). During the computation of  $\vec{E}(\vec{U}_0 + \Delta\vec{U}_i)$ , ATRGGR should be zero so that derivatives are examined on the same set of planes as  $\vec{E}_0$ . The trigger is restored to normal status for computing the new error vector  $\vec{E}^*$ .

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\*See Volume II.

## 8.2 LEAST SQUARES MINIMIZATION

In Section 8.1 it was shown that the merit function is essentially the length of an error vector  $\vec{E}$  whose components comprise a set of weighted aberrations or deviations. The magnitude of  $\varphi$  then becomes a measure of performance since decreasing  $\varphi$  corresponds to decreasing the various aberrations.

The procedure for minimizing  $\varphi$  is an iterative linearized least squares technique. Each iteration begins with a set of  $n$  design variables (lens system parameters) which are taken as independent components of a parameter vector  $\vec{U}$  so that:

$$\varphi \equiv \varphi(\vec{U}) = \vec{E}(\vec{U}) \cdot \vec{E}(\vec{U}). \quad (8.026)$$

The iteration commences with a nominal value or estimate for  $\vec{U}$ , say  $\vec{U}_0$ , and determines a new value  $\vec{U}^*$  such that  $\varphi(\vec{U}^*) < \varphi(\vec{U}_0)$ . This modified lens system is then used as the initial estimate for the next iteration.

### 8.2.1 Design Variables

The manner in which  $\vec{U}$  is established for each iteration is rather involved; consequently, it will be discussed in detail.

Let  $\tilde{v}$  represent a master set of  $N_D$  composite variables such that  $1 \leq N_D \leq 50$ . Each member  $\tilde{v}_j$  of  $\tilde{v}$  comprises  $|N_j|$  system parameters where  $1 \leq |N_j| \leq 6$ ; hence,  $\tilde{v}$  is more adequately described by a matrix  $V$  whose  $j$ th column,  $v_j$ , is given by:

$$v_j = \begin{bmatrix} N_j \\ \Delta_j^{(R)} \\ \Delta v_j \\ K_{1j} \\ K_{2j} \\ \cdot \\ \cdot \\ \cdot \\ K_{6j} \\ \Delta_j^{(c)} \end{bmatrix} \quad (8.027)$$

The parameters associated with  $\tilde{v}_j$ , denoted by  $v_{1j}$ ,  $v_{2j}$ ,  $\dots$ ,  $v_{N_jj}$ , are identified by a set of data indices  $K_{1j}$ ,  $K_{2j}$ ,  $\dots$ ,  $K_{N_jj}$ . The significance of  $\Delta_j^{(R)}$  and  $\Delta_j^{(c)}$  is discussed later.  $\Delta v_j$  is an increment which is applied to  $\tilde{v}_j$  during the computation of partial derivatives. If  $\tilde{v}_j'$  represents the perturbed value of  $\tilde{v}_j$ , then  $\Delta v_j$  is applied as follows:

$$\tilde{v}_j' = \tilde{v}_j + \Delta v_j \quad \text{if } N_j > 0 \quad (8.028)$$

$$(\tilde{v}_j')^{-1} = (\tilde{v}_j)^{-1} + \Delta v_j \quad \text{if } N_j < 0 \quad (8.029)$$

$N_j$  should only be assigned a negative value when  $\tilde{v}_j$  represents a radius of curvature parameter. Since  $\tilde{v}_j$  is actually a composite variable the perturbation by  $\Delta v_j$  actually applies to all  $|N_j|$  parameters simultaneously.

For example, suppose  $N_j > 0$ , then forming  $\tilde{v}_j + \Delta v_j$  actually consists of:

$$v'_{1j} = \begin{cases} v_{1j} + \Delta v_j & K_{1j} > 0 \\ v_{1j} - \Delta v_j & K_{1j} < 0 \end{cases} \quad (8.029)$$

$$v'_{2j} = \begin{cases} v_{2j} + \Delta v_j & K_{2j} > 0 \\ v_{2j} - \Delta v_j & K_{2j} < 0 \end{cases} \quad (8.030)$$

. . .  
. . .  
. . .

$$v'_{N_j j} = \begin{cases} v_{N_j j} + \Delta v_j & K_{N_j j} > 0 \\ v_{N_j j} - \Delta v_j & K_{N_j j} < 0 \end{cases} \quad (8.031)$$

By letting the variables  $\tilde{v}_j$  be composite rather than simple and allowing  $\Delta v_j$  to be either added or subtracted from the  $v_{ij}$  depending on the sign of  $K_{ij}$ , it is possible during an iteration to let system parameters change in such a way that their sum remains fixed. For example, suppose it is desirable to let  $S_2$  and  $S_3$  be design variables but constrained so that  $S_2 + S_3$  remains fixed. This is accomplished by setting  $N_1 = 2$  and letting  $K_{11} > 0$  and  $K_{21} < 0$  where  $v_{11} \equiv S_2$  and  $v_{21} \equiv S_3$ . Whenever  $\tilde{v}_1$  is perturbed by  $\Delta v_1$  the following is performed:

$$S_2' \equiv v_{11}' = S_2 + \Delta v_1$$

and

$$S_3' \equiv v_{21}' = S_3 - \Delta v_1$$

so that

$$S_2' + S_3' \equiv S_2 + S_3.$$

Matrix V which is referred to as the design matrix must be specified prior to initiating a sequence of design iterations. This is accomplished by supplying:

$$1 \leq N_D \leq 50,$$

and for  $1 \leq j \leq N_D$  the values  $N_j$ ,  $\Delta_j^{(R)}$ , and  $\Delta v_j$ . The data indices  $K_{ij}$  for  $1 \leq i \leq |N_j|$  are determined from a set of  $|N_j|$  parameter symbols which must be provided.

Each composite variable  $\tilde{v}_j$  is uniquely identified by the subscript j which is referred to as the design variable number or index.

For any arbitrary design iteration the applicable parameter vector  $\vec{U}$  is some proper subset  $U = \{u_i; i = 1, 2, \dots, n\}$  of the master set  $\tilde{v}$  such that  $1 \leq n \leq 10$ . The composite variables  $u_i$  are actually identified by a set,  $J = \{j_i; i = 1, 2, \dots, n\}$ , of design variable numbers such that  $u_i \equiv \tilde{v}_{j_i}$ .

It is also convenient to define an n-component parameter increment vector  $\vec{\Delta U}$  with components  $\Delta u_i$  such that  $\Delta u_i \equiv \Delta v_{j_i}$  for all  $1 \leq i \leq n$ . In general J, and therefore  $\vec{U}$ , changes, in a manner to be



specified below, from one design iteration to the next, that is, the design process operates with a different set of variables during each iteration.

If  $J^{(k)}$  represents the design variable number set for the  $k$ th iteration then  $J^{(k+1)}$  is determined as follows:

$$j_i^{(k+1)} = \begin{cases} j_i^{(k)} + 1 & \text{if } j_i^{(k)} < j_{\max} \\ j_{\min} & \text{if } j_i^{(k)} = j_{\max} \end{cases} \quad (8.032)$$

for  $1 < i < n$ , where  $j_{\min}$  and  $j_{\max}$  are bounds such that  $1 \leq j_{\min} \leq j_{\max} \leq N_D$ .  $J^{(1)}$ ,  $n$ , and the number of iterations  $N_I$  must be specified in order to start the process. The following constraints apply to  $J^{(1)}$ :

$$j_{\min} \leq j_i^{(1)} \leq j_{\max} \quad 1 \leq i \leq n,$$

and

$$j_i^{(1)} \neq j_k^{(1)} \quad i \neq k \quad 1 \leq i, k \leq n.$$

This incrementation procedure thus establishes a unique parameter vector  $\bar{U}$  for each iteration. It should be noted that this scheme actually cycles through the different design variable number sets, that is,  $J^{(k+P)} \equiv J^{(k)}$   $k = 1, 2, \dots$  where the period  $P$  is given by:

$$P = j_{\max} - j_{\min} + 1. \quad (8.033)$$

As an example, let:

$$j_{\min} = 4$$

$$j_{\max} = 8$$

$$n = 3$$

$$N_I = 15$$

$$J^{(1)} = \left\{ 4 \ 5 \ 7 \right\}$$

$$J^{(2)} = \left\{ 5 \ 6 \ 8 \right\}$$

$$J^{(3)} = \left\{ 6 \ 7 \ 4 \right\}$$

$$J^{(4)} = \left\{ 7 \ 8 \ 5 \right\}$$

$$J^{(5)} = \left\{ 8 \ 4 \ 6 \right\}$$

$$J^{(6)} = \left\{ 4 \ 5 \ 7 \right\} \equiv J^{(1)}$$

$$J^{(7)} = \left\{ 5 \ 6 \ 8 \right\} \equiv J^{(2)}$$

·        ·  
·        ·  
·        ·

$$J^{(15)} = \left\{ 8 \ 4 \ 6 \right\} \equiv J^{(5)}$$

### 8.2.2 Design Iteration

Having established the parameter vector  $\vec{U} = (u_1 \ u_2 \ \dots \ u_n)$ , a necessary condition that  $\varphi$  be a minimum is that its first partials vanish simultaneously:

$$\frac{\partial \varphi}{\partial u_j} = 0. \quad j = 1, 2, \dots, n. \quad (8.034)$$

In terms of  $\vec{E}$ , from equation (8.026),

$$\vec{E}(\vec{U}) \cdot \frac{\partial \vec{E}(\vec{U})}{\partial u_j} = 0. \quad j = 1, 2, \dots, n. \quad (8.035)$$

Equations (8.035) are a set of  $n$  non-linear equations in the  $n$  unknowns  $u_1, u_2, \dots, u_n$  whose solution is  $\vec{U}^*$ , the required value of  $\vec{U}$ . Since it is not possible, in general, to derive a closed formed expression for  $\vec{E} = \vec{E}(\vec{U})$  the procedure followed in solving (8.035) entails a linearized approximation of the non-linear system which can then be iterated upon.

With  $\vec{U}_0$  as the initial estimate of  $\vec{U}^*$ , expand  $\vec{E}(\vec{U}^*)$  in a Taylor Series about  $\vec{U}_0$ .

$$\begin{aligned} \vec{E}(\vec{U}^*) &= \vec{E}(\vec{U}_0 + \delta \vec{U}_0) = \vec{E}_0 + \sum_{j=1}^n \frac{\partial \vec{E}_0}{\partial u_j} \delta u_j \\ &+ \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n \frac{\partial^2 \vec{E}_0}{\partial u_j \partial u_k} \delta u_j \delta u_k + \dots + \end{aligned} \quad (8.036)$$

where:

$$\vec{E}_0 \equiv \vec{E}(\vec{U}_0)$$

$$\frac{\partial \vec{E}_0}{\partial u_j} \equiv \left. \frac{\partial \vec{E}}{\partial u_j} \right|_{\vec{U} = \vec{U}_0}$$

and  $\delta \vec{U}_0 \equiv (\delta u_1 \quad \delta u_2 \quad \dots \quad \delta u_n).$

Equation (8.036) is linearized by dropping all the high order terms, giving the approximation

$$\vec{E}(\vec{U}^*) \approx \vec{E}_0 + \sum_{j=1}^n \frac{\partial \vec{E}_0}{\partial u_j} \delta u_j. \quad (8.037)$$

This equation is then substituted into (8.035) producing the  $n \times n$  linear system

$$\sum_{j=1}^n \left( \frac{\partial \vec{E}_0}{\partial u_j} \cdot \frac{\partial \vec{E}_0}{\partial u_k} \right) \delta u_j = -\vec{E}_0 \cdot \frac{\partial \vec{E}_0}{\partial u_k} \quad k = 1, 2, \dots, n \quad (8.038)$$

in the  $n$  unknowns  $(\delta u_1 \quad \delta u_2 \quad \dots \quad \delta u_n)$ . The problem then is to invert (8.038) yielding the correction vector  $\delta \vec{U}_0$  such that  $\vec{U}_0 + \delta \vec{U}_0$  is an improved estimate of  $\vec{U}^*$  which, in principle, could be used as the starting point for a second iteration.

