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National Aeronautics and Space Administration
Goddard Space Flight Center
Contract No.NAS-5-12487

ST-GEO-PP-10797

"QUASI-ISOTROPIC" APPROXIMATION IN GEOMETRIC OPTICS

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13 JANUARY 1969

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Doklady A.N. SSSR, Fizika,
Tom 183, No.1, 74 - 76,
Izdatel'stvo "NAUKA", 1968

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SUMMARY

This paper expounds a modified geometric optics method for the solution of Maxwellian equations based upon waves in isotropic medium, earlier developed by Rytov. The equation arrived at is useful for possible applications in radioastronomy and radiophysics.

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1. As is well known, in feebly anisotropic media, for which the non-diagonal tensor components of dielectric constant ε_{ik} and the differences between the diagonal components are small by comparison with $\mu \sim 1/kl \ll 1$, where l is the characteristic scale of medium property variation, the geometric optics approximation in the form of independent normal waves is inapplicable [1]. However, in this case one may construct a geometric-optical approximation of Maxwellian equations, resting upon the representation on waves in an isotropic medium, considering the tensor $\nu_{ik} = \varepsilon_{ik} - \varepsilon\delta_{ik}$ as a perturbation $|\nu_{ik}| \ll \varepsilon$ (we may assume for ε, as an example, $\frac{1}{3}$ SP ε_{ik}), and applying the theory developed by S. M. Rytov [2, 3]. The corresponding modification of the geometric optics method, for which S. M. Rytov proposed the denomination of quasi-isotropic approximations, is expounded below.

2. In order to have the possibility to make use of formal field amplitude expansion by powers $1/k$ (the wave number $k \equiv \omega/c$ will be the great parameter of the problem), we shall ascribe to quantities ν_{ik} an order of smallness $1/k$, postulating $\nu_{ik} = \varepsilon_{ik}/k$. The Maxwellian equations will take the form

$$\mathrm{rot}\,\mathbf{H} + ik\varepsilon\mathbf{E} = -i\hat{\xi}\mathbf{E}, \quad \mathrm{rot}\,\mathbf{E} - ik\mathbf{H} = 0, \tag{1}$$

differing from equations for a field in an isotropic medium by the presence of the term $-i\xi E$, which is small by comparison with $k\varepsilon E$.

Searching for an asymptotic solution of Eq.(1) in the same form as for an inhomogenous isotropic medium, that is, assuming

$$\mathbf{E} = \sum_{m=0}^{\infty} (ik)^{-m}\mathbf{E}^{(m)}e^{ik\varphi}, \quad \mathbf{H} = \sum_{m=0}^{\infty} (ik)^{-m}\mathbf{H}^{(m)}e^{ik\varphi},$$

one may be convinced that the phase ϕ is subject, as in an isotropic medium, to the eikonal equation $(\nabla\phi)^2 = \varepsilon$, and that the vectorial amplitudes $E^{(0)}$ and $H^{(0)}$ (method's zero approximation) are tranvserse to one another and to the direction of wave normal $t = \nabla\phi/\ |\nabla\phi|$ This is why we may postulate

$$\vec{E}^{(0)} = \Phi_1\vec{n} + \Phi_2\vec{b}, \qquad \vec{H}^{(0)} = \sqrt{\varepsilon}(\Phi_1\vec{b}\ \ \Phi_2\vec{n}),$$

where $\vec{n}$ and $\vec{b}$ are unitary vectors of the main normal and binormal to the beam.

The compatibility conditions of first approximation equations having the form

$$\mathbf{H}^{(0)}\operatorname{rot}\mathbf{H}^{(0)} + \varepsilon\mathbf{E}^{(0)}\operatorname{rot}\mathbf{E}^{(0)} + i\mathbf{H}^{(0)}\hat{\xi}\mathbf{E}^{(0)} = 0,$$

$$\mathbf{H}^{(0)}\operatorname{rot}\mathbf{E}^{(0)} - \mathbf{E}^{(0)}\operatorname{rot}\mathbf{H}^{(0)} - i\mathbf{E}^{(0)}\hat{\xi}\mathbf{E}^{(0)} = 0, \tag{2}$$

represent in essence two equations for the determination of Φ_1 and Φ_2. The solution

$$\mathbf{E} = \mathbf{E}^{(0)}e^{ik\varphi} = (\Phi_1\mathbf{n} + \Phi_2\mathbf{b})e^{ik\varphi}, \qquad \mathbf{H} = \mathbf{H}^{(0)}e^{ik\varphi} = \sqrt{\varepsilon}(\Phi_1\mathbf{b} - \Phi_2\mathbf{n}), \tag{3}$$

in which Φ_1 and Φ_2 are found from (2), is the principal term of the asymptotic series in quasi-isotropic approximation.

