RATIONAL APPROXIMATIONS TO TRICOMI'S Ψ FUNCTION

INTERIM REPORT
June 1968

MRI Project No. 3162-P

Contract No. NAS 9-7641

For

NASA Manned Spacecraft Center
General Research Procurement Branch
Houston, Texas 77058

Attn: J.W. Carlson/BG731(48)

MRI

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by

Jerry Fields
Jet Wimp

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PREFACE

This report, written by Jerry Fields and Jet Wimp, Mathematics Branch, Midwest Research Institute, covers work performed from 20 March through 19 June 1968 on Contract No. NAS 9-7641.

Approved for:

MIDWEST RESEARCH INSTITUTE

Sheldon L. Levy, Director
Mathematics and Physics Division

1 July 1968

In this paper we derive closed form rational approximations to the Tricomi Ψ function ([1]),

$$\Psi(a,c;v) = \frac{1}{\Gamma(a)} \int_0^{\infty e^{i\varphi}} e^{-vt} t^{a-1} (1+t)^{c-a-1} dt \quad , \tag{1}$$

$$\text{Re } a > 0 \;,\; |\arg(e^{i\varphi} v)| < \pi/2 \;,\; -\pi < \varphi < \pi \quad ,$$

which converge uniformly on compact subsets of the sector $|\arg v| < \pi/2$, $v \neq 0$. As Tricomi's Ψ function can be written in terms of the Meijer G-function ([1])

$$\Psi(a,c;v) = \frac{v^{-a}}{\Gamma(a)\Gamma(1+a-c)} G_{2,1}^{1,2}\left(v^{-1}\middle|\begin{matrix}1-a,c-a\\0\end{matrix}\right) \quad , \tag{2}$$

we actually develop rational approximations to the G-function

$$E(v) = G_{2,1}^{1,2}\left(v^{-1}\middle|\begin{matrix}1-\alpha_1,1-\alpha_2\\0\end{matrix}\right)$$

$$= \frac{1}{2\pi i}\int_L \Gamma(-s)\Gamma(s+\alpha_1)\Gamma(s+\alpha_2)v^{-s}ds \;,\; |\arg v| < 3\pi/2 \quad , \tag{3}$$

where the contour L runs from $-i\infty$ to $+i\infty$, and separates the poles of $\Gamma(-s)$ from those of $\Gamma(s+\alpha_1)\Gamma(s+\alpha_2)$, see [1]. If $\alpha_1-\alpha_2$ is not an integer, it follows from the residue theorem that

$$E(v) = \Gamma(\alpha_2-\alpha_1)\Gamma(\alpha_1)v^{\alpha_1} {}_1F_1\left(\begin{matrix}\alpha_1\\1+\alpha_1-\alpha_2\end{matrix}\middle| v\right)$$

$$+ \Gamma(\alpha_1-\alpha_2)\Gamma(\alpha_2)v^{\alpha_2} {}_1F_1\left(\begin{matrix}\alpha_2\\1+\alpha_2-\alpha_1\end{matrix}\middle| v\right) \tag{4}$$

Our rational approximations for $E(v)$ are obtained as follows. If the contour L in (3) is moved k units to the right, we obtain

$$E(v) = \Gamma(\alpha_1)\Gamma(\alpha_2) \sum_{j=0}^{k-1} (\alpha_1)_j \ (\alpha_2)_j \ \frac{(-v)^{-j}}{j!} + R_k(v) \ ,$$

$$R_k(v) = \frac{(-v)^{-n}}{2\pi i} \int_L \frac{\Gamma(-s)\Gamma(s+1)\Gamma(s+n+\alpha_1)\Gamma(s+n+\alpha_2)}{\Gamma(s+1+n)} v^{-s} ds \ , \qquad (5)$$

$$(\sigma)_\mu = \frac{\Gamma(\sigma+\mu)}{\Gamma(\sigma)} \ .$$

As it can be shown that $R_k(v) = \mathcal{O}(v^{-k})$ as $v \to \infty$ and $|\arg v| < 3\pi/2$, (5) is a paraphrase of the statement

