MATHEMATICAL METHODS OF CONSTRUCTING NEW MODELS OF CONTINUA

L. I. Sedov

Uspekhi Matematicheskikh Nauk, Vol. 20,
No. 5(125), pages 121-180
September-October 1965, Moscow

Translated by
TECHTRAN CORP.
Glen Brunie, Md.
under Contract NAS 5-14826, GSFC 1/1968
#14

MATHEMATICAL METHODS OF CONSTRUCTING NEW MODELS OF CONTINUA[1]

Chapter 1. Introduction

The theoretical interpretation of the various existing phenomena is associated with the introduction of mathematical concepts and characteristics, from which the quantitative evaluation methods are established. In connection with this, it is necessary to introduce the descriptive models and processes with the aid of which various inherent trends are formulated to describe the events and their characteristics with the required degree of accuracy according to reality.

From the scientific viewpoint, it is important that the characteristics and the features of the models and processes being designed would be formulated distinctly on a rational basis.

In many modern problems, it is logical to avoid excessive complications, since as a rule the appropriate experiments and phenomena are associated with a variety in the experimental data, which makes it difficult to control the differences in the actual objects of study, for example the difference in the material specimens being tested, etc., by a difference in the conditions of conducting the tests, in their observations and in the errors in the classes of variation. Nevertheless, the question concerning the construction of the complicated models of material media with allowance for new and additional characteristics and effects has been placed on the agenda. As is known, often the allowance for slight effects, scarcely perceptible in the initial stage of research, subsequently becomes the basis for the development of progress in a more profound penetration into the essence of the nature of the phenomena and in the expansion of the field of applications.

Many fundamental properties of matter and its inherent tendencies were discovered as a result of discussing the effects that were revealed in the domain of empirical results; but as a rule, the presence of such effects has been subject to doubts not only by the skeptics, but sometimes by the authors themselves. As examples, let us recall the violations of the law of the conservation

---

[1]The basic results of this article were reported at the conference of the Moscow Mathematics Society on December 8, 1964.

of mass during the interaction of particles concerning the non-Euclidean state of space and time, the effects of viscosity in gases and liquids, the effect of creep for all metals, the anomalies in the heat capacities of solids, etc.

In practice, the correct approaches to these effects can and should be overlooked in many cases, thus developing a theory and analytical method without their detailed consideration; i.e., we can ignore the existence of the correct approaches. On the other hand, in other instances of such a type, the effects represent the very quintessence of the problem and must be taken into account. However, in any question, we can indicate other effects already discovered, or the effects which are still vague and under study, and which can occur in the cases being considered yet require further attention for a more detailed study. The significant progress in science, as a rule, is associated specifically with an ever more complete and detailed penetration into the nature of the properties of small microscopic particles and the mechanisms of their interaction and, on the other hand, into the nature of the macroscopic effects, which are being manifested at the forefront of the existing methods of observation and measurement.

Nevertheless, the history of science teaches us that the laws and concepts which we consider established at the present time will also preserve their significance in the future, e.g., the Newtonian mechanics, but these concepts and laws should be regarded as good approximations having a practical value, yet as approximations they are inadequate and unsatisfactory relative to their basic concepts for the more accurate problems. It is known that now it is necessary to utilize in the theory of atoms and molecules instead of the Newtonian mechanics.

It is clear to all that the conscious utilization of the methods, concepts and laws which are known to be unacceptable or simply untrue in case of a more detailed study, but which are quite satisfactory from the viewpoint of the problems which have been formulates, is fully permissible and useful in the study of many important problems.

We have every basis for thinking that in the next stages of the development of science, and especially in the study of the microscopic, physical and biological phenomena, such a situation will be repeated. Such is the state of affairs with which we are faced in the mechanics of deformed macroscopic liquid, solid and gaseous bodies within the framework of Newtonian mechanics.

In this manner, the aforesaid outline of the typical interaction between science and its object of study in the world surrounding us is applicable to an equal extent both to the past and to the present, and undoubtedly will also be applicable to the future, of scientific development.

In our days, in the study and solution of many basic problems of natural science and technology, we are required to examine the systems consisting of various interacting particles and bodies when their basic properties, the effects of the collective interactions and the typical features of the processes which are transpiring have a mechanical nature; this not only in the light of an analysis of the microscopic theories but also in a macroscopic description of the phenomena taking place.

The understanding of nature (astronomy, physics, chemistry, biology) and the development of various objects in technology is associated closely with the introduction of models of mechanical systems, and with the formulation and solution of various mechanical processes.

There exist many interesting phenomena and urgent problems which can and should be solved within the framework and with the aid of the already-introduced models of continuous media in the theory of ideal viscous liquids and gases, in the theory of elasticity, plasticity, etc. However, the discussion of the design of new models is useful in connection with the development of new important trends in the mechanics of gases, liquids and solids, and in the mechanics of the composite material media with a varying type of structure.

For illustration. let us recall briefly certain fields of mechanics which are currently being developed.

1. The intensive and numerous studies that are being conducted in the area of the theory of plasma.

2. The mechanics connected with the technology of production and with the application of polymer materials, which are interrelated extensively with the chemistry and physics of the internal structure of polymers and acquire ever-increasing practical and cognitive importance.

3. The problems of the motion of any type of heterogeneous bodies, mixtures, suspensions and cavitating liquids.

4. The problems of creep and plasticity, and also of the strength of metals and of many other materials under various conditions, particularly during changing or high temperatures.

5. The mechanical problems in case of the presence within the matter of high internal pressures or, on the other hand, the motion of intensively rarefied gases.

6. The problems of the motion and equilibrium of soils and the theory of the filtration of liquids and gases in porous media.

7. The motion of solid and liquid deforming media, with allowance for the electrical polarization and magnetization.

The theory of motion of multicomponent systems, with allowance for diffusion, radiation, chemical and phase transformations, becomes a very important basis for solving a number of practical problems.

In spite of its considerable practical and theoretical significance, the "statistical thermodynamics" of the turbulent motion in the material bodies is still only gradually being developed.

The new theories related to the motion of bodies at very low temperatures where the effects of quantum mechanics are significant in the macroscopic theory are interesting and important.

In recent times, considerable research has been undertaken in the field of biological mechanics, particularly for the description of the motion of blood in living organisms; for this, we require the introduction of models of liquids with unusual rheological properties.

The cited trends in mechanics clearly indicate the direct and close relationships of modern mechanics of a continuous medium with thermodynamics, with the statistical physical theories, with chemistry and with electrodynamics. In essence, the combination of mechanics and of these branches of science into one unified science is taking place.

The data provided by physics concerning the microscopic dimensions and the geometric forms of the particles, as well as the structures of their arrangement and their interactions, are quite useful and necessary for understanding the macroscopic properties of bodies. However, merely the microscopic data and the mechanisms are insufficient for establishing the macroscopic theories, and therefore the prevailing theory that no appreciable basic problems exist any longer after the establishment of the microscopic pattern of the arrangements of bodies in the theory of finite bodies is quite incorrect. In the simplest case, for a model of an ideal gas, a transition from the aggregation of a large number N of atoms, considered as a mobile mechanical system of elastic smooth small spheres interacting with prescribed forces and having 3N degrees of freedom, to a material continuum takes place, and the state of a physically small particle in the theories and in the experiments is determined only by two parameters, namely density and temperature. Such an approach is associated with the use of the important statistical regularities.

Another example of the macroscopic properties of the laws as having important applications is the concept of an absolute solid. The abstract, and of course generally false hypothesis concerning the invariance of the distances between any two specific points of a body determines the macroscopic mechanical properties and obviously constitutes an additional forced condition, reducing the number of degrees of freedom of a finite body to six.

The two examples listed typify the apparent simplicity of the macroscopic hypotheses. In other cases, such hypotheses have a more complex nature in essence and their formulation requires the use of complex ideas and characteristics concerning the deformation of particles and their internal state, which are assigned with the aid of scalar, tensor, spin and other functions.

The establishment of a supply of concepts and characteristics of the states is an important mathematical problem; its quantitative description is a necessary condition for the application of the scientific method. Specifically, we can require the involvement of basic physical concepts of an electromagnetic field and its methods of description. For example, this is the state of affairs in the macroscopic problems in describing the characteristics of dislocations in metals, in the mechanics of polymer materials, in describing the properties of the interaction of material media with the strongly-varying electromagnetic fields, in the beams of laser rays, in describing the phenomena of superconductivity and superfluidity, etc.

Many modern problems in the aforesaid subjects have not yet been solved, and have not even been formulated in a lucid manner.

In this manner, the thermodynamic properties of the macroscopic bodies should never be derived from the microscopic characteristics without additional significant hypotheses of a macroscopic nature.

In the construction of models of the bodies, one should never count on a complete clarity in the elementary microscopic relationships. Together with this, the necessity for additional hypotheses in the macroscopic theory serves as a basis for the intuitive phenomenological theories, in which the microscopic information is taken into account quite approximately, and in essence is replaced by the hypotheses based on the data obtained from the observations and measurements in the macroscopic experiments. Similar phenomenological methods are always utilized to some degree or other in all of the applications. In connection with this, we can understand the fruitfulness and successes in the past of the theory of the thermogenics in thermodynamics.

In biology, the improvement of the crops and animal husbandry can introduce considerable successes on the basis of the phenomenological laws, with quite scanty understandings of the significant internal microscopic mechanisms.

However, it is understandable that the role of the microscopic studies in biology is extensive and is growing every day; nevertheless, it is specifically in biology that the importance of the phenomenological laws is manifested rather clearly and distinctly. Even with a very slight understanding of the internal mechanisms, one can attain remarkable results with the aid of tests and intuition.

In a construction of the new models of the material continuous media, an important place is occupied by the data on the simplest rheological experiments, such as the simple elongation and torsion of spatial samples, the thorough expansion or compression of a medium, or the motion of the medium of the Couette type of flows, etc. However, it is evident that the data of such elementary experiments are quite inadequate for establishing the rheological characteristics of the models of material media, which is required for a consideration of a general case of motions with arbitrary stresses in various complex external conditions. The transition from the elementary regularities in the particular tests to the laws of a general nature is always associated with the use of a complicated mechanical and methematical apparatus, with the generalization of the concepts concerning the characteristics of the phenomena, and with a varying class of ideas concerning the nature of the interaction of the medium's particles with one another in a general case.

The construction of models of continuous media is always associated with the adoption of a number of hypotheses, which can be regarded as descriptive and concentrated data of the observations and tests. Often, such hypotheses are quite simple and natural, e.g., the hypothesis concerning the isotropic properties of space or of a given material body, but in the latter case the hypothetical nature of the isotropic characterization is especially clear since one often can and must introduce anisotropic bodies as well.

In the construction of the models, the question of an explicit formulation and rational selection of the noncontradictory, minimal system of hypotheses convenient for verification deserves attention both from the viewpoint of a physical-experimental or statistical substantiation of the model, and from the viewpoint of the general methods of the mathematical formulation of a system of closed equations and additional boundary or other conditions which specify the model for the theoretical researches. The revelation of the most convenient system of typical parameters and hypotheses for the formulation of the necessary regularities and for the subsequent experimental verification has significant importance from a procedural viewpoint. It is also theoretically useful to know the systems of equivalent hypotheses and available possibilities for varying the systems of hypotheses within the limits of accuracy of the problem formulation.

In the classical simplest cases, these questions have almost a trivial nature; however, in these cases the explicit formulations are also useful. In recent times, the rheological studies have become greatly complicated and therefore a discussion of the general methods and techniques which are useful in constructing the models of the continuous media has become necessary.

After the establishment of a test set-up and a system of measurement units, the characteristics of the states and processes are prescribed with the aid of numbers and certain operators of a mathematical nature; and corresponding laws and relationships are formulated with the help of the equations containing the numerical characteristics and operators.

For a given system, there can be variables and constants among the typical parameters while the number of the numerical parameters prescribing the position, state (condition) and process can be finite or infinite.

In their essence, the variables determining the parameters and their variations should be regarded as independent arguments, varying within certain limits in accordance with the full aggregation of all of the possible conditions and processes in the medium for which the model is applicable.

The concept concerning the control parameters and their number in a general case is a direct generalization of the concept concerning the degrees of freedom and the independent coordinates for the mechanical system in analytical mechanics and in classical thermodynamics.

It is evident that for the bodies which are being deformed, considering the space of finite dimensions, the number of controlling numbers, i.e., of the degrees of freedom, is infinite. For the infinitely small particles, in the typical examples, the number of the controlling parameters is finite and in general small.

Usually, if a closed system of equations is written, the control parameters can easily be discriminated and recalculated. If the equations of a system are differential and their number is finite, the number of the control parameters proves to be finite for an infinitely small particle.

On the other hand, in individual important cases when the equations of motion represent complex operator relationships, for example the integral-differential equations, the number of the control parameters for infinitely small particles still proves to be finite. In this respect, the problems of the motion or equilibrium of gaseous masses with consideration of the forces of mutual attraction of the gas molecules are typical examples in Newtonian mechanics.

We can also introduce and consider the mechanical models of continuous media when the number of degrees of freedom is finite, even for the arbitrarily small particles. Usually, in the actual cases of applications, which are always approximate in their nature, it is sufficient to consider the systems which have a finite number of degrees of freedom for an infinitely small particle.

The basic successes attained in mechanics and physics are associated with the examination of objects for which the number of described experimental and theoretical control characteristics is finite and small.

Let us emphasize that the number of the characteristics and of the parameters being determined can be arbitrary; we are speaking only of the number of the independent arguments, i.e., prescribed from the experiments and assigned according to the sense of the mathematical problems (coordinates, time or stress tensor, temperature, etc.) as variables or constants.

It is obvious that in the construction of the models of the material bodies, the control parameters that are significant are only those which are connected with the properties of the model and with the processes, and hence in a definite sense are invariant relative to the choice of the system of coordinates and the measurement units.

The control parameters can be scalar and tensor, dimensional and dimensionless, constants and variables.

In establishing the model for the small particles, the actual separation of the control parameters in and of itself greatly limits the possible arbitrariness.

A single list of parameters which can be used as independent arguments in the equations of states, in the kinetic equations, or in the determination of the various functions of state, is totally inadequate. However, the presence of tables of control parameters is quite necessary for the development of a general theory, for the formulation and presentation of the results of the necessary tests and for the theoretical formulation of the additional hypotheses required for a complete description of the model.

In certain particular cases, only the availability of a list of control parameters together with the simplest mathematical assumptions (e.g., the expansion of the functions of state into a series with respect to the control parameters, with the retention of only the first terms of the series) permits us to find a class of the functional dependences in the equations of state and the other physical dependences (relationships).

Now let us consider the typical and basic values which can be utilized in the corresponding tables of the control parameters in the construction of the models of the continuous media.

Well known are the examples of the dimensional physical constants, which in a number of cases can be included in a general listing of the control parameters, (speed of light, Boltzmann constant, acceleration of gravitational force for the gravitational constant, the modulus of elasticity, the factors of viscosity and heat conductivity, etc.).

In particular, we will also note the possible presence of the constant tensor among the control parameters prescribing the symmetry of the elementary particles of a continuous medium, which can and should be introduced for an object. It turns out that the properties of symmetry can be prescribed with the aid of a simple set of various tensors. The presence of symmetry reduces to the presence of the corresponding constant parametric tensors among the independent arguments in the unknown function. In other words, the presence of the properties of symmetry is associated with the presence, among the control parameters, of the tensors prescribing the appropriate groups of symmetry.

At the present time, we have established the simple systems of tensors for all of the point subgroups, i.e., the complete orthogonal group (the crystal groups, the group of an icosahedron, of texture, the prismatic groups). After the introduction of such tensor arguments, the necessity of verifying the feasability of the conditions of symmetry in the unknown mathematical equations is discarded; the appropriate conditions will be automatically satisfied.

The presence of a general algebraic theory of the structure of nonlinear tensor functions upon several tensor arguments often permits us to extract inferences. These conclusions are associated with the following equation for the tensor H which is being determined:

$$H = \sum_{s=1}^{p} k_s H_s, \tag{1.1}$$

where k_s equals the scalar functions and H_s equals the corresponding tensors, which are formed by way of the polyad products and contractions from the tensor arguments. The number p equalling the number of linearly independent tensors H_s is determined with the aid of the theory of the characters for the symmetry group, which is admitted by the system of tensor arguments. Currently, there exist all the necessary formulas of the type (1.1) for a 3-dimensional case when the tensor H being determined has first, second, third and fourth ranks.

As the simplest examples of the control parameters of a dynamic, and in general of a physical nature, we can introduce and consider temperature T or, for certain cases in the absence of local equilibrium, several temperatures T_1, T_2,... for the components; which in addition can be characterized by the densities ρ_1, ρ_2,..., who also can serve as the control parameters.

In a consideration of the electromagnetic effects, in the capacity of control parameters we can take the components of the vectors E^α and P^α of the electric field and electric polarization, and the components of the antisymmetric tensors $H_{\alpha\beta}$ and $M_{\alpha\beta}$ of the magnetic field strength and of the magnetic moment of the medium.

Let us emphasize that a typical feature of the modern theories is also associated with the fact that along with T_k, ρ_k, P^{α} and $M^{\alpha\beta}$, as the control parameters, it is necessary to introduce their derivatives with respect to the spatial coordinates and with respect to time.

The appearance of various structural parameters of a physical and chemical nature is associated with a quantitative description of the newly discovered or already known mechanisms which acquires considerable significance in the phenomena of energy exchange of the dissipation of energy, and of other interactions between the particles within the physical bodies being studied.

Specifically, such parameters can appear as the characteristics of the formation and distribution of the crystal and of other internal structures, as the characteristics of the distribution of fissures, porosity, etc., as the characteristics of plastic deformations, the effects of dislocation or of mechanisms of electrical and magnetic polarization.

The establishment of a system of control parameters is associated with a general description of the phenomenon, with the use of a varying class of research hypotheses, of experimental data, of statistical discoveries and with the problem of describing the studied objects and the processes by the exact or approximate equations and additional conditions.

The differentiation of a system of control parameters is associated with the penetration into the mechanism of the phenomena which are being studied, and constitutes the most important link in formulating the problem. As is known, in the theoretical construction of the models of continuous media, a reference base is provided by the basic dynamic equations concerning the conservation or variation of mass, of impulse, of momentum, and the universal laws of thermodynamics.

The variation principles and methods acquire particular significance in a construction of the models of continuous media in which the manifestation of the internal degrees of freedom is significant, or when the system of the control parameters contains successive derivatives of several typical orders. In all cases, in the theoretical assignment of the physical nature of the system, it is necessary to utilize several thermodynamic functions of the control parameters (the internal energy or entropy, etc.) or the Lagrange function based on the variation principles. In addition, in each of these approaches, it is necessary to prescribe the generalized forces and a number of other data (the various types of energy flows, the laws of distribution, etc.).

In the construction of a general theory, only the actual presence of the appropriate functions from the system of arguments which is being established is important. In the actual cases, exhaustive knowledge of such functions is required.

In the general theory, the basic source of the required initial information is provided by the hypotheses which should correspond in a certain definite sense to the test data in the description of the phenomena in the actual objects. After a sample test, the initial hypotheses can be reviewed and applied as the laws of nature.

In a large number of important mechanical theories, the simplest hypotheses or the test data can lead to the necessary macroscopic functions and to a macroscopic description in the result of complex additional statistical theories. The formulation and establishment in various cases of a convenient system of hypotheses or of experimental data and the technique of extracting from them the general equations comprise an important part of the general theory of mechanical models.

The general theory of the models of continuous media can be adapted to and compared with the general geometric theory of the multidimensional non-Euclidean sets.

The particular classes of the models of media are similar to the Riemann or the affine-connected sets, while the individual actual models are similar to the definite spaces or sets, such as spherical, ellipsoidal, toroida, etc.

In constructing models of continua it is advantageous to use the general conditions pertaining to continuity and differentiability as well as the postulates regarding the lack of relationships (between geometric or kinematic parameters), be they differential or any other type of relationships, which differ from their strict definition. An example of such a relationship is the condition of incompressability which nevertheless can sometimes be applied. The presence of the additional relationships leads to limitations in the laws of motion, to restrictions independent of the external conditions or the arbitrariness of the external mass or surface forces at the boundaries of finite volumes or small particles of the medium.

Chapter 2. Kinematic Characteristics of the Deformation of Small Particles

In recent times, we have introduced a more precise geometric and kinematic characterization of the deformation of small particles of material continuums into the mechanics of continuous media. Below, we provide a short description of the pertinent tensor concepts.

For the derivation of the geometric and kinematic characterization of the internal structure and the processes of deformation, we introduce the attached Lagrangian curvilinear system of coordinates (fixed into the medium) ξ^1, ξ^2, ξ^3, with the base vectors $\hat{\partial}_1, \hat{\partial}_2, \hat{\partial}_3$.

$$ds = d\xi^\alpha \hat{\partial}_\alpha, \quad ds^2 = \hat{g}_{\alpha\beta} d\xi^\alpha d\xi^\beta \quad (\hat{g}_{\alpha\beta} = (\hat{\partial}_\alpha \hat{\partial}_\beta)) \tag{2.1}$$

where ds equals the elements of length taken in an actual Euclidean space.

Let us denote by

$$ds_0 = d\xi^\alpha \overset{0}{\partial}_\alpha, \quad ds_0^2 = \overset{0}{g}_{\alpha\beta} d\xi^\alpha d\xi^\beta \quad (\overset{0}{g}_{\alpha\beta} = (\overset{0}{\partial}_\alpha \overset{0}{\partial}_\beta)) \tag{2.2}$$

the appropriate elements, vectors and bases which we introduce conceptually for the systematic ideal states, in which the internal stresses are lacking.

The three-dimensional aggregation of points with the coordinates ξ^1, ξ^2, ξ^3, of the vectors of the basis $\hat{\partial}_\alpha$, and of the elements ds determines the Euclidean space $\mathcal{E}$, wherein

$$\frac{\partial \hat{\partial}_\alpha}{\partial \xi^\beta} = \hat{\Gamma}^\gamma_{\alpha\beta} \hat{\partial}_\gamma, \text{ where } \hat{\Gamma}^\gamma_{\alpha\beta} = \frac{1}{2} \hat{g}^{\lambda\gamma} \left[\frac{\partial \hat{g}_{\alpha\lambda}}{\partial \xi^\beta} + \frac{\partial \hat{g}_{\beta\lambda}}{\partial \xi^\alpha} - \frac{\partial \hat{g}_{\alpha\beta}}{\partial \xi^\lambda} \right]. \tag{2.3}$$

Let us now assume that the three-dimensional combination of the coordinates ξa of the base vectors $\overset{0}{\ni}_\alpha$ and of the elements ds_0 determines the affine metric-bounded set K, for which the following relationships are valid:

$$\frac{\partial \overset{0}{\ni}_\alpha}{\partial \xi^\beta} = \overset{0}{\Gamma}{}^\gamma_{\alpha\beta} \overset{0}{\ni}_\gamma,$$

where

$$\overset{0}{\Gamma}{}^\gamma_{\alpha\beta} = \frac{1}{2} \overset{0}{g}{}^{\lambda\gamma} \left[\frac{\partial \overset{0}{g}_{\alpha\lambda}}{\partial \xi^\beta} + \frac{\partial \overset{0}{g}_{\beta\lambda}}{\partial \xi^\alpha} - \frac{\partial \overset{0}{g}_{\alpha\beta}}{\partial \xi^\lambda} \right] + S^\gamma_{\alpha\beta} - \overset{0}{g}_{\lambda\beta} \overset{0}{g}{}^{\gamma\mu} S^\lambda_{\alpha\mu} - \overset{0}{g}_{\lambda\alpha} \overset{0}{g}{}^{\gamma\mu} S^\lambda_{\beta\mu}, \tag{2.4}$$

hence

$$S^\gamma_{\alpha\beta} = \frac{1}{2} (\overset{0}{\Gamma}{}^\gamma_{\alpha\beta} - \overset{0}{\Gamma}{}^\gamma_{\beta\alpha}). \tag{2.5}$$

The tensor $S^\gamma_{\alpha\beta}(\xi^1, \xi^2, \xi^3) = -S^\gamma_{\beta\alpha}$ is said to be the torsion tensor of the set K, while $\overset{0}{\Gamma}{}^\gamma_{\alpha\beta}(\xi^1, \xi^2, \xi^3)$ — gives the coherence factors determining the parallel transport.