With  $\Delta \vec{U}$  as the given parameter increment vector define  $\vec{\lambda} = (\lambda_1 \quad \lambda_2 \quad \dots \quad \lambda_n)$  such that

$$\lambda_j = \delta u_j \Big|_{\Delta u_j}. \quad (8.039)$$

The linear system expressed by (8.038) can be written with  $\vec{\lambda}$  as the unknown instead of  $\delta\vec{U}_0$  by multiplying the  $k$ th equation by  $u_k$ .

$$\sum_{j=1}^n \left( \frac{\partial \vec{E}_0}{\partial u_j} \Delta u_j \right) \cdot \left( \frac{\partial \vec{E}_0}{\partial u_k} \Delta u_k \right) \lambda_j = - \vec{E}_0 \cdot \left( \frac{\partial \vec{E}_0}{\partial u_k} \Delta u_k \right)$$

$$k = 1, 2, \dots, n \quad (8.040)$$

Before inverting this system it is necessary to determine the coefficients:

$$\frac{\partial \vec{E}_0}{\partial u_j} \Delta u_j \quad j = 1, 2, \dots, n.$$

Let  $\Delta\vec{U}_j$  be an  $n$  vector whose  $j$ th component is  $\Delta u_j$  and whose remaining components are zero. By definition:

$$\frac{\partial \vec{E}_0}{\partial u_j} = \lim_{\Delta u_j \rightarrow 0} \frac{\vec{E}(\vec{U}_0 + \Delta\vec{U}_j) - \vec{E}_0}{\Delta u_j} \quad (8.041)$$

Therefore the term  $(\partial \vec{E}_0 / \partial u_j) \Delta u_j$  is approximated by  $\vec{I}_j$  where:

$$\vec{I}_j \equiv \vec{E}(\vec{U}_0 + \Delta\vec{U}_j) - \vec{E}_0 \quad (8.042)$$

$\vec{E}_0$  and  $\vec{E}(\vec{U}_0 + \Delta\vec{U}_j)$  are determined by the procedure discussed in Section 8.1 using  $\vec{U} = \vec{U}_0$  and  $\vec{U} = \vec{U}_0 + \Delta\vec{U}_j$ , respectively.

If any ray fails during the generation of  $\vec{E}_0$  this is considered to be sufficient justification for aborting the entire design process.

If, during the kth iteration ( $k \geq 1$ ) any ray fails while computing  $\vec{E}(\vec{U}_0 + \Delta\vec{U}_j)$  the system is restored ( $\vec{U} = \vec{U}_0$ ) and the program proceeds directly to the  $k + 1$  st iteration.

Equations (8.040) can be written in terms of the  $\vec{I}_j$  as follows:

$$\sum_{j=1}^n (\vec{I}_j \cdot \vec{I}_k) \lambda_j = -\vec{E}_0 \cdot \vec{I}_k \quad 1 \leq k \leq n \quad (8.043)$$

In matrix form:

$$I \wedge = C \quad (8.044)$$

where

$$I = \begin{bmatrix} \vec{I}_1 \cdot \vec{I}_1 & \vec{I}_1 \cdot \vec{I}_2 & \cdot & \cdot & \cdot & \vec{I}_1 \cdot \vec{I}_n \\ \vec{I}_2 \cdot \vec{I}_1 & \vec{I}_2 \cdot \vec{I}_2 & \cdot & \cdot & \cdot & \vec{I}_2 \cdot \vec{I}_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \vec{I}_n \cdot \vec{I}_1 & \vec{I}_n \cdot \vec{I}_2 & \cdot & \cdot & \cdot & \vec{I}_n \cdot \vec{I}_n \end{bmatrix} \quad (8.045)$$

$$\wedge = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \cdot \\ \cdot \\ \lambda_n \end{bmatrix} \quad (8.046)$$

$$C = \begin{bmatrix} -\vec{E}_0 \cdot \vec{I}_1 \\ -\vec{E}_0 \cdot \vec{I}_2 \\ \cdot \\ \cdot \\ -\vec{E}_0 \cdot \vec{I}_n \end{bmatrix} \quad (8.047)$$

If I is numerically singular the program should proceed immediately to the next iteration. If not the solution vector is given by

$$\Delta = I^{-1} C \quad (8.048)$$

The improved parameter vector  $\vec{U}^*$  is obtained by applying (8.039).

$$\vec{U}^* = \vec{U}_0 + \delta \vec{U}_0 = (u_1 + \lambda_1 \Delta u_1, u_2 + \lambda_2 \Delta u_2, \dots, u_n + \lambda_n \Delta u_n)$$

There are instances when one or more of the  $|\lambda_j|$  are so large that there is serious doubt regarding the reliability of  $\Delta$ . This can occur when the linearized equations represent a poor approximation or when the determinant of I is small (I is nearly singular). In an attempt to prevent "overshooting"  $\vec{U}^*$ , the components of  $\vec{\lambda}$  are scaled by a factor  $\xi < 1$ , prior to forming  $\vec{U}_0 + \delta \vec{U}_0$ .

Let

$$x = \text{Max} [\zeta, |\lambda_1|, |\lambda_2|, \dots, |\lambda_n|] \quad (8.049)$$

where  $\zeta > 0$ , is input, and define  $\xi_0$  so that

$$\xi_0 = 2 \frac{\zeta}{x} \quad (8.050)$$

Consider an ordered sequence of scale factors  $\xi_i$  given by

$$\xi_i = \frac{\xi_0}{2^i} \quad i = 1, 2, \dots \quad (8.051)$$

It is clear that  $1. > \xi_1 > \xi_2 \dots > \xi_i$  and  $\xi_i \rightarrow 0$ . as  $i \rightarrow \infty$ . Let  $\vec{U}_i^*$  represent the improved parameter vector given by.

$$\vec{U}_i^* = (u_1 + \xi_i \lambda_1 \Delta u_1, u_2 + \xi_i \lambda_2 \Delta u_2, \dots, u_n + \xi_i \lambda_n \Delta u_n) \quad (8.052)$$

Since  $\xi_i \rightarrow 0$ . as  $i$  increases it also follows that  $\vec{U}_i^* \rightarrow \vec{U}_0$  as  $i \rightarrow \infty$ .

The reason for considering a sequence of  $\xi_i$  and  $\vec{U}_i^*$  instead of just  $\xi_1$  is because of the possibility of boundary violations.\* A boundary violation occurs when one or more of the  $u_k$  (parameters) has been altered too much, i. e.,  $\xi_i \lambda_k \Delta u_k$  is too large. The mode of operation then is to try  $\xi_1$  and check for boundary violations. If there is a violation try  $\xi_2$  and repeat. This process is continued until the violations cease or until  $\xi_{10}$  is reached. If the boundary violations have not been eliminated by the time  $i = 10$ , the design iteration is considered to be a failure, the program restores  $\vec{U}_0$  and proceeds to the next design iteration.

Suppose then, that there is some  $1 \leq i \leq 10$  such that  $\vec{U}_i^*$  causes no boundary violations. Let  $\vec{E}^* = \vec{E}(\vec{U}_i^*)$  and let  $|\vec{E}|$  designate the length of  $\vec{E}$ . If there is any ray failure while computing  $\vec{E}^*$  restore  $\vec{U}$  to  $\vec{U}_0$  and proceed to the next design iteration.

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\* See Section 8.2.3



The "old" length and "new" length are, respectively,

$$|\vec{E}_0| = \left| \vec{E}_0 \cdot \vec{E}_0 \right|^{\frac{1}{2}} \quad (8.053)$$

and

$$|\vec{E}^*| = \left| \vec{E}^* \cdot \vec{E}^* \right|^{\frac{1}{2}}. \quad (8.054)$$

The relative improvement,  $\epsilon$ , is

$$\epsilon = \frac{|\vec{E}_0| - |\vec{E}^*|}{|\vec{E}_0|}. \quad (8.055)$$

Since the relations  $\partial \phi / \partial u_j = 0$  are equally valid for a local maximum it is possible for  $|\vec{E}^*| > |\vec{E}_0|$ , that is,  $\epsilon < 0$ . If this occurs set  $\vec{U}$  to  $\vec{U}_0$  (restore) and proceed to the next iteration.

### 8.2.3 Boundary Conditions

It is obvious that the procedure described above for computing the parameter correction vector  $\delta \vec{U}_0$  imposes no constraints on the individual components of  $\vec{U}$ . In practice it is often desirable to constrain some parameter  $u_i$  to lie in a specified interval  $u_{i \min} \leq u_i \leq u_{i \max}$ . A typical example is the surface separation between element  $i$  and element  $i + 1$ .

Boundary conditions on individual parameters are specified by a boundary condition matrix  $B$  ( $3 \times 100$ ) whose  $j$ th column contains  $r_j$ ,  $u_{j \min}$ , and  $u_{j \max}$ . The first entry  $r_j$  is a data index which uniquely identifies some parameter  $u_j$  constrained by  $u_{j \min} \leq u_j \leq u_{j \max}$ . There are several special cases which require explanation.

If  $u_j$  is the radius of curvature of some element then ( $u_{j \min}$ ,  $u_{j \max}$ ) actually applies to  $1/u_j$ , that is,  $u_j$  is constrained so that

$$u_{j \min} \leq \frac{1}{u_j} \leq u_{j \max}.$$

If  $u_j$  is the index of refraction of some material and  $N_c = 1$  the bounds apply in the usual way. However if  $N_c > 1$  then  $u_j$  should be chosen as the index of the material for the first color. Let  $n_1, n_2, \dots, n_{N_c}$  present the  $N_c$  indices of the material in question and suppose they are ordered such that  $n_i < n_{i+1}$ . Then  $u_j \equiv n_1$  and the bounds apply as follows:

$$u_{j \min} \leq n_i < n_{N_c} \leq u_{j \max}.$$

In addition to this constraint the set must always remain ordered. Because of the special nature of indices of refraction they will be identified by requiring  $r_j < 0$  when  $u_j$  is an index.

Suppose  $J = \{ j_i; i = 1, 2, \dots, n \}$  is the set of design variable numbers for the current iteration and let  $j \in J$ . Each of the parameters associated with the composite variable  $\tilde{v}_j$  must be checked for a possible boundary isolation. Let  $K_1, K_2, \dots, K_{|N|}$  be the data indices of the  $|N|$  parameters which comprise  $\tilde{v}_j$ . Finally, let  $k$  represent any of the  $K_i$  and  $v_{kj}$  the corresponding parameter.

To check  $v_{kj}$  for a boundary violation,  $|k|$  must be compared against  $|r_i|$   $i = 1, 2, \dots, NCOND$  where  $NCOND$  is the number of boundary conditions defined by matrix  $B$ . If neither of the conditions described below is satisfied for some  $1 \leq i \leq NCOND$  then  $v_{kj}$  is not subject to boundary constraints.

a) If  $r_i > 0$  and  $|k| = r_i$  for some  $i$  then  $v_{kj}$  must satisfy:

$$u_{i \text{ min}} \leq v_{kj} \leq u_{i \text{ max}} \quad \text{if } N > 0,$$

or

$$u_{i \text{ min}} \leq \frac{1}{v_{kj}} \leq u_{i \text{ max}} \quad \text{if } N < 0.$$

b) If  $r_i < 0$  and  $|r_i| \leq |k| \leq |r_i| + N_c - 1$  then  $v_{kj}$  is one of the indices of refraction of the material whose index of refraction for color  $l$  is specified by  $|r_i|$ . If  $N_c = 1$  then  $|r_i| \equiv |k|$  and  $v_{kj}$  must satisfy

$$u_{i \text{ min}} \leq v_{kj} \leq u_{i \text{ max}}.$$

If  $N_c > 1$  let  $\{l_1 \ l_2 \ \dots \ l_{N_c}\}$  be a set of integers which specifies the order of the indices prior to designing. For example, if  $N_c = 3$ ,  $n_1 = 1.77$ ,  $n_2 = 1.46$ , and  $n_3 = 1.62$  then  $\{l_1 \ l_2 \ l_3\} = \{3 \ 1 \ 2\}$ . Let  $\{l'_1 \ l'_2 \ \dots \ l'_{N_c}\}$  represent the current order. If  $n_1, n_2, \dots, n_{N_c}$  are the set of indices identified by  $|r_i|$  then the following conditions must be satisfied.

$$l'_m = l_m \quad 1 \leq m \leq N_c \quad \text{(same order),}$$

$$u_{i \text{ min}} \leq \text{Min}(n_1, n_2, \dots, n_{N_c}),$$

and

$$\text{Max}(n_1, n_2, \dots, n_{N_c}) \leq u_{i \text{ max}}.$$

In addition to the boundary conditions defined by matrix B which apply to individual parameters there are several boundary condition type constraints which apply to the lens system as a whole.

