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# Bandlimited Image Restoration by Linear Mean-Square Estimation

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A linear mean-square estimator optimum for data available only on a finite interval is derived for the restoration of images degraded by a system with a bandlimited spread function. The analysis is carried out in one dimension using a prolate spheroidal wavefunction expansion of the image process. If the noise is bandlimited to the same bandwidth as the spread function, the expansion represents the image process with zero mean square error on the entire interval, and the estimate of the geometrical image achieves the same mean-square error as the optimum estimate for data on the infinite interval. The rate at which the estimate converges is discussed and an example presented.

A solution to the image restoration problem using linear mean-quare estimation has been presented by Helstrom,[1] who demonstrated that data are available over an infinite interval, Fourier methods could be sed to obtain the restoring filter. Slepian has used the linear mean-quare method to obtain a restoring filter optimum for data from a system ith a stochastic spread function.[2] His method uses data only over a finite egion, but he does not assume that the system spread function is band-mited. Other authors have recently presented solutions to the restoration roblem for systems with bandlimited spread functions.[3,4,5]

In this paper a restoring filter is derived that is optimum for data vailable only over a finite interval. The analysis is performed in one imension for simplicity; this imposes no fundamental limitation on the echnique. In the special case where the noise is bandlimited to the same andwidth as the spread function, the optimum filter for data on the finite nterval achieves the same minimum mean-square error as the optimum lter for data on the infinite interval.

The analysis makes use of the fact that a bandlimited stationary tochastic process known only over a finite interval can be estimated with ero mean-square error at any point outside the interval. This can be done y expanding the process in a series of prolate spheriodal wave functions PSWF). A simple proof is given in the Appendix.

## 1. The Formulation of the Problem

The optical system will be modeled mathematically by the convolu-
onal integral equation

$$J(x) = \int_{-\infty}^{\infty} S(x-x')\, J_o(x')\, dx' + N(x). \qquad (1.1)$$

$_o(x)$, the illuminance of the geometrical image, will be taken to be a zero-
ιean stationary stochastic process with spectral density $\varphi_o(\omega)$ . $S(x)$ is
ιe point spread function of the system. The Fourier transform $s(\omega)$ of
(x) is bandlimited to the spatial frequency ($\omega$) interval $(-\Omega/2, \Omega/2)$. The
ystem noise $N(x)$ is also a zero-mean stationary stochastic process in this
ιodel. It has spectral density $\varphi_N(\omega)$, and for simplicity it is assumed to
e uncorrelated with the geometrical image. This model is discussed by
'elstrom.[1]

We assume that an experimenter has observed the image-plane
.luminance $J(x)$ over the finite interval $(-L/2, L/2)$. His task is to use
ιese data to form an estimate of the geometrical image $J_o(x)$ at some
oint $\xi$. He does so by performing a linear operation on the available
ata that minimizes the mean-square error. Hence, he must find a weighting
ınction $M(\xi, x)$ such that

$$\hat{J}_o(\xi) = \int_{-L/2}^{L/2} M(\xi, x')J(x')dx' \qquad (1.2)$$

ιinimizes

$$\epsilon(\xi) = \underset{\sim}{E}\left[\left|J_o(\xi) - \hat{J}_o(\xi)\right|^2\right]. \tag{1.3}$$

Here $\underset{\sim}{E}$ denotes mathematical expectation.

At this point we could proceed by formal methods to obtain an integral equation for $M(\xi, x)$. There is, however, an alternative procedure that makes use of the optimum filter for data on the infinite interval derived by Helstrom[1]. First it will be necessary to establish some properties of the PSWF's. There are taken directly from Slepian and Pollak.[6]

The PSWF's are solutions of the integral equation[7]

$$\int_{-L/2}^{L/2} \frac{\sin \frac{1}{2}\Omega\,(x - x')}{\pi\,(x - x')}\,\Psi_n(c, x')\,dx' = \lambda_n(c)\,\Psi_n(c, x) \tag{1.4}$$

where $c = \Omega L/4$. We will make use of the following two properties of the PSWF's:

PROPERTY I. The PSWF's $\Psi_n(c, x)$ form a complete orthonormal basis for the class of bandlimited functions.

PROPERTY II. The PSWF's $\Psi_n(c, x)$ are orthogonal over the interval $(-L/2, L/2)$ with

$$\int_{-L/2}^{L/2} \left|\Psi_n(c, x)\right|^2 dx = \lambda_n(c).$$

They form a complete orthogonal basis for the class of functions square integrable on the x-interval $(-L/2, L/2)$.

he following equation will also be used,[8]

$$\int_{-\infty}^{\infty} \Psi_n (c, x)\, e^{-i\omega x}\, dx = \begin{cases} i^{-n} \, (2\pi L/\Omega)^{\frac{1}{2}} \lambda_n^{-\frac{1}{2}}(c)\, \Psi_n(c, L\omega/\Omega) & |\omega| \leq \Omega/2 \\ 0 & |\omega| > \Omega/2 \end{cases} \tag{1.5}$$

