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ON SOME FURTHER PROPERTIES OF NONZERO-SUM DIFFERENTIAL GAMES

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By

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ON SOME FURTHER PROPERTIES OF NONZERO-SUM DIFFERENTIAL GAMES*

By

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ABSTRACT

The general nonzero-sum differential game has N players, each controlling a different set of inputs to a single nonlinear dynamic system and each trying to minimize a different performance criterion. Several interesting new phenomena arise in these general games which are absent in the two best-known special cases (the optimal control problem and the two person zero-sum differential game). This paper considers some of the difficulties which arise in attempting to generalize ideas which are well-known in optimal control theory, such as the "principle of optimality" and the relation between "open-loop" and "closed-loop" controls. Two types of "solutions" are discussed: the "Nash equilibrium" and the "noninferior set". Some simple multistage discrete (bimatrix) games are used to illustrate phenomena which also arise in the continuous formulation.

This work is a continuation of work reported in Harvard University Technical Report No. 564, May 1968.

I. Introduction

In the general N-player nonzero-sum differential game, the ith player chooses u_i, trying to minimize

$$J_i = \int_{t_o}^{t_f} L_i(x, t, u_1, \ldots, u_N)\, dt + K_i(x(t_f)) \qquad (1)$$

subject to the n-dimensional state equation (common to all players)

$$\dot{x} = f(x, t, u_1, \ldots, u_N), \qquad x(t_o) = x_o \qquad (2)$$

and possibly subject to various inequality or equality constraints on the state and/or control variables (which are omitted here for simplicity).

This problems, which includes the optimal control problem ($N = 1$) and the 2-person zero-sum differential game ($N = 2$, $J_1 = -J_2$) as special cases, is of interest in analysing a dynamic system with inputs controlled by several "players" with not entirely conflicting goals.

One would naturally expect that methods for computing solutions to these problems could be obtained by generalizing well-known methods of optimal control theory. While this is true to some extent, several difficulties arise which are absent in control problems and two-person zero-sum differential games. In this paper, we shall consider generalizations of two ideas which are of great use in solving optimal control problems:

1) The relation between "open-loop" and "closed-loop" optimal controls.

2) The "principle of optimality."

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Nonzero-sum differential games were discussed by Starr and Ho[1] who concluded that there was no single satisfactory definition of "optimality" for these problems. Depending upon the application, various types of solutions are relevant.

One interesting type of solution was the "Nash equilibrium." It is "optimal" in the sense that no player can achieve a better result by deviating from his "Nash" controls as long as the other players continue to use their "Nash" controls. Denoting the control strategy and the cost for the ith player by u_i and J_i respectively, the Nash equilibrium strategy set $\{u_1^*, \dots, u_N^*\}$ has the property that for $i = 1, \dots, N$,

$$J_i(u_1^*, \dots, u_N^*) = \min_{u_i} J_i(u_1^*, \dots, u_{i-1}^*, u_i, u_{i+1}^*, \dots, u_N^*)$$

Letting $u^* = \{u_1^*, \dots, u_N^*\}$ and $J = \{J_1, \dots J_N\}$, we sometimes refer to u^* as a "Nash saddle point" of $J(u)$.

Depending on the formulation of the problem, u_i may be one of a finite set of controls (static bimatrix game), a function of time (open-loop differential game), a function of the state vector and time (closed-loop differential game), etc.

In the analysis of competitive dynamic systems (e.g. several rival firms in an imperfectly competitive market) the restriction that no binding agreements can be made among the players leads naturally to the "secure" Nash solutions. One then would like to know what has been sacrificed to obtain this security, i.e. do solutions exist which reduce the costs of <u>all</u> players below their Nash costs? This leads us to a second type of interesting solution: the "set of noninferior strategies." If $\hat{u}$ is noninferior, then there exists no u such that

$$J_i(u) \leqq J_i(\hat{u}) \qquad \text{for } i = 1, \ldots, N$$

with the inequality strict for at least one i. Any "negotiated" solutions with all players cooperating but no transfer payments allowed should be chosen from this class. In most differential games, there is a single Nash solution but an (N - 1)-parameter family of noninferior, or "undominated," solutions.