$$E(v) \sim \Gamma(\alpha_1)\Gamma(\alpha_2) \ {}_2F_0\left(\alpha_1,\alpha_2 \middle| -\frac{1}{v}\right) \ ,$$

$$v \to \infty \ , \ |\arg v| < 3\pi/2 \qquad (6)$$

Multiplying the first line of (5) by arbitrary $A_{n,k}\gamma^k$, and summing from $k = 0$ to a fixed integer n , we obtain the equations

$$h_n(\gamma)E(v) = \psi_n(v,\gamma) + F_n(v,\gamma)$$

$$h_n(\gamma) = \sum_{k=0}^{n} \gamma^k A_{n,k} \ , \ F_n(v,\gamma) = \sum_{k=0}^{n} \gamma^k A_{n,k} R_k(v) \qquad (7)$$

$$\psi_n(v,\gamma) = \sum_{k=0}^{n} \gamma^k A_{n,k} \Gamma(\alpha_1)\Gamma(\alpha_2) \sum_{j=0}^{k-1} (\alpha_1)_j (\alpha_2)_j \ \frac{(-v)^{-j}}{j!}$$

Then we see that $\psi_n(v,\gamma)/h_n(\gamma)$ is a formal rational approximation to $E(v)$ and $F_n(v,\gamma)/h_n(\gamma)$ is its corresponding error. The triangular form $\psi_n(v,\gamma)$ can also be written as

$$\psi_n(v,\gamma) = \Gamma(\alpha_1)\Gamma(\alpha_2) \sum_{k=0}^{n} \sum_{j=0}^{n-k} A_{n,j+k} \quad \frac{(\alpha_1)_j(\alpha_2)_j(\gamma)^k(-\gamma/v)^j}{j!} \quad . \tag{8}$$

In Fields [2], the above formulation was shown to be equivalent to the Lanczos τ-method, see [3], and the following theorem was proved.

<u>Theorem 1.</u> If $|\arg v| < \pi/2$, $v \neq 0$, and

$$h_n(\gamma) = {}_2F_2\left(\begin{matrix}-n, n+\lambda \\ 1+\alpha_1, 1+\alpha_2\end{matrix}\middle| -\gamma\right), \ \lambda > 0 \ , \tag{9}$$

then

$$\lim_{n\to\infty} \frac{\psi_n(v,v)}{h_n(v)} = E(v) \ , \ \lim_{n\to\infty} \frac{F_n(v,v)}{h_n(v)} = 0 \quad . \tag{10}$$

As the asymptotic estimate

$${}_2F_2\left(\begin{matrix}-n, n+\lambda \\ 1+\alpha_1, 1+\alpha_2\end{matrix}\middle| -v\right) \sim \frac{\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)}{2\pi\sqrt{3}} (n^2v)^{\tau} \exp\left(3(n^2v)^{\frac{1}{3}} - \frac{v}{3}\right)$$

$$\times \left\{1 + \mathcal{O}(n^{-\frac{2}{3}})\right\} \quad ,$$

$$\tau = -(1+\alpha_1+\alpha_2)/3 \ ; \ n \to \infty \ , \ |\arg v| < \pi \ , \tag{11}$$

was already known, see [4], the proof of Theorem 1 reduced to obtaining a proper estimate for $F_n(v,v)$. This was effected by showing that the differential operator which annihilates $E(v)$,

$$\mathcal{N} = (\delta-\alpha_1)(\delta-\alpha_2)-v\delta \ , \ \delta = v\frac{d}{dv} \quad , \tag{12}$$

when applied to $F_n(v,\gamma)$ yields

$$\mathcal{N}\left\{F_n(v,\gamma)\right\} = -\sum_{k=0}^{n} A_{n,k}\,\frac{\Gamma(k+1+\alpha_1)\Gamma(k+1+\alpha_2)}{k!}\left(-\frac{\gamma}{v}\right)^k \quad . \tag{13}$$