## 2. The Solution for Noise Bandlimited to $(-\Omega/2, \Omega/2)$

If the noise $N(x)$ is bandlimited to the interval $(-\Omega/2, \Omega/2)$, $J(x)$ is rom Eq. (1.1) a bandlimited process. Now if $J(x)$ is available only over he finite interval $(-L/2, L/2)$, it can be estimated with zero mean-square error at any point by its PSWF expansion. This is proved in the appendix. Hence

$$\tilde{J}(x) = \sum_{n=0}^{\infty} J_n \Psi_n (c, x) \,, \tag{2.1}$$

where

$$J_n = \int_{-L/2}^{L/2} J(x') \Psi_n (c, x')\, dx'/\lambda_n(c) \,, \tag{2.2}$$

represents $J(x)$ with zero mean-square error <u>for all x</u>. For the purpose of linear mean-square estimation $\tilde{J}(x)$ is equivalent to $J(x)$. Hence the restoring filter for data on the infinite interval can be applied to (2.1). We have

$$\hat{J}_o(\xi) = \int_{-\infty}^{\infty} M_\infty (\xi - x')\, \tilde{J}(x')\, dx' = \sum_{n=o}^{\infty} J_n \int_{-\infty}^{\infty} M_\infty (\xi - x')\, \Psi_n (c, x')\, dx' \tag{2.3}$$

ɪere[9]

$$M_\infty(x) = \int_{-\Omega/2}^{\Omega/2} \frac{s^*(\omega)\varphi_o(\omega)}{|s(\omega)|^2 \varphi_o(\omega) + \varphi_N(\omega)} e^{i\omega x} d\omega/2\pi. \tag{2.4}$$

ɘt

$$q(\omega) = \frac{s^*(\omega)\varphi_o(\omega)}{|s(\omega)|^2 \varphi_o(\omega) + \varphi_N(\omega)}. \tag{2.5}$$

r using Eq. (1.5) and the convolution relation for Fourier transforms, ɉ. (2.3) can be simplified to

$$\hat{J}_o(\xi) = (2\pi L/\Omega)^{\frac{1}{2}} \sum_{n=0}^{\infty} \lambda_n^{\frac{1}{2}}(c) M_n(\xi) J_n \tag{2.6}$$

here $$M_n(\xi) = \left[i^{-n}/\lambda_n(c)\right] \int_{-\Omega/2}^{\Omega/2} q(\omega)\Psi_n(c, L\omega/\Omega) e^{i\omega\xi} d\omega/2\pi. \tag{2.7}$$

The minimum mean square error achieved by this estimate is

$$\epsilon = \int_{-\infty}^{\infty} \varphi_o(\omega) d\omega/2\pi - \int_{-\Omega/2}^{\Omega/2} \frac{|\varphi_o(\omega) s^*(\omega)|^2}{\varphi_o(\omega)|s(\omega)|^2 + \varphi_N(\omega)} d\omega/2\pi \tag{2.8}$$

his can be verified by substituting Eq. (2.6) into Eq. (1.3). Note that ı the analysis it was not necessary to assume that $J_o(x)$ itself was band-mited. Also, Eq. (2.8) is (as it must be from the analysis) the minimum ıean-square error achieved by $M_\infty(x)$ operating on $J(x)$.[10]

The fact that $\epsilon$ is independent of $\xi$ suggests that Eq. (2.6) could e used to estimate the illuminance of the geometrical image in the entire lane when image-plane data are available only over the finite interval. 'he practical difficulties in actually carrying this out are, however, ormidable. They will be discussed in section 4.

## 3. The General Solution and its Mean Square Error

When the noise term in Eq. (1.1) is not bandlimited to $(-\Omega/2, \Omega/2)$, he series of Eq. (2.1) no longer represents J(x) outside the interval -L/2, L/2). It is, however, only the noise term in Eq. (1.1) that is not epresented outside the interval (-L/2, L/2). One might suspect from his that Eq. (2.6) is still the optimum estimate. This is in fact correct. t can be established by showing that Eq. (2.6) does indeed still minimize ːq. (1.3). Now, unfortunately, the mean-square error is no longer inde-endent of $\xi$. It contains a $\xi$-dependent term that results from the imper-ect representation of N(x) outside (-L/2, L/2).

The mean square error can be written in the following form, which lisplays the contribution of the various terms. The function

$$W(\omega) = \begin{cases} 1 & |\omega| \le 1 \\ 0 & |\omega| > 1 \end{cases}$$

s used for notational convenience.

$$\varepsilon(\xi) = \int_{-\Omega/2}^{\Omega/2} \frac{\varphi_o(\omega)\varphi_N(\omega)}{|s(\omega)|^2 \varphi_o(\omega)+\varphi_N(\omega)} \, d\omega/2\pi$$

$$+\int_{-\infty}^{\infty} [1-W(2\omega/\Omega)]\varphi_o(\omega)d\omega/2\pi \tag{3.1}$$

$$+\int_{-\infty}^{\infty} [1-W(2\omega/\Omega)]\,|f(\omega)|^2 \varphi_N(\omega)d\omega/2\pi \; .$$

where

$$f(\omega) = \frac{2\pi L}{\Omega} \sum_{n=0}^{\infty} \int_{-\Omega/2}^{\Omega/2} q(\omega')\Psi_n(c, L\omega'/\Omega)e^{-i\omega'\xi}(d\omega'/2\pi)\Psi_n(c, L\omega/\Omega) \; . \tag{3.2}$$

Again this result can be derived by substituting Eq. (2.6) into Eq. (1.3) with due regard for the different limits involved. The first term in Eq. (3.1) is the minimum mean-square error achieved when both the geometrical image and the noise are bandlimited to $(-\Omega/2, \Omega/2)$. The second and third terms represent respectively the contributions of the object process spectrum and the noise spectrum outside the bandwidth of the spread function. From Eq. (3.2) the third term is $\xi$-dependent.