II. The relationship between "open-loop" and "feedback" Nash solutions.

In optimal control problems one often distinguishes between "open-loop" solutions, where the optimal control for a trajectory through a specified initial state x_0 is given as a function of time, and "closed-loop" or "feedback" solutions which give the optimal control as a function of the state x and time t everywhere in an appropriate region of the state-time space. It is well-known that in deterministic problems* the open-loop solution $u^o(t)$, $t_0 \leqq t \leqq t_f$, can be generated from the feedback solution $u^o(x, t)$ by simply integrating the state equation forward from the initial point (x_0, t_0). This would be a reasonable way to find the open-loop control if an algorithm (based on a dynamic programming approach) were available for computing the closed-loop optimal controls in a region containing the given initial point.

Alternatively, if a successful open-loop algorithm (based on a variational approach) is available for calculating $u^o(t)$ for a trajectory through (x_0, t_0), then the closed-loop control law can at least in principle be generated by successively solving the open-loop problem for each initial point (x_0, t_0).

In what appears to be the most interesting class of differential games, all players know the current state vector, so that a "closed-loop" Nash solution is required*. There may also be interesting "open-loop" problems where the entire sequence of controls for each player must be chosen prior to the initial time.

Whichever type of Nash solution is required, one could in principle solve for the Nash strategies for all the players in advance, since there are no "unpredictable" inputs to the system. One therefore is tempted to conclude that the same relation exists between the open-loop and closed-loop strategies as exists in the optimal control problem; i.e., that they are just different ways of describing the same outcome. The purpose of this section is to demonstrate that such a conclusion is false. Although our real interest is continuous differential games, we shall first illustrate the basic idea by considering a very simple discrete finite-state multistage game.

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[*] i. e., problems where all parameters and all inputs to the system over the time interval under consideration are known at the initial time.

[**] More realistically, they might have imperfect (noisy) measurements of the state vector, but here we assume exact knowledge of the state vector as well as all the system parameters including the cost functions for the other players.

In the two-player game in Fig. 1, each player has two possible controls, labeled 0 and 1. At each stage t, both players simultaneously choose a control. The resulting control pair determines the transition to the next stage. There are four possible transitions, leading to three possible stages x, and associated with each transition are costs c_1, c_2 (in circle) for the two players. Each player wants to minimize his total cost in reaching t = 2, the terminal stage.

Let us try to find the "closed-loop" Nash solution by following the "dynamic programming" approach. At stage t = 1 and state x = 2, the situation for the two players is represented by the bimatrix game in Fig. 2a. Clearly the controls 0, 0 are the only pair with the Nash property, since Player 1 would increase his cost from 2 to 3 by playing 1. (As far as the Nash equilibrium is concerned, it does not matter what would happen if both players played a non-Nash control.) The Nash costs are $c_1 = c_2 = 2$. Similarly, we see from Fig. 2b that the Nash controls at x = 1, t = 1 are 1, 1 with costs 0, 3 and from Fig. 2c we see that at x = 0, t = 1 the Nash control pair 1, 0 gives costs 4, 1. Moving back to the initial stage t = 0, we assume that the players will play their Nash controls at t = 1, so we add the Nash cost 2, 2 associated with state 2 to the costs of the transition leading to state 2, etc. The resulting situation is given in Fig. 2d. The Nash control pair is then 0, 1 with costs 4, 4 for the entire game. The "trajectory" is x(1) = 2, x(2) = 2.