It is obvious that for $\mathcal{E}$ the tensor of the Riemann curvature becomes zero:

$$\hat{R}_{\alpha\beta\gamma}{}^\lambda = \frac{\partial \hat{\Gamma}^\lambda_{\beta\gamma}}{\partial \xi^\alpha} - \frac{\partial \hat{\Gamma}^\lambda_{\alpha\gamma}}{\partial \xi^\beta} + \hat{\Gamma}^\lambda_{\alpha\mu} \hat{\Gamma}^\mu_{\beta\gamma} - \hat{\Gamma}^\lambda_{\beta\mu} \hat{\Gamma}^\mu_{\alpha\gamma} = 0.$$

For the set K, the tensor of Riemann curvature in general differs from zero:

$$R_{\alpha\beta\gamma\lambda} = \overset{0}{g}_{\nu\lambda} \left[\left(\frac{\partial \overset{0}{\Gamma}{}^\nu_{\beta\gamma}}{\partial \xi^\alpha} - \frac{\partial \overset{0}{\Gamma}{}^\nu_{\alpha\gamma}}{\partial \xi^\beta} \right) + \overset{0}{\Gamma}{}^\nu_{\alpha\mu} \overset{0}{\Gamma}{}^\mu_{\beta\gamma} - \overset{0}{\Gamma}{}^\nu_{\beta\mu} \overset{0}{\Gamma}{}^\mu_{\alpha\gamma} \right] \neq 0.$$

The set K can be regarded as a generalization of the concept of the unstressed state of a body in a Euclidean space, when such a state is possible for a finite body. The difference from zero of the tensors $S^\gamma_{\alpha\beta}$ and $R_{\alpha\beta\gamma\lambda}$ involves the impracticability of the equations of consistancy and hence the absence of displacements from the conceptually-introduced ideal state to the state under consideration.

The internal invariant properties of the set K can be prescribed fully by the two tensors $\overset{0}{g}_{\alpha\beta}$ and $S^{\gamma}_{\alpha\beta}$ or by the two tensors $R_{\alpha\beta\gamma\lambda}$ and $S^{\gamma}_{\alpha\beta}$.

In the basal plane in a Euclidean space, the tensors:

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\hat{g}_{\alpha\beta} - \overset{0}{g}_{\alpha\beta}), \quad \overset{0}{g}_{\alpha\beta}, \; S^{\gamma}_{\alpha\beta} \text{ and } R_{\alpha\beta\gamma\lambda}$$

can be regarded as the characteristics of the structure of the defects and of the geometric properties of the stresses (deformations).

Let us introduce the tensor:

$$E = E^{\alpha\beta\gamma}\overset{0}{\partial}_{\alpha}\overset{0}{\partial}_{\beta}\overset{0}{\partial}_{\gamma} = \sqrt{|\overset{0}{g}_{\alpha\beta}|}\,(\overset{0}{\partial}_{1}\overset{0}{\partial}_{2}\overset{0}{\partial}_{3} - \overset{0}{\partial}_{1}\overset{0}{\partial}_{3}\overset{0}{\partial}_{2} + \overset{0}{\partial}_{2}\overset{0}{\partial}_{3}\overset{0}{\partial}_{1} - \overset{0}{\partial}_{2}\overset{0}{\partial}_{1}\overset{0}{\partial}_{3} + \overset{0}{\partial}_{3}\overset{0}{\partial}_{1}\overset{0}{\partial}_{2} - \overset{0}{\partial}_{3}\overset{0}{\partial}_{2}\overset{0}{\partial}_{1}). \qquad (2.6)$$

As is known, Equation (2.6) for the tensor E retains its appearance under any transformation of the coordinates, wherein the components $E^{\alpha\beta\gamma}$ are invariant for any rotations of a three-dimensional space.

With the aid of the tensor $E^{\alpha\beta\gamma}$, we can introduce the tensor of second rank according to the formula:

$$K^{\alpha\beta} = E^{\gamma\lambda\alpha}S^{\beta}_{\gamma\lambda}. \qquad (2.7)$$

From Equation (2.7), the reciprocal formula follows:

$$S^{\gamma}_{\alpha\beta} = \frac{1}{2}E_{\alpha\beta\lambda}K^{\lambda\gamma}. \qquad (2.8)$$

It is obvious that in the three-dimensional space, the second rank tensor $K^{\alpha\beta}$ can be considered in place of the third rank tensor $S^{\gamma}_{\alpha\beta}$. In the absence of torsion, when $S^{\gamma}_{\alpha\beta} = 0$, the space K will be a Riemann one; in this case, the tensor $R_{\alpha\beta\gamma\lambda}$ is antisymmetrical relative to the first and second pairs of the indexes, and is symmetrical relative to the transposition of the indexes $\alpha\beta$ and $\gamma\lambda$. In connection with this, in a three-dimensional space, the following formulas will be valid:

$$R^{\alpha\beta} = E^{\gamma\lambda\alpha} E^{\mu\nu\beta} R_{\gamma\lambda\mu\nu} \tag{2.9}$$

and

$$R_{\alpha\beta\gamma\lambda} = \frac{1}{4} E_{\alpha\beta\mu} E_{\gamma\lambda\nu} R^{\mu\nu}. \tag{2.10}$$

In the general case of an affine, connected, metric space, the mutually reciprocal formulas (2.9) and (2.10) will not occur.

The covariant components $\varepsilon_{\alpha\beta}$ determine the tensor of the finite deformations.

As a result of the stresses of the medium and the different physical processes, the metric tensor $\overset{0}{g}_{\alpha\beta}$ (ξ^{α}, t) for the prescribed values of ξ^1, ξ^2, ξ^3 can in general be dependent on t. The components $\overset{0}{g}_{\alpha\beta}$ can be considered in place of the components of the tensor of the residual (plastic) deformations.

In establishing the link between the tensors $S^{\lambda}_{\alpha\beta}$ and $R_{\alpha\beta\gamma}{}^{\lambda}$ or between the tensors $K^{\alpha\beta}$ and $R^{\alpha\beta}$ with the mechanical defects (in the continual theory of dislocations), we note that in the establishment of the metrics and the coherence of the ideal set K, the following operations can be accomplished conceptually.

Let us take in an actual Euclidean space a certain curve C and let us examine the continuous aggregation of infinitely small elements of a material medium along the curve C; we obtain a certain infinitely-thin fiber.

Now let us separate conceptually this fiber from the entire remaining part of the body; if C equals a closed curve, let us cut this fiber in a certain section; let us release all of the elements of the fiber from the internal stresses and from the distortions in the structure of its elements, causing their mutual arrangement. After such an operation, conducted in the same Euclidean space, the unstressed fiber with the proper structure will change its initial form, while curve C will convert to a certain other curve C*. Along C*, the substantial points of the elements of the fiber will be determined by the same Lagrangian coordinates ξ^1, ξ^2, ξ^3, as for the fiber C.

If curve C were closed, then (generally speaking) curve C* would be open with a certain separation between the ends after relative rotation of the surfaces of the sectional area.

By definition, the metrics and the coherence of the elements being obtained along C* correspond to the metrics and to coherence of the set K of the affine connected space. As a basic assumption, it is admitted that $\overset{0}{g}_{\alpha\beta}$ and $\overset{0}{\Gamma}{}^{\gamma}_{\alpha\beta}$ depend only on the coordinates ξ^1, ξ^2, ξ^3, and do not depend on the choice of the curve C.

In the Euclidean space, if the base vectors along the curve C* are identical to the base vectors $\overset{0}{\partial}_{\alpha}$, then along C* within the limits of an infinitely thin fiber we will have:

$$\frac{\partial \mathbf{r}^*}{\partial \xi^{\alpha}} = \overset{0}{\partial}_{\alpha} \text{ and } d\overset{0}{\partial}_{\alpha} = \overset{0}{\Gamma}{}^{\gamma}_{\alpha\beta} d\xi^{\beta} \overset{0}{\partial}_{\gamma}, \tag{2.11}$$

where r* equals the radius vectors of the points along C*. These relationships written in the Lagrangian coordinates can also be regarded in an Euclidean space along the curve C, bearing in mind that the vectors $\overset{0}{\partial}_{\alpha}$ and the radius vector r* are taken on the curve C*.

It is obvious that in a general case of the integration along C, the radius vector r* and the base vectors $\overset{0}{\partial}_{\alpha}$ depend not only on the coordinates ξ^{α}, but also on the shape of the curve C.

Assuming that C is a closed curve and integrating Equation (2.11) along the curve C, we obtain:

$$\left.\begin{aligned} \Delta \mathbf{r}^* &= \oint_C \overset{0}{\partial}_{\alpha} d\xi^{\alpha}, \\ \Delta \overset{0}{\partial}_{\alpha} &= \oint_C \overset{0}{\Gamma}{}^{\gamma}_{\alpha\beta} \overset{0}{\partial}_{\gamma} d\xi^{\beta}. \end{aligned}\right\} \tag{2.12}$$

The vector Δr^* is said to be the Burgers vector. The vectors $\Delta\overset{0}{\partial}_{\alpha}$ typically specify the deformation and the relative rotation of the cross-sectional areas. If as the contour C in the Euclidean space we take an infinite parallelogram with the sides corresponding to the elements $d_1\xi^{\alpha}$ and $d_2\xi^{\alpha}$, we derive for the integrals (2.12)

$$\Delta \mathbf{r}^* = \overset{0}{\text{Э}}_\alpha d_1\xi^\alpha + \left(\overset{0}{\text{Э}}_\alpha + \frac{\partial \overset{0}{\text{Э}}_\alpha}{\partial \xi^\beta} d_1\xi^\beta\right) d_2\xi^\alpha - \overset{0}{\text{Э}}_\alpha d_2\xi^\alpha - \left(\overset{0}{\text{Э}}_\alpha + \frac{\partial \overset{0}{\text{Э}}_\alpha}{\partial \xi^\beta} d_2\xi^\beta\right) d_1\xi^\alpha =$$
$$= \frac{\partial \overset{0}{\text{Э}}_\alpha}{\partial \xi^\beta} d_1\xi^\beta d_2\xi^\alpha - \frac{\partial \overset{0}{\text{Э}}_\alpha}{\partial \xi^\beta} d_2\xi^\beta d_1\xi^\alpha = \left(\frac{\partial \overset{0}{\text{Э}}_\alpha}{\partial \xi^\beta} - \frac{\partial \overset{0}{\text{Э}}_\beta}{\partial \xi^\alpha}\right) d_1\xi^\beta d_2\xi^\alpha = 2S^\gamma_{\alpha\beta} d_1\xi^\beta d_2\xi^\alpha \overset{0}{\text{Э}}_\gamma \quad (2.13)$$

and similarly:

$$\Delta \overset{0}{\text{Э}}_\alpha = R_{\alpha\beta\gamma}{}^\lambda \overset{0}{\text{Э}}_\lambda d_1\xi^\beta d_2\xi^\gamma. \quad (2.14)$$

Equations (2.12), (2.13) and (2.14) together with the above-described process of the conceptual arrangement and unloading can serve as a source for the mechanical interpretation of the torsion tensor $S^\gamma_{\alpha\beta}$ through the Burgers vector Δr^* and the tensor of the Riemann curvature, by means of a variation in the base after enclosure with respect to curve C or C*.

These interpretations comprise the basis for the continual theory of dislocations.

If $R_{\alpha\beta\gamma}{}^\lambda = 0$ while $S^\gamma_{\alpha\beta} \neq 0$, the space K proves to be affine connected with absolute parallelism, i.e., $\Delta \overset{0}{\text{Э}}_\alpha = 0$ along the entire closed curve C. In this case, as the principal geometric characteristic in the sets, we can take the Burgers vector, depending on the form of the contours C. When $R_{\alpha\beta\gamma}{}^\lambda = 0$, we can assume that the senses of the vectors $\overset{0}{\text{Э}}_\alpha$ depend only on the coordinates ξ^α and do not depend on the lines C, along which these vectors are transported, and it is evident that in this case the equations of the following form will be valid:

$$\overset{0}{\text{Э}}_\alpha = A_\alpha{}^\beta \hat{\text{Э}}_\beta, \quad (2.15)$$

where the components of the tensor $A_\alpha{}^\beta$ depend only on the coordinates ξ^1, ξ^2, ξ^3.

From Equation (2.15), it follows

$$\overset{0}{g}_{\alpha\beta} = A_\alpha{}^\gamma A_\beta{}^\lambda \hat{g}_{\gamma\lambda},$$

and therefore

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\hat{g}_{\alpha\beta} - \hat{g}_{\gamma\lambda} A_\alpha{}^\gamma A_\beta{}^\lambda) = \frac{1}{2}\hat{g}_{\gamma\lambda}(\delta^\gamma_\alpha \delta^\lambda_\beta - A_\alpha{}^\gamma A_\beta{}^\lambda). \quad (2.16)$$

Further, on the basis of (2.15) and (2.3), we have

$$d\overset{0}{\partial}_\alpha = \left(\frac{\partial A_\alpha{}^\gamma}{\partial \xi^\beta} + A_\alpha{}^\lambda \hat{\Gamma}^\gamma_{\lambda\beta}\right)\hat{\partial}_\gamma d\xi^\beta = \left(\frac{\partial A_\alpha{}^\mu}{\partial \xi^\beta} + A_\alpha{}^\lambda \hat{\Gamma}^\mu_{\lambda\beta}\right) B_\mu{}^\gamma \overset{0}{\partial}_\gamma d\xi^\beta;$$

therefore

$$\overset{0}{\Gamma}{}^\gamma_{\alpha\beta} = \left(\frac{\partial A_\alpha{}^\mu}{\partial \xi^\beta} + A_\alpha{}^\lambda \hat{\Gamma}^\mu_{\lambda\beta}\right) B_\mu{}^\gamma, \tag{2.17}$$

where the matrixes $B_\mu{}^\gamma$ and $A_\alpha{}^\mu$ are mutually reciprocal, i.e., $B_\mu{}^\gamma A_\alpha{}^\mu = \delta^\gamma_\alpha$.

From (2.17), it follows

$$\left.\begin{aligned} R_{\alpha\beta\gamma}{}^\lambda &= 0, \\ S^\gamma_{\alpha\beta} &= -\frac{1}{2} B_\mu{}^\gamma \left(\frac{\partial A_\alpha{}^\mu}{\partial \xi^\beta} - \frac{\partial A_\beta{}^\mu}{\partial \xi^\alpha}\right) + \frac{1}{2} B_\mu{}^\gamma \left(A_\alpha{}^\lambda \hat{\Gamma}^\mu_{\lambda\beta} - A_\beta{}^\lambda \hat{\Gamma}^\mu_{\lambda\alpha}\right). \end{aligned}\right\} \tag{2.18}$$

In this manner, if in Equation (2.15), $A_\alpha{}^\beta$ depends only on ξ^α, the Riemann curvature of the set K becomes zero; otherwise, the factors $A_\alpha{}^\beta$ will depend functionally upon the selection of the curve C.

The affine transformation (2.15) reduces to a pure deformation and rotation in each point for any selected curve C. From the assumption to the fact that $\overset{0}{g}_{\alpha\beta}$ depends only on ξ^α and t, it follows that in each point with the coordinates ξ^α at the assigned t-value, the pure deformation corresponding to the unloading and the elimination of the defects, is determined independently of the form of curve C drawn through this point during the separation of the fiber.

Hence, the functional dependence of the matrix of the affine transformation $A_\alpha{}^\beta$ can be manifested only through an additional rotation of the deformation axes. In this manner, on the basis of the adopted assumptions relative to the process of the relief and systematization of the elements, only the rotation vectors during the transition from the base $\hat{\partial}_\alpha$ to the base $\overset{0}{\partial}_\alpha$, being determined in (2.15) by the matrix $A_\alpha{}^\beta$, can depend functionally upon the form of curve C in general.

The above-considered geometric tensor characteristics of the internal structure and states can be utilized as control parameters in the construction of the models of the material continuums.

For the derivation of the kinematic characteristics of the internal processes in the continuous media, we can utilize the derivatives with respect to time and taken in a certain definite sense from the above-introduced geometric tensors.

For example, the components of the tensor of the deformation rates in space of motion and in the set K can be determined respectively:

$$e_{\alpha\beta} = \frac{1}{2}\frac{d\hat{g}_{\alpha\beta}}{\partial t} \text{ и } \eta_{\alpha\beta} = \frac{1}{2}\frac{d\overset{0}{g}_{\alpha\beta}}{dt} \qquad (2.19)$$

It is obvious that $\eta_{\alpha\beta} = 0$, if the geometric properties of the set K remain unchanged.

As the typical control parameters, it is also possible to introduce the derivatives with respect to time in the appropriate sense (for example, relative to the attached base $\hat{\vartheta}_\alpha$) from the tensors $S^{\gamma}_{\alpha\beta}$ and $R_{\alpha\beta\gamma}{}^{\lambda}$, and accordingly the time derivatives of the following orders.

In certain instances, in the capacity of the kinematic characteristics of the given state, we can take the components of the eddy vector ω^{α} or the tensor $\nabla_\beta \omega^\alpha$, characterizing the distribution of the eddies, or accordingly the antisymmetric tensors $\omega_{\alpha\beta}$ and $\nabla_\gamma \omega_{\alpha\beta}$.

In the formulation of the problem and in the separation of a system of the control parameters, we will utilize further the assumption concerning the absence of purely kinematic holonomic or nonholonomic relationships.

Chapter 3. Dynamic and Thermodynamic Basic Equations

The equations of the impulses and of the momenta in a three-dimensional formulation have the form

$$\rho a^{\alpha} = \nabla_{\gamma} p^{\alpha\gamma} + \rho F^{\alpha} \tag{3.1}$$

and

$$\rho \frac{dm^{\alpha\beta}}{dt} = \rho h^{\alpha\beta} + \nabla_{\gamma} Q^{\alpha\beta\gamma} + p^{\beta\alpha} - p^{\alpha\beta}$$
$$(\alpha,\ \beta,\ \gamma = 1,\ 2,\ 3), \tag{3.2}$$

where ρ equals density a^{α} and F^{α} equal the components of the vectors of the acceleration and mass forces, $p^{\alpha\beta}$ equals the components of the tensor of stresses, $m^{\alpha\beta}$ equals the components of the tensor of the internal moments of the parameters of motion, referred to a unit of mass, while $dm^{\alpha\beta}/dt$ equals the derivatives with respect to time taken relative to the inertial frame of reference, $h^{\alpha\beta}$ equals the components of the internal mass moments, while $Q^{\alpha\beta\gamma}$ equals the components of the internal surface moments. In Equations (3.1) and (3.2), it is assumed that in a general case

$$p^{\alpha\beta} \neq p^{\beta\alpha}.$$

From Equations (3.2) and (3.1), these equations follow:

$$\frac{d}{dt}\left(\frac{v^2}{2}\right) = F^{\alpha} v_{\alpha} + \frac{1}{\rho} \nabla_{\gamma} (p^{\alpha\gamma} v_{\alpha}) - \frac{1}{\rho} p^{\alpha\beta} e_{\alpha\beta} - \frac{1}{\rho} p^{\alpha\beta} \omega_{\alpha\beta} \tag{3.3}$$

and

$$\frac{1}{\varrho} p^{\alpha\beta}\omega_{\alpha\beta} = -\frac{1}{2} h^{\alpha\beta}\omega_{\alpha\beta} + \frac{1}{2\varrho}\nabla_\gamma (Q^{\alpha\beta\gamma}\omega_{\alpha\beta}) - -\frac{1}{2}\omega_{\alpha\beta}\frac{dm^{\alpha\beta}}{dt} - \frac{1}{2\varrho} Q^{\alpha\beta\gamma}\nabla_\gamma\omega_{\alpha\beta}, \qquad (3.4)$$

where v equals the value of the velocity of the particles of the medium, while v_α equals the components of the velocity vector.

For a long time, Equation (3.2) has been reduced by the condition of symmetry $p^{\alpha\beta} = p^{\beta\alpha}$ on the basis of the assumption $m^{\alpha\beta}$ = constant and $h^{\alpha\beta} = Q^{\alpha\beta\gamma} = 0$; and in accordance with this, Equation (3.4) is identically satisfied. At the present time, in many reports, Equations (3.2) and (3.4) are introduced as the basic equations in connection with the introduction of the material media and phenomena when $p^{\alpha\beta} \neq p^{\beta\alpha}$.

The value

$$dA^{(e)} = (F^\alpha v_\alpha \, dm + \nabla_\gamma p^{\alpha\gamma} v_\alpha \, d\tau) \, dt$$

(dm and $d\tau$ are infinitely small elements of mass and volume) can be regarded as the inflow of the macroscopic energy to a particle from the elementary work of the external mass and surface forces. Obviously, the inflow of energy caused by the elementary work of the external mass moments $h^{\alpha\beta}$ and of the surface pairs $Q^{\alpha\beta\gamma}$ does not enter into the equation of the kinetic forces (3.3).

Along with the kinetic energy of the particle $E = \frac{\varrho v^2}{2} d\tau$, we introduce into the consideration the intrinsic energy $U_m = \varrho U \, d\tau$, where U equals the intrinsic energy computed per unit of mass. The specific intrinsic energy U can be regarded as a certain function of the specific entropy S and of other parameters determining the physical and chemical state of the separated particle.

For an element of any process, the equation of the first law of thermodynamics can be written in the form:

$$dE + dU_m = dA^{(e)} + dQ^{(e)} + dQ^{**}, \qquad (3.5)$$

where

$$dQ^* = dQ^{(e)} + dQ^{**} -$$

is the total external flow of energy being added to the work of the macroscopic forces $dA^{(e)}$. In turn, the value dQ^* can be divided into an external flow of heat energy $dQ^{(e)}$ and the inflow of the nonthermal energy dQ^{**} which occurs as the result of various interactions of the given particles with the external bodies and with the adjacent particles of the medium under consideration. The flow of energy dQ^{**} can be caused by the work of the mass and surface pairs of forces, by the presence of an exchange of diffusion energies, by an inflow of electromagnetic energy, and by other mechanisms.