Thus, if the $A_{n,k}$ are chosen as indicated in (9), the right-hand side of (13) is essentially a Jacobi polynomial which has a uniform algebraic rate of growth in n , $\mathcal{O}(n^\sigma)$, for $0 \le \gamma/v \le 1$. A variation of parameters' technique then implies

$$F_n(v,v) = \mathcal{O}(n^\sigma)\ ,\ n \to \infty\ ,\ v \ \text{fixed} \quad . \tag{14}$$

Note that in Theorem 1, the parameter λ is essentially unspecified. By specializing λ and considering difference instead of differential operators, we obtain, among other benefits, a more convenient formulation of the error $F_n(v,v)$.

Let

$$U(\mu,n,\lambda) = \frac{(n+\lambda-1)(n+\mu)}{2n+\lambda-1}E^0 - \frac{n(n+\lambda-1-\mu)}{2n+\lambda-1}E^{-1} \ ,$$

$$U^*(n,\lambda) = \lim_{\mu\to\infty}\frac{U(\mu,n,\lambda)}{\mu} \ , \tag{15}$$

where E^{-j} is the shift operator on n , i.e., $E^{-j}\left\{f(n)\right\} = f(n-j)$, and

$$M(\gamma) = U(0,n,\lambda-2)U(\alpha_1,n,\lambda-1)U(\alpha_2,n,\lambda)-n(n+\lambda-3)\gamma E^{-1}U^*(n,\lambda) \ ,$$

$$= A_0\left[E^0 + \sum_{j=1}^{3}\left[A_j+\gamma B_j\right]E^{-j}\right],\ A_0 = \frac{n(n+\alpha_1)(n+\alpha_2)(n+\lambda-3)_3}{(2n+\lambda-3)_3} \ ,$$

$$A_1 = \frac{(n-1)(2n+\lambda-2)_2(n+\alpha_1-1)(n+\alpha_2-1)}{(n+\lambda-1)(2n+\lambda-4)(n+\alpha_1)(n+\alpha_2)} - \frac{n(2n+\lambda-2)}{(n+\lambda-1)} \ ,$$

Equation (16) concluded next page.

$$A_2 = \frac{(n-1)(2n+\lambda-1)(n+\lambda-\alpha_1-2)(n+\lambda-\alpha_2-2)}{(n+\lambda-1)(n+\alpha_1)(n+\alpha_2)}$$

$$- \frac{(n-1)(n+\lambda-3)(2n+\lambda-2)_2(n+\lambda-\alpha_1-3)(n+\lambda-\alpha_2-3)}{(n+\lambda-2)_2(2n+\lambda-5)(n+\alpha_1)(n+\alpha_2)} ,$$

$$A_3 = \frac{(n-2)_2(2n+\lambda-2)_2(n+\lambda-\alpha_1-3)(n+\lambda-\alpha_2-3)}{(2n+\lambda-5)_2(n+\lambda-2)_2(n+\alpha_1)(n+\alpha_2)} ,$$

$$B_1 = - \frac{(2n+\lambda-2)_2}{(n+\lambda-1)(n+\alpha_1)(n+\alpha_2)} , \quad B_2 = - \frac{(n-1)(2n+\lambda-2)_2}{(n+\lambda-2)_2(n+\alpha_1)(n+\alpha_2)} , \quad B_3 = 0 \quad . \tag{16}$$