3. One may write Eqs.(2) in different form, provided the denotations $\chi = \ln(\sqrt{\varepsilon}\Phi^2)$, $\Phi^2 = \Phi_1^2 + \Phi_2^2$, $\theta = \operatorname{arctg}(\Phi_2/\Phi_1)$ are introduced:

$$\frac{d\chi}{d\sigma} + \operatorname{div}\mathbf{t} = \frac{i}{2\sqrt{\varepsilon}}[(\xi_{nn} + \xi_{bb}) + (\xi_{nn} - \xi_{bb})\cos 2\theta + (\xi_{nb} + \xi_{bn})\sin 2\theta], \tag{4}$$

$$\frac{d\theta}{d\sigma} = \frac{1}{T} + \frac{i}{4\sqrt{\varepsilon}}[(\xi_{bn} - \xi_{nb}) + (\xi_{nb} + \xi_{bn})\cos 2\theta - (\xi_{nn} - \xi_{bb})\sin 2\theta]. \tag{5}$$

Here $\xi_{nn}, \xi_{nb}, \xi_{bn}, \xi_{bb}$ are the components of tensor ξ_{ik} in a system of coordinates whose axes coincide with orts of the natural trihedrons $\vec{n}$, $\vec{b}$, $\vec{t}$; $d\sigma$ is the element of length, and T is the torsion radius of the ray: $2/T = \vec{n}\ \mathrm{rot}\ \vec{n} + \vec{b}\ \mathrm{rot}\ \vec{b}$. It is essential that the angle $\theta = \mathrm{arc\ tg}\ (\Phi_2/\Phi_1)$ may be found from Eq.(5) independently of the amplitude Φ. In the general case this angle is complex: $\theta = \theta' + i\theta''$. The real part of θ determines the orientation of the polarization ellipse relative to orts $\vec{n}$ and $\vec{b}$, while the imaginary part is the ellipse's degree of stretching: the minor to major axis ratio is equal to $\mathrm{th}\theta''$.

Within the bounds of vanishingly small anisotropy ($\nu_{ik} = \xi_{ik}/k \to 0$), Eqs. (4) and (5) obviously yield results earlier obtained for the isotropic medium. and namely, from them is derived the law of intensity conservation $\mathrm{div}\ (\sqrt{\varepsilon}\Phi^2\vec{t}) = 0$ and the Rytov law for the rotation of the polarization plane $d\theta/d\sigma = 1/T$ [2, 3]. It may be shown that, as the anisotropy increases (when $|\nu_{ik}| \gg \mu \approx 1/kl$, but $|\nu_{ik}| \ll \varepsilon$) solution (3) passes to the sum of two so called shortened normal waves, i.e.,independent waves, in which terms of the order $|\nu_{ik}|^2 \ll |\nu_{ik}|$ in amplitudes and phases are rejected. Owing to this, the solution (3) joins naturally with normal waves (consideration of the latter was conducted, for example, in [1,4,5] for the one-dimenional case, and in [6,7], for the three-dimensional case).

As a result of this, the investigation of wave propagation at the outset of the aritrary three-dimensionally inhomogenous anistropic layer is significantly simplified. In connection with this it should be stated more precisely that the conclusion derived in [1,4,5] on the inapplicability of geometric optics in feebly anisotropic medium (or at the origin of the anisotropic layer) refers in reality only to the approximation in the form of independent normal waves, and not to the quasi-isotropic approximation.