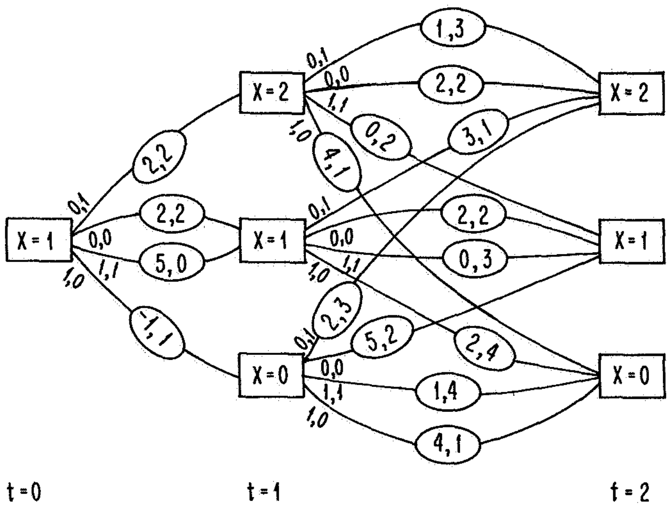

FIG. 1 A DISCRETE MULTISTAGE GAME

PLAYER 2

(a) PLAYER 1

	0	1	
0	2,2	1,3	x = 2
1	4,1	0,2	t = 1

PLAYER 2

(b) PLAYER 1

	0	1	
0	2,2	3,1	x = 1
1	2,4	0,3	t = 1

PLAYER 2

(c) PLAYER 1

	0	1	
0	5,2	2,3	x = 0
1	4,1	1,4	t = 1

PLAYER 2

(d) PLAYER 1

	0	1	
0	2,5	4,4	x = 0
1	3,2	5,3	t = 0

FIG. 2 SITUATION AT EACH STATE x AND TIME t.

PLAYER 2

PLAYER 1

	00	01	10	11
00	4,4	5,3	4,4	3,5
01	4,6	2,5	6,3	2,4
10	4,3	1,4	7,2	8,1
11	3,2	0,5	7,4	5,3

NASH CLOSED-LOOP

NASH OPEN-LOOP

FIG. 3 OPEN LOOP COST TABLE FOR GAME IN FIG. 1

Can we then conclude that this trajectory with its associated control sequences 00, 10 is also the open-loop Nash trajectory? In Fig. 3 the costs are tabulated for each pair of open-loop control sequences. Inspection of this bimatrix game shows that only the control sequence pair 11, 00 has the Nash property (giving costs 3, 2). The closed-loop Nash solution 00, 10 does <u>not</u> have the Nash property in the open-loop table. The open-loop Nash trajectory is $x(1) = 0, x(2) = 0$.

One reason for this difference between the open- and closed-loop solutions is the fact that several control sequences were eliminated from consideration at $t = 0$ by the assumption that the player would only choose Nash controls at $t = 1$ (based on knowledge of state at $t = 1$). This assumption that the players will always attempt to "optimize" the remaining part of the trajectory based on current state regardless of previous actions is the natural extension of the basic principle of optimalty found in all dynamic programming type of calculations. Yet it is NOT always safe to employ such assumptions in the nonzero-sum case. Another interesting point to note is that the Nash open loop costs (3, 2) in Fig. 3 is strictly superior to the closed loop costs (4, 4) calculated via "dynamic programming." This casts further doubt in the applicability of the principle of optimalty. We shall have more to say on this in section III.

It should be pointed out that the two-stage game with closed-loop control in Fig. 1 can also be represented as a single bimatrix game, but not the same one as was obtained in Fig. 3 for open-loop controls. Since each player has eight possible feedback strategies, the closed-loop bimatrix game will be an 8 x 8 table. In this array, only the closed loop strategy pair

$$\begin{array}{ll} u_1(0,0) = 0 & u_2(0,0) = 1 \\ u_1(1,1) = 1 & u_2(1,1) = 1 \\ u_1(2,1) = 0 & u_2(2,1) = 0 \end{array}$$

has the Nash property. Obviously this would be a very cumbersome way to find closed-loop Nash strategies, especially with a larger number of states, stages, controls or players.