The complication of the models and a more detailed consideration of the internal interactions and particularly of the interaction of the physical bodies with an electromagnetic field, i.e., the effects of polarization and magnetization, leads to the necessity of the explicit introduction of energy inflows caused by the microscopic processes referred to dQ^{**}.

From (3.5) and (3.3), we obtain the equation of heat inflow

$$dU = \frac{1}{\varrho} p^{\alpha\beta} e_{\alpha\beta}\, dt + \frac{1}{\varrho} p^{\alpha\beta} \omega_{\alpha\beta}\, dt + \frac{dQ^{(e)}}{dm} + \frac{dQ^{**}}{dm}, \tag{3.6}$$

where $e_{\alpha\beta}\, dt = \frac{1}{2} d\hat{g}_{\alpha\beta}$ or $e_{\alpha\beta}\, dt = d\hat{\varepsilon}_{\alpha\beta}$, if $\overset{0}{g}_{\alpha\beta}$ equals a constant.

With the aid of (3.4), from Equation (3.6), we can exclude the term $\frac{1}{\varrho} p^{\alpha\beta}\omega_{\alpha\beta}$ and in certain cases, the work of the force couples from dQ^{**}/dm.

Further, let us consider in general the nonequilibrium processes for which we can introduce the absolute temperature T and the free energy F for each small particle according to the formula

$$F(T, \mu^1, \mu^2, \ldots, \mu^n) = U - TS,$$

where $T, \mu^1, \mu^2, \ldots, \mu^n$ is the system of control parameters selected by the condition that the following equation is satisfied:

$$\frac{\partial F}{\partial T} = -S.$$

Among the control parameters $\mu^1, \mu^2, \ldots, \mu^n$, in a general case it is necessary to include $\hat{g}_{\alpha\beta}$ or $\hat{\varepsilon}_{\alpha\beta}$, $\omega_{\alpha\beta}$ and the other parameters upon which F and dQ^{**}/dm can depend.

In conformity with the second law of thermodynamics, for a small particle we can write

$$T\,dS = \frac{dQ^{(e)}}{dm} + dq',$$

where $dq' \geqslant 0$ equals the noncompensated heat, computed per unit of mass.

In many cases, for the irreversible processes, we can assume that:

$$dq' = \alpha_k\,d\mu^k = \Phi\,dt, \tag{3.7}$$

where α_k and Φ equal certain functions of μ^k and $d\mu^k/dt$.

In a number of important cases, it is assumed that $\Phi = c_{ks}\frac{d\mu^k}{dt}\frac{d\mu^s}{dt}$, where $c_{ks} = c_{sk}$ are certain functions of μ^k or are simply constants.

Equation (3.6) can be represented in the form

$$\left(\Lambda_k - \frac{\partial F}{\partial \mu^k}\right) d\mu^k - dq' = 0, \tag{3.8}$$

where Λ_k equals the pertinent factors during the increments of $d\mu^k$ in the right-hand part of (3.6).

For the reversible processes when $dq' = 0$, $\alpha_k = 0$, or for the irreversible processes when Equation (3.7) is fulfilled, Equation (3.8) is represented in the form

$$\pi_k\,d\mu^k = 0, \text{ where } \pi_k = \Lambda_k - \frac{\partial F}{\partial \mu^k} - \alpha_k. \tag{3.9}$$

In accordance with the statement of the problem, we may assume that the parameters $\mu^1, \mu^2, \ldots, \mu^n$ and their increments $d\mu^1, d\mu^2, \ldots, d\mu^n$ can acquire all possible values in certain ranges of the various processes, whereupon there are no linear homogeneous nonholonomic relationships between the increments $d\mu^1, d\mu^2, \ldots, d\mu^n$ which are satisfied regardless of the processes under consideration.

The possible linear independence of the increments $d\mu^k$ permits us to make certain conclusions concerning the values for π_k being attained at all possible processes, according to the condition that Λ_k, $\frac{\partial F}{\partial \mu^k}$ and α_k comprise functions only of $\mu^1, \mu^2, \ldots, \mu^n$.

1. If among $\mu^1, \mu^2, \ldots, \mu^n$ there are no derivatives of $\frac{d\mu^s}{dt}$, from this it follows that, for the reversible processes, π_k does not depend on $\frac{d\mu^s}{dt}$ and therefore the following equations should be satisfied.

$$\pi_k = 0 \quad \text{or} \quad \Lambda_k = \frac{\partial F}{\partial \mu^k}. \tag{3.10}$$

The equations (3.10) are either identically satisfied; and hence can serve simply as a determination of Λ_k, or at prescribed Λ_k-values (accordingly non-arbitrary) these can be the relationships for determining the derivatives of $\frac{\partial F}{\partial \mu^k}$, or at the prescribed function F (T, $\mu^1, \mu^2, \ldots, \mu^n$) they yield the equations of state, relating to the parameters which are being determined, for example, the stresses or the phase concentrations, to the control parameters. Equation (3.10) can be regarded as generalized equations of the theory of elasticity; specifically, they can simply coincide with the equations of state of the elasticity theory.

Among the Equations (3.10), we also derive the equations of state for determining the components of the tensor of stresses as a function of the components of the tensor of deformation and of other control parameters.

Equations (3.10), and their aforesaid interpretations for the reversible processes, preserve their validity also in the case when among the control parameters $\mu^1, \mu^2, \ldots, \mu^n$, their derivatives with respect to the coordinates, for example, if along with $\hat{\varepsilon}_{\alpha\beta}$ we include the derivatives $\nabla_\gamma \varepsilon_{\alpha\beta}$, etc.

In this manner, in the theory of elasticity, the assignment of the free energy as a function of the control parameters, excluding their derivatives with respect to time, permits us to fully determine the values for Λ_k.

From the basic Equation (3.9), the relationships (3.10) do not follow if the values for π_k can depend on $d\mu^s/dt$, specifically if among the control parameters $\mu^1, \mu^2, \ldots, \mu^n$, we include the derivatives

$$\frac{d\mu^1}{dt}, \frac{d\mu^2}{dt}, \ldots, \frac{d\mu^p}{dt}, p<n.$$

It is obvious that in the irreversible processes, the π_k-values can depend on the time-derivative control parameters.

We can consider such models for which in the reversible processes the π_k-values can also depend on the time derivatives from several control parameters.

2. Let the π_k-values depend on $d\mu^s/dt$ $(s = 1, 2,\ldots, p)$. From (3.9), we have

$$\sum_{k=1}^{p} \pi_k \frac{d\mu^k}{dt} + \sum_{k=p+1}^{n} \pi_k \frac{d\mu^k}{dt} = 0;$$

and since $\frac{d\mu^{p+1}}{dt}, \ldots, \frac{d\mu^n}{dt}$ can acquire arbitrary values, at $\frac{d\mu^{p+1}}{dt} = \frac{d\mu^{p+2}}{dt} = \ldots = \frac{d\mu^n}{dt} = 0$ we have

$$\sum_{k=1}^{p} \pi_k \frac{d\mu^k}{dt} = 0. \tag{3.11}$$

However, it is evident that this equation is always satisfied since, by definition, the π_k-values do not depend on $\frac{d\mu^{p+1}}{dt}, \ldots, \frac{d\mu^n}{dt}$; from this it also follows that the following equalities are always valid:

$$\sum_{k=p+1}^{n} \pi_k \frac{d\mu^k}{dt} = 0 \quad \text{и} \quad \pi_k = 0 \quad (k = p+1, \ldots, n). \tag{3.12}$$

If $\pi_1, \pi_2, \ldots, \pi_q$ contain unknown values while $\pi_{q+1}, \ldots, \pi_p$ are known, for example, $\Lambda_m = \alpha_m$ while $\frac{\partial F}{\partial \mu^m}$ $(m = q + 1,\ldots, p)$ are assigned, then relationship (3.11) can be written in the form

$$\sum_{k=1}^{q} \pi_k \frac{d\mu^k}{dt} = -\sum_{k=q+1}^{p} \pi_k \frac{d\mu^k}{dt} = \gamma \cdot \sum_{k=1}^{q} \left(\frac{d\mu^k}{dt}\right)^2. \tag{3.13}$$

In (3.13), the right-hand part is known, while the γ-value is determined from the last equation.

It is easy to demonstrate that the most general solution of Equation (3.13) of the values $\pi_1, \pi_2, \ldots, \pi_q$ depends on the arbitrary functions and has the form

Chapter 4. Variational Principles and Their Consequences

If among the control parameters we include the successive derivatives of several parameters with respect to time and coordinates, the derivation of the equations of state and the exclusion of the pertinent arbitrariness of (γ_k) can prove to be more convenient with the aid of the variational principles.

Further, having in mind the consideration of the electromagnetic effects and the utilization of the Maxwell equations, let us consider the applications of the variational principles and their results within the framework of the special theory of relativity.

We assume that x^1, x^2, x^3, x^4, = t are the Cartesian coordinates of a four-dimensional pseudo-Euclidean space, in which the metrics are determined by the quadratic equation:

$$ds^2 = -dx^{1^2} - dx^{2^2} - dx^{3^2} + c^2 dt^2 = g_{ij} dx^i dx^j, \tag{4.1}$$

where c equals the speed of light.

Let us denote by Λ the density of the Lagrange function. According to the assumption, the value $\Lambda c\, dx^1 dx^2 dx^3 dt$ constitutes a four-dimensional scalar, wherein Λ is the prescribed function of a system of control parameters comprised of the generalized coordinates $\mu^1, \mu^2, \ldots, \mu^n$ and their first derivatives $\mu^k{}_i = \frac{\partial \mu^k}{\partial x^i}$.

Among the generalized coordinates, there can be the scalars and certain independent components of the tensors or the invariant tensor functions[1].

---

[1]In the utilization of only the Cartesian coordinates, it can be assumed that a certain part of the generalized parameters μ^s can form a system of independent components of the spins. In this case, it can be shown that in the transition to a curvilinear system of coordinates, the invariant arguments containing the components of the spins, can be replaced by arguments equal to them and containing the components of properly selected tensors with a large number of components (which, however, are interdependent).

Here and below, we adopt the following condition: in the general expressions and in the summation, the Latin indexes i, j, l, m,... acquire the values 1, 2, 3, 4; the Greek indexes α, β, γ,... acquire the values 1, 2, 3.

$$
\left.\begin{aligned}
\pi_1 &= \gamma\dot{\mu}^1 - \gamma_1 \frac{\dot{\mu}^2}{\dot{\mu}^1}, \\
\pi_2 &= \gamma_1 + \gamma\dot{\mu}^2 - \gamma_2 \frac{\dot{\mu}^3}{\dot{\mu}^2}, \\
\pi_3 &= \qquad \gamma_2 + \gamma\dot{\mu}^3 - \gamma_3 \frac{\dot{\mu}^4}{\dot{\mu}^3}, \\
&\ldots \quad \ldots \quad \ldots \\
\pi_q &= \qquad\qquad\qquad \gamma_{q-1} + \gamma\dot{\mu}^q,
\end{aligned}\right\} \tag{3.14}
$$

where $\gamma_1, \gamma_2, \ldots, \gamma_{q-1}$ are arbitrary functions of any parameters. In this manner, the values $\pi_1, \pi_2, \ldots \pi_q$ are not determined unequivocally in this instance. The assignment or the determination of $\gamma_1, \gamma_2, \ldots, \gamma_{q-1}$ should be associated with additional data. If q = 1, i.e., there is only one unknown (for example, pressure p), Equation (3.13) is solved unequivocally by the formula

$$
\pi_1 = -\sum_{k=2}^{p} \pi_k \frac{\dot{\mu}^k}{\dot{\mu}^1}.
$$

Let us consider as an example the reversible processes $T\,ds = dQ^{(e)}$, $dQ^{**} = 0$ in an ideal liquid, for which the free energy has the form

$$
F(T, \varrho, \dot{\varrho}, \ddot{\varrho}, \ldots, \varrho^{(p-1)}).
$$

In this case, the equation for the heat inflow yields:

$$
\frac{\partial F}{\partial \varrho} d\varrho + \frac{\partial F}{\partial \dot{\varrho}} d\dot{\varrho} + \ldots + \frac{\partial F}{\partial \varrho^{(p-1)}} d\varrho^{(p-1)} = -\frac{p}{\varrho^2} d\varrho,
$$

from which we obtain the generalization of the known thermodynamic equation:

$$
p = -\varrho^2 \left(\frac{\partial F}{\partial \varrho} + \frac{\partial F}{\partial \dot{\varrho}} \frac{\ddot{\varrho}}{\dot{\varrho}} + \ldots + \frac{\partial F}{\partial \varrho^{(p-1)}} \frac{\varrho^{(p)}}{\dot{\varrho}} \right). \tag{3.15}
$$

The corresponding model of an ideal compressible liquid (when $F(\rho, \dot{\rho}, T)$) in application to the motion of water with bubbles changing its volume during cavitation has been described in the report B. Ṡ. Kogarko [198].

For simplicity, we also assume below that Λ depends only on the independent components of the scalars and tensors from μ^k, $\mu^k{}_i$ and g_{ij} only through the invariants comprised of these values and possibly from the functions of Λ (present in the arguments) of the parametric constants of the tensors forming the system of "physical constants"[2].

In conformity with the general theory of the calculus of variations, the introduction of the arguments of the Λ function of the higher-order derivatives into the system is quite admissible [226], and all of the following conclusions can be generalized accordingly.

By variational principles in the mechanics of a continuous medium and in the field theory, we can connote the functional equations obtained by equating to zero the sum of certain volume and surface integrals containing the variations of the subintegral functions and, generally speaking, the variations of the domain of integration.

In the formulation of the variational principle, it is necessary to establish the independent functions of μ^k which are being varied and the variations of the dependent values, specifically their derivatives with respect to the coordinates and time $\mu^k{}_i$.

---

[2]In conformity with the latter assumption, it follows that in the application of the curvilinear coordinates in the invariant arguments of the function Λ, the derivatives of the components of the tensors $\mu^k{}_i$ should be replaced by the covariant derivatives.

As is known, we can consider the density of the Lagrange function in a fixed Cartesian or curvilinear system of coordinates η^i as a function of the parameters μ^k and of the derivatives $\frac{\partial \mu^k}{\partial \eta^i}$, which in general do not comprise tensor values. Many of the following conclusions also maintain their validity in this instance. However, in this case, difficulties can develop in the introduction of the local concepts concerning the impulse and concerning the energy, and accordingly upon the introduction of the concept of the tensor of impulse energy.

Such difficulties developed in the theory of gravitation. For a gravitational field, we introduced the function Λ, which in its nature depends only on the control parameters g_{ij} and on their first derivatives with respect to the coordinates, which do not constitute tensor values (the second derivatives of g_{ij}, which can enter into Λ, introduce additional terms of a divergent nature, not influencing the basic equations of the general theory of relativity [224]).

It is also necessary to assign the subintegral functions which are not determined as a result of the application of the variational principle; specifically, this pertains to the density of the Lagrangian function Λ as a function of the control parameters μ^k and their derivatives μ^k_i.

The general variational equation can be written in the form:

$$\int_V \delta\Lambda\, d\tau + \Lambda\delta\, d\tau + \int_V \left(Q_k\delta\mu^k + \frac{\partial Q_k{}^j\delta\mu^k}{\partial x^j}\right) d\tau + \int_\Sigma P_k\delta\mu^k\, d\sigma = 0. \tag{4.2}$$

The integration is extended to an arbitrary four-dimensional volume V, wherein $d\tau = d\tau^* c\, dt = c\, dx^1\, dx^2\, dx^3\, dt$. The volume V is bounded by the three-dimensional surface Σ, the element of which is denoted by $d\sigma = d\sigma^* c\ dt$, where $d\sigma^*$ equals the element of the two-dimensional surface. The three-dimensional spatial volume V^* with the element $d\tau^*$ is bounded by the two-dimensional spatial surface Σ^* with the element $d\sigma^*$. In (4.2) the summation from one to n is conducted based on the index k.

It is necessary to assign the generalized mass forces Q_k and the surface forces $Q_k{}^j$, based on additional assumptions in consideration of the nonconservative systems or allowing for certain influences of the bodies external to volume V and separated by surface Σ from objects which are confined within V, but not included in the system which is under review. The generalized surface forces P_k must be introduced in consideration of the interactions along surface Σ between the conceptually-separated part of the medium and the part of the medium separated by the surface Σ.

According to the definition for the conservative systems, we can assume that $Q_k = Q_k{}^j = 0$. The surface integral in (4.2) is balanced separately with the surface integral which is encountered during the transformation of the variation $\delta \int_V \Lambda\, d\tau = \int_V \delta\Lambda\, d\tau + \Lambda\delta\, d\tau$ and $\int_V \frac{\partial Q_k{}^j\delta\mu^k}{\partial x^j}\, d\tau$; the relationships thus derived are used for determining the generalized surface forces P_k; therefore, these values are found from the basic equation (4.2).

Many authors omit the consideration of the surface integrals since, utilizing the arbitrariness of the variations, they consider only such variations of $\delta\mu^k$, for which Σ becomes zero.

However, the consideration of the surface integrals and the pertinent equations ensuing from the arbitrariness of the variations for Σ leads to additional significant physical conclusions.

The consideration of the special variations for Σ corresponding to the group of symmetries which (group) is admitted by the value $\Lambda d\tau$ permits us to derive the theorem of Noether, which can be regarded as an explicit basis for the formulation of the laws of conservation.

Equation (4.2) is invariant relative to the selection of a system of coordinates. In the consideration of the motion of a continuous medium, we always have two significantly important systems of coordinates.

In the special theory of relativity, this is a certain inertial frame of reference $K: x^1, x^2, x^3, t$, in which the motion of the medium is determined, and the attached system of coordinates L: ξ^1, ξ^2, ξ^3, $\xi^4 = \hat{t}$ ($\hat{t}$ equals the actual time). The coordinates ξ^1, ξ^2, ξ^3 are constant for the individual points. The construction of the theory of motion of a continuum as an aggregation of individual points is necessarily based on the introduction and use of the Lagrangian coordinates ξ^1, ξ^2, ξ^3.

The basic Equation (4.2) can be considered in any system of coordinates. The variations of the different parameters and functions in the case of the conceptual introduction of the adjacent states and the comparable processes can be computed in fixed points of space, assigned by the coordinates x^i in the inertial system K, i.e., the Euler viewpoint, or in the individual points of the Lagrangian system L, assigned by the coordinates ξ^i, i.e., the Lagrange viewpoint. The infinitely small increments, namely the variations of a certain value ϕ, will be denoted by the symbol $\partial\phi$ in the first case, and by the symbol $\delta\phi$ in the second case.

In the basic motion of a continuous medium, the law of motion is represented by the functions

$$x^i = x^i(\xi^1, \xi^2, \xi^3, \xi^4), \tag{4.3}$$

which can also be regarded as equations of the transformation of the coordinates during the transition from the Euler to the Lagrangian system of coordinates.

For the variational motions we have

$$x'^i = x'^i(\xi^1, \xi^2, \xi^3, \xi^4).$$

According to the definition, we set

$$x'^i - x^i = \delta x^i(\xi^i). \quad (4.4)$$

Assume that $\varphi(x^i) = \varphi(x^i(\xi^i)) = \tilde{\varphi}(\xi^i)$ is a certain function; the variations of this function caused by the variation in the actual function and in its arguments will be determined by the equalities

$$\delta\tilde{\varphi}(\xi^i) = \varphi'(x'^i) - \varphi(x^i) = \varphi'(x'^i) - \varphi(x'^i) + \varphi(x'^i) - \varphi(x^i) = \\ = \partial\varphi(x'^i) + \frac{\partial\varphi}{\partial x^i}\delta x^i. \quad (4.5)$$

If the form of the function $\phi(x^i)$ is not varied, $\varphi'(x'^i) = \varphi(x'^i)$, and therefore the local variation $\partial\phi(x'^i) = 0$. It is evident that with an accuracy of a higher order up to small values, the following equality is valid

$$\varphi'(x'^i) - \varphi(x'^i) = \varphi'(x^i) - \varphi(x^i),$$

i.e.

$$\partial\varphi(x'^i) = \partial\varphi(x^i),$$

wherein this variation differs from zero only because of the variation of the actual function ϕ, and not because of the variation of the arguments x'.

From the definition of the variations, we have

$$\left.\begin{aligned} \frac{\partial}{\partial\xi^i}\delta\varphi &= \frac{\partial\tilde{\varphi}'(\xi^i)}{\partial\xi^j} - \frac{\partial\tilde{\varphi}(\xi^i)}{\partial\xi^j} = \delta\frac{\partial\tilde{\varphi}}{\partial\xi^j} \\ \frac{\partial}{\partial x^j}\partial\varphi &= \frac{\partial\varphi'(x^i)}{\partial x^j} - \frac{\partial\varphi(x^i)}{\partial x^j} = \partial\frac{\partial\varphi}{\partial x^j}. \end{aligned}\right\} \quad (4.6)$$

However, noting that the relationships (4.6) indicate the possibility of transposing the symbols of the derivatives and the corresponding symbols of variation, we can also write the following equations:

$$\frac{\partial}{\partial x^j}\,\delta\varphi = \frac{\partial\varphi'(x'^i)}{\partial x'^l}\,\frac{\partial x'^l}{\partial x^j} - \frac{\partial\varphi(x^i)}{\partial x^j} =$$

$$= \frac{\partial\varphi'(x'^i)}{\partial x'^j} - \frac{\partial\varphi(x^i)}{\partial x^j} + \frac{\partial\varphi'(x'^i)}{\partial x'^l}\,\frac{\partial\delta x^l}{\partial x^j} = \delta\,\frac{\partial\varphi}{\partial x^j} + \frac{\partial\varphi}{\partial x^l}\,\frac{\partial\delta x^l}{\partial x^j} \qquad (4.7)$$

and

$$\frac{\partial}{\partial\xi^j}\,\delta\varphi = \frac{\partial\varphi'(x^i)}{\partial\xi^j} - \frac{\partial\varphi(x^i)}{\partial\xi^j} = \frac{\partial x^l}{\partial\xi^j}\,\delta\left(\frac{\partial\varphi}{\partial x^l}\right). \qquad (4.8)$$

In the construction of the various models, it is necessary to introduce a varying type of characteristics and their derivatives, taken in different senses. Specifically, the function Λ can be regarded as a function of certain parameters μ^k and their various derivatives $\frac{\partial\mu^k}{\partial x^i}$ or $\frac{\partial\mu^k}{\partial\xi^i}$.

Further, for precision and simplicity, let us assume that the function Λ depends

$$\text{on } x^i = \mu^i, \text{ on } \frac{\partial x^i}{\partial\xi^j} = x^i_j \ (i,\ j = 1,\ 2,\ 3,\ 4),$$

and upon the parameters μ^k and their first derivatives

$$\mu^k_i = \frac{\partial\mu^k}{\partial x^i} \ (k = 5,\ 6,\ \ldots,\ n;\ i = 1,\ 2,\ 3,\ 4).$$

In addition, the function Λ can depend on ξ^i, which will be regarded as nonvarying independent parameters.