## 4. Practical Aspects of the Solution

For applications we need to know the rate at which Eq. (2.6) con-erges, since only a finite number of terms can be evaluated. Unfortunately his rate is strongly dependent on $\xi$ even though the $\xi$-dependence of the nean square error Eq. (3.1) might be small. This can be seen as follows.

A comparison of Eq. (1.2) and Eq. (2.6) leads to the conclusion hat

$$M(\xi, x) = (2\pi L/\Omega)^{\frac{1}{2}} \sum_{n=0}^{\infty} \lambda_n^{-\frac{1}{2}}(c) M_n(\xi) \Psi_n(c, x). \tag{4.1}$$

f $M(\xi, x)$ is represented with negligible error by N terms in the series (4.1),

$$\hat{J}_{oN}(\xi) = (2\pi L/\Omega)^{\frac{1}{2}} \sum_{n=0}^{N} \lambda_n^{\frac{1}{2}}(c) M_n(\xi) J_n \tag{4.2}$$

nust achieve a mean-square error close to the minimum possible error ;iven by Eq. (3.1). Hence consider

$$\delta_N(\xi) = \int_{-L/2}^{L/2} \left| M(\xi, x) - (2\pi L/\Omega)^{\frac{1}{2}} \sum_{n=0}^{N} \lambda_n^{-\frac{1}{2}}(c) M_n(\xi) \Psi_n(c, x) \right|^2 dx$$

$$= \frac{2\pi L}{\Omega} \int_{-L/2}^{L/2} \left| \sum_{n=N+1}^{\infty} \lambda^{-\frac{1}{2}}(c) M_n(\xi) \Psi_n(c, x) \right|^2 dx$$

$$= \frac{2\pi L}{\Omega} \sum_{n=N+1}^{\infty} |M_n(\xi)|^2 . \tag{4.3}$$

$_N(\xi)$ is the mean-square error (not to be confused with the statistical ean square error) that results when only N terms are used in Eq. (4.1).

We now want to know when $\delta_N(\xi)$ is small. This is determined by e behavior of $M_n(\xi)$. Now for most applications $q(\omega)$ is a slowly varying nction of $\omega$. From Eq. (2.7), then, the gross behavior of $|M_n(\xi)|$ is :oportional to

$$\left(1/\lambda_n(c)\right) \left| \int_{-L/2}^{L/2} \Psi_n(c, \frac{\Omega\omega}{L}) e^{i\omega\xi} \, d\omega/2\pi \right| .$$

ut from Eq. (1.5) this is proportional to $\lambda_n^{-\frac{1}{2}}(c)\,|\Psi_n(c,\omega)|$. In Fig. 1 $_n(c,x)$ is plotted for c = 8 for n = 0, 5, and 8. The behavior of $\Psi_n(c,x)$ easily understood when we recall that $\lambda_n(c)$ represents the fraction of .e total energy in the n-th PSWF that is in the interval (-L/2, L/2). .epian and Pollak[6] have shown that for $n < n_{crit}$, where[11] $n_{crit} = [2c/\pi]$, $_n(c) \approx 1$, and for $n > n_{crit}$, $\lambda_n(c) \approx 0$. This behavior is clearly displayed Fig. 1. From this we conclude that for $N > n_{crit}$, $\delta_N(\xi)$ will be small r $|\xi| < L/2$. $\hat{J}_{oN}(\xi)$ will achieve a minimum mean-square error close the theoretical value given by Eq. (3.1) for the same N and range of $\xi$.

›r $|\xi| > L/2$ terms of increasing order will be necessary.

## 5. An Example

To test the solution 1024 data points in the interval (-1, 1) were ›mputed by first convolving the geometrical image function

$$J_o(x) = (2\pi\sigma^2)^{-\frac{1}{2}}\left\{\exp\left[-(x-M)^2/2\sigma^2\right] + \exp\left[-(x+M)^2/2\sigma^2\right]\right\} \tag{5.1}$$

.th a spread function for a slit-diffracted image. The Fourier transform the spread function is

$$s(\omega) = \begin{cases} 1-2|\omega|/\Omega & |\omega| \leq \Omega/2 \\ 0 & |\omega| > \Omega/2 \,. \end{cases} \tag{5.2}$$

dependent, zero-mean gaussian random variates, $N_k$ and $N_{k'}$, were then lded according to the formula

$$J_k = [(J_{o_k} * S_k + N_k)^2 + N_{k'}{}^2]^{\frac{1}{2}}. \tag{5.3}$$

:he asterisk means convolution.) The convolution was carried out by ›ing the Cooley-Tukey Fourier transform algorithm on a CDC 3600 .gital computer. The normal variates were computed from a pseudo- ındom number sequence by standard methods. Data so constructed .olate our initial assumptions of stationarity and additive noise. The ıta are, however, non-negative and serve to represent a simple geometrical ıage.