We then have

Theorem 2. If the $A_{n,k}$ are chosen so that

$$h_n(\gamma) = {}_2F_2\left(\begin{matrix}-n, n+\lambda \\ 1+\alpha_1, 1+\alpha_2\end{matrix}\middle| -\gamma\right) ,$$

$$\psi_n(v,\gamma) = \Gamma(\alpha_1)\Gamma(\alpha_2) \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{(-n)_{k+j}(n+\lambda)_{k+j}(\alpha_1)_j(\alpha_2)_j(-\gamma)^k(\gamma/v)^j}{(1+\alpha_1)_{k+j}(1+\alpha_2)_{k+j}(k+j)!j!} , \tag{17}$$

then

$$M(\gamma)\left\{h_n(\gamma)\right\} = 0$$

$$M(\gamma)\left\{\psi_n(v,\gamma)\right\} = -n(n+\lambda-3)\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)(\gamma/v)\ {}_2F_1\left(\begin{matrix}-n+1, n+\lambda-2 \\ 2\end{matrix}\middle|\frac{\gamma}{v}\right) , \tag{18}$$

$$M(v)\left\{\psi_n(v,v)\right\} = (-1)^n \frac{\Gamma(n+\lambda-2)\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)}{\Gamma(n)\Gamma(\lambda-2)}$$

Proof. All these results follow directly by computation from the operator equations

$$U(\mu,n,\lambda)\left\{(-n)_s(n+\lambda)_s\right\} = (-n)_s(n+\lambda-1)_s(s+\mu) \quad ,$$

$$U^*(n,\lambda)\left\{(-n)_s(n+\lambda)_s\right\} = (-n)_s(n+\lambda-1)_s \tag{19}$$

<u>Corollary 2.1</u>. If in Theorem 2, $\lambda-3$ is a negative integer, then

$$M(v)\left\{F_n(v,v)\right\} = 0 \tag{20}$$

Hence, to analyze the error $F_n(v,v)$, in this case, it is sufficient to analyze the equation

$$M(v)\left\{g_n(v)\right\} = 0 \tag{21}$$

To do this, we introduce some recent results of Wimp, [5]. Let

$$g_n(w) = \frac{\Gamma(n+1)}{\Gamma(n+\lambda)} G^{3,1}_{2,3}\left(w \,\middle|\, \begin{matrix} 1-n-\lambda, n+1 \\ 0, -\alpha_1, -\alpha_2 \end{matrix}\right) \quad ,$$

$$= \frac{1}{2\pi i}\int_L \frac{\Gamma(-s)\Gamma(-s-\alpha_1)\Gamma(-s-\alpha_2)(n+\lambda)_s}{(n+1)_{-s}} w^s ds \quad , \tag{22}$$

then Wimp's work* shows that

* This is not quite the function that Wimp treated in [5]. But a close inspection of that work shows that his analysis is actually applicable. The identification process with Wimp's work is made by replacing his λ, n and γ by w, $n+\lambda$ and $-\lambda$, respectively. Also, the application of Lemma 3 in this reference is made easier by employing the fact $2d_1-d_2 = 1+\gamma$.

$$g_n(w) \sim \frac{(2\pi)}{\sqrt{3}} \left[n^2 w\right]^{\tau} \exp\left(-3\left[n^2 w\right]^{\frac{1}{3}} + \frac{w}{3}\right)\left\{1 + \mathcal{O}(n^{-\frac{1}{3}})\right\} \quad , \tag{23}$$

$$\tau = -(1+\alpha_1+\alpha_2)/3 \ ; \ n \longrightarrow +\infty \ , \ |\arg w| < 3\pi/2 \quad .$$

This leads to our main result,

Theorem 3. If $|\arg v| < \pi/2$, $v \neq 0$, then $g_n(ve^{\pi i})$, $g_n(ve^{-\pi i})$ and $h_n(v)$ as defined by (17) form a basis of solutions for the difference equation (21).

Proof. It follows from (11) and (23) that the three functions are linearly independent as functions of n in the right half plane. A direct computation using the integral representation in (22) and an analog of (19) shows that $g_n(ve^{\pm\pi i})$ satisfy (21).