It is evident that the derivatives of the type $\frac{\partial\mu^k}{\partial\xi^j}$ can be expressed through μ^k_i and x^i_j.

For a comparison of the variation in $\delta\int\limits_V \Lambda\,d\tau$, we note that

$$d\tau = \sqrt{-g}\,dt\,dx^1\,dx^2\,dx^3 \qquad (t = x^4,\ g = |g_{ij}|),$$

wherein

$$\delta\,d\tau = \left[\left|\frac{\partial x'^i}{\partial x^j}\right| - 1\right]d\tau = \frac{\partial\delta x^i}{\partial x^i}\,d\tau.$$

On the strength of this and of the noted relationships we can write for the variations:

$$\delta \int_V \Lambda\, d\tau = \int_V \left(\frac{\partial \Lambda}{\partial x^i} \delta x^i + \frac{\partial \Lambda}{\partial x^i{}_j} \delta x^i{}_j + \frac{\partial \Lambda}{\partial \mu^k} \delta \mu^k + \frac{\partial \Lambda}{\partial \mu^k{}_j} \delta \mu^k{}_j + \Lambda \delta^j{}_i \frac{\partial \delta x^i}{\partial x^i} \right) d\tau =$$
$$= \int_V \left\{ -\left[\frac{\partial \Lambda}{\partial x^l{}_j} \frac{\partial x^l{}_j}{\partial x^i} + \frac{\partial}{\partial x^l} \left(\frac{\partial \Lambda}{\partial x^i{}_j} x^l{}_j \right) \right] \delta x^i + \left[\frac{\partial \Lambda}{\partial \mu^k} - \frac{\partial}{\partial x^j} \left(\frac{\partial \Lambda}{\partial \mu^k{}_j} \right) \right] \partial \mu^k \right\} d\tau +$$
$$+ \int_V \frac{\partial}{\partial x^j} \left(\frac{\partial \Lambda}{\partial x^i{}_l} x^j{}_l \delta x^i + \frac{\partial \Lambda}{\partial \mu^k{}_j} \partial \mu^k + \Lambda \delta^j{}_i \delta x^i \right) d\tau; \qquad (4.9)$$

where in the transformations under the sign of the integral we have taken into account (4.5), (4.6) and the following equality:

$$\frac{d\Lambda}{dx^i} = \frac{\partial \Lambda}{\partial x^i} + \frac{\partial \Lambda}{\partial x^h{}_j} \frac{\partial x^h{}_j}{\partial x^i} + \frac{\partial \Lambda}{\partial \mu^k} \mu^k{}_i + \frac{\partial \Lambda}{\partial \mu^k{}_j} \mu^k{}_{ij}.$$

Substituting (4.9) into (4.2), with allowance for (4.5), we obtain

$$\int_V \left[Q_i - \frac{\partial \Lambda}{\partial x^l{}_j} \frac{\partial x^l{}_j}{\partial x^i} - \frac{\partial}{\partial x^l} \left(\frac{\partial \Lambda}{\partial x^i{}_j} x^l{}_j \right) - \left(\frac{\partial \Lambda}{\partial \mu^k} - \frac{\partial}{\partial x^j} \frac{\partial \Lambda}{\partial \mu^k{}_j} \right) \mu^k{}_i \right] \delta x^i\, d\tau +$$
$$+ \int_V \left(\frac{\partial \Lambda}{\partial \mu^k} - \frac{\partial}{\partial x^j} \frac{\partial \Lambda}{\partial \mu^k{}_j} + Q_k \right) \delta \mu^k\, d\tau +$$
$$+ \int_\Sigma \left\{ \left[P_i + \left(\frac{\partial \Lambda}{\partial x^i{}_l} x^j{}_l - \mu^k{}_i \frac{\partial \Lambda}{\partial \mu^k{}_j} + \Lambda \delta^j{}_i + Q_i{}^j \right) n_j \right] \delta x^i + \right.$$
$$\left. + \left(P_k + \frac{\partial \Lambda}{\partial \mu^k{}_j} n_j + Q_k{}^j n_j \right) \delta \mu^k \right\} d\sigma = 0 \qquad (4.10)$$

(n_j are the components of the unit vector of a perpendicular to surface Σ).

Equation (4.10) should be satisfied in the case of arbitrary variations in δx^i and $\delta \mu^k$.

Let us first consider those variations where $\delta x^i = \delta \mu^k = 0$ for Σ and are arbitrary within V; utilizing this, we derive the following system of Lagrangian equations

$$\frac{\partial}{\partial x^j} \left(\frac{\partial \Lambda}{\partial x^i{}_l} x^j{}_l \right) + \frac{\partial \Lambda}{\partial x^l{}_j} \frac{\partial x^l{}_j}{\partial x^i} = Q_i + Q_k \mu^k{}_i \quad (i = 1,\ 2,\ 3,\ 4) \qquad (4.11)$$

and

$$\frac{\partial}{\partial x^j} \frac{\partial \Lambda}{\partial \mu^k{}_j} - \frac{\partial \Lambda}{\partial \mu^k} = Q_k \quad (k = 5,\ 6,\ \ldots,\ n). \qquad (4.12)$$

If the function Λ does not depend on x^i_j, Equations (4.11) yield

$$Q_i = -Q_k \mu^k_i. \qquad (4.13)$$

At the assigned values for Q_i and Q_k, Equations (4.11) form a complete system of n equations for finding the n functions $x^i(\xi^j)$ and $\mu^k(x^i)$.

On the basis of (4.11) and (4.12), it follows that for the arbitrary variations differing from zero on any surface Σ conceptually separated in the medium, the following equality should be satisfied

$$\int_\Sigma \left[\left(P_i{}^j - \mu^k{}_i \frac{\partial \Lambda}{\partial \mu^k{}_j} + \frac{\partial \Lambda}{\partial x^i{}_l} x^j{}_l + \Lambda \delta^j_i + Q_i{}^j \right) n_j \delta x^i + \right.$$
$$\left. + \left(P_k{}^j + \frac{\partial \Lambda}{\partial \mu^k{}_j} + Q_k{}^j \right) n_j \delta \mu^k \right] d\sigma, \qquad (4.14)$$

where

$$P_i = P_i{}^j n_j \text{ и } P_k = P_k{}^j n_j. \qquad (4.15)$$

Equation (4.14) is satisfied for any surface Σ at arbitrary δx^i and $\delta \mu^k$, therefore, from (4.14), it follows that

$$P_i{}^j = \mu^k{}_i \frac{\partial \Lambda}{\partial \mu^k{}_j} - x^j{}_l \frac{\partial \Lambda}{\partial x^i{}_l} - \Lambda \delta^j_i - Q_i{}^j \qquad (4.16)$$

and

$$P_k{}^j = -\frac{\partial \Lambda}{\partial \mu^k{}_j} - Q_k{}^j. \qquad (4.17)$$

Obviously, the specifying of a model of a continuous medium is associated with the assignment of the following functions: Λ, Q_i, Q_k, $Q_i{}^j$, $Q_k{}^j$. The necessity of specifying the parameters for Q_r and $Q_r{}^j$ introduces a considerable arbitrariness, however for the conservative systems, by definition

$$Q_r = Q_r{}^j = 0,$$

while in certain other cases, these parameters can be introduced with the aid of simple hypotheses of a physical nature associated with a consideration of the irreversible effects.

The assignment of the function Λ is analogous but not equivalent to the specification of the specific density of the intrinsic energy or the free energy for the infinitely small elements of a continuous medium. Similar to the assignment of the density of free energy, the assignment of the function Λ in a general case is also insufficient for establishing a closed system of equations determining the actual model of a continuous medium.

However, with the aid of the prescribed function Λ, the arguments of which contain various derivatives from the control parameters with respect to the coordinates and time, we derive a closed system of equations determining the model of the medium in the case of the conservative systems.

The system of n Equations (4.11) and (4.12) and of 4n Equations (4.16) and (4.17) consists of the equations of motion, the equations of state and the kinetic equations. These equations contain the regularities describing the processes in the medium caused by the presence of the internal degrees of freedom.

In this manner, the establishment of the regularities for conservative systems in the case of a large number of degrees of freedom can be reduced to the problem of establishing the form of the Lagrangian function depending on the control parameters.

For the derivation of the connecting links with the statistical physics, with the phenomenological thermodynamic relationships and, in this way, with the experimental data, it is necessary to clarify the relationship of the system of Equations (4.11), (4.12), (4.16) and (4.17), and its corresponding functions, with the basic physical laws, with the characteristic functions of the thermodynamic and with other laws of nature.

Specifically, Equations (4.16) and (4.17) contain the equations of the classical theory of elasticity and hydrodynamics, and the Maxwell equations for an electromagnetic field.

At Q_i and Q_i^j-values differing from zero and determined in the appropriate manner, Equations (4.11), (4.12), (4.16) and (4.17) reduce to the equations of the theory of a viscous liquid, with consideration of the phenomena of heat conductivity, and to other equations in the theory of irreversible phenomena.

In this manner, the utilization of the variational principles, together with the sum of the data and the relationships of the functions Λ, Q_r and Q_r^j with the thermodynamic and other physical functions, can serve as an initial basis for the expression of the continuous media, in specific terms of the models, which in turn can serve for the introduction of the characteristic parameters and functions, with the aid of which we can formulate a varying class of hypotheses of a physical nature.

The development of a theory in these problems is closely associated with the use of various results ensuing from (4.11) and (4.12) in a number of cases under certain assumptions of a very general nature. Among such important results, we include the laws of conservation.

The laws of conservation can be derived based on the theorem expressed by E. Noether, which consists of the following:

Let us consider a case when $Q_i = Q_k = 0$. Assume that the integral $I = \int_V \Lambda \, d\tau$ is invariant relative to a certain m-parametric continuous group of transformations G $(\alpha^1, \alpha^2, \alpha^3, \ldots, \alpha^m)$ of the variables x^i and of the corresponding transformations μ^k $(\alpha^1, \alpha^2, \ldots, \alpha^m$ equals a system of parameters independent of the coordinates x^i). The group G forms an m-parametric group of symmetry for the integral I. In this case, there occur m laws of conservation.

In actuality, from the invariant state of the integral

$$I(\alpha^1, \alpha^2, \ldots, \alpha^m) = I(0, 0, \ldots, 0)$$

it follows that for an infinitely small transformation of an element of group G, at which

$$\delta x^i = C^i_q \delta\alpha^q \quad \text{and} \quad \delta\mu^k = S^k_q \delta\alpha^q,$$

we have

$$\delta I = 0. \tag{4.18}$$

The first integral in Expression (4.9) for the variation of the integral becomes zero on the basis of (4.11) and (4.12); therefore, on the basis of (4.18), equating to zero the second integral in (4.9), we obtain

$$\int_V \frac{\partial}{\partial x^j}\left[\left(\frac{\partial \Lambda}{\partial x^i{}_l} x^j{}_l - \mu^k{}_i \frac{\partial \Lambda}{\partial \mu^k{}_j} + \Lambda \delta^j_i\right) C^i{}_q + \frac{\partial \Lambda}{\partial \mu^k{}_j} S^k{}_q\right] \delta a^q \, d\tau = 0. \tag{4.19}$$

Since the volume V is arbitrary and $\delta\alpha^q$ is arbitrary, we obtain from (4.19) the following m laws of conservation:

$$\frac{\partial}{\partial x^j}\left[\left(\frac{\partial \Lambda}{\partial x^i{}_l} x^j{}_l - \mu^k{}_i \frac{\partial \Lambda}{\partial \mu^k{}_j} + \Lambda \delta^j_i\right) C^i{}_q + \frac{\partial \Lambda}{\partial \mu^k{}_j} S^k{}_q\right] = 0. \tag{4.20}$$

If the integral I is invariant relative to the translation group G_4, it follows that

$$\delta x^i = \delta a^i, \quad C^i{}_q = \delta^i_q, \quad \delta\mu^k = 0, \quad S^k{}_q = 0;$$

and therefore Equation (4.20) acquires the form

$$\frac{\partial T_i{}^j}{\partial x^j} = 0 \quad (i = 1, 2, 3, 4), \tag{4.21}$$

where

$$T_i{}^j = P_i{}^j + Q_i{}^j = \mu^k{}_i \frac{\partial \Lambda}{\partial \mu^k{}_j} - x^j{}_q \frac{\partial \Lambda}{\partial x^i{}_q} - \Lambda \delta^j_i. \tag{4.22}$$

It is easy to observe that, subtracting Equation (4.11) from Equation (4.12), multiplied times $\mu^k{}_i$ and added with respect to the index k, we obtain

$$\frac{\partial T_i{}^j}{\partial x^j} = -Q_i - \frac{\partial \Lambda}{\partial x^i}. \tag{4.23}$$

Equations (4.23), being satisfied in the general case, convert to the law of conservation (4.21), provided that $Q_i = 0$ and the partial derivative $\frac{\partial \Lambda}{\partial x^i}$ also becomes zero. The latter condition is associated with the invariant state of the integral I relative to the group G_4. Subsequently, let us assume $\frac{\partial \Lambda}{\partial x^i} = 0$. The dependence of Λ on ξ^i can be preserved.

It is evident that in the general derivation of the laws of conservation (4.20), in place of the conditions $Q_i = Q_k = 0$, it is sufficient to require the fulfillment of the less stringent condition:

$$Q_i C^i_q + Q_{ik} S^k_q = 0 \quad (q = 1, 2, \ldots, m). \tag{4.24}$$

Along with the laws of conservation corresponding to the group of translations, we can consider the laws of conservation corresponding to the complete or natural Lorentz group [240]. In connection with this, we can introduce the tensor $\Theta_i^j = T_i^j + \Sigma_i^j$ $(\Sigma_i^j \neq 0)$ as a function of the control parameters; and the following laws of conservation are then satisfied:

$$\frac{\partial \Theta_i^j}{\partial x^j} = 0 \quad \text{and} \quad \Theta^{ij} = \Theta^{ji}. \tag{4.25}$$

The laws of conservation (4.25) can be regarded as the equations of the moments of quantities of motion, corresponding to a special form of internal degrees of freedom.

Equation (4.23) can be written in the form

$$\frac{\partial P_i^j}{\partial x^j} = -\left(Q_i + \frac{\partial Q_i^j}{\partial x^j}\right) = Q'_i. \tag{4.26}$$

If $Q'_i = 0$, it follows that

$$Q_i = -\frac{\partial Q_i^j}{\partial x^j},$$

which yields

$$Q_i \delta x^i + \frac{\partial Q_i^j \delta x^i}{\partial x^j} = Q_i^j \frac{\partial \delta x^i}{\partial x^j}; \tag{4.27}$$

in this case, Equation (4.26) also acquires the usual form as for the conservative systems. In this manner, the equation

$$\frac{\partial P_i^j}{\partial x^j} = 0 \tag{4.28}$$

is satisfied for the nonconservative systems for which P_i^j is determined by Equation (4.16), in which $Q_i^j \neq 0$; consequently $P_i^j \neq T_i^j$.

Equations (4.21) and (4.22) occur in any inertial Cartesian frame of reference. In (4.21) and (4.22), we assume that

$$\Lambda\left(\xi^i, \frac{\partial x^i}{\partial \xi^j}, \mu^k, \frac{\partial \mu^k}{\partial x^j}, g_{ij}\right). \tag{4.29}$$

In a consideration of all possible inertial systems of coordinates, the function can be regarded as a scalar; therefore, in the functional relationship (4.29), and in the capacity of the significant arguments, the indicated variables can occur only through their combined invariant combinations which can be taken in the form of polynomials. The number of the functionally independent variables of the combined invariants of the control parameters in any case does not exceed the total number of the variable arguments indicated in (4.29).

After the establishment of the inertial frame of reference with the coordinates x^i and the attached system with the coordinates ξ^i, the variables x^i and x^i_j can be regarded as components of the vector $r = x^i\mathfrak{Э}_i$ and of the tensor of the second rank $A = \frac{\partial r}{\partial \xi^j} = x^i{}_j\mathfrak{Э}_i\mathfrak{Э}^j$, where $\mathfrak{Э}_i$ and $\mathfrak{Э}^j$ are covariant and antivariant base vectors in an inertial system, which are constant with respect to the x^i coordinates.

Let us now introduce the arbitrary, generally speaking, mobile system of coordinates η^i with the bases $\mathfrak{Э}^*_i$ and $\mathfrak{Э}^{*i}$, which are related to the system of coordinates x^i by the relations

and
$$\left.\begin{aligned} x^i &= x^i(\eta^1, \eta^2, \eta^3, \eta^4) \\ ds^2 &= g_{ij}\,dx^i\,dx^j = g^*_{ij}\,d\eta^i\,d\eta^j. \end{aligned}\right\} \tag{4.30}$$

For the metric tensor g^*ij, we can introduce the Christoffel symbols Γ^{*l}_{ij} and the operation of covariant differentiation ∇_q. According to the transformation (4.30), we have

$$r = x^i\mathfrak{Э}_i = r^i\mathfrak{Э}^*_i \text{ and } A = x^i{}_j\mathfrak{Э}_i\mathfrak{Э}^j = A^i{}_j\mathfrak{Э}^*_i\mathfrak{Э}^{*j},$$

where

$$r^i = x^l\frac{\partial \eta^i}{\partial x^l}, \quad A^i{}_j = x^l{}_m\frac{\partial \eta^i}{\partial x^l}\frac{\partial x^m}{\partial \eta^j} \text{ и } g^*_{ij} = g_{lm}\frac{\partial x^l}{\partial \eta^i}\frac{\partial x^m}{\partial \eta^j}.$$

Certain of the values from μ^k can form tensors of the type:

$$B = B^{il}{}_m \partial_i \partial_l \partial^m = B^{*il}{}_m \partial_i^* \partial_l^* \partial^{*m}.$$

In connection with the derivatives $\frac{\partial \mu^k}{\partial x^j}$, we can introduce the tensor

$$C = \frac{\partial B}{\partial x^j} \partial^j = \frac{\partial B^{il}{}_m}{\partial x^j} \partial_i \partial_l \partial^m \partial^j = \nabla_j B^{*il}{}_m \partial_i^* \partial_l^* \partial^{*m} \partial^{*j}.$$

The components of the tensor B $B^{il}{}_m$ and $B^{*il}{}_m$, and also the components of the tensor $\frac{\partial B^{il}{}_m}{\partial x^j}$ and $\nabla_j B^{*il}{}_m$ are interrelated by the general formulas of the tensor transformations during the conversion from x^i to η^i.

In the use of any system of coordinates, the function Λ can be determined as a scalar if we replace the arguments $\frac{\partial x^l}{\partial \xi^j}$ by $A^i{}_l$, the arguments $\mu^k = B^{il}{}_m$ by $B^{*il}{}_m$, the relation $\frac{\partial \mu^k}{\partial x^j} = \frac{\partial B^{il}{}_m}{\partial x^j}$ by $\nabla_j \mu^k = \nabla_j B^{*il}{}_m$, and g_{ij} by g^*_{ij}.

After this, the system of components T_i^{*j}, determined in any system of coordinates η^i by the equation

$$T_i^{*j} = P_i^{*j} + Q_i^{*j} = \nabla_i \mu^k \frac{\partial \Lambda}{\partial \nabla_j \mu^k} - A^j{}_l \frac{\partial \Lambda}{\partial A^i{}_l} - \Lambda \delta_i^j, \tag{4.31}$$

forms the tensor satisfying the equation

$$\nabla_j T_i^{*j} = -Q_i^* = -Q_l \frac{\partial x^l}{\partial \eta^i} \quad (i = 1, 2, 3, 4). \tag{4.32}$$

By P_i^{*j}, Q_i^{*j} and the Q_i^*, we denote the components $P_i{}^j$, $Q_i{}^j$ and Q_i transformed on the basis of the tensor formulas.

The conclusions obtained concerning the form of Equation (4.31) and Equation (4.32) are related to the basic assumption indicated above to the extent that the arguments in the specified function (4.29) can be regarded as scalars and as components of the tensors not only in an inertial system of coordinates, but also in any other curvilinear system of coordinates.

Specifically, the system of coordinates η^i can coincide with the attached system of coordinates ξ^i. In this case, we will have[1]:

[1] Equations (4.33) constitute the result of the pseudo-Euclidean state of space-time in the special theory of relativity. In the general theory of relativity, the determination of $\hat{g}_{ij}$ involves the integration of the gravitational equation.

$$g^{\circ}_{ij} = \hat{g}_{ij} = g_{lm} \frac{\partial x^l}{\partial \xi^i} \frac{\partial x^m}{\partial \xi^j} = c^2 x^4{}_i x^4{}_j - \sum_{\alpha=1}^{3} x^\alpha{}_i x^\alpha{}_j, \tag{4.33}$$

since $g_{44} = c^2$, $g_{\alpha\alpha} = -1$ and $g_{ij} = 0$ at $i \neq j$.

In addition, in this case, the formula $A^i{}_j = \frac{\partial x^i}{\partial \xi^j}$, is valid and hence $\frac{\partial x^i}{\partial \xi^j} \partial_i \partial^j = \frac{\partial x^i}{\partial \xi^j} \hat{\partial}_i \hat{\partial}^j$.

In Equation (4.29), after the transition to the attached system of coordinates, the components of the tensor $\hat{g}_{ij}$ will appear as the transformed components of the tensor g_{ij}; but in addition, the argument $\partial x^i/\partial \xi^j$ may appear in combinations (4.33) of equal $\hat{g}_{ij}$, which remain invariant after conversion to the attached system of coordinates.

The convenience in applying the attached system of coordinates is also in part connected with the fact that in many important cases, the derivatives $\partial x^i/\partial \xi^j$ enter only into Equation (4.29) through the combinations $\hat{g}_{ij}$, and consequently the number of the significant arguments is reduced in (4.29) in the attached system of coordinates.

Among the parameters μ^k and $\nabla_{j\mu}{}^k$, there can exist various tensors which can characterize the states and processes for the medium's particles.

The tensor $\hat{g}_{ij}$ is present among the control parameters as a natural characteristic, since the quadratic form:

$$ds^2 = \hat{g}_{ij}\, d\xi^i\, d\xi^j$$

serves for the determination of the distances and time intervals between the various particles and events; it is obvious that these kinematic characteristics are basic ones in the formulation and description of the physical regularities.

As is known, the kinematic (geometric) parameters are of considerable and comparable importance, since they permit the configurations and processes under review to be compared with certain standard or conceptually introduced states and processes.

In conformity with the ideas developed in Chapter 2 for the comparative evaluation of events, we can introduce for the aggregation of values ξ^1, ξ^2, ξ^3 and ξ^4 the ideal sets of state and process, which can be regarded as the affine-connected metric space D_0. The geometric characteristic D_0 can be provided by the quadratic form specifying the metrics:

$$ds^2 = -\overset{0}{g}_{ij}\, d\xi^i\, d\xi^j, \qquad \overset{0}{g}_{ij}(\xi^1, \xi^2, \xi^3, \xi^4),$$

and the torsion tensor $S^l_{ij}(\xi^1, \xi^2, \xi^3, \xi^4)$. The space of the initial state D_0, the significant properties of which are prescribed by the tensors $\overset{0}{g}_{ij}$ and S^l_{ij}, can be introduced based on hypotheses of a physical nature.