The coefficients $J_n$ were calculated using Simpson's rule for n = 0 rough 8 with c = 8. The PSWF's were computed by using a Legendre ılynomial expansion with tabulated coefficients from Stratton et al.[12] ıe PSWF's were normalized to correspond to Slepian and Pollak's def- itions.[6] The eigenvalues $\lambda_n(c)$ are tabulated.[13]

To evaluate the weights $M_n(\xi)$, $\varphi_o(\omega)$ and $\varphi_N(\omega)$ must be specified. In e absence of any prior information they can be taken to be constant on e interval $(-\Omega/2, \Omega/2)$.[14] With these assumptions, by Eq. (2.5),

$$q(\omega) = \frac{1-2|\omega|/\Omega}{(1-2|\omega|/\Omega)^2+B} \tag{5.4}$$

here B is a constant that depends on the signal-to-noise ratio.[15] Now om Eq. (2.6) $M_n(\xi)$ is just $i^{-n}/\lambda_n(c)$ times the inverse Fourier transform : $q(\omega)\Psi_n(c, \omega)$.

With data on the interval (-1, 1) and $c = \Omega L/4 = 8$, $\Omega = 16$. By ıing these parameters the weights $M_n(\xi)$ were calculated for n = 0 ırough 8 using the Cooley-Tukey algorithm. The series (4.2) was then ʼaluated with 9 terms and the results plotted on a Calcomp automatic .otter. An example with $\sigma = 0.4$, M = 1.0 and B = 0.5 is reproduced ı Fig. 2.

## 6. Discussion

The example demonstrates that the estimate given by Eq. (4.2) works ell for a deterministic function. The reason for this is clear if we observe ıat with B equal to zero in Eq. (5.4), Eq. (2.5) results from the analytic ontinuation of the function J(x) to the infinite interval followed by the econvolution of Eq. (1.1) with N(x) = 0. This can be carried out as long s $J_o(x)$ is effectively bandlimited to $(-\Omega/2, \Omega/2)$, as was the case in the xample.

If, on the other hand, the Fourier transform of $J_o(x)$ is much roader than $\Omega$, $J_o(x)$ will normally be nearly zero outside (-L/2, L/2). data were available on the infinite interval, we would have the conditions r attempting super-resolution. We know, however, that super-resolution s frustrated by the slightest amount of noise.[3,4,5] In the estimate given y Eq. (4.2) the resolution is limited to that allowed by the bandwidth of e spread function. [Recall section 4 and the fact that the PSWF's emselves are bandlimited to $(-\Omega/2, \Omega/2)$] Hence for super-resolution ere is nothing new.

The conclusion seems to be that although super-resolution is not ossible when noise is present, extension of an image a small distance utside a finite interval is possible.

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## Appendix

### The Spheroidal Wavefunction Expansion of Bandlimited Stationary Stochastic Processes

A stationary stochastic process X(x) is bandlimited if its spectral nsity is non-zero only over a finite interval. Let the covariance function the process be K(x). Since

$$K(x) = \int_{-\Omega/2}^{\Omega/2} \varphi(\omega) e^{i\omega x} d\omega / 2\pi ,$$

ere $\varphi(\omega)$ is the spectral density of the process, K(x) is a bandlimited ction. Because of properties I and II of the PSWF's (Sec. 1.), the WF expansion[16]

$$K(x-x') = \underset{\sim}{E}\,[X(x)X(x')] = \sum_{n=0}^{\infty} \beta_n(x')\,\Psi_n(c, x), \qquad \text{(A1)}$$

ere

$$\beta_n(x') = \int_{-L/2}^{L/2} K(x'-x'')\,\Psi_n(c, x'')\, dx''/\lambda_n(c) \qquad \text{(A2)}$$

nverges in the mean of order two for $-\infty < x < \infty$ with x' fixed. For a al process K(x-x') = K(x'-x). Hence we also have convergence in the ıean of order two for $-\infty < x' < \infty$ with x fixed.

Now from Eq. (1.4) the PSWF's are eigenfunctions of the Hermitian ıtegral operator

$$Ly(x) = \int_{-L/2}^{L/2} \frac{\sin \frac{1}{2}\Omega(x-x')}{\pi(x-x')} y(x')dx', \quad -\infty < x < \infty. \tag{A3}$$

ſoreover, the kernel of Eq. (A3) is square integrable over the unbounded quare $-\infty < x, x' < \infty$, and there exists a positive dominating constant M uch that

$$\int_{-\infty}^{\infty} \left| \frac{\sin \frac{1}{2}\Omega(x-x')}{\pi(x-x')} \right|^2 dx \le M \tag{A4}$$

or $-\infty < x < \infty$. In fact we have strict equality in (A4) with $M = \Omega/2\pi$. We urther observe that any function z(x) bandlimited to $(-\frac{1}{2}\Omega, \frac{1}{2}\Omega)$ can be written

$$z(x) = \int_{-L/2}^{L/2} \frac{\sin \frac{1}{2}\Omega(x-x')}{\pi(x-x')} y(x')dx' \tag{A5}$$

ʳhere y is a function square integrable over $(-L/2, L/2)$. The represen- ation in Eq. (A5) and the Hermitian property of the operator L are suf- icient to establish the <u>pointwise</u> convergence of the series in Eq. (A1) for $-\infty < x < \infty$ and x' fixed. The additional property, Eq. (A4), of the kernel ɛnsures that the convergence is <u>uniform</u> for $-\infty < x < \infty$ and x' fixed.[17]

;ain, from the symmetry of $K(x-x')$ the convergence is pointwise and iform in $x'$ as well as $x$.