4. As an example of application of Eq.(5), we shall consider the problem of polarization variation of a high frequency field in a gyrotropic plasma, taking account simultaneously of linear and quadratic magneto-optical effects. In high frequencies, $v \equiv 4\pi e^2 N / m\omega^2 \ll 1$ and $\sqrt{u} = eH_0/mc\omega \ll 1$ (H_0 is a permanent magntic field). It is natural to take for ε the quantity $1 - v$. Having computed the tensor components $\xi_{ik} = k\nu_{ik} = k(\varepsilon_{ik} - \varepsilon\delta_{ik})$ with the help of expressions for ε_{ik} available, for example, in [1], and having preserved in them the linear and quadratic terms with respect to H_0 , we shall obtain for θ

$$\frac{d\theta}{d\sigma} = \frac{1}{T} + \frac{1}{2} kv\sqrt{u}\cos\alpha - \frac{i}{4} kvu \sin^2\alpha \sin 2(\theta + \psi),$$

where α is the angle between vectors $\vec{t}$ and $\vec{H}_0$ and ψ is the angle between the main normal to the beam $\vec{n}$ and the plane $(\vec{t}, \vec{H}_0)$.

Eq.(6) describes the field polarization at quasi-longitudinal ($\alpha \ll 1$), as well as at quasi-transverse ($|\pi/2 - \alpha| \ll 1$) propagation. In case of quasi-longitudinal propagation we may neglect the last term in (6), thus obtaining

$$\theta(\sigma) = \theta(0) + \int_0^\sigma \frac{d\sigma}{T} + \theta_F(\sigma), \tag{7}$$

where

$$\theta_F(\sigma) = \frac{1}{2} k \int_0^\sigma v \sqrt{u} \cos\alpha \, d\sigma = \frac{2\pi e^3}{m^2 c^2 \omega^2} \int_0^\sigma N H_0 \cos\alpha \, d\sigma$$

is the Faraday rotation angle. Therefore, when $\alpha \ll 1$, the resulting rotation angle of the polarization plane is determined by beam torsion and Faraday effect (this result was obtained in [7] by another, not quite legitimate method, i.e. from the consideration of the superimposition of two independent circularly-polarized waves).

The last term in Eq.(6), which is quadratic with respect to H_0, describes the Cotton-Mouton effect. Explicit expressions for θ in the form cf quadratures may be obtained at not too great distances σ, taking into account that in high frequencies $u \ll \sqrt{u}$, and applying to (6) the theory of perturbations. Having assumed (7) for the zero approximation, we shall have in the first approximation

$$\theta'(\sigma) = \mathrm{Re}\,\theta(\sigma) = \theta(0) + \int_0^\sigma \frac{d\sigma}{T} + \theta_F(\sigma),$$

$$\theta''(\sigma) = \mathrm{Im}\,\theta(\sigma) = -\frac{1}{4} k \int_0^\sigma uv \sin^2\alpha \sin[2(\theta'(\sigma) + \psi)] \, d\sigma. \tag{8}$$

As stated above, the quantity $\mathrm{th}\theta'' \sim \theta''$ determines the minor to major axis of polarization ellipse's axes, that is, the degree of field depolarization. Formulas (8) may be useful for radioastronomical and radiophysical applications.

The author expresses his deep gratitude tp S. M. Rytov, L. L. Goryshkin and Yu. Ya. Yashin, for the discussion and valuable counsel.

*** T H E E N D ***

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Manuscript received on
8 March 1968

REFERENCES

1. V.L. GINSBURG, Propagation of Electromagnetic waves in a Plasma. "Nauka", 1967.
2. S.M. RYTOV, DAN* 18, 263, 1938.
3. S.M. RYTOV, Tr. Fiz. Inst. in the name of P.A. Lebedeva AN SSSR, 2, 1, 1940.
4. J.A. RATCLIFF, Magneto-ionic Theory and its Application to Ionosphere, IL** 1962
5. K.G. BUDDEN, Radio Waves in the Ionosphere, Cambridge, 1961.
6. R.M. LEWIS, Arch. Rat. Mech. Anal., 20, 191, 1965.
7. Yu.A. ZAYTSEV, Yu.A. KRAVTSOV, Yu.Ya. YASHIN, IZVVUZ, Radiofizika, 11, No.12, 1968.

CONTRACT NO.NAS-5-12487
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Translated by
Dr. Andre L. Brichant
13 January, 1969.

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* DAN stands for Doklady A.N. SSSR
** IL means "Foreign Literature"