Specifically, sometimes we can postulate that $S^l_{ij} = 0$ and that in the given attached system of coordinates

$$\overset{0}{g}_{ij}(\xi^1, \xi^2, \xi^3) = \hat{g}_{ij}(\xi^1, \xi^2, \xi^3, \xi^{04}), \tag{4.34}$$

where the constant coordinate ξ^{04} corresponds to a certain initial time instant In conformity with the definition of (4.34), the space D_0 is stationary, i.e., all of the spatial distances between the individual fixed particles are identical at various given time instants.

The system of coordinates ξ^1, ξ^2, ξ^3, $\hat{t}$ in space D_0 can be regarded as a stationary nondeforming Lagrangian system of coordinates. The conceptual space D_0 is generally non-Euclidean within the scope of the special theory of relativity (just as the space of the unstressed non-defect states in the continual theory of dislocations [Section 2] in Newtonian mechanics). At the instant ξ^{04}, the space D_0 can be in contact along a three-dimensional space with the actual pseudo Euclidean space connected with the moving medium. If $S^l_{ij} = 0$, D_0 and its three-dimensional subspace form, generally speaking, the Riemann (elliptic) spaces. It is evident that the non-Euclidean state of the space D_0 is permissible; moreover, many of the laws controlling the values of the parameters characterizing the internal degrees of freedom can be formulated as the equations containing the tensors S^l_{ij}, $\overset{0}{g}_{ij}$, R_{ijlm}, introduced as the geometric characteristics of the space D_0.

The concept of deformation in the theory of relativity has been reviewed by a number of authors [236, 228, 229, 234], who introduced this concept with the aid of the comparison of the motion considered with the motion of a medium as a solid. The basic difficulty consisted in determining the solid motions of a medium with the conservation of the property of a pseudo Euclidean state for the space D_o. In conformity with the definition made by Born, the space D_o in which the solid motion is achieved is pseudo-Euclidean; moreover, its metrics $\overset{0}{g}_{ij}$ $(\xi^1, \xi^2, \xi^3, \xi^4)$ are determined from the condition of a constant distance between any two fixed universal lines for each point prescribed for each of them. The Born definition complicates the problem of finding the $\overset{0}{g}_{ij}$-value, moreover the coordination of the concepts of hardness in Newtonian mechanics and in the special theory of relativity becomes complicated.

The assumption (4.34) is associated with the choice of the attached system of coordinates. In the case of the general transformations of attached Lagrangian coordinates, if we have $\overset{0}{g}_{ij}(\xi^1, \xi^2, \xi^3)$ in one system of coordinates, then in the other system we will have $\overset{0}{g}'_{ij}(\xi'^1, \xi'^2, \xi'^3, \xi'^4)$, i.e., the field of the fundamental metric tensor can become nonstationary.

However, it is evident that if in a certain given system the field $\overset{0}{g}_{ij}(\xi^1, \xi^2, \xi^3)$ is stationary, the property of a stationary state in the components $\overset{0}{g}_{ij}$ is maintained in the case of any transformation of only the spatial coordinates.

In the capacity of a generalized four-dimensional tensor of the finite deformations, we can introduce the tensors $\overset{0}{\mathcal{E}} = \varepsilon_{ij}\overset{0}{\partial}{}^i\overset{0}{\partial}{}^j$ in the space D_o or $\hat{\mathcal{E}} = \varepsilon_{ij}\hat{\partial}^i\hat{\partial}^j$ in the space of the motion of the medium, with the components ε_{ij} determined by the equation

$$\varepsilon_{ij} = \frac{1}{2}(\hat{g}_{ij} - \overset{0}{g}_{ij}). \tag{4.35}$$

Along with the tensor ε_{ij} introduced by Equation (4.35), let us consider also the tensors $\overset{0}{E} = E_{ij}\overset{0}{\partial}{}^i\overset{0}{\partial}{}^j$ and $\hat{E} = E_{ij}\hat{\partial}^i\hat{\partial}^j$, characterizing the deformation and with the components E_{ij} determined by the equality:

$$E_{ij} = \frac{1}{2}(\hat{g}_{ij} - u_i u_j - \overset{0}{g}_{ij} + \overset{0}{u}_i\overset{0}{u}_j), \tag{4.36}$$

where u_i and $\overset{0}{u}_j$ equal the covariant components of the four-dimensional velocity vector, respectively, for a moving continuum and for an ideal space of comparison D_o. In the attached system of coordinates, we have

$$\hat{u}_i = \hat{g}_{ij}\hat{u}^j = g_{i4}\hat{u}^4 = \frac{\hat{g}_{i4}}{\sqrt{\hat{g}_{44}}} \text{ и } \overset{0}{u}_i = \frac{\overset{0}{g}_{i4}}{\sqrt{\overset{0}{g}_{44}}},$$

therefore in the attached system of coordinates Equation (4.36) can be rewritten in the form:

$$E_{ij} = -\frac{1}{2}\left(\hat{g}_{ij} - \frac{\hat{g}_{i4}\hat{g}_{j4}}{\hat{g}_{44}} - \overset{0}{g}_{ij} + \frac{\overset{0}{g}_{i4}\overset{0}{g}_{j4}}{\overset{0}{g}_{44}}\right). \tag{4.37}$$

It is evident that in the attached system of coordinates, only the components $E_{\alpha\beta}$ (α, β = 1, 2, 3) can differ from zero, whereas

$$E_{\alpha 4} = E_{4\alpha} = E_{44} = 0.$$

In this manner, the four-dimensional tensor E_{ij} in the attached system of coordinates reduces to the three-dimensional tensor $E_{\alpha\beta}$. The determination of the tensor E_{ij} as the characteristics of the deformations is a natural generalization of the usual determination of the tensor of residual deformations in the classical theory of elasticity, because for the calculation of the spatial distances between the points of a small particle of the medium at dt = 0, it is necessary to utilize the quadratic form [224]

$$-ds^2 = dl^2 = -\left(g_{\alpha\beta} - \frac{g_{\alpha 4}g_{\beta 4}}{g_{44}}\right) d\xi^\alpha d\xi^\beta = \gamma_{\alpha\beta} d\xi^\alpha d\xi^\beta. \tag{4.38}$$

Therefore

$$2E_{ij}\, d\xi^i\, d\xi^j = dl^2 - dl_0^2.$$

The tensor components ε_{ij} or E_{ij} can be regarded as the control parameters (representing the known combinations from $x^i_{\ j}$), upon which the density of the Lagrangian function in (4.29) can depend.

Chapter 5. Equation of Energy and Equation of Heat Inflow

Let us consider the equation of energy ensuing from (4.11) and (4.12), and let us clarify the relationship between the functions entering the basic variational Equation (4.2) and the thermodynamic functions.

The components of the four-dimensional velocity vector of the medium's points (the fixed coordinates ξ^1, ξ^2, ξ^3) in the inertial system K are determined by the derivatives

$$u^\alpha = \frac{dx^\alpha}{ds} = \frac{\partial x^\alpha}{\partial t}\frac{\partial t}{\partial s} = \frac{v^\alpha}{\sqrt{c^2-v^2}}, \quad u^4 = \frac{dx^4}{ds} = \frac{\partial t}{\partial s} = \frac{1}{\sqrt{c^2-v^2}}, \tag{5.1}$$

where

$$v^2 = \left(\frac{\partial x^1}{\partial t}\right)^2 + \left(\frac{\partial x^2}{\partial t}\right)^2 + \left(\frac{\partial x^3}{\partial t}\right)^2 \text{ и } \left(\frac{\partial s}{\partial t}\right)^2_{\xi^\alpha} = c^2 - v^2.$$

In the system L for the components of the four-dimensional velocity, we have

$$\hat{u}^\alpha = \frac{d\xi^\alpha}{ds} = u^i \frac{\partial \xi^\alpha}{\partial x^i} = 0 \text{ и } \hat{u}^4 = \frac{d\hat{t}}{ds} = u^i \frac{\partial \hat{t}}{\partial x^i} = \frac{1}{c} \tag{5.2}$$

In Equation (5.2), the coordinate $\xi^4 = \hat{t}$ is determined as the intrinsic (proper) time; in conformity with this, the metrics in the system L are determined by the formula

$$ds^2 = \hat{g}_{ij}\, d\xi^i\, d\xi^j, \qquad \text{in which} \qquad \hat{g}_{44} = c^2. \tag{5.3}$$

At any time instant t in any mobile point of the medium M, we can select an intrinsic inertial system of coordinates K* such that at instant t, the three-dimensional velocity of the point M is equal to zero in the system K*.

By way of the convolution of Equation (4.26) with the vector u^i, we derive the scalar equation

$$u^i \frac{\partial P_i^j}{\partial x^j} = Q_i^j u^i. \qquad (5.4)$$

Equation (5.4) also is applicable in the case where the function Λ depends on the derivatives with respect to the time and coordinates of any order from the control parameters using the somewhat complicated formula for the tensor components

$$T_i^j = P_i^j + Q_i^j.$$

The invariant Equation (5.4), which can be written in any system of coordinates and particularly in the attached system of coordinates L, is the energy equation and the transformation of this equation to the heat equation in the usual form has been given in our report [230] for the attached system of coordinates; we present below another elementary transformation of this equation in an inertial natural system of coordinates.

Let us consider the corresponding natural inertial system of coordinates K* in every point of a mobile medium for and at time instant. In this manner, for each time instant t taken in K, we have the aggregation of the systems of coordinates of K* with the parallel spatial axes, each of which will move progressively with a constant velocity, equalling the velocity of the pertinent point of the medium.

For every point and at any time instant, let us consider the components of the tensor P^{*j}_i, forming a tensor field determined in the system K*.

Let us signify by $x^{*\alpha}, x^{*4} = t^*$ the coordinates of the pseudo-Euclidean space in the system K*. The Lorentz transformation connecting x^i and x^{*i} has the form

$$x^i = c^i{}_j x^{*j} \quad \text{и} \quad x^{*j} = d^j{}_i x^i. \qquad (5.5)$$

The matrices $c^i{}_j$ and $d^j{}_i$ are mutually reciprocal. For $d^j{}_i$, we can write the following matrix [238]:

$$\|d^j{}_i\| = \left\| \begin{matrix} 1+k(v^1)^2 & kv^1v^2 & kv^1v^3 & -\dfrac{cv^1}{\sqrt{c^2-v^2}} \\ kv^2v^1 & 1+k(v^2)^2 & kv^2v^3 & -\dfrac{cv^2}{\sqrt{c^2-v^2}} \\ kv^3v^1 & kv^3v^2 & 1+k(v^3)^2 & -\dfrac{cv^3}{\sqrt{c^2-v^2}} \\ -\dfrac{v^1}{c\sqrt{c^2-v^2}} & -\dfrac{v^2}{c\sqrt{c^2-v^2}} & -\dfrac{v^3}{c\sqrt{c^2-v^2}} & \dfrac{c}{\sqrt{c^2-v^2}} \end{matrix} \right\|, \tag{5.5'}$$

where $k = \frac{1}{v^2}\left[\frac{c}{\sqrt{c^2-v^2}} - 1\right]$; at v = 0, k = 1/2.

The matrix c^i_j is obtained from the matrix d^i_j after the substitution of v^i for $-v^i$.

Based on the equations of the tensor transformation, we have

$$P_i{}^j = P^{*m}_l d^l{}_i c^j{}_m. \tag{5.6}$$

Substituting (5.6) at j = β = 1, 2, 3 into (5.4), we obtain

$$u^i \frac{\partial P_i{}^4}{\partial x^4} + u^i \frac{\partial P^{*m}_l}{\partial x^\beta} d^l_i c^\beta{}_m + u^i P^{*m}_l \frac{\partial d^l{}_i}{\partial x^\beta} c^\beta{}_m + u^i P^{*m}_l d^l{}_i \frac{\partial c^\beta{}_m}{\partial x^\beta} = Q_i u^i. \tag{5.7}$$

For any point of the medium and at any time instant, let us now consider Equation (5.7) in the system K*. In this case, it is clearly necessary to assume:

$$u^\alpha = v^\alpha = 0, \quad u^4 = \frac{1}{c}, \quad d^l{}_i = \delta^l_i, \quad c^\beta{}_m = \delta^\beta_m, \tag{5.8}$$

but it is also necessary to consider that for the matrices $\frac{\partial d^l{}_i}{\partial x^\beta}$ and $\frac{\partial c^l{}_i}{\partial x^\beta}$, the following formulas are valid

$$\left\| \frac{\partial d^l{}_i}{\partial x^\beta} \right\| = \left\| \begin{matrix} 0 & 0 & 0 & -\dfrac{\partial v^1}{\partial x^\beta} \\ 0 & 0 & 0 & -\dfrac{\partial v^2}{\partial x^\beta} \\ 0 & 0 & 0 & -\dfrac{\partial v^3}{\partial x^\beta} \\ -\dfrac{1}{c^2}\dfrac{\partial v^3}{\partial x^\beta} & -\dfrac{1}{c^2}\dfrac{\partial v^2}{\partial x^\beta} & -\dfrac{1}{c^2}\dfrac{\partial v^3}{\partial x^\beta} & 0 \end{matrix} \right\| = -\left\| \frac{\partial c^l{}_i}{\partial x^\beta} \right\|. \tag{5.9}$$

Based on (5.8), and (5.9), Equation (5.7) acquires the form

$$\frac{\partial P_4{}^4}{\partial t}+\frac{\partial P_4^{*\beta}}{\partial x^\beta}-P_\alpha{}^\beta\frac{\partial v^\alpha}{\partial x^\beta}+P_4{}^4\frac{\partial v^\beta}{\partial x^\beta}=Q_4'. \tag{5.10}$$

Since in the point under consideration, $v^\alpha = 0$, from (5.5) and (5.5') there follow the equalities

$$\left(\frac{\partial P_4{}^4}{\partial t}\right)_{x^\alpha=\text{const}}=\left(\frac{\partial P_4{}^4}{\partial t}\right)_{x^{*\alpha}=\text{const}}=\left(\frac{\partial P_4{}^4}{\partial t^*}\right)_{x^{*\alpha}=\text{const}}. \tag{5.11}$$

Let us introduce the density ρ, determined by the equation

$$\frac{d\varrho}{dt^*}+\varrho\frac{\partial v^\beta}{\partial x^\beta}=0, \qquad \text{from which} \qquad \varrho\, d\tau^*=\text{const}, \tag{5.12}$$

where $d\tau^*$ equals the three-dimensional element of volume.

Further, let us assume

$$P_4{}^4=\varrho U, \quad P_\alpha^{*\beta}=-P^{*\alpha\beta}=p^{\alpha\beta}, \tag{5.13}$$

where U equals the intrinsic energy, while $p^{\alpha\beta}$ equals the three-dimensional tensor of the internal stresses, and let us introduce the vector of energy flow

$$\mathbf{Q}=P_4{}^1\mathfrak{Э}_1+P_4{}^2\mathfrak{Э}_2+P_4{}^3\mathfrak{Э}_3. \tag{5.14}$$

Taking into account Equation (5.11) - (5.14), Equation (5.10) can be given the form

$$dU=\frac{p^{\alpha\beta}}{\varrho}\frac{\partial v_\alpha}{\partial x^\beta}d\hat{t}-\frac{1}{\varrho}\operatorname{div}\mathbf{Q}\,d\hat{t}+\frac{1}{\varrho}Q_4'\,d\hat{t}. \tag{5.15}$$

It is obvious that in a spatial Cartesian system of coordinates x^1, x^2, and x^3, the equality $v^\alpha = v_\alpha$ is valid.

Equation (5.15) is an equation of heat flow in the standard thermodynamic form[1]. This equation is applicable in the general case of an irreversible process. For the reversible processes, i.e., for the conservative system, in this equation it is necessary to set $P_i^{\ j} = T_i^{\ j}$ and $Q'_4 = 0$. In the reversible processes, in a general case, div $Q \neq 0$ owing to the internal surface interactions of the adjacent particles of the medium with one another.

Comparing (5.15) with (3.6), we obtain an expression for the total energy flow:

$$dQ^* = dQ^{(e)} + dQ^{**} = -\operatorname{div} Q\, d\hat{t}\, d\tau^* + Q'_4\, d\hat{t}\, d\tau^*. \tag{5.16}$$

In many models, both for the reversible and for the irreversible processes, we can assume that $Q'_4 = 0$. When $Q'_4 = 0$ Equation (5.16) leads to an important conclusion concerning the relation of the external energy flows with the four-dimensional tensor $P_i^{\ j}$. With the aid of this tensor and according to (5.14), we can introduce the three-dimensional vector Q for a material medium and for a field, similar to the Poynting vector. In connection with this result, the four-dimensional treatment of the laws of conservation within the scope of the special theory of relativity indicates the cases when the energy flow dQ^* is attained owing to the flow of the three-dimensional spatial vector Q on the two-dimensional spatial boundary of the medium's particle. The energy flow dQ^{**} is obtained owing to the energy flow through the spatial boundary also in the case when $Q'_4 \neq 0$, but we know from additional data that the energy flow $Q'_4 d\hat{t} d\tau^*$ is a flow of heat energy.

The division of div Q into the flow of heat and nonheat energy is associated with the nature of the internal mechanisms and the properties of the different forms of energy participating in the energy exchange between the adjacent particles.

We will now write the relationships (5.13) in a more explicit form.

From (5.13) and (4.31) the following formula follows for the components $\hat{p}^{\alpha\beta}$ in the attached system of coordinates:

---

[1] We also take special note that the arrangement of the indexes $P_4^{\ 4}$ in Equation (5.13) is significant. If we set $\rho U_1 = P_{44}$ or we take $\frac{\partial P_4^{*4}}{\partial t}$ in place of $\frac{\partial P_4^{\ 4}}{\partial t}$, in (5.7), Equation (5.16) is modified, and in it there appear the additional terms depending on acceleration.

$$\hat{p}^{\alpha\beta} = -\hat{g}^{\alpha l}\hat{p}_l{}^{\beta} = -\hat{g}^{\alpha l}p^{*j}_i \frac{\partial\xi^\beta}{\partial x^j}\frac{\partial x^i}{\partial\xi^l} =$$

$$= -\hat{g}^{\alpha l}\nabla_l\mu^k \frac{\partial\Lambda}{\partial\nabla_j\mu^k}\frac{\partial\xi^\beta}{\partial x^{*j}}\frac{\partial x^{*i}}{\partial\xi^l} + \hat{g}^{\alpha l}\frac{\partial x^{*i}}{\partial\xi^l}\frac{\partial\Lambda}{\partial\frac{\partial x^{*i}}{\partial\xi^\beta}} + \Lambda\hat{g}^{\alpha\beta} + \hat{Q}^{\alpha\beta}. \tag{5.17}$$

In addition, for ρU, we have

$$\rho U = P_4{}^4 = P^{*j}_i \frac{\partial x^{*i}}{\partial t}\frac{\partial t}{\partial x^{*j}}. \tag{5.18}$$

Equations (5.17) and (5.18) permit us to find $\hat{p}^{\alpha\beta}$ and ρU as a function of the control parameters, provided that the function Λ and the tensor $Q_i^{\ j}$ are known.

By way of an example, let us consider the application of Equations (5.17) and (5.18) to the theory of an elastic body, for which we assume

$$\Lambda(\mu^k,\ \hat{E}_{\alpha\beta},\ \overset{\circ}{g}_{ij},\ \xi^i), \tag{5.19}$$

wherein the components of the tensor $\overset{\circ}{g}_{ij}(\xi^1, \xi^2, \xi^3)$ determine the stationary metrics in the system of coordinates being applied. In Equations (5.17) and (5.18), $\overset{\circ}{g}_{ij}, \mu^k$ should be regarded as constant parameters.

For the components of the tensor $\hat{E}_{ij}$ in the attached system of coordinates expressed through the coordinates x^* in a natural system of coordinates, according to (4.37) and (4.33), the simple formula follows:

$$\hat{E}_{\alpha\beta} = \frac{1}{2}\left[\sum_{\gamma=1}^{3} x^{*\gamma}{}_{\alpha} x^{*\gamma}{}_{\beta} + \overset{\circ}{g}_{\alpha\beta} - \frac{\overset{\circ}{g}_{\alpha 4}\overset{\circ}{g}_{\beta 4}}{\overset{\circ}{g}_{44}}\right]. \tag{5.20}$$

The substitution of (5.19) into Equation (5.17) yields

$$\hat{p}^{\alpha\beta} = \hat{g}^{\alpha l}\frac{\partial x^{*i}}{\partial\xi^l}\frac{\partial\Lambda}{\partial\hat{E}_{\lambda\mu}}\frac{\partial\hat{E}_{\lambda\mu}}{\partial\frac{\partial x^{*i}}{\partial\xi^\beta}} + \Lambda\hat{g}^{\alpha\beta} + \hat{Q}^{\alpha\beta};$$

and allowing for (4.37), (4.33) and the fact that

$$\frac{\partial\hat{E}_{\lambda\mu}}{\partial\left(\frac{\partial x^{*i}}{\partial\xi^\beta}\right)} = -\frac{1}{2}\left(g^*_{iq}\delta^\beta_\lambda\frac{\partial x^{*q}}{\partial\xi^\mu} + g^*_{pi}\delta^\beta_\mu\frac{\partial x^{*p}}{\partial\xi^\lambda}\right),$$

we obtain

$$\hat{p}^{\alpha\beta} = -\frac{\partial\Lambda}{\partial\hat{E}_{\alpha\beta}} + \Lambda\hat{g}^{\alpha\beta} + \hat{Q}^{\alpha\beta} = -\varrho\frac{\partial\frac{\Lambda}{\varrho}}{\partial\hat{E}_{\alpha\beta}} + \hat{Q}^{\alpha\beta}. \qquad (5.21)$$

In the transformation in the last equation, we take into consideration the continuity equation which yields $\varrho\sqrt{\gamma_{\alpha\nu}}$ = const or $\varrho\sqrt{|\hat{g}_{ij}|} = \text{const}\cdot\sqrt{\hat{g}_{44}}$ and, consequently, $\frac{\partial\varrho}{\partial g_{\alpha\beta}} = -\frac{1}{2}\frac{\partial\varrho}{\partial\hat{E}_{\alpha\beta}} - \frac{\varrho}{2}\hat{g}^{\alpha\beta}$.

With allowance for the fact that we always have $\hat{E}_{4\alpha} = 0$, by a simple calculation we obtain from (5.18) and (5.19)

$$pU = P_4^{\ 4} = -\Lambda - Q_4^{\ 4}, \qquad \text{or} \qquad \frac{\Lambda}{\varrho} = -U - \frac{1}{\varrho}Q_4^{\ 4}. \qquad (5.22)$$

Equations (5.22) and (5.21) at $Q_i^{\ j} = 0$ represent the classical equations of the nonlinear theory of elasticity.