We can now prove the following theorem.

HEOREM: The PSWF expansion of the process $X(x)$

$$\tilde{X}(x) = \sum_{n=0}^{\infty} a_n \Psi_n(c, x) \tag{A6}$$

ıere

$$a_n = \int_{-L/2}^{L/2} X(x')\Psi_n(c, x')dx' / \lambda_n(c) \tag{A7}$$

ınverges in the mean <u>for all</u> $x$, i.e.

$$\lim_{N\to\infty} \underset{\sim}{E} [\,|X(x) - \sum_{n=0}^{N} a_n(c, x)|^2] = 0 \quad \forall x \in (-\infty, \infty)$$

roof: Let

$$\tilde{X}_N(x) = \sum_{n=0}^{N} a_n \Psi_n(c, x)$$

$$\lim_{N\to\infty} \underset{\sim}{E}[|X(x)-\tilde{X}_N(x)|^2] = \underset{\sim}{E}[|X(x)|^2]-2\lim_{N\to\infty} \underset{\sim}{E}[\tilde{X}_N(x)X(x)] + \lim_{N\to\infty} \underset{\sim}{E}[|\tilde{X}_N(x)|^2]. \tag{A8}$$

ow

$$\underset{\sim}{E}[|X(x)|^2] = K(0),$$

nd (A9)

$$\underset{\sim}{E}[\tilde{X}_N(x)X(x')] = \sum_{n=0}^{N} \underset{\sim}{E}[a_n X(x')]\Psi_n(c,x).$$

ʾut from Eq. (A7) and Eq. (A2)

$$\underset{\sim}{E}[a_n X(x')] = \int_{-L/2}^{L/2} K(x'-x'')\Psi_n(c,x'')dx''/\lambda_n(c)$$

$$= \beta_n(x') \tag{10}$$

Jsing the pointwise convergence of the series in Eq. (A1)

$$\lim_{N\to\infty} \underset{\sim}{E}[\tilde{X}_N(x)X(x)] = \sum_{n=0}^{\infty} \beta_n(x)\Psi_n(c,x) = K(0) \tag{A11}$$

o evaluate the last term in Eq. (A8) consider

$$\underset{\sim}{E}\,[\tilde{X}_N(x)\tilde{X}_N(x')] = \sum_{m=0}^{N}\sum_{n=0}^{N} \underset{\sim}{E}\,[a_n a_m]\,\Psi_n(c,x)\,\Psi_m(c,x') \qquad \text{(A12)}$$

sing Eq. (A7) and Eq. (A10)

$$\underset{\sim}{E}\,[a_n a_m] = \int_{-L/2}^{L/2} \underset{\sim}{E}\,[a_n X(x'')]\,\Psi_m(c,x'')dx''/\lambda_m(c)$$

$$= \int_{-L/2}^{L/2} \beta_n(x'')\,\Psi_m(c,x'')dx''/\lambda_m(c) \qquad \text{(A13)}$$

ow

$$\lim_{N\to\infty} \underset{\sim}{E}\,[\,|\tilde{X}_N(x)|^2] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\int_{-L/2}^{L/2} \beta_n(x'')\Psi_m(c,x'')dx''/\lambda_m(c)\right]\Psi_n(c,x)\Psi_m(c,x)$$

$$= \sum_{m=0}^{\infty}\left[\int_{-L/2}^{L/2}\sum_{n=0}^{\infty}\beta_n(x'')\Psi_n(c,x)\Psi_m(c,x'')dx''/\lambda_m(c)\right]\Psi_m(c,x)$$

'he interchange of summation and integration is justified by the uniform onvergence of the series in Eq. (A1). By using Eq. (A2) this can be ırther simplified to

$$\sum_{m=0}^{\infty} \left[ \int_{-L/2}^{L/2} K(x-x'')\Psi_m(c,x'')dx''/\lambda_m(c) \right] \Psi_m(c,x)$$

$$= \sum_{m=0}^{\infty} \beta_m(x)\Psi_m(c,x)$$

.nally using the pointwise convergence of the series in Eq. (A1)

$$\lim_{N\to\infty} \underset{\sim}{E} [\,|\tilde{X}_N(x)|^2] = K(0) \tag{A14}$$

ɔw substituting Eq. (A14), Eq. (A11), and Eq. (A9) into Eq. (A8)

$$\lim_{N\to\infty} E [\,|X(x)-\tilde{X}_N(x)|^2] = 0 \qquad \forall x \in (-\infty, \infty)$$

Q. E. D.

ɔte that from Eq. (A13) the coefficients $a_n$ are in general correlated.

Footnotes

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Note that Slepian and Pollak[6] define the frequency interval $(-\Omega, \Omega)$ rather than $(-\Omega/2, \Omega/2)$ as in this paper. Hence in their formulas $\Omega$ should be replaced by $\Omega/2$. With this slight change their definitions are used directly.

This is derived from Slepian and Pollak Ref. 6, Eq. (29).

Ref. 1, Eq. (2.5).