Corollary 3.1. If

$$h_n(v) = {}_2F_2\left(\begin{matrix}-n, n+\lambda \\ 1+\alpha_1, 1+\alpha_2\end{matrix}\middle| -v\right) \quad , \ \lambda = 1 \text{ or } 2 \quad ,$$

$$\psi_n(v,v) = \Gamma(\alpha_1)\Gamma(\alpha_2) \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{(-n)_{k+j}(n+\lambda)_{k+j}(\alpha_1)_j(\alpha_2)_j(-v)^k}{(1+\alpha_1)_{k+j}(1+\alpha_2)_{k+j}(k+j)!j!} \tag{24}$$

and $|\arg v| < \pi/2$, $v \neq 0$, then there exist well defined analytic functions $C^-(v)$, $C^+(v)$ independent of n such that

$$G^{1,2}_{2,1}\left(v^{-1}\middle| \begin{matrix}1-\alpha_1, 1-\alpha_2 \\ 0\end{matrix}\right) - \frac{\psi_n(v,v)}{h_n(v)}$$

$$= C^-(v) \frac{g_n(ve^{-\pi i})}{h_n(v)} + C^+(v) \frac{g_n(ve^{+\pi i})}{h_n(v)} \tag{25}$$

$$\sim \sum_{\epsilon=+,-} \frac{C^{\epsilon}(v)4\pi^2}{\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)} \exp\left(-3\sqrt{3}(n^2 ve^{i\epsilon\pi/2})^{\frac{1}{3}} + i\pi\epsilon\tau\right)\left\{1 + \mathcal{O}(n^{-\frac{1}{3}})\right\} , \ n \longrightarrow \infty \ .$$

Proof. Clearly, the left-hand side of (25) is just the error $F_n(v,v)/h_n(v)$. From Corollary 2.1 and Theorem 3 it follows that there exist functions $C^\epsilon(v)$, $\epsilon = +,-,0$, in $|\arg v| < \pi/2$, such that

$$F_n(v,v) = C^+(v)g_n(ve^{+\pi i}) + C^-(v)g_n(ve^{-\pi i}) + C^0(v)h_n(v) \quad . \qquad (26)$$

The functions $C^\epsilon(v)$ can be found in theory by setting n equal to zero, one and then two in the first line of (25) and then solving the resulting equations. The linear independence of $g_n(ve^{-\pi i})$, $g_n(ve^{+\pi i})$ and $h_n(v)$ implies the analytic character of the $C^\epsilon(v)$ in $|\arg v| < \pi/2$, $v \neq 0$. From Theorem 1, we deduce that $C^0(v)$ is identically zero in $|\arg v| < \pi/2$. Equation (26) then reduces to the first line of (25). The last line of (25) follows from the preceding asymptotic estimates and the simple fact, $\sqrt{3}e^{i\epsilon\pi/6} = 1 + e^{\epsilon\pi i/3}$, $\epsilon = \pm 1$.

Corollary 3.2. The sequence of rational approximations in Corollary 3.1 converges uniformly to $G_{2,1}^{1,2}\left(v^{-1}\middle|\begin{matrix}1-\alpha_1, 1-\alpha_2\\ 0\end{matrix}\right)$ on compact subsets of $|\arg v| < \pi/2$, $v \neq 0$.

Finally, we reiterate the main advantages of the rational approximations in Corollary 3.1. First, they are explicit, as opposed to mini-max rational approximations which, in general, can only be given numerically. Second, they can be computed fairly easily, due to the fact that both numerator and denominator polynomials, $\psi_n(v,v)$ and $h_n(v)$, satisfy the same third order recursion relation. And last, an explicit form for the error is known.

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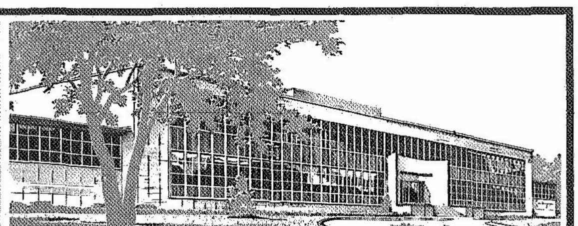

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