For the determination of the metrics of the space D_o stationary in an attached system of coordinates, we can determine the components of two tensors of the deformation rates with the components in the attached system by the formulas

$$e_{ij} = \frac{\partial\hat{\varepsilon}_{ij}}{\partial\hat{t}} = -\frac{1}{2}\frac{\partial\hat{g}_{ij}}{\partial\hat{t}} \qquad \text{and} \qquad \dot{\hat{E}}_{ij} = \frac{\partial\hat{E}_{ij}}{\partial\hat{t}} = -\frac{1}{2}\left(\frac{\partial\hat{g}_{ij}}{\partial\hat{t}} - \hat{u}_i\frac{\partial\hat{u}_j}{\partial\hat{t}} - \hat{u}_j\frac{\partial\hat{u}_i}{\partial\hat{t}}\right). \qquad (5.23)$$

On the basis of (4.33), (4.35) and (5.20), we obtain

$$e^*_{\alpha\beta} = \frac{\partial v^*_\alpha}{\partial x^{*\beta}} + \frac{\partial v^*_\beta}{\partial x^{*\alpha}}, \quad e^*_{\alpha 4} = -\frac{\partial v^*_\alpha}{\partial t}, \quad e^*_{44} = 0$$

and

$$\dot{E}^*_{\alpha\beta} = \frac{\partial v^*_\alpha}{\partial x^{*\beta}} + \frac{\partial v^*_\beta}{\partial x^{*\alpha}}, \quad \dot{E}^*_{i4} = 0,$$

where v^*_α gives the components of the three-dimensional velocity in the frame of reference K*. It is obvious that in the case of the arbitrary three-dimensional transformations of the coordinates $x^{*\alpha}$ and in the attached system of coordinates ξ^α, the following formulas are valid

$$\dot{\hat{E}}_{\alpha\beta} = e_{\alpha\beta} = \nabla_\beta \hat{v}_\alpha + \nabla_\alpha \hat{v}_\beta. \tag{5.24}$$

Here the symbols of the covariant derivative ∇ are taken in the spatial three-dimensional sense.

In this manner, we have examined the tensors of the finite deformation and the tensors of the deformation rates within the framework of the special theory of relativity.

In a general case, when

$$\Lambda\left(\mu^k, \frac{\partial\mu^k}{\partial x^j}, E_{ij}, \xi^i\right), \tag{5.25}$$

where the x^j-values are taken in the system K, Equation (5.22) acquires the form

$$\rho U = \frac{\partial\mu^k}{\partial t} \frac{\partial\Lambda}{\partial \frac{\partial\mu^k}{\partial t}} - \Lambda - Q_4^4. \tag{5.26}$$

If the function Λ of (5.25) and $\dot{Q}_4^4$ are assigned, U is determined from (5.26).

The inverse problem of determining Λ as a function of the arguments (5.25), if ρU and Q_4^4 are assigned, reduces to the integration of the simple linear equation with the partial derivatives (5.26).

It is evident that a solution of Equation (5.26) for Λ contains arbitrariness. Let us consider in the examples the direct and inverse problem.

Assume that Λ is represented by a formula of the form

$$\Lambda = \varkappa_{rs}\left(\mu^k, \frac{\partial\mu^k}{\partial x^\alpha}, \frac{\partial\mu^k}{\partial t}\right) \frac{\partial\mu^r}{\partial t} \frac{\partial\mu^s}{\partial t} + \chi_s\left(\mu^k, \frac{\partial\mu^k}{\partial x^\alpha}, \frac{\partial\mu^k}{\partial t}\right) - W\left(\mu^k, \frac{\partial\mu^k}{\partial x^\alpha}\right), \tag{5.27}$$

where we conduct the summation from 1 to n with respect to r, s. Let us assume that the functions κ_{rs} and χ_s equal the arbitrary functions of the first two arguments and the arbitrary homogeneous functions of zero order with respect to the derivatives $\frac{\partial\mu^k}{\partial t}$. For example, at any λ-value, the following equation is fulfilled

$$\chi_s\left(\mu^k, \frac{\partial\mu^k}{\partial x^a}, \lambda\frac{\partial\mu^k}{\partial t}\right) = \chi_s\left(\mu^k, \frac{\partial\mu^k}{\partial x^a}, \frac{\partial\mu^k}{\partial t}\right). \tag{5.28}$$

Substituting (5.27) into (5.26), we obtain

$$\rho U = W + \varkappa_{rs}\frac{\partial\mu^r}{\partial t}\frac{\partial\mu^s}{\partial t} - Q_4^4. \tag{5.29}$$

Let us consider the problem of determining Λ when $\rho U + Q_4^4$ is assigned as a function of the arguments indicated in (5.25). Assume that Λ* equals a certain particular solution of Equation (5.26); then the general solution will be represented in the form

$$\Lambda = \Lambda^* + \chi_s\left(\mu^k, \frac{\partial\mu^k}{\partial x^a}, \frac{\partial\mu^k}{\partial t}\right)\frac{\partial\mu^s}{\partial t}, \tag{5.30}$$

where χ_s (s = 1,...,n) equal arbitrary functions of their own arguments, satisfying the zero homogeneity condition (5.28) of with respect to $\frac{\partial\mu^k}{\partial t}$.

Thus, if on the basis of certain data for a conservative system for $Q_4^4 = 0$, the intrinsic energy U is given as a function of $E_{\alpha\beta}$, μ^k and $\partial\mu^k/\partial x^i$, then the system of equations (4.11) and (4.12) will contain the arbitrary functions χ_s; it is necessary to rely on additional data for the selection of these functions and accordingly for establishing the model.

If in the basic relationship (5.26), the derivatives $\frac{\partial\mu^k}{\partial t}$ cam be replaced by certain functions $g^\sigma = g^\sigma\left(\mu^k, \frac{\partial\mu^k}{\partial t}\right)$ and we can represent Equation (5.26) in the form

$$\rho U + Q_4^4 = \sum_\sigma g^\sigma \frac{\partial\Lambda}{\partial g^\sigma} - \Lambda,$$

after the substitution of $\frac{\partial\mu^k}{\partial t}$ and g^σ, Equations (5.27) - (5.30) are still satisfied in the variables g^σ in place of $\frac{\partial\mu^k}{\partial t}$

Among the parameters of the arguments μ^k in the functions Λ and in the intrinsic energy U, there can be present various tensor or scalar variables (particularly, entropy), which enter only through their values, while their derivatives with respect to the coordinates and time do not affect the values for Λ and U. Such variables can be regarded as constant parameters in (5.26); their presence involves the appearance of additional equations in (4.12). In the calculation of the complete individual increments of the function U, it is necessary to take into account all of the variable arguments in (5.15). In particular, owing to the presence of entropy S among the μ^k-values, there will appear a term of the form

$$\frac{\partial U}{\partial S} dS = T\, dS,$$

which for the reversible processes is balanced with the external heat flow, calculated per unit of mass $\frac{1}{\varrho}\frac{dQ^{(e)}}{dt}$. For the irreversible processes

$$T\, dS\, \varrho\, d\tau^* - dQ^{(e)} = dQ' > 0; \tag{5.31}$$

where dQ' equals the uncompensated heat.

Equation (5.31) expresses the second law of thermodynamics; the value dQ' is associated with the mechanisms of dissipation of mechanical energy.

We note further that if the tensor of the impulse energy or the tensor P_i^j is represented in the form of the sum of several tensors

$$P_i^j = P_{(1)i}^j + P_{(2)i}^j + P_{(3)i}^j + \dots,$$

it is then evident that the Poynting vector Q and the corresponding energy flow $dQ^{(l)}$ + dQ** also can be represented in the form of a sum.

If $Q = P_4^\alpha \partial_\alpha = 0$ and $Q_i d\tau^*$ determines the flow of heat energy, it follows that dQ** = 0. In the examples of the classical reversible models, we have Q = 0 and dQ** = 0. However, in the new complicated examples of the models of material media, and particularly in the case of interaction of the material medium with an electromagnetic field, we had dQ** ≠ 0.

Chapter 6. Pondermotive Forces of the Interaction of an Electromagnetic Field with a Mobile Material Continuum

Let us consider the macroscopic continuous motion of a material medium, interacting with an electromagnetic field. Let us take into account the interaction of a moving and deforming medium with the electromagnetic field, occasioned by the presence, in the medium, of electrical currents and the phenomena of electrical polarization and magnetization of the medium.

For the pondermotive forces acting from the side of the field upon the material medium, various authors propose different equations and only for individual particular cases. The possibility of a different definition of the energy tensor of an impulse of the electromagnetic field and the complexity of the physical problem concerning the property of a material medium are the reasons for the vaguenesses and the difference in the approaches to the treatment of this problem.

For a description of the electromagnetic field in the medium, we introduce the following electromagnetic characteristics:

$$E,\ H,\ D = E + 4\pi P, \quad B = H + 4\pi M, \quad j,\ \rho_e, \tag{6.1}$$

where E, H equal the vectors of the electrical and magnetic field intensity, D and B equal the vectors of the electrical and magnetic induction, P and M equal the vectors of electrical polarization and magnetization, j equals the vector of the density of the electrical field and ρ_e equals the scalar density of the distribution of charges. The enumerated parameters, introduced for the inertial systems of coordinates, satisfy the closed system of Maxwell equations.

As is known, for writing the transformed Maxwell equations in any curvilinear accelerated moving system of coordinates, it is convenient to utilize the tensor form of Maxwell equations written in the four-dimensional form in a pseudo-Euclidean Minkowski space.

In the Cartesian coördinates x^1, x^2, x^3, $x^4 = t$, the Minkowski metric space is connected with the quadratic form

$$ds^2 = -dx^{1^2} - dx^{2^2} - dx^{3^2} + c^2 dt^2. \tag{6.2}$$

As is known, any transformation

$$x^i = x^i (y^1, y^2, y^3, y^4 = t'), \tag{6.3}$$

for which the following equation is fulfilled

$$ds^2 = -dy^{1^2} - dy^{2^2} - dy^{3^2} + c^2 dt'^2, \tag{6.4}$$

is linear and is said to be the Lorentz transformation.

The three-dimensional vectors (6.1) can be determined in any inertial system of coordinates. For the derivation of the formulas of the transformation of these three-dimensional vectors into the four-dimensional Lorentz transformations, it is necessary to introduce two antisymmetrical tensors of the second order, F and H, the components of which in the inertial Cartesian systems are determined by the matrices

$$\left.\begin{aligned} F = \| F_{ij} \| &= \left\| \begin{matrix} 0 & B^3 & -B^2 & cE_1 \\ -B^3 & 0 & B^1 & cE_2 \\ B^2 & -B^1 & 0 & cE_3 \\ -cE_1 & -cE_2 & -cE_3 & 0 \end{matrix} \right\|, \\ H = \| H_{ij} \| &= \left\| \begin{matrix} 0 & H^3 & -H^2 & cD_1 \\ -H^3 & 0 & H^1 & cD_2 \\ H^2 & -H^1 & 0 & cD_3 \\ -cD_1 & -cD_2 & -cD_3 & 0 \end{matrix} \right\|. \end{aligned}\right\} \tag{6.5}$$

The Maxwell equations can be written in the forms

$$\begin{aligned} \operatorname{rot} E &= -\frac{1}{c}\frac{\partial B}{\partial t}, \\ \operatorname{div} B &= 0 \end{aligned} \quad \text{or} \quad \nabla_l F_{ij} + \nabla_j F_{li} + \nabla_i F_{jl} = 0 \tag{6.6}$$

and

$$\text{rot}\, H = \frac{4\pi}{c} j + \frac{1}{c}\frac{\partial D}{\partial t},$$

$$\text{div}\, D = 4\pi\rho_e \qquad \text{or} \qquad \nabla_k H_i^{\ k} = \frac{4\pi}{c} J_i, \qquad (6.7)$$

where $J_1 = j_1$, $J_2 = j_2$, $J_3 = j_3$, $J_4 = \rho_e c^2$ and equals the covariant components of the four-dimensional vector of electrical current.

In the Lorentz transformation and in any transformations of only the spatial coordinates to the antisymmetric tensors F_{ij} and H_{ij}, we can set the vectors E, H and B, D, correspondingly.

The transition from the components of these vectors in the system x^α, t to the analogous components of the vectors in the system y^α, t' is derived from the general rules for the transformation of the tensor components F_{ij} and H_{ij}. In distinction from the vectors E, H, B, D; the tensors F and H, their components F_{ij}, H_{ij} and the tensor Equations (6.6) and (6.7) have meaning for any noninertial system. In this manner, the tensor Equations (6.6) and (6.7) express the invariant physical laws independent of the choice of the system of coordinates, which in the inertial systems of coordinates are represented by the Maxwell equations.

In the noninertial systems of coordinates, for example in the system of coordinates obtained from the given inertial systems with the aid of the Galilean transformation in the Newtonian sense (without the Lorentz reductions of links and time), the transformed components in the matrices (6.5) can also be regarded as certain corresponding vectors E*, H* and B*, D*. However, these can be regarded as the vectors E, H and B, D only in an approximate sense in the case of low velocity of the mobile system.

For the determination of the pondermotive forces, it is necessary to introduce the tensor of the energy-impulse with the components $S_i^{\ j}$ for an electromagnetic field. The general equations for the components of four-dimensional pondermotive force in any system of coordinates have the form

$$F_i = -\nabla_j S_i^{\ j}. \qquad (6.8)$$

The laws of the variation of impulse and energy for a system composed of the field plus the material medium can be represented in the form

$$\nabla_j T_i^j = F_i + Q'_i \qquad \text{or} \qquad \nabla_j (T_i^j + S_i^j) = Q'_i, \tag{6.9}$$

where Q'_i equals the components of the four-dimensional vector of external forces. In many cases we can assume that $Q'_i = 0$. The components of the tensor of the energy-impulse of the medium and of the field, as of one system, are represented by the sum

$$\Sigma_i^j = T_i^j + S_i^j.$$

In a general case, the tensor T_i^j of the medium's energy-impulse characterizes the physical properties and the internal interactions in the medium; this tensor also has an electromagnetic nature since the internal stresses in a material medium are caused either by the collision of the particles or by the direct interaction of the atoms and molecules at distances which are large compared with the dimensions of the medium elementary particles. As is known, in both cases these microscopic interactions have an electromagentic nature. According to the appropriate definitions of the model of a continuous medium, the components T_i^j are connected with the metric tensor, with the vector of the four-dimensional velocity of the medium's points, with the thermodynamic functions of state, and with the characteristics of the dissipated mechanisms in the medium[1].

The division of the general tensor Σ_i^j of the energy-impulse into the sum $T_i^j + S_i^j$ for a material medium and a field is associated directly with the separation of the total electromagnetic force acting upon the conceptually-separated particle of the medium, on the mass force and on the surface force. The internal surface stresses in a medium are determined by the components of the tensor T_α^β ($p^{\alpha\beta} = - T^{\alpha\beta}$), while the mass electromagnetic forces are determined by the vector components $F_\alpha = - \nabla_j S_\alpha^j$.

---

[1]The tensor T and its components T_i^j can be regarded as functions of the constant and variable tensor and scalar parameters which determine the structure, physical state and internal processes for infinitely small particles.

It is evident that in an unequivocal determination of the tensor $\mathfrak{T}_i^j$, which is physically signif cant, the tensors T_i^j and S_i^j can nevertheless be determined differently and this is essentially associated with the various methods of dividing one electromagnetic system into two interacting electromagnetic systems.

It is significant that after the selection of S_i^j for the field the tensor T_i^j for a material medium should be determined by a standard method with consideration of the selection of S_i^j.

On the basis of what has been said, it is evident that we can establish the tensor S_i^j with a known arbitrariness; this fact served as a basis for numerous discussions and for the derivation by various authors of different equations for the pondermotive forces, wherein this question has often been regarded quite independently of the selection of the tensor T_i^j for a material medium.

Let us consider below the formulas for the pondermotive forces when the tensor S_i^j in any system of coordinates is determined according to the Minkowski tensor equation:

$$S_i^j = -\frac{1}{4\pi}\left[F_{il}H^{lj} - \frac{1}{4}\delta_i^j F_{lm}H^{lm}\right]. \tag{6.10}$$

In a general case, the Minkowski tensor is nonsymmetrical, i.e.

$$S_{ij} \neq S_{ji}.$$

Utilizing Equation (6.10) and the conditions of antisymmetry for F_{ij} and H_{ij}, based on Equations (6.6) and (6.7), we obtain[1].

$$F_i = \frac{1}{c} F_{il} J^l = \frac{1}{16\pi}\left[F_{lj}\nabla_i H^{lj} - H^{lj}\nabla_i F_{lj}\right]. \tag{6.11}$$

The tensor Equations (6.6) and (6.7) and Formulas (6.10) and (6.11) are valid in any mobile and in general curvilinear system of coordinates.

Along with the tensors F and H, we can also introduce the antisymmetrical tensor P, determined by the equalities

[1]In the derivation of (6.11), it was considered that on the basis of (6.6),

$$H^{lj}\nabla_j F_{il} = \frac{1}{2} H^{lj}\nabla_i F_{lj}.$$

$$P = \frac{1}{4\pi}(F - H), \qquad \|P_{ij}\| = \left\| \begin{matrix} 0 & M^3 & -M^2 & -cP_1 \\ -M^3 & 0 & M^1 & -cP_2 \\ M^2 & -M^1 & 0 & -cP_3 \\ cP_1 & cP_2 & cP_3 & 0 \end{matrix} \right\|. \tag{6.12}$$

In an inertial system of coordinates, P is formed with the aid of the three-dimensional vectors of electrical polarization $P(P_1, P_2, P_3)$ and magnetization $M(M^1, M^2, M^3)$. With the aid of the tensor P, Equation (6.11) can be written in the following form:

$$F_i = \frac{1}{c} F_{il} J^l + \frac{1}{4} [F_{lj} \nabla_i P^{lj} - P^{lj} \nabla_l F_{ij}]; \tag{6.13}$$

where the first term in Equations (6.11) and (6.13) determines the Lorentz force; the second term becomes zero in the absence of polarization and magnetization; and the symbols ∇_i denote the covariant four-dimensional derivatives.

If the system of reference is inertial, we can introduce a system of three-dimensional vectors (6.1) in connection with the tensors F and P. In an inertial system of coordinates, Equation (6.13) can be rewritten in the form

$$F^\alpha = \varrho_e E_\alpha + \frac{1}{c}[\mathbf{j}, \mathbf{B}]_\alpha + \frac{1}{2}\left[\mathbf{P}\frac{\partial \mathbf{E}}{\partial x^\alpha} - \mathbf{E}\frac{\partial \mathbf{P}}{\partial x^\alpha} + \mathbf{M}\frac{\partial \mathbf{B}}{\partial x^\alpha} - \mathbf{B}\frac{\partial \mathbf{M}}{\partial x^\alpha}\right], \tag{6.14}$$

where the four-dimensional anti-variant components of the force F^α correspond to the spatial three-dimensional covariant force components. Equation (6.14) preserves its form during the transition from the Cartesian to the curvilinear spatial system of coordinates.

In the inertial system of coordinates, using Equation (6.13) with i = 4, we derive[1]:

$$F^4 c^2 = F_4 = (\mathbf{E}, \mathbf{j}) + E_\beta \frac{\partial P^\beta}{\partial t} + B_\beta \frac{\partial M^\beta}{\partial t} - \frac{\partial}{\partial t}\left(\frac{E_\beta P^\beta + B_\beta M^\beta}{2}\right). \tag{6.15}$$

---

[1]Equation (6.15) is obtained as the result of the vector equality for the Poynting vector $\Pi = S_4^{\alpha} \vartheta_\alpha = \frac{c}{4\pi}[\mathbf{E}, \mathbf{H}]$, which is valid for the various definitions of the tensors of the energy-impulse of a field, particularly both for the Minkowski and the Abraham definitions.

Equations (6.14) and (6.15) are directly suitable for determining the pondermotive forces through a system of vectors (6.1) when the material medium is quiescent or is in a state of inertial translational motion. In the latter case, if the vectors E, P, B and M are determined in the inertial frame of reference K associated with the body in Equation (6.15), the term (E, j) yields a Joule heat, the term $E_\beta \frac{\partial P^\beta}{\partial t} + B_\beta \frac{\partial M^\beta}{\partial t}$ can be regarded as a macroscopic flow of energy from the field towards the body owing to the microscopic mechanisms of polarization and magnetization, while the value $\frac{1}{2}(E_\beta P^\beta + B_\beta M^\beta)$ can conveniently be included in the intrinsic energy of the material medium.

It is easy to observe that in Equation (6.14) and in the second term in (6.15), we can replace everywhere the components of the vector B by the components of the vector H.

If the body moves at an accelerated rate and becomes deformed, we can use Equation (6.13) which is applicable in any system of coordinates, specifically in an attached mobile Lagrangian system of coordinates L, in which the three-dimensional volocities of all points of the medium always equal zero.

In a number of cases, the components of the energy tensor of the impulse of a material medium can conveniently be prescribed and considered in the attached system of coordinates L, whereas the components of the tensor $S_i^{\ j}$ of the impulse energy of an electromagnetic field can conveniently be prescribed in the inertial system of coordinates K.

In the application of the natural system K, we can introduce the three-dimensional vectors of the characteristics of an electromagnetic field and the pertinent Maxwell equations in a vector form. At the same time, in each point, the three-dimensional vectors introduced for the natural system K in this point can be regarded in the spatial coordinates of the attached system of coordinates L. In this manner, the introduction of the system K can be regarded as an additional method for determining the usual vector characteristics of an electromagnetic field. If for an electromagnetic field, we can restrict ourselves to the tensors F, H, P and S, we can consider all of the tensors only in the attached system of coordinates. In this case, the introduction of the inertial system K may be necessary for determining the coordinates of the tensor $\hat{g}_{ij}$ [Equation (4.33)] and of the vector of four-dimensional velocity. Generally speaking, one or the other is necessary for determining the tensors of the impulse energy of an electromagnetic field and of a material medium.

For the pondermotive forces and for the energy influx, in a general case of the motion of a medium which is being deformed, we can utilize Equations (6.14) and (6.15) in which the vector components E^{α}, B^{α}, P^{α}, M^{α} are determined in a spatial system of coordinates of the inertial system K. In Equations (6.14) and (6.15), the coordinate axes in the system K can be regarded as a curvilinear. Equation (6.14) preserves its form when using the vectors $\hat{E}^{\alpha}\hat{\partial}_{\alpha}$, $\hat{P}^{\alpha}\hat{\partial}_{\alpha}$, $\hat{B}^{\alpha}\hat{\partial}_{\alpha}$ and $\hat{M}^{\alpha}\hat{\partial}_{\alpha}$, in the attached system of coordinates L[1].