Continuous Differential Games

A general conceptual method for finding the "closed-loop" Nash equilibrium control $u_1^*(x,t), \ldots, u_n^*(x,t)$ was presented in [1]. One finds the "remaining cost functions" $V_i(x,t)$, $i = 1, \ldots, N$, by solving a set of coupled partial differential equations

$$-\frac{\partial V_i}{\partial t} = \min_{u_i} H_i(x; t; u_1, \ldots, u_N; \frac{\partial V_i}{\partial x}), \qquad i = 1, \ldots, N \tag{3}$$

where the Hamiltonian for the ith player is

$$H_i(x; t; u_1, \ldots, u_N; \frac{\partial V_i}{\partial x}) = L_i(x, t, u_1, \ldots, u_N) + \frac{\partial V_i}{\partial x} f(x, t, u_1, \ldots, u_N) \tag{4}$$

On the terminal surface.

$$V_i(x(t_f), t_f) = K_i(x(t_f)) \tag{5}$$

The Nash controls are the u_i which achieve the required minima. If the functions L_i and f are continuously differentiable in u_i and if the minimum is in the interior of the set of admissible controls, then u_i can be found by solving

$$\frac{\partial H_i}{\partial u_i} = 0, \qquad i = 1, \ldots, N \tag{6}$$

to obtain u_i explicitly as a function of x, t, and $\frac{\partial V}{\partial x}$. One must then solve the set of partial differential equations for the $V_i(x, t)$, from which one finally obtains the $u_i^*(x, t)$.

To find the open-loop Nash solutions, one first uses a variational method to derive necessary conditions. Case[2] obtained the following conditions, which hold only if the controls are all open-loop:

$$\dot{x} = f(x, t, u_1, \ldots, u_N) \tag{7}$$

$$\dot{\lambda}_i^T = - \frac{\partial H_i}{\partial x} \tag{8}$$

$$\lambda_i^T(t_f) = \frac{\partial}{\partial x(t_f)} K_i(x(t_f)) \tag{9}$$

$$u_i \text{ minimizes } H_i(x;t;u, \ldots, u_N; \lambda_i^T) \tag{10}$$

where

$$H_i(x;t;u_1, \ldots, u_N;\lambda_i^T) = L_i(x, t, u_1, \ldots, u_N) + \lambda_i^T f(x, t, u_1, \ldots, u_N) \tag{11}$$

Computational algorithms can be obtained from these necessary conditions. Necessary conditions for the closed-loop Nash controls $\Psi_1(x,t),\ldots,\Psi_N(x,t)$ were obtained by Starr and Ho[1] by replacing (8) by

$$\dot{\lambda}_i^T = -\frac{\partial H_i}{\partial x} - \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\partial H_i}{\partial u_j} \frac{\partial \Psi_j}{\partial x}(x,t) \tag{12}$$

The presence of the summation term in (12) makes the necessary conditions (7), (12), (9), (10) virtually useless for deriving computational algorithms. Note that this troublesome term is absent in the optimal control problem (because N = 1), in the two-person zero-sum game (because $H_1 = -H_2$ so $\frac{\partial H_1}{\partial u_2} = -\frac{\partial H_2}{\partial u_2} = 0$), and in the open-loop nonzero sum problem (because $\frac{\partial \Psi_j}{\partial x} = 0$). One certainly expects the open- and closed-loop solutions to be different whenever this term is nonzero.

Using reasoning familiar from optimal control theory, one may interpret (12) as follows: λ_i is the "influence function" for the ith player, i. e., the sensitivity of his cost to a perturbation in the state vector. If the other players are using feedback strategies, any perturbation δx of the state vector will cause them to change their controls by an amount $\frac{\partial \Psi_j}{\partial x}\delta x$. If the ith Hamiltonian were already extremized with respect to the control u_j, $j \neq i$, this would not affect the ith player's cost, but since generally $\frac{\partial H_i}{\partial u_j} \neq 0$ for $i \neq j$, the reactions of the other players to the perturbation will influence the ith player's cost, and the ith player must account for this effect in considering variations of the trajectory.