Cf. Ref. 1, Eq. (2.6).

The expression $[x]$ means the smallest integer larger than x.

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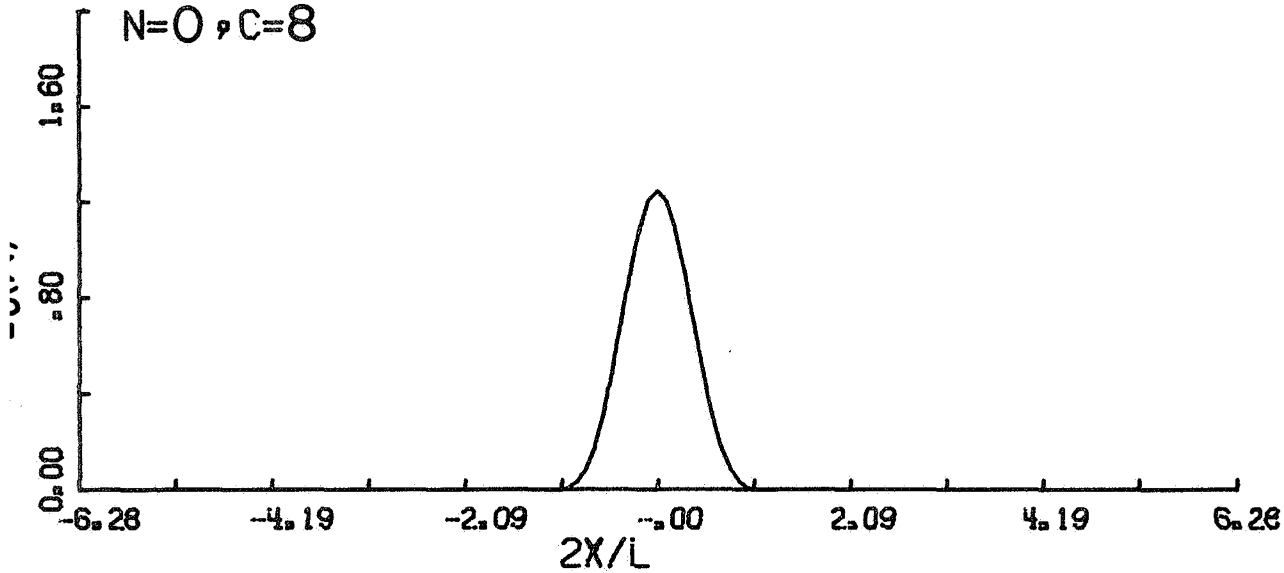
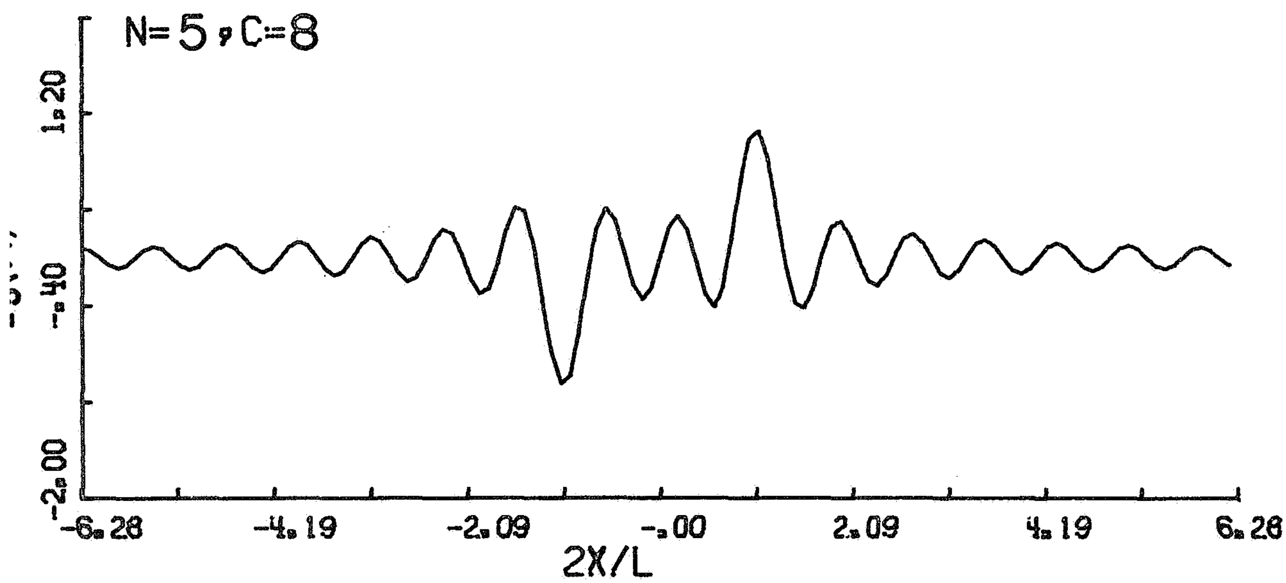
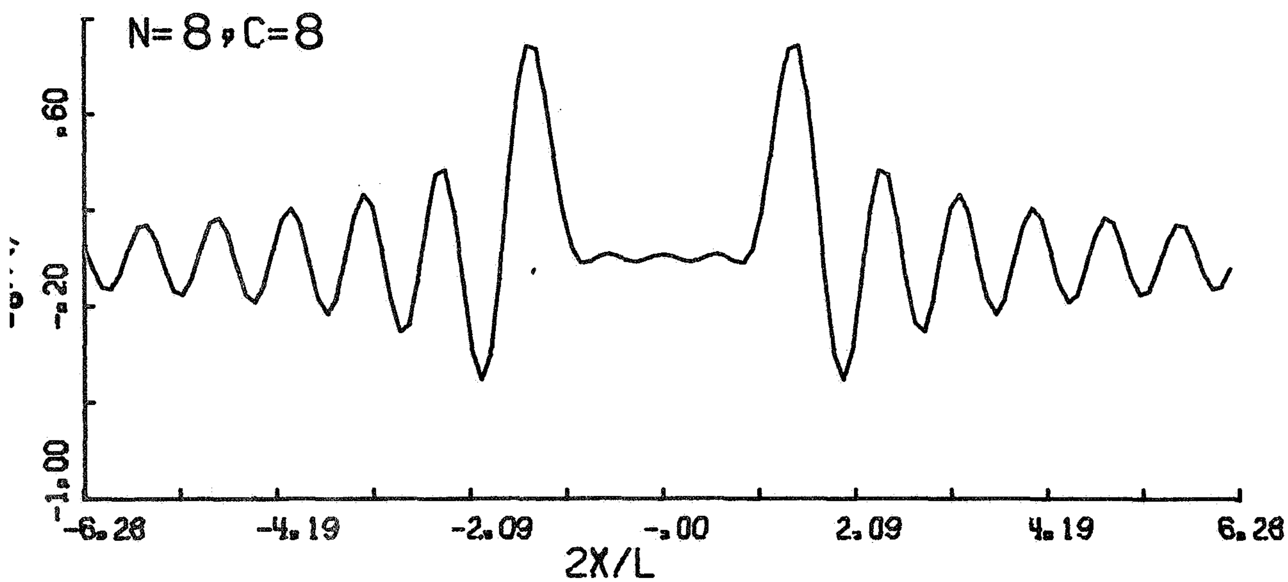
Figure Captions