If we introduce $\hat{E}^{\alpha}$, $\hat{P}^{\alpha}$, $\hat{B}^{\alpha}$ and $\hat{M}^{\alpha}$ into Equation (6.15), it is necessary to allow for the equation

$$\left.\begin{aligned}\left(\frac{\partial P^{\alpha}}{\partial t}\right)_{x^{\alpha}} &= \left(\frac{\partial \hat{P}^{\alpha}}{\partial \hat{t}}\right)_{\xi^{\alpha}} + \hat{P}^{\beta}\left(\hat{e}^{\alpha}{}_{\beta} + \hat{\omega}^{\alpha}{}_{\beta}\right),\\ \left(\frac{\partial M^{\alpha}}{\partial t}\right)_{x^{\alpha}} &= \left(\frac{\partial \hat{M}^{\alpha}}{\partial \hat{t}}\right)_{\xi^{\alpha}} + \hat{M}^{\beta}\left(\hat{e}^{\alpha}{}_{\beta} + \hat{\omega}^{\alpha}{}_{\beta}\right).\end{aligned}\right\} \tag{6.16}$$

Here $\hat{e}_{\alpha\beta}$ and $\hat{\omega}_{\alpha\beta} = 1/2\left(\frac{\partial \hat{v}_{\alpha}}{\partial \xi^{\beta}} - \frac{\partial \hat{v}_{\beta}}{\partial \xi^{\alpha}}\right)$ equal the components of the three-dimensional tensors of rates of deformation and eddy, determined for a three-dimensional velocity vector v at the points of the attached system L relative to the system K.

On the basis of (6.16), Equation (6.15) acquires the form

$$F_{i} = -\nabla_{j}S_{i}{}^{j} = (\mathrm{E}, \mathrm{j}) + \hat{E}_{\beta}\frac{\partial \hat{P}^{\beta}}{\partial \hat{t}} + \hat{B}_{\beta}\frac{\partial \hat{M}^{\beta}}{\partial \hat{t}} + (\hat{E}^{\alpha}\hat{P}^{\beta} + \hat{B}^{\alpha}\hat{M}^{\beta})(\hat{e}_{\alpha\beta} + \hat{\omega}_{\alpha\beta}) - \frac{\partial}{\partial \hat{t}}\left(\frac{\hat{E}_{\beta}\hat{P}^{\beta} + \hat{B}_{\beta}\hat{M}^{\beta}}{2}\right). \tag{6.17}$$

The scalar equation of energy for a system from a material medium and for the field in any system of coordinates can be written in the form

$$u^{i}\nabla_{j}T_{i}{}^{j} = -u^{i}\nabla_{j}S_{i}{}^{j} + u^{i}Q_{i}. \tag{6.18}$$

In order to modify Equation (5.18), if we assume

$$\varrho U = T_{4}{}^{4} + \frac{1}{2}(E_{\beta}P^{\beta} + B_{\beta}M^{\beta}), \tag{6.19}$$

[1]This conclusion follows from the equations of the transformation of the vectors (6.1) in the introduction of the systems K in each point of the medium, Equation (6.14) are preserved during the transformation of the form $\eta'^{\alpha} = \eta^{\alpha}(\xi^{1}, \xi^{2}, \xi^{3})$ and $\hat{t}' = \hat{t}$.

in the case under consideration, the energy Equation (5.15) can be written in the form

$$\frac{\partial U}{\partial t} = \Big[\frac{\hat{p}^{\alpha\beta}}{\rho} - \frac{1}{2} (\hat{E}_\gamma \hat{\pi}^\gamma + \hat{B}_\gamma \hat{m}^\gamma) \hat{g}^{\alpha\beta} + \hat{E}^\alpha \hat{\pi}^\beta + \hat{B}^\alpha \hat{m}^\beta \Big] \nabla_\beta \hat{v}_\alpha + \\ + \hat{E}_\beta \frac{\partial \hat{\pi}^\beta}{\partial \hat{t}} + \hat{B}_\beta \frac{\partial \hat{m}^\beta}{\partial \hat{t}} + \frac{1}{\rho} (\mathrm{E}, \mathrm{j}) - \frac{1}{\rho} \operatorname{div} \mathrm{Q} + \frac{1}{\rho} Q'_4. \tag{6.20}$$

Here we use the notation $\pi^\alpha = \frac{p^\alpha}{\rho}$, $m^\alpha = \frac{M^\alpha}{\rho}$ and $\mathrm{Q} = T_4{}^\alpha \hat{\partial}_\alpha$. In Equations (6.19) and (6.20) and in the initial equations of conservation for models with irreversible processes in a material medium (separated from an electromagnetic field) presented in this chapter we may either define the tensor components $T_i{}^j$ or tensor components $P_i{}^j$ introduced for the material medium in the preceding chapter, or we may introduce additional terms to the correspondingly modified value Q'_4, preserving the definition of $T_i{}^j$ in (4.22).

Equation (6.20) can be regarded as Equation (5.15), in which we take into account the terms determining the interaction of a material medium with the electromagnetic field, represented explicitly in Equation (6.20) by the terms containing the vector components E and B, which in the general Equation (5.15) can be regarded as included in the overall external specific heat flow Q'_4.

The right-hand part in the heat flow Equation (6.20) is written in an attached system of coordinates; the value U, defined by Equation (6.19), can be regarded as the specific internal energy per unit of mass of the quiescence of the material medium. The value U, just like the specific entropy S, the absolute temperature T and $dm_0 = \rho d\tau_0^*$, can be regarded as a scalar value.

Along with the value U, it is convenient to use the specific free energy F determined by the equation[1]

$$F = U - TS.$$

With the aid of the function F, Equation (6.20) can be rewritten in the form

---

[1]In the following, we shall consider the reversible processes or only such irreversible processes where the concepts of temperature and free energy are meaningful.

$$(dF)_{\xi^a} = -S\,dT + \left[\frac{\hat{p}^{\alpha\beta}}{\varrho} - \frac{1}{2} (\hat{E}_\gamma \hat{\pi}^\gamma + \hat{B}_\gamma \hat{m}^\gamma) \hat{g}^{\alpha\beta} + \hat{E}^\alpha \hat{\pi}^\beta + \hat{B}^\alpha \hat{m}^\beta \right] (\hat{e}_{\alpha\beta} + \hat{\omega}_{\alpha\beta})\, d\hat{t} +$$
$$+ \hat{E}_\alpha d\hat{\pi}^\alpha + \hat{B}_\alpha d\hat{m}^\alpha + \frac{1}{\varrho} (E, j)\, d\hat{t} - \frac{1}{\varrho} \operatorname{div} Q\, d\hat{t} + \frac{1}{\varrho} Q_4'\, d\hat{t} - T\,dS. \tag{6.21}$$

Further, we shall use the three-dimensional treatment of Equation (6.21); all of the values entering this heat flow equation will be regarded as three-dimensional scalars, vectors and tensors.

The vectors E and j are taken in the natural system of coordinates K, therefore the energy flow $\frac{1}{\varrho}(E, j)$ represent Joule heat.

The energy flow $-\frac{1}{\varrho} \operatorname{div} Q\, d\hat{t}$ can be represented in the form of the sum of the flow of heat energy and the nonheat energy; this inflow is expressed through the flow of the vector $Q\,d\hat{t} = \hat{T}_4^{\ \alpha} \partial_\alpha\, d\hat{t}$ at the boundary of a small particle. It is obvious that the vector Q, just like the components $T_4^{\ 4}$, can depend only on the same control parameters as the tensor of impulse energy $T_i^{\ j}$.

The energy inflow independent of the tensor of impulse energy, for instance owing to the inflow of radiant energy, will be contained in the term $\frac{1}{\varrho} Q_4'\, d\hat{t}$.

Equation (6.21) is satisfied for all possible processes in the medium occurring under the effect of arbitrary external forces in case of arbitrary changes in the control parameters. Thus, Equation (6.21) can be used as the basis of the conclusions of the equations of state and of the kinetic equations being fulfilled during the arbitrary processes. These physical relationships can be derived when the free energy F and the entropy increments ds = $d_e s + d_i s$ are given as functions of the control parameters. ($d_e s$ is the entropy flow through the boundary surface of the volume of a small particle).

Let us consider Equation (6.21) under the assumption that the free energy F can be regarded as a function of the following parameters[1]:

[1]Further, the components of all vectors and tensors are taken in an attached system of coordinates. For the sake of simplification, we will drop the symbol "^" at the top.

The further discussions and the formulas are simplified if in place of the system of control parameters (6.22), we take the system T, $\mathring{g}_{ij}$, g_{ij}, π^α, m^α, $\overset{0}{\nabla}_\beta \pi^\alpha$, $\overset{0}{\nabla}_\beta m^\alpha$, $\overset{0}{\nabla}_\gamma g_{\alpha\beta}$ ($\overset{0}{\nabla}$ equals the symbol of the covariant derivative in the space of the initial states).

In the following, we will not consider the case of saturated magnetization, where |m| = constant.

$$T,\ \overset{0}{g}_{\alpha\beta},\ \hat{g}_{\alpha\beta},\ \hat{\pi}^{\alpha},\ \hat{m}^{\alpha},\ \hat{\nabla}_{\beta}\hat{\pi}^{\alpha},\ \hat{\nabla}_{\beta}\hat{m}^{\alpha},\ \frac{\partial\hat{g}_{\alpha\gamma}}{\partial\xi^{\beta}},\ \frac{\partial\overset{0}{g}_{\alpha\gamma}}{\partial\xi^{\beta}},\tag{6.22}$$

where $\overset{0}{g}_{\alpha\beta}$ is the three-dimensional metric tensor of a certain initial state. Since $\hat{\nabla}_{\gamma}\hat{g}_{\alpha\beta}=0$, and $\hat{\nabla}_{\beta}\overset{0}{g}_{\alpha\gamma}=\frac{\partial\overset{0}{g}_{\alpha\gamma}}{\partial\xi^{\beta}}-\overset{0}{g}_{\alpha\lambda}\hat{\Gamma}^{\lambda}_{\beta\gamma}-\overset{0}{g}_{\lambda\gamma}\hat{\Gamma}^{\lambda}_{\alpha\beta}$, the functions F in the arguments can be indicated as the time-related variables $\frac{\partial\hat{g}_{\alpha\gamma}}{\partial\xi^{\beta}}$ and as the constants $\overset{0}{g}_{\alpha\beta}$ and $\frac{\partial\overset{0}{g}_{\alpha\gamma}}{\partial\xi^{\beta}}$ (system [6.22] can be supplemented by other parameters and may include certain derivatives with respect to time; in these more general cases, the development of a complicated subsequent theory is also possible).

Let us further assume that

$$\hat{Q}^{\beta}\,d\hat{t}=\hat{R}_{\alpha}{}^{\beta}d\hat{\pi}^{\alpha}+\hat{N}_{\alpha}{}^{\beta}d\hat{m}^{\alpha}+\hat{\Lambda}^{\beta\alpha\gamma}\,d\hat{g}_{\alpha\gamma}+\hat{\Omega}^{\beta}\,d\hat{t},\tag{6.23}$$

where the coefficients $\hat{R}_{\alpha}{}^{\beta}$, $\hat{N}_{\alpha}{}^{\beta}$, $\hat{\Lambda}^{\beta\alpha\gamma}$ and $\hat{\Omega}^{\beta}$ depend on the parameters (6.22), and in a general case upon certain other values.

It is easy to verify the validity of the equality

$$d\nabla_{\beta}\pi^{\alpha}=\nabla_{\beta}\,d\pi^{\alpha}+\pi^{\gamma}\,d\Gamma^{\alpha}_{\gamma\beta},$$

where

and

$$\left.\begin{aligned} d\Gamma^{\alpha}_{\gamma\beta}&=-\Gamma^{\lambda}_{\gamma\beta}g^{\alpha\mu}\,dg_{\lambda\mu}+\frac{1}{2}g^{\alpha\mu}\left(d\frac{\partial g_{\gamma\mu}}{\partial\xi^{\beta}}+d\frac{\partial g_{\beta\mu}}{\partial\xi^{\gamma}}-d\frac{\partial g_{\gamma\beta}}{\partial\xi^{\mu}}\right)\\ \nabla_{\beta}\,dg_{\alpha\gamma}&=d\frac{\partial g_{\alpha\gamma}}{\partial\xi^{\beta}}-dg_{\alpha\mu}\Gamma^{\mu}_{\gamma\beta}-dg_{\mu\gamma}\Gamma^{\mu}_{\alpha\beta}.\end{aligned}\right\}\tag{6.24}$$

On the basis of (6.22) and (6.24), Equation (6.21) can be written in the form

$$\begin{aligned}&\varphi\,dT+2\Psi^{\alpha\beta}\,dg_{\alpha\beta}+\Omega^{\alpha\beta}\omega_{\alpha\beta}\,dt+\chi_{\alpha}\,d\pi^{\alpha}+\varkappa_{\alpha}\,dm^{\alpha}+\Theta_{\alpha}{}^{\beta}\,d\nabla_{\beta}\pi^{\alpha}+\Xi_{\alpha}{}^{\beta}\,d\nabla_{\beta}m^{\alpha}+\\&+\Phi^{\beta\alpha\gamma}\,d\frac{\partial g_{\alpha\gamma}}{\partial\xi^{\beta}}-\frac{1}{\varrho}\nabla_{\beta}\Omega^{\beta}\,dt+\frac{1}{\varrho}(\mathrm{E},\ \mathrm{j})\,dt+\frac{Q_1'}{\varrho}\,dt-T\,dS=0,\end{aligned}\tag{6.25}$$

where φ, $\Psi^{\alpha\beta}$, $\Omega^{\alpha\beta}$, χ_{α}, $\varkappa_{\alpha}$, $\Theta_{\alpha}{}^{\beta}$, $\Xi_{\alpha}{}^{\beta}$ and $\Phi^{\beta\alpha\gamma}$ are determined by the equations:

$$
\left.
\begin{aligned}
-S &= \frac{\partial F}{\partial T} + \varphi \\
\frac{p^{\alpha\beta} + p^{\beta\alpha}}{2} &= 2\varrho \frac{\partial F}{\partial g_{\alpha\beta}} + \frac{1}{2}(E_\gamma P^\gamma + B_\gamma M^\gamma) g^{\alpha\beta} - \\
&- \frac{1}{2}(E^\alpha P^\beta + E^\beta P^\alpha + B^\alpha M^\beta + B^\beta M^\alpha) - \\
&- \frac{2}{\sqrt{g}} \frac{\partial \left[\varrho \sqrt{g} \dfrac{\partial F}{\partial \frac{\partial g_{\alpha\beta}}{\partial \xi^\lambda}} \right]}{\partial \xi^\lambda} + \frac{1}{2} \nabla_\lambda [(N^{\beta\lambda} - N^{\lambda\beta}) m^\alpha + \\
&+ (N^{\alpha\lambda} - N^{\lambda\alpha}) m^\beta + (N^{\beta\alpha} + N^{\alpha\beta}) m^\lambda] + \\
&+ \frac{1}{2} \nabla_\lambda [(R^{\beta\lambda} - R^{\lambda\beta}) \pi^\alpha + (R^{\alpha\lambda} - R^{\lambda\alpha}) \pi^\beta + \\
&+ (R^{\beta\alpha} + R^{\alpha\beta}) \pi^\lambda] - \frac{2}{\sqrt{g}} \frac{\partial [\varrho \sqrt{g}\, \Phi^{\lambda\alpha\beta}]}{\partial \xi^\lambda} + \varrho \Psi^{\alpha\beta}, \\
p^{\alpha\beta} - p^{\beta\alpha} &+ E^\alpha P^\beta - E^\beta P^\alpha + B^\alpha M^\beta - B^\beta M^\alpha = 2\varrho \Omega^{\alpha\beta}, \\
E_\alpha &= \frac{\partial F}{\partial \pi^\alpha} + \frac{1}{\varrho} \nabla_\beta R_\alpha{}^\beta + \chi_\alpha, \quad R_\alpha{}^\beta = -\varrho \frac{\partial F}{\partial \nabla_\beta \pi^\alpha} + \varrho \Theta_\alpha{}^\beta, \\
B_\alpha &= \frac{\partial F}{\partial m^\alpha} + \frac{1}{\varrho} \nabla_\beta N_\alpha{}^\beta + \varkappa_\alpha, \quad N_\alpha{}^\beta = -\varrho \frac{\partial F}{\partial \nabla_\beta m^\alpha} + \varrho \Xi_\alpha{}^\beta, \\
\Lambda^{\beta\alpha\gamma} &= -\varrho \frac{\partial F}{\partial \frac{\partial g_{\alpha\gamma}}{\partial \xi^\beta}} + \frac{1}{4} [(R^{\gamma\beta} - R^{\beta\gamma}) \pi^\alpha + \\
&+ (R^{\alpha\beta} - R^{\beta\alpha}) \pi^\gamma + (R^{\gamma\alpha} + R^{\alpha\gamma}) \pi^\beta] + \\
&+ \frac{1}{4} [(N^{\gamma\beta} - N^{\beta\gamma}) m^\alpha + (N^{\alpha\beta} - N^{\beta\alpha}) m^\gamma + \\
&\qquad + (N^{\gamma\alpha} + N^{\alpha\gamma}) m^\beta] \cdot \varrho \Phi^{\beta\alpha\gamma}.
\end{aligned}
\right\} \qquad (6.26)
$$

The tensor $\Psi^{\alpha\beta}$ is symmetrical while the tensor $\Omega^{\alpha\beta}$ is antisymmetrical. The components $\Lambda^{\beta\alpha\gamma}$ and $\Phi^{\beta\alpha\gamma}$ are symmetrical with respect to the two last indeces.

If we assume that the energy inflows $-\frac{1}{\varrho} \nabla_\beta \Omega^\beta \, dt$ and $\frac{1}{\varrho} Q_4' \, dt$ correspond to the heat energy inflow in the case of the reversible and certain irreversible processes (for example, in the consideration of the heat conductivity and radiation), the following equation will be fulfilled:

$$
T\, dS = \frac{1}{\varrho}(\mathrm{E}, \mathrm{j}) + \frac{1}{\varrho} Q_4' \, dt - \frac{1}{\varrho} \nabla_\beta \Omega^\beta \, dt = dq^{(e)}. \qquad (6.27)
$$

If we assume in addition to this that the values φ, $\Psi^{\alpha\beta}$ $\Omega^{\alpha\beta}$, χ_α, $\varkappa_\alpha$, $\Theta_\alpha{}^\beta$, $\Xi_\alpha{}^\beta$ and $\Phi^{\beta\alpha\gamma}$ determined by Equation (6.26) are independent of the time derivatives[1] of the control parameters (6.22), then using Equation (6.25) and the assumption concerning the linear independence of increments of the control parameters with respect to time[2], we obtain

$$\varphi = \Psi^{\alpha\beta} = \Omega^{\alpha\beta} = \chi_\alpha = \varkappa_\alpha = \Theta_\alpha{}^\beta = \Xi_\alpha{}^\beta = \Phi^{\beta\alpha\gamma} = 0. \qquad (6.28)$$

In this manner, based on (6.27) and (6.28), we find that the Equation (6.26) determine the equations of state for a material medium. These equations comprise a generalization of the standard equations in the theory of elasticity for the case where the free energy depends on the gradients of the polarization vector, the magnetization vector and the gradients of the tensor of deformations.

If F depends only on T, $\overset{0}{g}_{\alpha\beta}$, $g_{\alpha\beta}$, π^α and m^α, and does not depend on their gradients, it follows that

$$R_\alpha{}^\beta = N_\alpha{}^\beta = \Lambda^{\beta\gamma\alpha} = 0;$$

and in this case, the vector components Qdt, $Q^\alpha dt = \Omega^\alpha dt$ determine the inflow of heat, while Equation (6.26) convert to the equations of state of the theory of elasticity with allowance for electrical polarization and magnetization intensity.

The further complication of the models of a continuous medium with allowance for the electromagnetic effects in the cases of reversible and irreversible processes can be connected with the consideration of the dependence of the factors ϕ, $\Psi^{\alpha\beta}$, $\Omega^{\alpha\beta}$, χ_α, $\varkappa_\alpha$, $\Theta_\alpha{}^\beta$, $\Xi_\alpha{}^\beta$ and $\Phi^{\beta\alpha\gamma}$ in Equation (6.25) on the time derivatives of the control parameters, with allowance for the linear dependence of these derivatives (the nonholonomoic state is physical), and also with the introduction into Equation (6.27) of uncompensated heat in the case of the irreversible processes. Certain concepts along these lines can be found in [230], [162].

[1]What is only significant is the assumption concerning the independence of the time derivatives. The hypotheses concerning the dependence or independence of the coefficients of any space derivatives are not necessary.

[2]It is possible to construct models in which the time derivatives of the control parameters can be linearly dependent [162].

LISTING OF CERTAIN REPORTS PERTAINING TO THE GENERAL THEORY OF MODELS OF CONTINUOUS MEDIA

In the reports given in this list, one can find more complete references to the pertinent literature.

This list was compiled with reference to the indicated branches of mechanics. In certain reports, problems from various branches are considered; nevertheless they are cited only once in this list. The author recognizes that the listing is quite incomplete and imperfect.

I. NONLINEAR TENSOR FUNCTIONS AND GENERAL GEOMETRIC AND DYNAMIC RELATIONSHIPS OF THE MECHANICS OF CONTINUOUS MEDIA

[1] Adkins, J.E., Dynamic Properties of Resilient Materials: Constitutive Equations. Phil. Trans. Roy. Soc., A250, 519-541, 1958.

[2] Adkins, J.E., Symmetry Relations for Orthotropic and Transversely Isotropic Materials. Arch. Rational Mech. Anal. 4, No. 3, 193-213, 1960.

[3] Adkins, J.E., Further Symmetry Relations for Transversely Isotropic Materials. Arch. Rational Mech. Anal. 5, 263, 1960.

[4] Coleman, B.D., Kinematical Concepts with Applications in the Mechanics and Thermodynamics of Incompressible Viscoelastic Fluids. Arch. Rational Mech. Anal. 9, No. 4, 273-300, 1962.

[5] Coleman, B.D., Substantially Stagnant Motions. Trans. Soc. Rheol. 6, 293-300, New York - London, Interscience. 1962.

[6] Green, A.E., Anistropic Simple Fluids. Proc. Roy. Soc., A250, No.1379, 437-445, 1964.

[7] Green, A.E., A Continuum Theory of Anisotropic Fluids. Proc. Camb. Phil. Soc. 60, No. 1, 123-128, 1964.

[8] Green, A.E., and Rivlin, R.S., The Mechanics of Non-linear Materials with Memory, Part 1. Arch. Rational Mech. Anal. 1, No. 1, 1-81, 1957.

[9] Green, A.E., and Rivlin, R.S., The Mechanics of Non-Linear Materials with Memory, Part 3, Arch. Rational Mech. Anal. 4, No. 5, 387-404, 1960.

[10] Green, A.E., Rivlin, R.S., and Spencer, A.J.M., The Mechanics of Non-Linear Materials with Memory, Part 2, Arch. Rational Mech. Anal. 3, No. 1, 82-90, 1959.

[11] Lokhin, V.V., and Sedov, L.I. Nonlinear Tensor Functions of Several Tensor Arguments, Translation of Prikl. Mat. Mech. 27, 3, 393-417, by Pergamon Press, 122 East 55th Street, New York 22, N.Y. 1963.