The first constraint is on the length of the lens system denoted by  $L$ . Let  $L_{\max}$  be an input. If  $L_{\max} = 0$ , then  $L$  is not constrained; however, if  $L_{\max} > 0$ , then  $L$  must satisfy  $L \leq L_{\max}$ . Let  $\delta_i$  ( $1 \leq i \leq N$ ) designate the signed axial distance from  $\sigma_0$  (entrance pupil) to  $\sigma_i$ , where  $N$  is the number of surfaces. Remembering that  $S_i$  is the signed axial distance from  $\sigma_{i-1}$  to  $\sigma_i$  it follows that:

$$\delta_1 = S_1$$

and 
$$\delta_i = \delta_{i-1} + S_i \quad 2 \leq i \leq N. \quad (8.056)$$

With  $\delta_{\min} = \text{Min}(\delta_i; 1 \leq i \leq N)$  and  $\delta_{\max} = \text{Max}(\delta_i; 1 \leq i \leq N)$ , the length  $L$  is given by:

$$L = |\delta_{\max} - \delta_{\min}|. \quad (8.057)$$

A second boundary violation which can occur is known as feathering which is a condition that results when two consecutive surfaces intersect one another as a consequence of certain parameters being altered too drastically during the design iteration. To check for feathering it is necessary to trace a ray (color 1) through the lens system and compare thickness at each element against a specified minimum  $\tau_{\min}$ .

From Section 5.1, thickness at  $\sigma_i$  is given by

$$\tau_i = \frac{n_{i+1}}{|n_{i+1}|} \lambda_i, \quad 1 \leq i \leq N - 1 \quad (8.057)$$

where

$$\lambda_i = z_{i+1} - (z_i - S_{i+1}). \quad (8.058)$$

If the ray in question misses or reflects at any surface it is treated as a boundary violation. If not then feathering has occurred if  $\tau_i \leq \tau_{\min}$  for some  $1 \leq i \leq N - 1$ . Feather checking should be performed for the following 4 rays where  $\Omega_F$  and  $\Omega_2$  are input and  $h_{\max} = h_0 + (N_H - 1)\Delta h$ .

Ray 1

$$(h, \theta) = (h_{\max}, 0.) \quad (\hat{x}_0, \hat{y}_0) = (0., -\Omega_F)$$

Ray 2

$$(h, \theta) = (h_{\max}, 0.) \quad (\hat{x}_0, \hat{y}_0) = (0., \Omega_F)$$

Ray 3

$$(h, \theta) = (0., 0.) \quad (\hat{x}_0, \hat{y}_0) = (0., \Omega_2)$$

Ray 4

$$(h, \theta) = (0., 0.) \quad (\hat{x}_0, \hat{y}_0) = (0., \Omega_F)$$

8.2.4 Termination of Iteration

Let  $K$  be the number of the current iteration where  $K \geq 1$ . The iteration commences with a set of design variables identified by  $J^{(K)} = [j_1 \ j_2 \ \dots \ j_n]$ , a nominal parameter vector  $\vec{U} = \vec{U}_0$ , and the corresponding error vector  $\vec{E}_0 \equiv \vec{E}(\vec{U}_0)$ .

The first step in the iteration is to recompute  $\Delta v_j$  as follows:

$$\Delta v_j = \text{Max}[|\Delta v_j|, \Delta u_{\min}] \quad 1 \leq j \leq j_{\max} \quad (8.059)$$

where  $\Delta u_{\min}$  is an input parameter.

There are two classes of failures which can be defined in connection with the design iteration.

1) Failures which occur prior to the inversion of matrix I. This class includes one or more ray failures during the computation of  $\vec{E} (\vec{U}_0 + \Delta \vec{U}_j)$   $1 \leq j \leq n$ , and unsuccessful inversion of matrix I due to its being singular.

2) Failures which occur following the inversion of matrix I. This class includes a negative improvement ( $\epsilon < 0$ ), a boundary violation, and a ray failure while computing  $\vec{E}^* \equiv \vec{E} (\vec{U}^*)$ . Before proceeding to the next iteration it is necessary to restore  $\vec{U}$  to  $\vec{U}_0$  and to modify  $\Delta v_{j_i}$  ( $1 \leq i \leq n$ ) as follows. With  $\Delta_{j_i}^{(c)} = \vec{I}_i \cdot \vec{I}_i$ , if  $\Delta_{j_i}^{(c)} \neq 0$  then:

$$\Delta v_{j_i} = \left| \Delta v_{j_i} \begin{bmatrix} (R) \\ \Delta_{j_i} \\ \Delta_{j_i}^{(c)} \end{bmatrix} \frac{1}{2} \right|. \quad (8.060)$$

Regardless of the type of failure the program should print  $J^{(K)}$  and a comment indicating the cause of the failure.

Let  $P_c$  be a counter which is initially set to zero. Each time an iteration fails set  $P_c = P_c + 1$  and compare it against  $2P$  where  $P$  is the incrementation period defined by equation (8.033). If  $P_c \geq 2P$  the design process should be aborted with a comment indicating there were  $2P$  successive failures. In order to insure that the  $2P$  failures are truly successive whenever the iteration is successful set  $P_c = 0$ . This procedure prevents wasteful looping when all possible design variable combinations result in a failure.

It is possible to remove some combination of design variables say  $J^{(K)} = [j_1 \ j_2 \ \dots \ j_n]$  from the iteration sequence when it results in a failure. This is accomplished by sending it to "jail" with a specified "sentence" and keeping it there until it has served its sentence at which time it is paroled. A maximum of 10 different offenders at any one time is permitted. Consider the following definitions.

$N_J$  The number of cycles (sentence) that each offender is jailed (input).  $N_J \geq 0$

$N_{OF}$  The current number of offenders which is initially zero.  
 $0 \leq N_{OF} \leq 10$ .

$H$  The jail matrix such that  $h_{ij}$   $1 \leq i \leq n$  contains the design variable numbers for the  $j$ th offender.

$G$  The sentence vector such that  $g_j$  contains the current sentence of the  $j$ th offender.

Suppose it has been determined that  $J^{(K)}$  resulted in a failure.

If  $N_J = 0$  or  $N_J > 0$  but  $N_{OF} = 10$  then  $J^{(K)}$  is not jailed. If  $N_J > 0$  and  $N_{OF} < 10$ ,  $J^{(K)}$  is jailed by setting:

$$N_{OF} = N_{OF} + 1$$

$$h_{iN_{OF}} = j_i \quad 1 \leq i \leq n$$

$$g_{N_{OF}} = N_J$$

Following the generation of  $J^{(K)}$  from  $J^{(K-1)}$  and prior to iterating with  $J^{(K)}$  it must be determined if  $J^{(K)}$  is currently in jail as the result of some previous offense (failure). If  $N_J = 0$  or  $N_{OF} = 0$  iterate with  $J^{(K)}$ . If  $N_J > 0$  and  $N_{OF} > 0$  check  $J^{(K)}$  against the  $N_{OF}$  columns of matrix H. If  $J^{(K)}$  does not match any column of H (not in jail) iterate with  $J^{(K)}$ .

Suppose  $j_i = h_{i,r}$  for all  $1 \leq i \leq n$  which means that  $J^{(K)}$  is the  $r$ th offender in jail. Reduce the sentence by setting  $g_r = g_r - 1$ . If  $g_r \neq 0$  do not iterate with  $J^{(K)}$ , instead proceed directly to  $J^{(K+1)}$ . If  $g_r = 0$  then  $J^{(K)}$  is paroled by setting:

$$h_{ij} = h_{i,j+1} \quad 1 \leq i \leq n \quad 1 \leq j \leq 9$$

$$g_j = g_{j+1} \quad 1 \leq j \leq 9$$

$$N_{OF} = N_{OF} - 1$$



After parolling  $J^{(K)}$  it is the used in the iteration.

Thus far the discussion has been limited to iterations which fail for one reason or another. When the iteration is successful the following quantities are printed:  $J^{(K)}$ ,  $\xi$ ,  $|I|$ ,  $|\vec{E}_0|$ ,  $|\vec{E}^*|$ , and  $\epsilon$ .  $\vec{U}^*$  replaces  $\vec{U}_0$  and  $\vec{E}^*$  should replace  $\vec{E}_0$ . In addition set  $P_c = 0$  and recompute  $\Delta v_{ji}$  as per equation (8.060).

## Section 9

### PROFILE DIAGRAM

The lens system is described by a set of  $N$  surfaces designated  $\{ \sigma_i ; 1 \leq i \leq N \}$ . The center of the entrance pupil ( $\sigma_0$ ) is taken as the origin of a  $z$ - $y$  coordinate system where  $z$  is along the optic axis and  $y$  is vertical. Let  $\delta_0 = 0$ , and let  $(\delta_i, 0)$  be the coordinates of the vertex of  $\sigma_i$ . From Section 8.2.3.

$$\delta_i = \delta_{i-1} + S_i \quad 1 \leq i \leq N. \quad (9.001)$$

With  $S^*$  is the directed distance from the vertex of  $\sigma_n$  to the vertex of the image surface. The image surface vertex has coordinates  $(\delta^*, 0)$  where:

$$\delta^* = \delta_n + S^* \quad (9.002)$$

One of the first steps is to establish the set of lens elements and the set of air spaces which taken together comprise the lens system.

#### 9.1 ELEMENT AND AIR SPACE MATRICES

An element is either a single surface (mirror) or a pair of consecutive surfaces between which the index of refraction exceeds one. An air space is a pair of consecutive surfaces between which the index of refraction is one.

The elements are described by a  $2 \times N_e$  matrix  $E$  of the form:

$$E = \begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1N_e} \\ e_{21} & e_{22} & \cdots & e_{2N_e} \end{pmatrix} \quad (9.003)$$

where  $N_e$  is the number of elements. The pair  $(e_{1k}, e_{2k})$  represents the surface numbers for element  $k$  such that:

$$e_{2k} = \begin{cases} e_{1k} + 1 & k \text{ is not a mirror} \\ e_{1k} & k \text{ is a mirror} \end{cases} \quad (9.004)$$

Similarly, the air spaces are described by a  $2 \times N_a$  matrix  $A$  of the form:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N_a} \\ a_{21} & a_{22} & \cdots & a_{2N_a} \end{pmatrix} \quad (9.005)$$

where  $N_a$  is the number of air spaces. The pair  $(a_{1k}, a_{2k})$  represents the surface numbers for the  $k$ th air space such that:

$$a_{2k} = a_{1k} + 1. \quad (9.006)$$

If  $e_{2, k-1} \neq e_{1, k-1}$  then  $e_{1k}$  satisfies:

$$e_{1k} = \begin{cases} e_{2, k-1} \\ e_{2, k-1} + 1 \end{cases} \quad (9.007)$$

If  $e_{1k} = e_{2, k-1}$  then  $(e_{1, k-1}, e_{2, k-1})$  and  $(e_{1k}, e_{2k})$  comprise a cemented doublet. If  $e_{1k} = e_{2, k-1} + 1$  there is an air space between  $(e_{1, k-1}, e_{2, k-1})$  and  $(e_{1k}, e_{2k})$ .

Let  $n_i$  represent the index of refraction (hereafter simply called index) between  $\sigma_{i-1}$  and  $\sigma_i$  and suppose  $n_1 = 1$ .