In fact, a rather peculiar situation arises when the ith player makes a small change δu_i in his control in the vicinity of the Nash trajectory. Since $\frac{\partial H_i}{\partial u_i} = 0$, the effect of δu_i on the ith player's cost is only second order in δu_i, but the effects on all the other player's costs are first because $\frac{\partial H_j}{\partial u_i} \neq 0$ for $i \neq j$. In making fine adjustments to reach his minimum cost, the ith player thus may cause wild fluctuations (either beneficial or harmful) in his rivals' costs. If they are able to react to this change (i.e., they have closed-loop control) they in turn cause first order changes in the ith player's cost, so that another second order term in δu_i (due to the reactions of the rivals) must be added to the "direct" second order effect of δu_i on the ith cost. It is thus easy to see that the equilibrium conditions (and consequently the trajectories which satisfy them) are not the same in the open- and closed-loop problems. Even for the simplest nonzero sum differential game, the "linear-quadratic" case, entirely different Nash solutions have been obtained by the authors for the open-loop and closed-loop formulations.

III. The Optimality Principle

The well-known "principle of optimality" has been of great use in providing a conceptual framework for solving optimal control problems. The same principle, which Isaacs called the "tenet of transition," is the basis of a general method for finding optimal strategies in zero-sum two-person differential games. It is thus naturally interesting to inquire what principle of optimality, if any, holds for more general N-person nonzero-sum differential game. In this section we shall discuss the relation between the noninferior solutions, the Nash solution, and the optimality principle.

In a static nonzero-sum game, we shall speak of a "prisoners' dilemma" situation* whenever the Nash solution does not belong to the noninferior set. For example. in Fig. 2 the "prisoners' dilemma" occurs in bimatrix games a and d, but not in b or c. It should also be clear what is meant by the statement that the vector Hamiltonian

$$H = \{H_1, \ldots, H_N\} \qquad \text{(with } H_i \text{ defined in (11))}$$

has a "prisoners' dilemma" for some particular values of $x, t, \lambda_1, \ldots, \lambda_N$.

Now consider a dynamic game (either a differential game or a multistage game) whose closed-loop Nash solution is obtained via the "dynamic programming" approach used in Section II. One is tempted to guess that if no "prisoners' dilemma" occurs at any stage or state during the computation of the Nash equilbrium, then the Nash solution is noninferior. But this conjecture is false, as we shall see below. Again we start with a discrete multistage game. The game in Fig. 4 is almost trivial; it is really a single static bimatrix game played twice. Since there is only one state, there is no difference between "open-loop" and "closed-loop" †. One can see by inspection that the "prisoners'

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[*] See footnote in introduction of ref. [1].

[†] A more complicated counter example where "state" is important can also be constructed, but the game in Fig. 4 is adequate for our purposes.

dilemma" does not occur at either stage in the Nash solution. The pair of control sequences 00, 11 gives the Nash solutions. At no stage did the "prisoners dilemma" situation occur; i. e. , the Nash solution at each stage was noninferior. Can we conclude from this that the Nash solution is noninferior globally over 2 stages? In other words is there no "cooperative" solution by which both players can reduce their costs? To answer this, we tabulate the costs for all possible pairs of control sequences in Fig. 5.

Inspection of Fig. 5 shows that there are eight noninferior solutions (marked with *) but the Nash solution is not among them. By playing either 01 against 01 or 10 against 10, the costs are 5, 5, compared to the Nash costs 8, 8 obtained by playing 00 against 11. But to obtain the costs 5, 5 by the sequence 01, 01, Player 2 must trust Player 1 not to try to optimize (by playing control 0) at t = 1. Similarly, if the costs 5, 5 are to be obtained by the sequences 10, 10, then Player 1 must trust Player 2.

This very simple game has illustrated two basic points about nonzero-sum multistage games:

(i) The absence of a "prisoners' dilemma" situation at every stage in solving for the Nash controls does not guarantee that the Nash solution is noninferior over all stages.

(ii) Noninferior solutions generally require trusting the rivals to play nonoptimal controls, not only at the present stage but at all future stages as well.

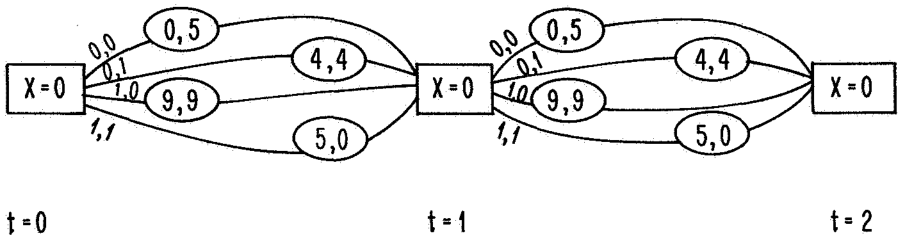

FIG. 4. A SIMPLE MULTISTAGE GAME.