g. 1. Prolate spheroidal wave functions for $c = 8$, $n = 0, 5, 8$.

g. 2. Geometrical image, computed data, and geometrical image estimate with $\sigma = 0.4$, $M = 1.0$, and $B = 0.5$.


N=0 , C=8
1.60
.80
0.00
-6.28
-4.19
-2.09
-.00
2.09
4.19
6.28
2X/L


N=5 , C=8
1.20
-.40
-2.00
-6.28
-4.19
-2.09
-.00
2.09
4.19
6.28
2X/L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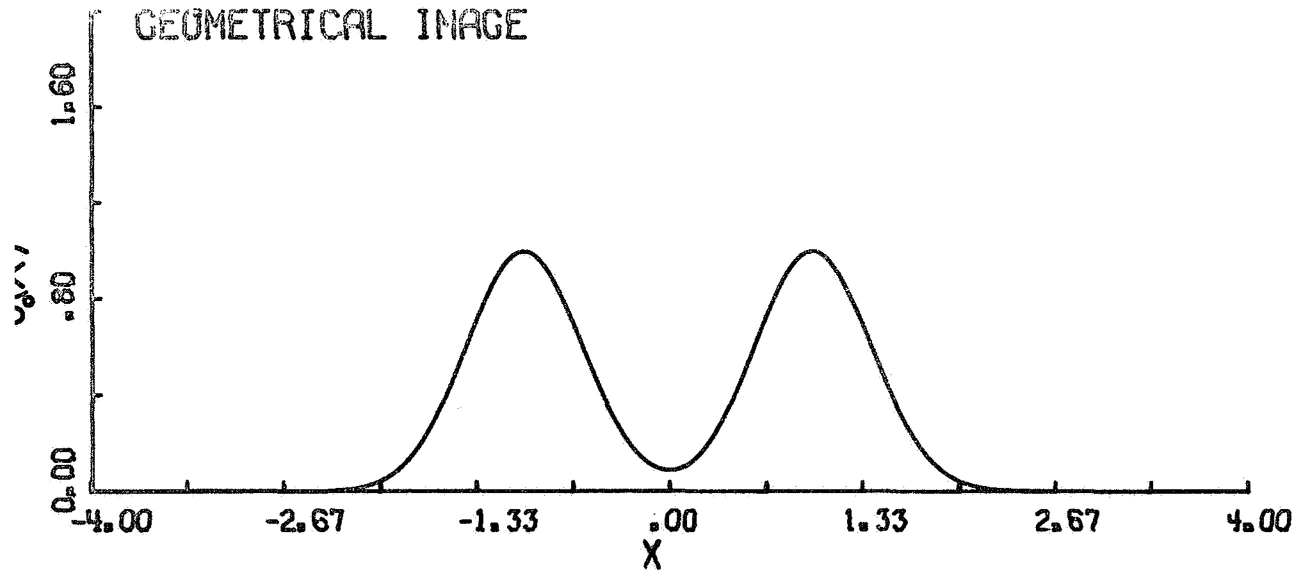
.60
-.20
-1.00
-6.28
-4.19
-2.09
-.00
2.09
4.19
6.28
2X/L