To establish E and A,  $N_e$  and  $N_a$  are initially set to zero. Starting with  $i = 1$  and continuing to  $i = n$  the pair  $(n_i, n_{i+1})$  is examined with the following possibilities:

$$1) \quad \underline{n_i = 1. \text{ and } n_{i+1} > 1.}$$

$(\sigma_i, \sigma_{i+1})$  is an element thus set:

$$N_e = N_e + 1$$

$$e_{1N_e} = i \quad e_{2N_e} = i + 1$$

$$2) \quad \underline{n_i = 1. \text{ and } n_{i+1} = -1.}$$

$\sigma_i$  is a mirror and there is an air space between  $\sigma_i$  and  $\sigma_{i+1}$  thus set:

$$N_e = N_e + 1$$

$$e_{1N_e} = i \quad e_{2N_e} = i$$

$$N_a = N_{a+1}$$

$$a_{1N_a} = i \quad a_{2N_a} = i+1$$

3)  $n_i > 1$  and  $n_{i+1} = 1$ .

There is an air space between  $\sigma_i$  and  $\sigma_{i+1}$  thus set:

$$N_a = N_{a+1}$$

$$a_{1N_a} = i \quad a_{2N_a} = i+1$$

4)  $n_i > 1$  and  $n_{i+1} > 1$ .

$(\sigma_i, \sigma_{i+1})$  is an element thus see 1.

5)  $n_i > 1$  and  $n_{i+1} = -n_i$

$\sigma_i$  is silvered glass and  $(\sigma_i, \sigma_{i+1})$  is an element  
thus see 1.

6)  $n_i = -1$  and  $n_{i+1} < -1$ .

$(\sigma_i, \sigma_{i+1})$  is an element thus see 1.

$$7) \quad \underline{n_i = -1. \text{ and } n_{i+1} = 1.}$$

$\sigma_i$  is a mirror and there is an air space between  $\sigma_i$  and  $\sigma_{i+1}$  thus see 2.

$$8) \quad \underline{n_i < -1. \text{ and } n_{i+1} = -1.}$$

There is an air space between  $\sigma_i$  and  $\sigma_{i+1}$  thus see 3.

$$9) \quad \underline{n_i < -1. \text{ and } n_{i+1} < -1.}$$

$(\sigma_i, \sigma_{i+1})$  is an element thus see 1.

$$10) \quad \underline{n_i < -1. \text{ and } n_{i+1} = n_i}$$

$\sigma_i$  is silvered glass and  $(\sigma_i, \sigma_{i+1})$  is an element thus see 1.

The following 4 conditions are included in order to allow for dummy surfaces which will not be plotted. A dummy surface is a surface with the same index of refraction on both sides.

$$11) \quad \underline{n_i = 1. \text{ and } n_{i+1} = 1.}$$

$\sigma_i$  is a dummy surface and it is necessary to correct the current air space. If  $N_a > 0$  set:

$$a_{2N_a} = i + 1$$

$$12) \quad \underline{n_i = -1. \text{ and } n_{i+1} = -1.}$$

$\sigma_i$  is a dummy surface (in air) thus see 11.

$$13) \quad \underline{n_i > 1. \text{ and } n_{i+1} = n_i}$$

$\sigma_i$  is a dummy surface in glass thus set:

$$e_{2N_e} = i + 1$$

$$14) \quad \underline{n_i < -1. \text{ and } n_{i+1} = n_i}$$

$\sigma_i$  is a dummy surface (in glass) thus see 13.

Because of the possibility (in a reflective system) that one or more elements may be repeated it is necessary to edit matrix E by deleting all equivalent elements. Each element is identified by a pair (i, j) where i and j are the surface numbers of the bounding surfaces.

Let  $(i_1, j_1)$  and  $(i_2, j_2)$  be two arbitrary elements. These two elements are equivalent,  $(i_1, j_1) \sim (i_2, j_2)$ , if they both describe the same physical lens element.

It follows that:

$$(i, j) \equiv (i, j) \quad (9.008)$$

and

$$(i, j) \equiv (j, i). \quad (9.009)$$

For arbitrary  $\sigma_i$  let  $\delta_i$ ,  $\tau_i$ ,  $R_i$ , and  $b_i$  represent vertex coordinate, type, cu.vature and eccentricity, respectively.

An element  $(i, j)$  is ordered if  $\delta_i \leq \delta_j$ . Any arbitrary element may be expressed in ordered form by invoking equation (9.009).

Suppose that  $(i_1, j_1)$  and  $(i_2, j_2)$  are both ordered elements. If  $i_1 = i_2$ , and  $j_1 = j_2$  they are (trivially) equivalent. They are also (non-trivially) equivalent if the following conditions are satisfied:

$$\delta_{i1} = \delta_{i2}$$

$$\tau_{i1} = \tau_{i2}$$

$$R_{i1} = R_{i2}$$

$$b_{i1} = b_{i2}$$

$$\delta_{j1} = \delta_{j2}$$

$$\tau_{j1} = \tau_{j2}$$

$$R_{j1} = R_{j2}$$

$$b_{j1} = b_{j2}$$



The editing process applied to E should retain only one member from each set of equivalent elements.

A similar procedure is used to edit matrix A except that the conditions for non-trivial equivalence are simply:

$$\delta_{i1} = \delta_{i2}$$

$$\delta_{j1} = \delta_{j2}$$

Because of the complexity entailed with translated and/or tilted elements only those systems composed of surface types 1, 2, and 3 will be considered where Type 1 is a plane or standard conic of revolution, Type 2 is a horizontal conic cylinder and Type 3 is a vertical conic cylinder.

The z-y trace of a Type 1 ( $R = 0$ ) or Type 3 surface is simply the local y axis, that is,

$$z = 0 \quad (9.010)$$

The z-y trace of a Type 1 ( $R \neq 0$ ) or Type 2 surface is a plane conic specified by

$$y^2 + (z - R)^2 = R^2 + bz^2. \quad (9.011)$$

where the origin (0, 0) is at the vertex of the conic.

Since (9.010) and (9.011) are symmetric in the z-axis only the upper half ( $y \geq 0$ ) need be generated which is then reflected in the z-axis to produce the lower half.

## 9.2 RANGE MATRICES

Let  $\hat{z}$  and  $\hat{y}$  be  $2 \times N_e$  range matrices of the form:

$$\hat{z} = \begin{bmatrix} \hat{z}_{11} & \hat{z}_{12} & \cdots & \hat{z}_{1N_e} \\ \hat{z}_{21} & \hat{z}_{22} & \cdots & \hat{z}_{2N_e} \end{bmatrix} \quad (9.012)$$

$$\hat{y} = \begin{bmatrix} \hat{y}_{11} & \hat{y}_{12} & \cdots & \hat{y}_{1N_e} \\ \hat{y}_{21} & \hat{y}_{22} & \cdots & \hat{y}_{2N_e} \end{bmatrix} \quad (9.013)$$

The pairs  $(\hat{z}_{1i}, \hat{z}_{2i})$  and  $(\hat{y}_{1i}, \hat{y}_{2i})$  apply to the  $i$ th element,  $(e_{1i}, e_{2i})$ , in particular,  $(\hat{z}_{1i}, \hat{y}_{1i})$  refers to  $e_{1i}$  and  $(\hat{z}_{2i}, \hat{y}_{2i})$  to  $e_{2i}$ . Let  $f_1(z, y) = 0$  be the  $z - y$  trace of  $e_{1i}$ . The pair  $(\hat{z}_{1i}, \hat{y}_{1i})$  specifies the range of  $z$  and  $y$  over which  $f_1(z, y)$  is defined.

If  $f_1(z, y) = z = 0$ , then  $\hat{z}_{1i} = 0$ , and  $\hat{y}_{1i} > 0$ .

Suppose that  $e_{1i}$  is a plane conic, that is,  $R_1 \neq 0$ . If  $R_1 > 0$ , then  $f_1(z, y)$  is defined for  $0 < z \leq \hat{z}_{1i}$ . If  $R_1 < 0$ , the  $f_1(z, y)$  is defined for  $\hat{z}_{1i} \leq z \leq 0$ . The range of  $y$  is always  $0 \leq y \leq \hat{y}_{1i}$ .

Let  $\hat{C}$  be a  $2 \times N_e$  code matrix of the form:

$$\hat{C} = \begin{bmatrix} \hat{C}_{11} & \hat{C}_{12} & \cdots & \hat{C}_{1N_e} \\ \hat{C}_{21} & \hat{C}_{22} & \cdots & \hat{C}_{2N_e} \end{bmatrix} \quad (9.014)$$

where  $\hat{C}_{1i}$  refers to  $e_{1i}$  and  $\hat{C}_{2i}$  to  $e_{2i}$ . Each element in  $\hat{C}$  is a code or flag whose setting is related to some constraint on the corresponding curve which causes it to be confined to some range specified by  $\hat{z}$ . For example, if  $e_{1i}$  is a circle (or ellipse)  $f_i(z, y)$  is not defined beyond the maximum point (see Figure 22). In this case  $\hat{C}_{1i} = 1$ .

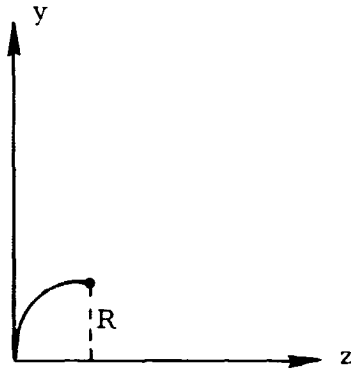


Figure 22. Code 1

### 9.3 MAXIMUM APERTURE CONSTRAINT

The first constraint to be applied to the system is the maximum aperture constraint whereby all curves are limited so that  $-y_{\max} \leq y \leq y_{\max}$  where  $y_{\max} > 0$ . is an input parameter. The code ( $\hat{C}$ ) for this type of constraint is 2. Each type of curve is considered below.

#### A. Plane

The equation is  $z = 0$ . so that:

$$\hat{z}_{1i} = 0.$$

B. Circle (b = 0.)

The equation is:

$$y^2 + (z - R)^2 = R^2 \quad (9.016)$$

If  $R > 0$ :

$$\hat{z}_{li} = \begin{cases} R & y_{\max} \geq R \\ R - \left\{ R^2 - y_{\max}^2 \right\}^{1/2} & y_{\max} < R \end{cases} \quad (9.017)$$

$$\hat{y}_{li} = \begin{cases} R & y_{\max} \geq R \\ y_{\max} & y_{\max} < R \end{cases} \quad (9.018)$$

$$\hat{C}_{li} = \begin{cases} 1 & y_{\max} \geq R \\ 2 & y_{\max} < R \end{cases} \quad (9.019)$$

If  $R < 0$ :

$$\hat{z}_{li} = \begin{cases} R & y_{\max} \geq |R| \\ R + \left\{ R^2 - y_{\max}^2 \right\}^{1/2} & y_{\max} < |R| \end{cases} \quad (9.020)$$

$$\hat{y}_{li} = \begin{cases} |R| & y_{\max} \geq |R| \\ y_{\max} & y_{\max} < |R| \end{cases} \quad (9.021)$$

$$\hat{C}_{li} = \begin{cases} 1 & y_{\max} \geq |R| \\ 2 & y_{\max} < |R| \end{cases} \quad (9.022)$$