		PLAYER 2				
		00	01	10	11	
PLAYER 1	00	*0,10	*4,9	*4,9	8,8	NASH
	01	9,14	*5,5	13,13	*9,4	
	10	9,14	13,13	*5,5	*9,4	
	11	18,18	14,9	14,9	*10,0	

FIG. 5. COST TABLE FOR ALL CONTROL SEQUENCES FOR THE GAME IN FIG.4.

More basically, the principle of optimality, which is obvious in control problems, also applies in zero-sum differential game problems because it is reasonable to base the choice of action at one time on an assumed mode of behavior of the players at later times, i. e. they will seek a minimum or a saddle point. The fact that the "Nash" solution for the simple game at $t = 0$ was noninferior was <u>dependent on the assumption</u> that a Nash solution would be used at $t = 1$. In nonzero sum games, since the meaning of "optimality" is nonunique, it is natural but not necessarily desirable to assume that the rivals will continuously seek one particular form of solution, in this case the Nash equilibrium. Cooperation should thus be considered not only at any given stage but over several stages.

The noninferior solutions to the general differential game were also presented in [1]. They could be obtained by solving the (N-1)-parameter set of scalar optimization problems

$$\min_{u_1, \ldots, u_N} \sum_{i=1}^{N} \mu_i J_i \quad \text{where} \sum_{i=1}^{N} \mu_i = 1 \text{ and } \mu_i > 0 \tag{13}$$

provided that certain convexity conditions are satisfied*.

For a given time-invariant weighting vector μ, the associated noninferior trajectory can be found by solving the Hamilton-Jacobi equation

$$-\frac{\partial \hat{V}}{\partial t}(x, t, \mu) = \min_{u_1, \ldots, u_N} \hat{H}(x;t;u_1, \ldots, u_N; \frac{\partial \hat{V}}{\partial x}; \mu) \tag{14}$$

---

*See footnote on next page.

where

$$\hat{H} = \sum_{i=1}^{N} \mu_i L_i(x, t, u_1, \ldots, u_N) + \frac{\partial \hat{V}}{\partial x} f(x, t, u_1, \ldots, u_N) \tag{15}$$

and

$$\hat{V}(x(t_f), t_f) = \sum_{i=1}^{N} \mu_i K_i(x(t_f), t_f) \tag{16}$$

Let us now assume that the closed-loop Nash solution has been found by solving eq. (3). Generally the Nash solution will not belong to the noninferior set. But suppose our game has the special property that the controls for the Nash trajectory through any initial point are also the controls for the noninferior trajectory for some time-invariant weighting vector μ*. Then the remaining noninferior cost must be related to the remaining Nash costs by

$$\hat{V}(x, t, \mu^*) = \sum_{i=1}^{N} \mu_i^* V_i(x, t) \tag{17}$$

---

[*] It is sufficient that the set of (N + n)-vectors

$$\begin{bmatrix} L_1(x, t, u_1, \ldots u_N) \\ \cdots \\ L_N(x, t, u_1, \ldots, u_N) \\ f(x, t, u_1, \ldots, u_N \end{bmatrix}$$

generated by all the admissible controls be convex for all admissible x. A weaker sufficient condition and a rigorous derivation are given (for the discrete-time control problem with vector cost criterion) in Ref. 4.

Substituting (17) into (15) (with $\mu = \mu^*$), the Nash Hamiltonians H_i are related to the noninferior Hamiltonian $\hat{H}$ by

$$\hat{H}(\mu^*) = \sum_{i=1}^{N} \mu_i^* H_i \tag{18}$$

Thus the assumption that the Nash solution is noninferior implies that, at each time t on the trajectory, the set of controls which satisfied the Nash condition for the (static) vector function $[H_1, \ldots, H_N]$ also minimizes some time-invariant positive weighted linear combination of the H_i, $i = 1, \ldots, N$. In other words, as we solve the infinite sequence of "static Nash saddle-point problems" (to get the Nash trajectory) we never encounter the "static prisoners' dilemma situation."