GEOMETRICAL IMAGE
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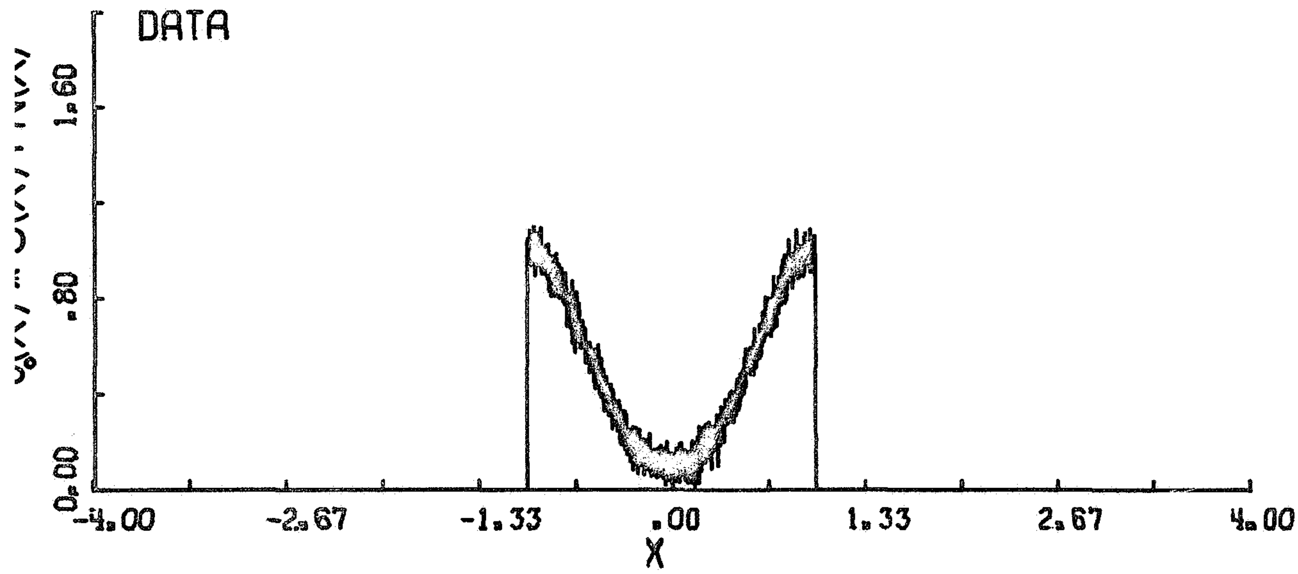
-4.00
-2.67
-1.33
.00
1.33
2.67
4.00
X


DATA
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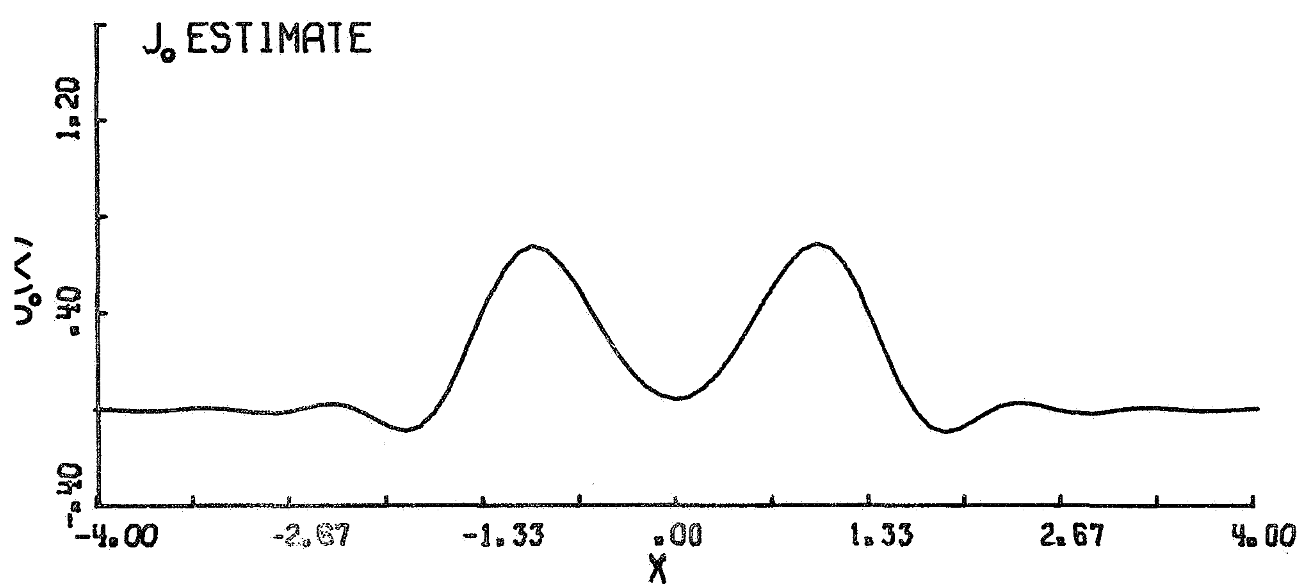
-2.67
-1.33
.00
1.33
2.67
4.00
X


J₀ ESTIMATE
1.20
.40
-.40
-4.00
-2.67
-1.33
.00
1.33
2.67
4.00
X