C. Parabola (b = 1)

The equation is

$$y^2 = 2 R z \quad (9.023)$$

Thus:

$$\hat{z}_{li} = \frac{y_{\max}^2}{2R} \quad (9.024)$$

$$\hat{y}_{li} = y_{\max} \quad (9.025)$$

$$\hat{C}_{li} = 2 \quad (9.026)$$

D. Ellipse ( $0 < |b| < 1$ )

The equation is

$$y^2 + (z - R)^2 = R^2 + bz^2. \tag{9.027}$$

Let

$$a_1 = \frac{|R|}{(1 - b)^{1/2}} \tag{9.028}$$

$$a_2 = \frac{R}{1 - b} \tag{9.029}$$

If  $R > 0$  then:

$$\hat{z}_{li} = \begin{cases} a_2 & y_{\max} \geq a_1 \\ a_2 - \left\{ \frac{a_1^2 - y_{\max}^2}{1 - b} \right\}^{1/2} & y_{\max} < a_1 \end{cases} \tag{9.030}$$

$$\hat{y}_{li} = \begin{cases} a_1 & y_{\max} \geq a_1 \\ y_{\max} & y_{\max} < a_1 \end{cases} \tag{9.031}$$

$$\hat{C}_{li} = \begin{cases} 1 & y_{\max} \geq a_1 \\ 2 & y_{\max} < a_1 \end{cases} \tag{9.032}$$

If  $R < 0$  then:

$$\hat{z}_{li} = \begin{cases} a_2 & y_{\max} \geq a_1 \\ a_2 + \left\{ \frac{a_1^2 - y_{\max}^2}{1 - b} \right\}^{1/2} & y_{\max} < a_1 \end{cases} \quad (9.033)$$

$$\hat{y}_{li} = \begin{cases} a_1 & y_{\max} \geq a_1 \\ y_{\max} & y_{\max} < a_1 \end{cases} \quad (9.034)$$

$$\hat{C}_{li} = \begin{cases} 1 & y_{\max} \geq a_1 \\ 2 & y_{\max} < a_1 \end{cases} \quad (9.035)$$

E. Hyperbola ( $b > 1$ )

The equation is

$$y^2 + (z - R)^2 = R^2 + b z^2 \quad (9.036)$$

Let:

$$a_1 = \frac{|R|}{(b - 1)^{1/2}} \quad (9.037)$$

$$a_2 = \frac{-R}{b - 1} \quad (9.038)$$

$$\hat{z}_{1i} = \begin{cases} a_2 + \left\{ \frac{a_1^2 + y_{\max}^2}{b-1} \right\}^{1/2} & R > 0. \\ a_2 - \left\{ \frac{a_1^2 + y_{\max}^2}{b-1} \right\}^{1/2} & R < 0. \end{cases} \quad (9.039)$$

$$\hat{y}_{1i} = y_{\max} \quad (9.040)$$

$$\hat{C}_{1i} = 2 \quad (9.041)$$

#### 9.4 INTERSECTION CONSTRAINTS

Consider some arbitrary element, say  $(e_{1i}, e_{2i})$ , and suppose  $e_{1i} \neq e_{2i}$ . Assume that  $R_1 > 0$  and  $R_2 > 0$  so that the element is equiconvex. It is clear that, should the two curves describing the element intersect at a point, the range matrices should be edited so that the curves are not allowed to proceed beyond the point of intersection. This situation is illustrated in Figure 23.

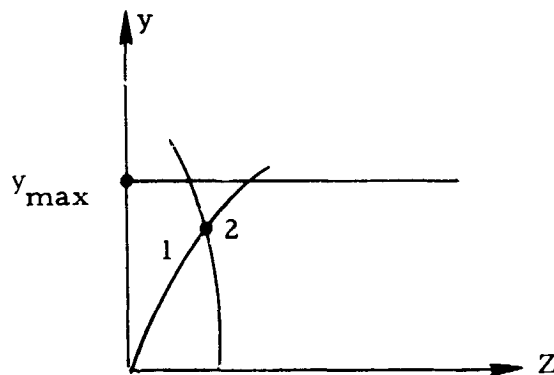


Figure 23. Equiconvex Element  
Illustrating Intersection Constraint



Each curve in the lens system is described by an equation of the form

$$f(z, y) = 0. \quad (9.042)$$

where  $z_{\min} \leq z \leq z_{\max}$  is the domain of definition specified by matrix  $\hat{z}$ .

Consider the problem of determining the point of intersection (if it exists) of two such curves, say  $f_1(z, y) = 0$  and  $f_2(z, y) = 0$ .

Let:

$$i_1 = \text{surface number of curve 1}$$

$$\delta_1 = \text{coordinate of vertex of curve 1}$$

$$[z_{1, \min}, z_{1, \max}] = \text{Interval of definition of curve 1 referenced to its vertex}$$

$$i_2 = \text{surface number of curve 2}$$

$$\delta_2 = \text{coordinate of vertex of curve 2}$$

$$[z_{2, \min}, z_{2, \max}] = \text{Interval of definition of curve 2 referenced to its vertex.}$$

Choose a  $z - y$  coordinate system with the origin at the vertex of curve 1. The vertex of curve 2 is at the point  $(S, 0.)$  where

$$S = \delta_2 - \delta_1. \quad (9.043)$$

The two curves are given by:

$$f_1(z, y) = 0. \quad z_{1, \min} \leq z \leq z_{1, \max} \quad (9.044)$$

$$f_2(z - S, y) = 0. \quad z'_{2, \min} \leq z \leq z'_{2, \max} \quad (9.045)$$

where

$$z'_{2, \min} = z_{2, \min} + S \quad (9.046)$$

$$z'_{2, \max} = z_{2, \max} + S \quad (9.047)$$

Interpreting the closed interval  $[z_1, z_2]$  as a point set, let:

$$[z_A, z_B] = [z_{1, \min}, z_{1, \max}] \cap [z'_{2, \min}, z'_{2, \max}] \quad (9.0471)$$

If  $[z_A, z_B] = \phi$  (void set) then  $f_1$  and  $f_2$  do not intersect. Suppose then, that  $[z_A, z_B] \neq \phi$  and consider those cases when  $z_A = z_B$ .

If  $z_A = z_B = 0$ . it follows that  $f_1(z, y) = z = 0$ . and the point of intersection,  $(z_I, y_I)$ , is given by:

$$z_I = 0. \quad (9.048)$$

$$y_I = \left\{ (b_2 - 1) S^2 - 2R_2 S \right\}^{1/2} \quad (9.049)$$

If  $z_A = z_B = S$  then  $f_2(z, y) = z = 0$ . and the point of intersection is given by:

$$z_I = S \quad (9.050)$$

$$y_I = \left\{ (b_1 - 1) S^2 + 2R_1 S \right\}^{1/2} \quad (9.051)$$

These two cases which involve a plane are illustrated in Figure 24.

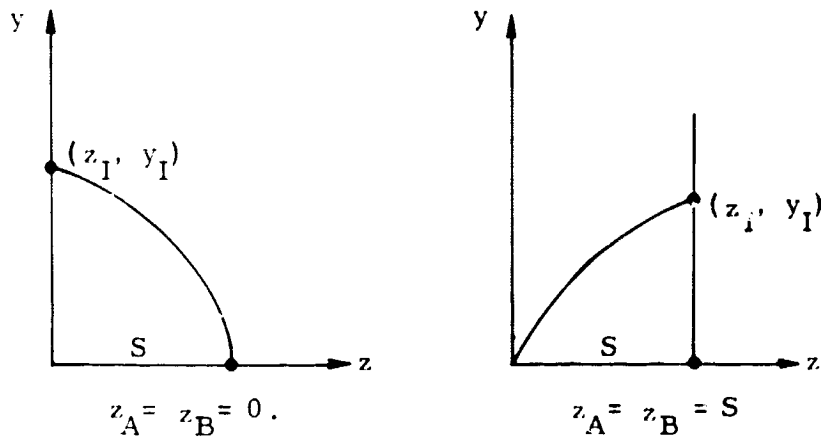


Figure 24. Intersection of Conic and Plane

All other cases are such that

$$f_1(z, y) = y^2 + (1 - b_1) z^2 - 2R_1 z = 0. \quad (9.052)$$

$$f_2(z - S, y) = y^2 + (1 - b_2)(z - S)^2 - 2R_2(z - S) = 0. \quad (9.053)$$

Suppose that  $z_A = z_B = C$  where  $0 < C < S$ . From equations (9.052) and (9.053):

$$y_1(C) = \left\{ (b_1 - 1)C^2 + 2R_1 C \right\}^{1/2}$$

$$y_2(C) = \left\{ (b_2 - 1)(C - S)^2 + 2R_2(C - S) \right\}^{1/2}$$

The curves intersect only if  $y_1(C) = y_2(C)$  in which case

$$z_I = C \quad (9.054)$$

$$y_I = y_I(C). \quad (9.055)$$

Finally, suppose  $z_A < z_B$  so that  $(z_I, y_I)$  is the simultaneous solution of equations (9.052) and (9.053). Subtracting the two equations and simplifying yields:

$$\alpha z^2 + 2\beta z + \gamma = 0. \quad (9.056)$$

with

$$\alpha = b_1 - b_2 \quad (9.057)$$

$$\beta = R_1 - R_2 + S(b_2 - 1) \quad (9.058)$$

$$\gamma = 2R_2S - S^2(b_2 - 1) \quad (9.059)$$

Suppose  $\alpha = 0$  and let:

$$z_I = \frac{-\gamma}{2\beta} \quad (9.060)$$

If  $z_I \notin [z_A, z_B]$  the curves do not intersect; otherwise,

$$z_I = z_1 \quad (9.061)$$

$$y_I = \left\{ (b_1 - 1) z_1^2 + 2R_1 z_1 \right\}^{1/2} \quad (9.062)$$

Suppose  $\alpha \neq 0$  and define  $\Delta$  so that:

$$\Delta = \beta^2 - \alpha\gamma \quad (9.063)$$

There is no intersection if  $\Delta < 0$ . If  $\Delta = 0$  let:

$$z_1 = -\frac{\beta}{\alpha}. \quad (9.064)$$

If  $z_A \leq z_1 \leq z_B$  the point of intersection is given by equations (9.061) and (9.062).

Finally, suppose  $\Delta > 0$  and let:

$$z_1 = -\frac{\beta}{\alpha} + \frac{\Delta^{1/2}}{\alpha} \quad (9.065)$$

$$z_2 = -\frac{\beta}{\alpha} - \frac{\Delta^{1/2}}{\alpha} \quad (9.066)$$

If  $z_1 \notin [z_A, z_B]$  and  $z_2 \notin [z_A, z_B]$  there is no point of intersection.

If  $z_1 \in [z_A, z_B]$  but  $z_2 \notin [z_A, z_B]$  the point is given by (9.061) and (9.062). If  $z_1 \notin [z_A, z_B]$  but  $z_2 \in [z_A, z_B]$  the point is given by (9.061) and (9.062) using  $z_2$  in place of  $z_1$ .

Suppose that  $z_1 \in [z_A, z_B]$  and  $z_2 \in [z_A, z_B]$ , that is, there are two points of intersection. If  $R_1 > 0$  then  $[z_{1, \min}, z_{1, \max}] = [0., z_{1, \max}]$  and, consequently,  $0. \leq z_A < z_B$ . Thus  $z_1 \geq 0.$  and  $z_2 \geq 0.$  so that:

$$z_I = \min(z_1, z_2) \quad (9.067)$$

$$y_I = \left\{ (b_1 - 1) z_I^2 + 2R_1 z_I \right\}^{1/2} \quad (9.068)$$

If  $R_1 < 0$ , then  $[z_{1, \min}, z_{1, \max}] = [z_{1, \min}, 0]$  and  $z_A < z_B \leq 0$ .

Therefore  $z_1 \leq 0$ , and  $z_2 \leq 0$ , so that:

$$z_I = \max(z_1, z_2) \quad (9.069)$$

and  $y_I$  is given by (9.068).

In order to edit  $\hat{z}$  and  $\hat{y}$  by subjecting the elements to intersection constraints, consider the  $i$ th element and suppose  $e_{1i} \neq e_{2i}$  and let the equation of  $e_{1i}$  be (9.044) and the equation of  $e_{2i}$  be (9.045).