This is a necessary condition for the Nash solution to be noninferior. In effect, it says that it is impossible for all players to gain by playing "cooperative" controls in the time interval $[t, t + dt]$ and then reverting to the local Nash controls in the interval $[t + dt, t_f]$. Without the requirement that μ^* be time invariant, it would not be sufficient that "the static prisoners' dilemma situation" never occurs along the Nash trajectory.

Suppose the Nash solution has already been obtained for a given game. We wish to determine whether or not this solution is noninferior. A simple way to check this would be to start at the terminal time and compute the controls which minimize (at time t_f) the linear combination

$$\sum_{i=1}^{N} \mu_i H_i$$

for some arbitrary positive weighting μ. By iteration we then attempt to find a μ^* satisfying

$$\sum_{i=1}^{N} \mu_i^* = 1 \quad \text{and} \quad \mu_i^* > 0, \; i = 1, \ldots, N$$

which gives controls coinciding (at time t_f) with the Nash controls. Three results are possible:

(i) No such μ^* exists, in which case the Nash solution is not noninferior.

(ii) A unique μ^* is obtained.

(iii) μ^* is not uniquely determined, in which case more conditions are obtained by repeating this procedure at earlier times.

If a unique μ^* is found, one can then solve the optimal control problem with the scalar cost criterion

$$\hat{J} = \sum_{i=1}^{N} \mu_i^* J_i$$

starting at the terminal point of the Nash trajectory. The resulting noninferior trajectory (holding μ^* constant) will coincide with the Nash trajectory if an only if the latter is noninferior.

IV. Conclusions

The previous two sections have illustrated some of the interesting phenomena which arise when the optimal control problem (or alternatively, the "strictly competitive" zero-sum differential game) is generalized by allowing several controllers with different cost criteria. If one seeks a Nash equilibrium trajectory, one must specify whether or not the controlers have instantaneous access to the state vector, since the "open-loop" and "closed-loop" formulations lead to entirely different solutions. If one wonders whether a different solution exists which produces a better result for all "players" than the "secure" closed-loop Nash set of control strategies, it is not sufficient to examine the set of Hamiltonians at each point on the Nash trajectory. This "vector Hamiltonian" contains the information necessary for computing the closed-loop Nash controls at time t, provided the problem has already been solved for the remaining time interval, but it does not contain information about noninferior solutions, open-loop Nash solutions, or any other solutions which may be of interest.

Also central to the discussion in Sections II and III was the fact that on a Nash trajectory each player's cost is minimized with respect to his own control but not with respect to the other players' controls. Generally there will be no set of controls which simultaneously minimizes all the players' costs. If such a set of controls did exist, the problem would degenerate into N uncoupled optimal control problems, with each player controlling all N controls. All players would arrive at the same set of N optimal controls, and the Nash solution would thus be noninferior (for every positive weighting vector μ).

Because his cost is not minimized with respect to the jth player's control (i.e., $\frac{\partial H_i}{\partial u_j} \neq 0$) the ith player will be very sensitive to changes in his rivals' controls. This fact is the cause of considerable difficulty in developing algorithms for computing Nash controls for nonlinear problems.

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13. ABSTRACT

The general nonzero-sum differential game has N players, each controlling a different set of inputs to a single nonlinear dynamic system and each trying to minimize a different performance criterion. Several interesting new phenomena arise in these general games which are absent in the two best-known special cases (the optimal control problem and the two person zero-sum differential game). This paper considers some of the difficulties which arise in attempting to generalize ideas which are well-known in optimal control theory, such as the "principle of optimality" and the relation between "open-loop" and "closed-loop" controls. Two types of "solutions" are discussed: the "Nash equilibrium" and the "noninferior set". Some simple multistage discrete (bimatrix) games are used to illustrate phenomena which also arise in the continuous formulation.

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