The interval  $[z_{1, \min}, z_{1, \max}]$  which is relative to the vertex of  $e_{1i}$  is given by:

$$[z_{1, \min}, z_{1, \max}] = \begin{cases} [0., \hat{z}_{1i}] & \hat{z}_{1i} \geq 0. \\ [\hat{z}_{1i}, 0.] & \hat{z}_{1i} < 0. \end{cases} \quad (9.070)$$

Similarly:

$$[z_{2, \min}, z_{2, \max}] = \begin{cases} [0., \hat{z}_{2i}] & \hat{z}_{2i} \geq 0. \\ [\hat{z}_{2i}, 0.] & \hat{z}_{2i} < 0. \end{cases} \quad (9.071)$$

Suppose  $e_{1i}$  and  $e_{2i}$  intersect at  $(z_I, y_I)$  relative to the vertex of  $e_{1i}$ .

The coordinates relative to the vertex of  $e_{2i}$  are:

$$z'_I = z_I - S \quad (9.072)$$

$$y_I^1 = y_I \quad (9.073)$$

where  $S$  is the coordinate of the vertex of  $e_{2i}$  relative to the vertex of  $e_{1i}$ .

The point of intersection is valid provided  $z_{1, \min} \leq z_I \leq z_{1, \max}$ ,  $z_{2, \min} \leq z_I \leq z_{2, \max}$ ,  $0 \leq y_I \leq \hat{y}_{1i}$ , and  $0 \leq y_I \leq \hat{y}_{2i}$ . In this instance set:

$$\hat{z}_{1i} = z_I$$

$$\hat{z}_{2i} = z_I$$

$$\hat{y}_{1i} = \hat{y}_{2i} = y_I$$

$$\hat{C}_{1i} = \hat{C}_{2i} = 3$$

where "3" is the code for an intersection.

### 9.5 SEPARATION CONSTRAINTS

Consider two elements, say A and B, which are separated by an air space but shaped so that they intersect (see Figure 25).

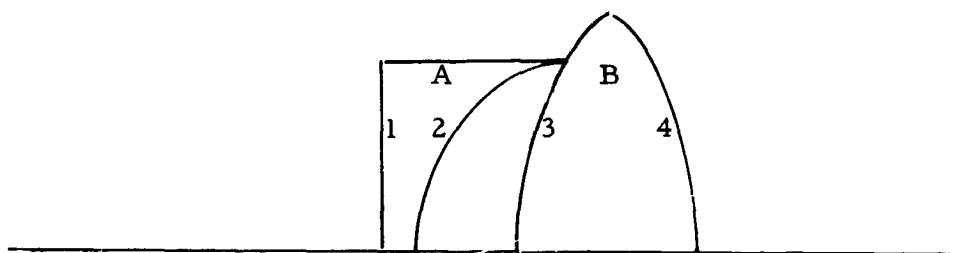


Figure 25. Intersecting Elements

This type of intersection is not detected when the system is subjected to aperture and intersection constraints; therefore, further editing is required which is designed to prevent elements from coming to close, namely within  $\Delta z_{\min}$  of one another in the z direction. (See Figure 26).

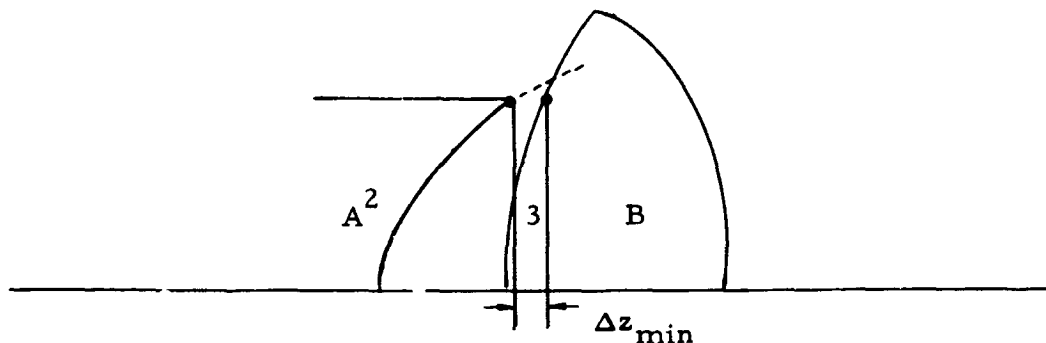


Figure 26. Separation Constraint



Let equations (9.043), (9.044), and (9.045) and the definitions pertaining to them represent two curves separated by an air space such that  $S > 0$ . With the origin at the vertex of curve 1, let:

$P_I : (z_I, y_I)$  be the point of intersection,

$P_1 : (z_1, y_1)$  be a point on curve 1,

and  $P_2 : (z_2, y_2)$  be a point on curve 2, such that

$$z_2 - z_1 = \Delta z_{\min} \quad (9.074)$$

where  $\Delta z_{\min} > 0$  is a specified number.

The problem is to determine  $P_1$  and  $P_2$  given the equations of the curves.

Consider first the case  $z_A = z_B = 0$ , or  $f_1(z, y) = z = 0$ , so that:

$$z_1 = 0. \quad (9.075)$$

$$z_2 = \Delta z_{\min} \quad (9.076)$$

$$y_1 = \left\{ (b_2 - 1)(z_2 - S)^2 + 2R_2(z_2 - S) \right\}^{1/2} \quad (9.077)$$

as illustrated in Figure 27.

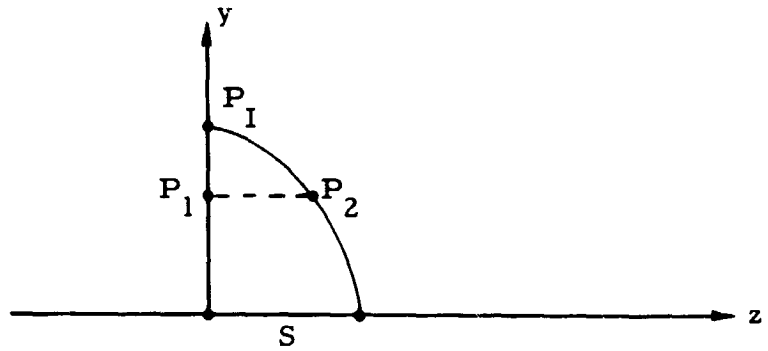


Figure 27.  $P_1$  and  $P_2$  when

$$f_1(z, y) = z = 0.$$

Next, suppose that  $z_A = z_B = S$  or  $f_2(z - S, y) = z - S$  so that:

$$z_1 = S - \Delta z_{\min} \tag{9.078}$$

$$z_2 = S \tag{9.079}$$

$$y_1 = \left\{ (b_1 - 1) z_1^2 + 2R_1 z_1 \right\}^{1/2} \tag{9.080}$$

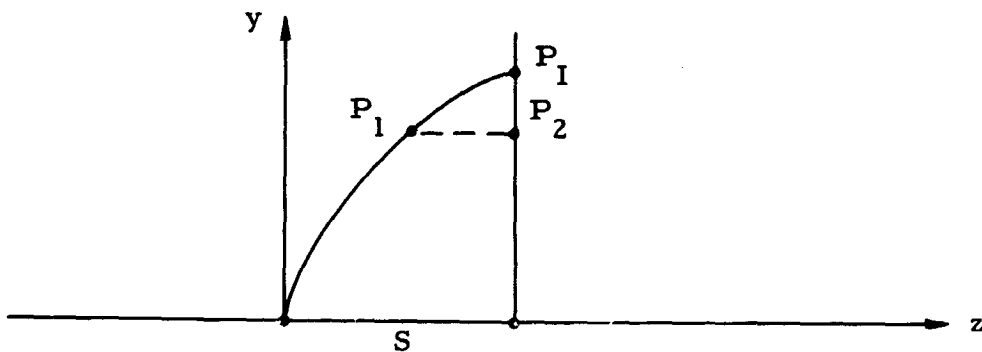


Figure 28.  $P_1$  and  $P_2$  when

$$f_2(z - S, y) = z - S = 0.$$

Finally, suppose  $z_A \leq z_B$  and solve equation (9.052) for  $y = y(z_1)$  and (9.053) for  $z_2 = z_2(y)$  to give:

$$y(z_1) = \left\{ (b_1 - 1) z_1^2 + 2 R_1 z_1 \right\}^{1/2} \quad (9.081)$$

$$z_2(y) = S + \frac{y^2}{R_2 + \epsilon \left\{ R_2^2 - (1 - b_2) y^2 \right\}^{1/2}} \quad (9.082)$$

$$\epsilon = \begin{cases} 1. & R_2 > 0. \\ -1. & R_2 < 0. \end{cases} \quad (9.083)$$

Treating  $z_1$  as an unknown let:

$$g(z_1) = \left[ z_2(y) - z_1 \right] - \Delta z_{\min} \quad (9.084)$$

At the point of intersection  $P_I$ :

$$z_1 = z_2 = z_I$$

so that  $g(z_I) = -\Delta z_{\min} < 0$ .

At the origin (0., 0.) we have  $z_1 = y = 0$ . and, from (9.082),  $z_2 = S$  so that  $g(0) = S - \Delta z_{\min}$ . The parameter  $\Delta z_{\min}$  must be chosen so that  $\Delta z_{\min} < S$ ; otherwise, the elements are spaced too close axially. Therefore  $g(0) > 0$ .

The problem is to determine that value of  $z_1$ , say  $z_T$ , such that  $g(z_T) = 0$ . Let  $z_p$  be any value of  $z_1$  such that  $g(z_p) > 0$ . and  $z_M$  any value such that  $g(z_M) < 0$ . , in particular,  $z_p = 0$ . and  $z_M = z_1$ . Having determined  $z_T$ ,  $P_1$  and  $P_2$  are given by:

$$z_1 = z_T$$

$$y_1 = y(z_T)$$

$$z_2 = z_2(y_1)$$

With  $(0, z_1)$  as starting values for  $(z_p, z_M)$  consider the following 7 step iterative procedure for establishing  $z_T$ ,  $P_1$ , and  $P_2$ .

1) SET

$$z_T = \frac{z_p + z_M}{2}$$

2) COMPUTE

$$y(z_T) \quad (\text{Equation 9.081})$$

$$z_2 [y(z_T)] \quad (\text{Equation 9.082})$$

$$g(z_T) = (z_2 - z_T) - \Delta z_{\min}$$

- 3) IF: (a)  $g(z_T) = 0$ . ; (7)  
(b)  $g(z_T) > 0$ . ; (4)  
(c)  $g(z_T) < 0$ . ; (5)

4) SET

$$z_p = z_T$$

Go to (6)

5) SET

$$z_M = z_T$$

- 6) IF: (a)  $\left| \frac{z_p - z_M}{z_M} \right| \geq 10^{-6}$ ; (1)  
(b)  $\left| \frac{z_p - z_M}{z_M} \right| < 10^{-6}$ ; (7)

7) SET

$$P_1 = \{ z_T, y(z_T) \}$$
$$P_2 = \{ z_2(y), y(z_T) \}$$

The  $z$  coordinate of  $P_2$  relative to the vertex of curve 2 is  
given by:

$$z'_2 = z_2 - S \quad (9.085)$$

If the curves do not intersect, that is,  $P_I$  does not exist, it is assumed that they are sufficiently well separated for all  $z$ . The effect of separation constraints on  $\hat{z}$ ,  $\hat{y}$ , and  $\hat{C}$  may now be described by considering an example of two elements:

a) Element A composed of surfaces  $i_1$  and  $i_2$

and

b) Element B composed of surfaces  $i_3$  and  $i_4$ .

The range and code data for element A is given by  $\hat{z}_{1A}, \hat{z}_{2A}, \dots, \hat{C}_{2A}$  and for element B by  $\hat{z}_{1B}, \hat{z}_{2B}, \dots, \hat{C}_{2B}$ .

Suppose curves  $i_2$  and  $i_3$  were found to intersect and that  $P_1$  and  $P_2$  have been computed employing the scheme described above.

$$9.5.1 \quad \underline{\hat{C}_{2A} = 3 \text{ and } \hat{C}_{1B} \neq 3}$$

In this instance A is an intersecting element and B is not.

Set:

$$\hat{z}_{1B} = z'_2$$

$$\hat{y}_{1B} = y_1$$

$$S = \delta_{i_4} - \delta_{i_3}$$

Ultimately, each element must be a closed curve composed of one or more arcs and wherever possible the arcs should proceed to the boundary  $y = y_{\max}$ .

Suppose B is not a mirror and consider the case  $R_{i_2} \leq 0$ . (See Figure 29).

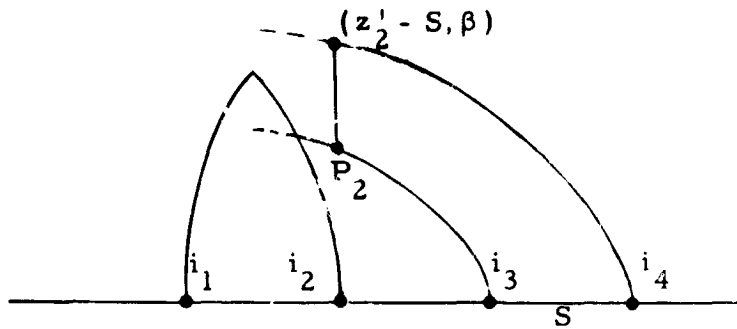


Figure 29. A is Intersecting and  $R_{i_2} \leq 0$ .

In this case set  $\hat{C}_{1B} = 4$  which states that a vertical line from  $P_2$  towards the line  $y = y_{\max}$  is permissible. It is necessary to determine if curve  $i_4$  intersects the line  $z = z_2' - S$ . Let  $(z_2' - S, \beta)$  be the point of intersection (if it exists) and set:

$$\hat{z}_{2E} = z_2' - S$$

$$\hat{y}_{2B} = \beta$$

$$\hat{C}_{2B} = 1$$

If  $R_{i_2} > 0$ , set  $\hat{C}_{1B} = 5$  which states that a vertical line from  $P_2$  towards  $y = y_{\max}$  is not permissible. Let  $(\beta, y_1)$  be the point where  $i_4$  intersects  $y = y_1$  (if it exists) and set:

$$\hat{z}_{2B} = \beta$$

$$\hat{y}_{2B} = y_1$$

$$\hat{C}_{2B} = 1$$

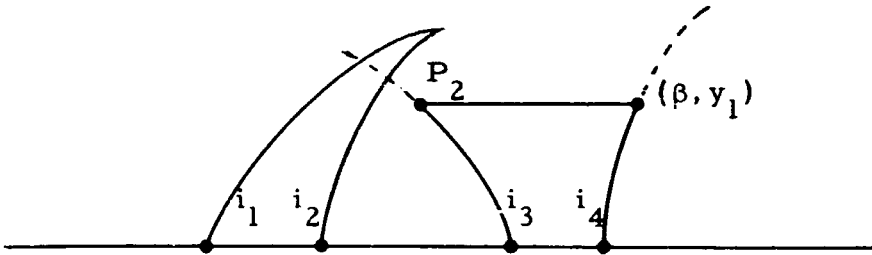


Figure 30. A is Intersecting and  $R_{i_2} > 0$ .

9.5.2  $\hat{C}_{2A} \neq 3$  and  $\hat{C}_{1B} = 3$

Since B is intersecting and A is not set:

$$\hat{z}_{2A} = z_1$$

$$\hat{y}_{2A} = y_1$$

$$S = \delta_{i_2} - \delta_{i_1}$$

Suppose A is not a mirror and  $R_{i_3} \geq 0$ , as illustrated in Figure 31.

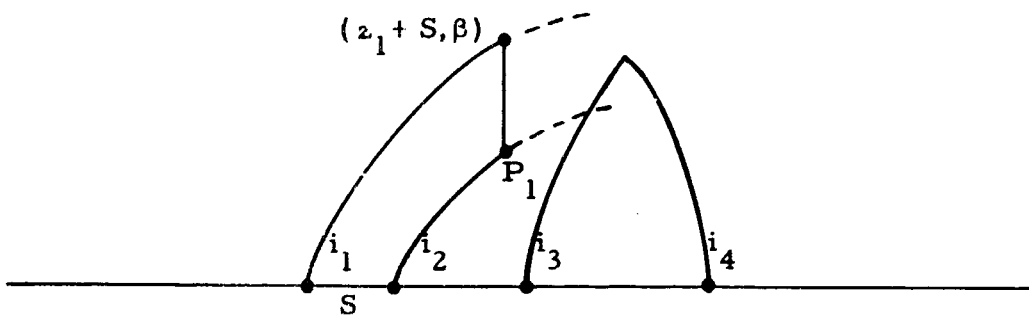


Figure 31. B is Intersecting and  $R_{i_3} \geq 0$ .



Set  $\hat{C}_{2A} = 4$  and let  $(z_1 + S, \beta)$  be the point where  $i_1$  intersects  $z = z_1 + S$ .

If this point exists set:

$$\hat{z}_{1A} = z_1 + S$$

$$\hat{y}_{1A} = \beta$$

If  $R_{i_3} < 0$ . set  $\hat{C}_{2A} = 5$  and  $(\hat{z}_{1A}, \hat{y}_{1A}) = (\beta, y_1)$  where  $(\beta, y_1)$  is the point of intersection of  $i_1$  and  $y = y_1$  (see Figure 32).

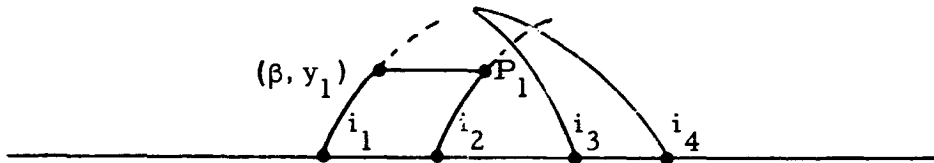


Figure 32. B is Intersecting and  $R_{i_3} < 0$ .

9.5.3  $\hat{C}_{2A} = 3$  and  $\hat{C}_{1B} = 3$

The case where both elements are intersecting is treated in exactly the same manner as the case where A is and B is not which is discussed in Section 9.5.1.

9.5.4  $\hat{C}_{2A} \neq 3$  and  $\hat{C}_{1B} \neq 3$

When neither element is intersecting there are several special cases to consider.

If  $R_{i_2} \leq 0$ . set:

$$\hat{z}_{1B} = z'_2$$

$$\hat{y}_{1B} = y_1$$

$$S = \delta_{i_4} - \delta_{i_3}$$

$$\hat{C}_{1B} = 4$$

and  $(\hat{z}_{2B}, \hat{y}_{2B}) = (z'_2 - S, \beta)$  which is the point where  $i_4$  intersects  $z = z'_2 - S$ .

If  $R_{i_2} > 0$ . and  $R_{i_3} \geq 0$ . set  $(\hat{z}_{2A}, \hat{y}_{2A}) = (z_1, y_1)$  and

$S = \delta_{i_2} - \delta_{i_1}$ . If A is not a mirror set  $\hat{C}_{2A} = 4$  and  $(\hat{z}_{1A}, \hat{y}_{1A})$

$= (z_1 + S, \beta)$  which is the point where  $i_1$  intersects  $z = z_1 + S$ .

If  $R_{i_2} > 0$ . and  $R_{i_3} < 0$ . set  $(\hat{z}_{2A}, \hat{y}_{2A}) = (z_1, y_1), (\hat{z}_{1B}, \hat{y}_{1B})$

$= (z'_2, y_1)$ .  $\hat{C}_{2A} = 4$ , and  $\hat{C}_{1B} = 4$ . If A is not a mirror set  $(\hat{z}_{1A}, \hat{y}_{1A})$

$= (z_1 + S, \beta)$  which is the point where  $i_1$  intersects  $z = z_1 + S$ . If B is not a mirror set  $(\hat{z}_{2B}, \hat{y}_{2B}) = (z'_2 - S, \beta)$  which is the point where  $i_4$  intersects  $z = z'_2 - S$ .

9.6 ARCS MATRIX

Let A be a 2 x 5 x 100 arcs matrix whose ith layer contains the end points of each of the arcs which comprise element i, that is,

$$\begin{bmatrix} z_1 & z_2 & \dots & z_N \\ y_1 & y_2 & \dots & y_N \end{bmatrix}.$$

If element i is bounded by surfaces  $i_1$  and  $i_2$  then each point is relative to the vertex of  $i_1$ . The jth arc has end points  $(z_j, y_j)$  and  $(z_{j+1}, y_{j+1})$  with  $1 \leq j \leq N-1$  and  $(z_1, y_1) = (0., 0.)$ .

If i is a mirror ( $i_1 = i_2$ ) then  $N = 2$  and  $(z_2, y_2) = (\hat{z}_{1i}, \hat{y}_{1i})$ .

If i is an intersecting element ( $\hat{C}_{1i} = 3$ ) then  $N = 3$ ,  $(z_2, y_2) = (\hat{z}_{1i}, \hat{y}_{1i})$ , and  $(z_3, y_3) = (S, 0.)$  where  $S = \delta_{i2} - \delta_{i1}$ .

With  $S = \delta_{i2} - \delta_{i1}$ , suppose first that  $\hat{z}_{1i} = \hat{z}_{2i} + S$ . Then  $N = 4$ ,  $(z_2, y_2) = (\hat{z}_{1i}, \hat{y}_{1i})$ ,  $(z_3, y_3) = (\hat{z}_{1i}, \hat{y}_{2i})$  and  $(z_4, y_4) = (S, 0.)$ . If  $\hat{z}_{1i} \neq \hat{z}_{2i} + S$  but  $\hat{y}_{1i} = \hat{y}_{2i}$  then, analogously,  $N = 4$ ,  $(z_2, y_2) = (z_{1i}, y_{1i})$ ,  $(z_3, y_3) = (z_{2i} + S, y_{1i})$  and  $(z_4, y_4) = (S, 0.)$ . These two cases are illustrated in Figure 33.

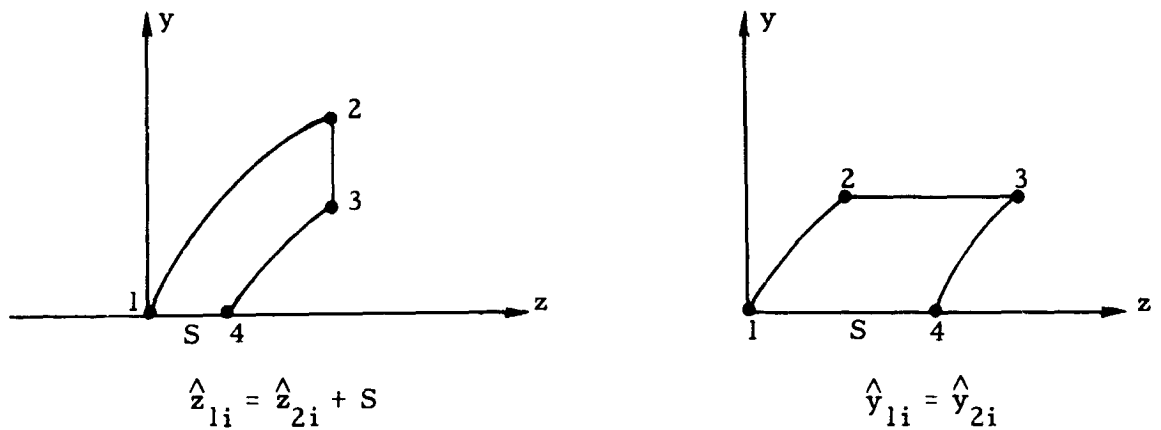


Figure 33. Three Arc Elements

Suppose then, that  $\hat{y}_{1i} < \hat{y}_{2i}$ . If  $\hat{C}_{1i} \neq 5$  then  $N = 5$ ,  $(z_2, y_2) = (\hat{z}_{1i}, \hat{y}_{1i})$ ,  $(z_3, y_3) = (\hat{z}_{1i}, \hat{y}_{2i})$ ,  $(z_4, y_4) = (\hat{z}_{2i} + S, \hat{y}_{2i})$ , and  $(z_5, y_5) = (S, 0.)$ . If  $\hat{C}_{1i} = 5$  cannot proceed vertically from  $(\hat{z}_{1i}, \hat{y}_{1i})$  so let  $(\beta, \hat{y}_{1i})$  be the point (relative to  $i_2$ ) where  $i_2$  intersects  $y = \hat{y}_{1i}$ . It follows that  $N = 4$ ,  $(z_2, y_2) = (\hat{z}_{1i}, \hat{y}_{1i})$ ,  $(z_3, y_3) = (\beta + S, \hat{y}_{1i})$ , and  $(z_4, y_4) = (S, 0.)$ . The case for  $\hat{y}_{1i} < \hat{y}_{2i}$  is illustrated in Figure 34.

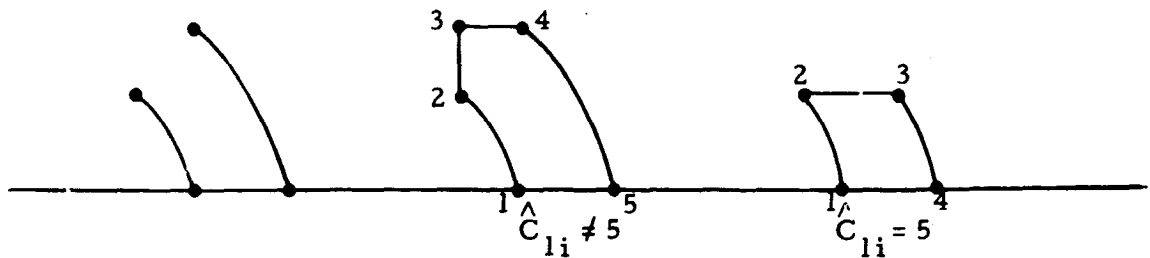


Figure 34.  $\hat{y}_{1i} < \hat{y}_{2i}$

The case  $\hat{y}_{1i} > \hat{y}_{2i}$  is very similar to the case  $\hat{y}_{1i} < \hat{y}_{2i}$ . If  $\hat{C}_{2i} \neq 5$  then  $N = 5$ ,  $(z_2, y_2) = (\hat{z}_{1i}, \hat{y}_{1i})$ ,  $(z_3, y_3) = (\hat{z}_{2i} + S, \hat{y}_{1i})$ ,  $(z_4, y_4) = (\hat{z}_{2i} + S, \hat{y}_{2i})$ , and  $(z_5, y_5) = (S, 0.)$ . However, if  $\hat{C}_{2i} = 5$  cannot proceed vertically from  $(\hat{z}_{2i} + S, \hat{y}_{2i})$ , therefore  $N = 4$ ,  $(z_2, y_2) = (\beta, \hat{y}_{2i})$ ,  $(z_3, y_3) = (\hat{z}_{2i} + S, \hat{y}_{2i})$ , and  $(z_4, y_4) = (S, 0.)$  where  $(\beta, \hat{y}_{2i})$  is the point where  $i_1$  intersects  $y = \hat{y}_{2i}$ .

## 9.7 POINT LIST

In order to produce a  $z - y$  profile plot of the system it is necessary to generate a list of points for each element. Plotting will consist of joining each consecutive pair with a straight line; consequently, the points must be dense (closely spaced) along conic arcs.

Let the list to be generated for the  $i$ th element be given by,  $\{(z_j^*, y_j^*); 1 \leq j \leq N_p\}$ , where  $z_1^* = y_1^* = 0$ , and  $N_p$  is the number of points in the list. The list is assumed to be arranged so that  $(z_j^*, y_j^*)$  and  $(z_{j+1}^*, y_{j+1}^*)$  are consecutive points on the curve, and is produced by examining the corresponding arcs matrix (defined in Section 9.6).

Each curve is composed of from 1 to 4 arcs where an arc is one of:

a) A vertical line of the form  $z = z_1$  where  $y_1 \leq y \leq y_2$  so that  $(z_1, y_1)$  and  $(z_1, y_2)$  are added as consecutive points to the list.

b) A horizontal line of the form  $y = y_1$  where  $z_1 \leq z \leq z_2$  so that  $(z_1, y_1)$  and  $(z_2, y_1)$  are added to the list.

c) A portion of the conic  $y^2 - \left[ (b - 1) z^2 + 2Rz \right] = 0$ , for  $z_A \leq z \leq z_B$ . The following set of  $M$  points ( $2 \leq M \leq 149$ ) is added to the list:

$$z_i = z_A + (i - 1) \Delta z \quad (9.086)$$

$$y_i = \left[ (b - 1) z_i^2 + 2Rz_i \right]^{\frac{1}{2}} \quad 1 \leq i \leq M \quad (9.087)$$

$$\Delta z = \frac{z_B - z_A}{M - 1} \quad (9.088)$$

The point list is such that  $y_j^* \geq 0$ , for all  $j$ , that is, only the upper half of the curve is represented; however, the lower half may be produced by reflecting the data in the  $z$  (optic) axis. All data must be referenced to a single coordinate system namely the one whose origin is at the center of the entrance pupil. This is accomplished by replacing  $z_j^*$  with  $z_j^* + \delta_{i_1}$ , where  $\delta_{i_1}$  is the coordinate of the vertex of surface  $i_1$ .

A profile diagram is produced by generating a point list for each element and displaying the results as a two dimensional ( $z - y$ ) plot.

The image plane and the entrance pupil may be included on the profile diagram by attaching two additional point lists each of which contains 2 points describing a vertical line. The image plane is the line  $z = \delta^*$  with  $0 \leq y \leq y_{\max}$  and the entrance pupil is the line  $z = 0$ , with  $0 < y < \rho_0$ .

Suppose some arbitrary ray is traced through the system and let its coordinates on  $\sigma_i$  be  $(x_i, y_i, z_i)$  where  $0 \leq i \leq (N + 1)$  with

$i = 0$  designating the entrance pupil and  $i = N + 1$  the image plane. This ray is superimposed on the profile diagram by including the point list given by:

$$y_j^* = y_j$$

$$z_j^* = z_j + \delta_j$$

where

$$0 \leq j \leq N + 1.$$

Section 10

SPOT DIAGRAM PLOTS

Using the notation of Section 8, let  $(x_{ijk}, y_{ijk})$  be the coordinates of the point of intersection of ray  $k$  and the image plane where  $1 \leq i \leq N_c$  refers to color and  $1 \leq j \leq N_H$  refers to object point.

The point set,  $\{(x_{ijk}, y_{ijk}); k = 1, 2, \dots, M_R\}$ , describes a monochromatic spot whose center  $(\bar{x}_{ij}, \bar{y}_{ij})$  is given by equations (8.012) and (8.013). Translating the origin to  $(\bar{x}_{ij}, \bar{y}_{ij})$  the pattern becomes  $\{(\tilde{x}_{ijk}, \tilde{y}_{ijk}); 1 \leq k \leq M_R\}$  as defined by equations (8.014) and (8.015).

Each of the  $N_H N_c$  monochromatic spots is to be plotted in the  $x$ - $y$  plane using a universal scale which is chosen so that all plots can be accommodated. Let:

$$x_{\min} = \min \left[ \tilde{x}_{ijk} \right] \quad (10.001)$$

$$x_{\max} = \max \left[ \tilde{x}_{ijk} \right] \quad (10.002)$$

$$y_{\min} = \min \left[ \tilde{y}_{ijk} \right] \quad (10.003)$$

$$y_{\max} = \max \left[ \tilde{y}_{ijk} \right] \quad (10.004)$$

The minima and maxima are taken over all values of  $i$ ,  $j$ , and  $k$ . Define:

$$s_{\min} = \min \left[ x_{\min}, y_{\min} \right] \quad (10.005)$$

$$\text{and } s_{\max} = \max \left[ x_{\max}, y_{\max} \right] \quad (10.006)$$



Since all coordinates are relative to spot center then  $s_{\min} \leq 0$ .  
and  $s_{\max} \geq 0$ . Define  $s$  as follows:

$$s = \kappa \max \left[ |s_{\min}|, s_{\max} \right] \quad (10.007)$$

where  $\kappa > 0$ . is a scale factor to be applied to all  $x$  and  $y$ . Both the  $x$  and the  $y$  scales are chosen to extend over the interval  $-[s, s]$  whose length,  $\lambda$ , is

$$\lambda = 2s \quad (10.008)$$

The term, image coordinates, is used to specify the scaled  $x$ - $y$  values.

The plotting is to be performed on a printer where the horizontal coordinate is a character position and the vertical coordinate is a line position. By choosing 103 character positions and 62 line positions the grid is essentially square which, of course, is necessary if distortion is to be avoided. This character-line system will be referred to as the printer coordinates. The two frames of reference are illustrated in Figures 35 and 36, respectively.

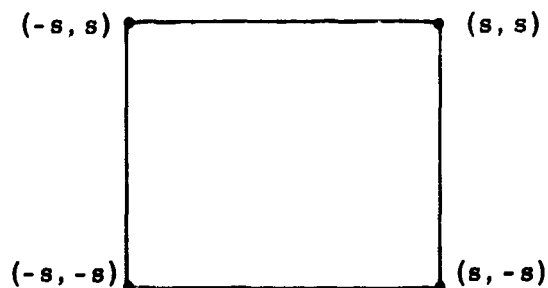


Figure 35 Image Coordinates

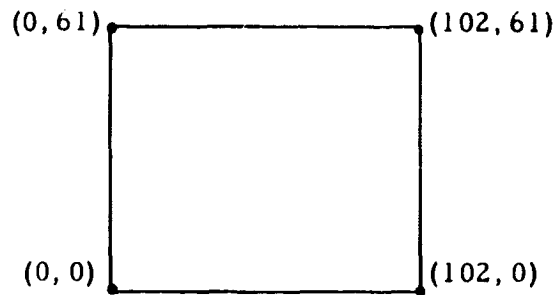


Figure 36 Printer Coordinates

Consider a transformation from image to printer coordinates which takes  $(x, y)$  into  $(C, L)$ . The transformation is obviously linear so that:

$$C = a_1 x + a_2 \quad (10.009)$$

and

$$L = b_1 y + b_2 \quad (10.010)$$

It is uniquely determined by the requirements:

- a)  $(-s, s)$  maps into  $(0, 102)$  which is the horizontal axis.
- b)  $(-s, s)$  maps into  $(0, 61)$  which is the vertical axis.
- c)  $C$  and  $L$  are integers where  $0 \leq C \leq 102$  and  $0 \leq L \leq 61$ .

Let:

$$\Delta C = \frac{\lambda}{102} \quad (10.011)$$

$$\Delta L = \frac{\lambda}{61} \quad (10.012)$$

$$C' = \frac{x + s}{c} \quad (10.013)$$

$$L' = \frac{y + s}{\Delta L} \quad (10.014)$$

$$C^* = \left[ C' \right] \quad (10.015)$$

$$L^* = \left[ L' \right] \quad (10.016)$$

$\left[ X \right]$  is the largest integer which does not exceed X.

It follows that:

$$C = \begin{cases} C^* & (C' - C^*) \leq .5 \\ C^* + 1 & (C' - C^*) > .5 \end{cases} \quad (10.017)$$

$$L = \begin{cases} L^* & (L' - L^*) \leq .5 \\ L^* + 1 & (L' - L^*) > .5 \end{cases} \quad (10.018)$$

It is perhaps more convenient to number the lines from the top of the page instead of from the bottom. In this case replace L as computed by (10.018) with  $61-L$ .

To plot the monochromatic spot specified by  $\left\{ (\tilde{x}_{ijk}, \tilde{y}_{ijk}) \right\}; 1 \leq k \leq M_R \left\{ \right.$  determine the set of printer coordinates  $\left\{ (C_k, L_k); 1 \leq k \leq M_R \right\}$  where  $C_r \equiv C_r (\kappa \tilde{x}_{ijr})$  and  $L_s \equiv L_s (\kappa \tilde{y}_{ijs})$ . If  $M_R \leq 35$  the characters 1, 2, 3, ..., 9, A, B, ..., Z are used as plot symbols so that each ray retains a unique identity. If  $M_R > 35$  the character X is used as a single plot symbol for all points.