[12] Lyubarskiy, G.Ya., Teoriya grupp i yeye primeneniye v fizike (Theory of Groups and Its Application in Physics), Gostekhizdat, Moscow, 1957.

[13] Pipkin, A.C., and Rivlin, R.S., The Formulation of Constitutive Equations in Continuum Physics, Part 1, Arch. Rational Mech. Anal. 4, No. 2, 129-144, 1959.

[14] Pipkin, A.C., and Rivlin, R.S., Electrical Conduction in Deformed Isotropic Materials. Journ. Math. Phys. 1, 127, 1960.

[15] Pipkin, A.C., and Rivlin, R.S., Galvanomagnetic and Thermomagnetic Effects in Isotropic Materials. Journ. Math. Phys. 1, 542, 1960.

[16] Pipkin, A.C., and Wineman, A.S., Material Symmetry Restrictions on Non-polynomial Constitutive Equations, Arch. Rational Mech. Anal. 12, No. 5, 420-426, 1963.

[17] Rivlin, R.S., The Constitutive Equations for Certain Classes of Deformations, Arch. Rational Mech. Anal. 3, No. 4, 304-311, 1959.

[18] Rivlin, R.S., The Formulation of Constitutive Equations in Continuum Physics, Part 2, Arch. Rational Mech. Anal. 4, No. 3, 262-272,1960.

[19] Rivlin, R.S., The Constitutive Equations for Certain Classes of Deformations, "Viscoelasticity: Phenomenoligical Aspects", Acad. Press, Inc., New York, 93-108, 1960.

[20] Rivlin, R.S., The Constitutive Equations Involving Functional Dependence on one Vector on Another. Journ. Appl. Math. Phys. (ZAMP), No. 5, 447-452, 1961.

[21] Sedov, L.I., and Lokhin, V.V., Description with Aid of Tensors of Point Groups of Symmetry, Doklady Akademii Nauk SSSR (Proceedings of USSR Academy of Sciences), Vol. 149, No. 4, pp. 796-797, 1963.

[22] Sirotin, Yu. I., Anisotropic Tensors, Dokl. Akad. Nauk SSSR, Vol. 133, No. 2., pp. 321-324, 1960.

[23] Sirotin, Yu. I., Group Tensor Spaces, Kristallografiya (Crystallography) Vol. 5, No. 2, pp. 171-179, 1960.

[24] Sirotin, Yu. I., Construction of Tensors of Prescribed Symmetry, Kristallografiya, Vol. 6, No. 3, pp. 331-340, 1961.

[25] Sirotin, Yu. I., Integrity Rational Bases of Tensor Invariants of Crystallographic Groups, Dokl. Akad. Nauk SSSR, Vol. 151, No. 3, pp. 564-566, 1963.

[26] Sirotin, Yu. I., Tensor Functions of a Polar and Axial Vector Compatible with Symmetry of Textures, Prikladnaya Matematika i Mekhanika (Applied Mathematics and Mechanics) Vol. 28, No. 4, pp.653-663, 1964.

[27] Smith, G.F., On the Minimality of Integrity Bases for Symmetric 3 X 3 Matrices, Arch. Rational Mech. Anal. 5, No. 3, 263-274, 1960.

[28] Smith, G.F., Further Results on the Strain-energy Function for Anisotropic Elastic Materials, Arch. Rational Mech. Anal. 10, No. 2, 108-118, 1962.

[29] Smith, G.F., Smith, M.M., and Rivlin, R.S., Integrity Bases for a Symmetric Tensor and a Vector. The Crystal Classes. Arch. Rational Mech. Anal. 12, No. 2, 93-133, 1963.

[30] Smith, G.F., and Rivlin, R.S., The Anisotropic Tensors. Quart. Appl. Math. 15, No. 3, 308-314, 1957.

[31] Smith, G.F., and Rivlin, R.S., Stress-deformation Relations for Anisotropic Solids. Arch. Rational Mech. Anal. 1, 107, 1957.

[32] Smith, G.F., and Rivlin, R.S., The Strain-energy Function for Anisotropic Elastic Materials. Trans. Amer. Math. Soc. 88, No. 1, 175-193, 1958.

[33] Smith, G.F., and Rivlin, R.S., Integrity Bases for Vectors--The Crystal Classes, Arch. Rational Mech. Anal. 15, No. 3, 169-221, 1964.

[34] Spencer, A.J.M., The Invariants of Six Symmetric 3 X 3 Matrices. Arch. Rational Mech. Anal. 7, No. 1, 64-77, 1961.

[35] Spencer, A.J.M., and Rivlin, R.S., The Theory of Matrix Polynomials and its Application to the Mechanics of Isotropic Continua. Arch. Rational Mech. Anal. 2, No. 4, 309-336, 1959.

[36] Spencer, A.J.M., and Rivlin, R.S., Finite Integrity Bases for Five or Fewer Symmetric 3 X 3 Matrices. Arch. Rational Mech. Anal. 2, No. 5, 435-446, 1959.

[37] Spencer, A.J.M., and Rivlin, R.S., Further Results in the Theory of Matrix Polynomials, Arch. Rational Mech. Anal. 4, No. 3, 214-230,1960.

[38] Spencer, A.J.M., and Rivlin, R.S., Isotropic Integrity Bases for Vectors and Second-order Tensors. Part 1, Arch. Rational Mech. Anal. 9, No. 1, 45-63, 1962.

[39] Toupin, R.A., and Rivlin, R.S., Linear Functional Electromagnetic Constitutive Relations and Plane Waves in Hemihedral Isotropic Materials. Arch. Rational Mech. Anal. 6, No. 3, 188-197, 1960.

[40] Toupin, R.A., and Rivlin, R.S., Electro-magneto-optical Effects. Arch. Rational Mech. Anal. 7, 434, 1961.

II. MODELS OF RESILIENT MEDIA WITH ALLOWANCE FOR FINITE DEFORMATIONS

[41] Amiel, R., Mechanics of Continuous Media. Rotational Force of Capillary Media. Compt. Rend. Acad. Sci. (Paris) Vol. 258, pp.1709-1711, 1964.

[42] Biot, M.A., Theory of Elasticity with Large Displacements and Rotations. Proc. 5, Inter. Congr. Appl. Mech., 1939.

[43] Biot, M.A., Non-Linear Theory of Elasticity and the Linearized Case for a Body Under Initial Stress. Phil. Mag. 27, No. 183, pp. 449-452, 468-489, 1939.

[44] Bondar', V.D., On the Gensor Characteristics of Finite Deformations of a Continuous Medium, Prikl. Matem. i Mekh., Vol. 25, No. 3, pp.508-518, 1961.

[45] Casal, P., Internal Capillarity in the Mechanics of Continuous Media, Comptes Rendus (Proceedings), Vol. 256, No. 18, pp.3820-3822, 1963.

[46] Casal, P., Internal Capillarity. Report of French Study Group on Rheology. Vol. 6, No. 3, 1961.

[47] Casal, P., General Mechanics of Continuous Media.

[48] Ericksen, J.L., and Rivlin, R.S., Large Elastic Deformations of Homogeneous Anisotropic Elastic Materials. Journ. Rational Mech. and Anal. 3, No. 3, pp. 281-301, 1954.

[49] Eringen, A.C., Nonlinear Theory of Continuous Media. New York, McGraw-Hill Book Company, 1964.

[50] Green, A.E., and Adkins, J.E., Large Elastic Deformations and Nonlinear Continuum Mechanics. Oxford, Clarendon Press, 1960.

[51] Green, A.E., and Rivlin, R.S., Simple Force and Stress Multipoles. Arch. Rational Mech. Anal. Vol. 16, No. 5, pp.325-353, 1964.

[52] Green, A.E., and Rivlin, R.S., Multipolar Continuum Mechanics. Arch. Rational Mech. Anal. Vol. 17, No. 2, pp. 113-147, 1964.

[53] Green, A.E., and Zerna, W., Theoretical Elastity. Oxford Univ. Press, 1954.

[54] Hanin, M., and Reiner, M., On Isotropic Tensor Functions and the Measure of Deformation. Journ. Appl. Math. and Phys. (ZAMP) Vol. 11, No. 5, pp. 377-393, 1956.

[55] Ivlev, D.D., On the Construction of a Theory of Elasticity. Dokl. Akad. Nauk SSSR, Vol. 138, No. 6., pp. 1321-1324, 1961.

[56] Jaumann, G., Principals of Mechanics of Motion. Leipzig, 1905. Conference Proceedings of Academy of Sciences in Vienna (110), 120, 1911.

[57] Yoskmaru, Y., On the Definition of Stress in the Finite Deformation Theory. Journ. Phys. Soc. Japan, Vol. 8, No. 5, 1953.

[58] Kappus, R., On the Elasticity Theory of Finite Displacements. ZAMM (Journal of Applied Math. and Mech.), Vol. 19, No. 5. pp. 271-285,1939.

[59] Kirchhoff, G., Lectures on Mathematical Physics, Vol. 1, Mechanics, 2nd Edition, Leipzig, 1877; 3rd Ed., 1883.

[60] Lyav, A., Mathematical Theory of Elasticity, Translated from English. ONTI, Moscow-Leningrad, 1935.

[61] Murnaghan, F.D., Finite Deformation of an Elastic Solid. Amer. Journ. Math. Vol. 59, No. 2, pp. 235-260, 1937.

[62] Murnaghan, F.D., Finite Deformation of an Elastic Solid. John Wiley, Chapman, New York, 1951.

[63] Novozhilov, V.V., Teoriya uprugosti (Theory of Elasticity), Sudpromgiz, 1958.

[64] Oldroyd, J.G., On the Formulation of Rheological Equation of State. Proc. Roy. Soc. (A) 200, p.523, 1950.

[65] Prager, W., On an Elementary Determination of Stress Rates. Collection of translations and reviews of foreign periodical literature. "Mechanics", No. 4, 1960.

[66] Prager, W., Introduction to Continuum Mechanics, Birkhäuser Publishing House, Basel and Stuttgart, 1961.

[67] Rivlin, R.S., Large Elastic Deformations of Isotropic Materials, IV. Further Development of the General Theory. Phil. Trans. Roy. Soc., (A) 241, pp.379-397, 1948.

[68] Rivlin, R.S., Some Topic in Finite Elasticity. Structural Mechanics. Perg. Press, 169-198, 1960.

[69] Rivlin, R.S., and Saunders, D.M., Large Elastic Deformation of Isotropic Materials, VII. Experiments on the Deformation of Rubber. Phil. Trans. Roy. Soc. London (A) Vol. 234, pp.251-288, 1951.

[70] Roy, M., Mechanics of Continuous and Deformable Media, Vol. 1-2, G.V., 1950.

[71] Sedov, L.I., On the Basic Concepts of Continuous Media Mechanics. In the book: "Certain Problems of Mathematics and Mechanics", Siberian Department of USSR Academy of Sciences, pp. 227-235, Novosibirsk, 1961.

[72] Sedov, L.I., Concepts of Various Rates of Variation of Tensors., Prikl. Matem. i Mekh. Vol. 24, No. 3, pp. 393-398, 1960.

[73] Sedov, L.I., Concerning the Basic Principles of Mechanics of Continuous Media, Moscow State University Press, Moscow, 1961.

[74] Sedov, L.I., Vvedeniye v mekhaniku sploshnoy sredy (Introduction to the Mechanics of Continuous Media), Fizmatgiz, Moscow, 1962.

[75] Seth, B.R., Finite Strain in Elastic Problems.,Phil. Trans. Royal Soc. (London), Ser. A, Vol. 234, pp. 231-264, 1935.

[76] Signorini, A., Problems of Elasticity of Nonlinearized Materials. Rome, 1960.

[77] Sokolnikoff, J.S., Tensor Analysis. Theory and Application of Geometry and Mechanics of Continua., Second Edition, John Wiley and Sons, Inc., New York-London-Sidney, 1964.

[78] Truesdell, C., The Mechanical Foundations of Elasticity and Fluid Dynamics., Journ. Rational Mech. and Anal. Vol. 1, p. 125, 1952, Vol. 2, No. 3, pp. 593-616, 1953.

[79] Truesdell, C.T., and Toupin, R.A., The Classical Field Theories, in: S. Flugge, Handbuch der Ohysik, Band III/I (Springer-Verlag, Berlin) pp. 226-793, 1960.

[80] Eglit, M.E., On the Tensor Characteristics of Finite Deformations, Prikl. Matem. i Mekh. Vol. 24, No. 5, pp. 947-950, 1960.

[81] Eglit, M.E., On Determining the Elastic Potentials from an Experiment. Prikl. Matem. i Mekh. (PMM), Vol. 25, No. 4, 1961.

III. MODELS WITH ASYMMETRICAL TENSOR OF INTERNAL STRESSES

[82] Aero, E.L., and Kuvshinskiy, Ye.V., Basic Equations of Theory of Elastic Media with Rotational Interaction of Particles. Fizika Tverdogo Tela (Solid State Physics), Vol. 2, p. 1399, 1960.

[83] Vlasov, K.B., Equations of State Determining the Magneto-elastic Properties of Ferromagnetic Monocrystals. ZHETF (JETP = Journal of Experimental and Theoretical Physics), Vol. 38, No. 3, pp. 889-894, 1960.

[84] Vlasov, K.B., and Ishmykhametov, B.Kh., Equations of Motion and State of Magneto-elastic Media. JETP, Vol. 46, No. 1, p. 201, 1964.

[85] Corre, J.L., Asymmetry of Tensor Stresses and Its Results. Journ. Phys. Radium, Vol. 17, p. 934, 1956.

[86] Cosserat, E., and Cosserat, F., Theory of a Deformable Body. Paris, A. Hermann and Son, 1909.

[87] Grad, H., Statistical Mechanics, Thermodynamics and Fluid Dynamics of Systems with an Arbitrary Number of Integrals. Comm. Pure Appl. Math. Vol. 5, pp. 455-494, 1952.

[88] Grioli, G., Asymmetrical Elasticity. (Italian) Ann. Matem. Pura ed Appl. (Annals of Pure and Applied Mathematics), Ser. IV, 50, pp.389-417,1960.

[89] Halbwachs, F., Relativistic Theory of Spin Fluids. Gauthier-Villars, Paris, 1960.

[90] Kaliski, S., On a Model of the Continuum with an Essentially Nonsymmetric Tensor of Mechanical Stress. Arch. Mech. Stosowaney, 15, 1, 1963.

[91] Kaliski, S., Pfochocki, Z., and Rogula, D., The Asymmetry of the Stress Tensor and the Angular Momentum Conservation for a Combined Mechanical and Electromagnetic Field in a Continuous Medium. Bull. of Polish Academy of Sciences, Technical Science Series, Vol. 10, No. 4, pp. 135-141, 1962.

[92] Krishnan, R.S., and Rajagopal, E.S., The Atomic and the Continuum Theory of Crystal Elasticity. Ann. Physik, Vol. 4, No. 3, 1961.

[93] Kröner, E., On the Physical Reality of Torque stress in Continuum Mechanics. Int. Journ. Engng. Sci. Vol. 1, pp. 261-278; Pergamon Press 1963, Printed in Great Britain, 1963.

[94] Lubanski, J., New Motion Equations of Material Systems in a Minkowski World. Acta Physica Pol., Vol. 6, pp. 356-370, 1937.

[95] Mathisson, M., New Mechanics of Material Systems (in German). Acta Phys. Pol. Vol. 6, pp. 163-200, 1937.

[96] Meixner, J., Rotational Pulse Theorem in Thermodynamics of Irreversible Processes (in German). Z. Physik (Journal of Physics), Vol. 164, No.2, 1961.

[97] Mindlin, R.D., Effect of Moment Stresses on Stress Concentrations. Mekhanika (Mechanics), 4, 86, 1964.

[98] Mindlin, R.D., Micro-structure in Linear Elasticity. Arch. Rational Mech. Anal. Vol. 16, pp. 51-78, 1964.

[99] Mindlin, R.D., and Tiersten, H.F., Effects of Couple-stresses in Linear Elasticity. Arch. Rational Mech. Anal. Vol. 11, pp. 415-448, 1962.

[100] Raman, C.V., and Viswanathan, K.S., On the Theory of Elasticity of Crystals. Proc. Ind. Acad. Sci., Ser. A., Vol. 51, No. 42, 1955.

[101] Toupin, R.A., The Elastic Dielectric. Journ. Rational Mech. Anal. Vol. 5, pp. 849-916, 1956.

[102] Toupin, R.A., Stress Tensor in Elastic Dielectric. Arch. Rational Mech. Anal. Vol. 5, p. 440, 1960.

[103] Toupin, R.A., Elastic Materials with Couple-stress. Arch. Rational Mech. Anal. Vol. 11, pp. 385-414, 1962.

[104] Weyssenhoff, J., and Raabe, A., Relativistic Dynamics of Spin-fluids and Spin-particles. Acta Phys. Polonica, Vol. 9, pp. 19-25, 1947-1948.

IV. CONTINUAL THEORY OF DISLOCATIONS

[105] Bilby, B.A., Continuous Distributions of Dislocations. Progr. Solid Mech. Vol. 1/ p.329; edited by L.N. Sneddon and R. Hill, North Holland Publishing Company, 1960.

[106] Bilby, B.A., Bullough, R., and Smith, E., Continuous Distributions of Dislocations. A New Application of the Methods of Non-Riemannian Geometry. Proc. Roy. Soc. (London) Vol. A231, pp. 263-273, 1955.

[107] Holländer, E.F., The Geometric Equation of Dislocation Dynamics. Szech. Journ. Phys. Vol. 12B, No. 1, p.35, 1962.

[108] Indenbom, V.L., Scientific Results. Fiziko-matematicheskiye nauki (Physical-Mathematical Sciences), No. 3, p. 117, 1960.

[109] Indenbom, V.L., and Orlov, A.N., Physical Theory of Plasticity and Strength. Uspekhi Fizicheskikh Nauk (Advances in Physical Sciences), Vol. 26, p. 557, 1962.

[110] Kestin, J., On the Correct Application of the Principles of Thermodynamics in Strained Solid Materials. Brown University, Providence, Rhode Island, 1964.

[111] Kondo, K., Memoirs of the Unifying Study of the Basic Problems in Engineering Science by Means of Geometry. Vol. I,II,III. Tokyo, Gakujussu Bunken Fukyu-Kai, 1955,1958,1962.

[112] Kröner, E., Continuum Theory of Displacements of Eigen-stresses. Berlin, 1958.

[113] Kröner, E., Continuum Theory of Displacements and Eigen Stresses. Ergebnisse Angew. Math. (Results of Applied Mathematics), Vol. 5, p. 1-179, 1958.

[114] Kröner, E., General Continuum Theory of Displacements and Natural Stresses. (in German). Arch. Rational Mech. Anal., Vol. 4, No. 4, pp. 273-334, 1960.

[115] Kröner, E., Remark on the Geometric Basic Principle of the General Continuum Theory of Dislocations and Natural Stresses (in German). Arch. Rational Mech. Anal., Vol. 7, No. 1, pp. 78-80, 1961.

[116] Kröner, E., Physica status solidi (Solid State Physics), Vol. 1, No. 1, 1961.

[117] Kröner, E., Dislocations and Continuum Mechanics. Applied Mechanics Reviews, Vol. 15, No. 8, pp. 599-606, 1962.

[118] Kröner, E., A New Concept in the Continuum Theory of Plasticity. Journ. Math. Phys. Vol. 42, No. 1, pp. 27-37, 1963.

[119] Schaefer, H., Stress Functions of Three-Dimensional Continuum and of an Elastic Body (in German). ZAMM (Journ. of Appl. Math. and Mech.), Vol. 33, 10/11, pp. 356-362, 1953.

[120] Eshelby, J., Continual Theory of Dislocations, Moscow, IL (Foreign Literature Press), 1963; translations of articles: J.D. Eshelby, Solid State Physics 3, New York, 1956, 79; J.D. Eshelby, Proc. Roy. Soc. A241, No. 1226, 1957, 376; J.D. Eshelby, Proc. Roy. Soc. A 252, No. 1271, 1959, 561; J.D. Eshelby, F.C. Frank, F.R.N. Nabarro, Phil. Mag. Vol. 42, No. 327, 1951, 351; R. de Wit, Solid State Physics, New York, Vol. 10, 249, 1960; A.K. Head, P.F. Thomson, Phil. Mag., Vol. 7, No. 75, 439, 1962.

V. PLASTICITY OF CRYSTALS AND POLYCRYSTALS

[121] Bishop, J.F.W., and Hill, R., A Theory of the Plastic Distortion of a Polycrystalline Aggregate Under Combined Stresses. Phil. Mag., Vol.42, 414, 1951.

[122] Bishop, J.F.W., and Hill, R., A Theoretical Derivation of the Plastic Properties of Polycrystalline Face Centred (sic), Metal, Vol. 42, 1298, 1951.

[123] Budiansky, B., Hashin, Z., and Sanders, J.L., The Stress Field of a Slipped Crystal and the Early Plastic Behavior of Polycrystalline Materials. Proc. 2nd Symp. Naval Struc. Mech., 239, 1960.

[124] Budiansky, B., and Wu, T.T., Theoretical Prediction of Plastic Strains of Polycrystals. Harvard University Nonr. 1866 (02), TR12, 1961.

[125] Budiansky, B., and Wu, T.T., Theoretical Prediction of Plastic Strains of Polycrystals. Proc. Fourth U.S. National Congress of Mechanics, New York, 1175-1185.

[126] Yashimaru, Yoshimura, Comment on the Slip Theory of Batdorf and Budiansky. Bull. of JSME 1, No. 2, 109-113, 1958.

[127] Kröner, E., On the Plastic Deformation of Polycrystal (in German). Acta Metal., Vol.9, 155, 1961.

[128] Lin, T.H., Analysis of Elastic and Plastic Strains of a Face-Centred Cubic Crystal. Journ. Mech. Phys. Solids, Vol.5, 143, 1957.

[129] Taylor, G.J., Plastic Strain in Metals. Journ. Inst. Metal., Vol. 62, 307, 1938.

VI. THEORY OF PLASTICITY AND CREEP

[130] Budiansky, B., A Reassessment of Deformation Theory of Plasticity. Trans. ASME, Ser. E., Journ. of Appl. Mech., Vol.26, No.1-2, pp. 259-264, 1959.

[131] Drucker, D.C., A More Fundamental Approach to Plastic Stress-Strain Relations. Proc. Inst. U.S. Nat. Cong. Appl. Mech., 487, 1951.

[132] Drucker, D.C., Structural Mechanics. Proc. Ist. Symp. Naval Structural Mech., Pergamon Press, 407, 1960.

[133] Drucker, D.C., On the Postulate of Stability of Material in the Mechanics of Continua. Journ. de Mecan., Vol. 3, No. 2, pp. 235-249, 1964.

[134] Edelmann, F., and Drucker, D., Some Extensions of Elementary Plasticity Theory. Journ. of the Franklin Institute, Vol.251, No.6, pp. 581-607, 1951.

[135] Ericksen, J.L., Singular Surfaces in Plasticity. Journ. Math. Phys., Vol.34, No.1, pp. 74-79, 1955.

[136] Goodier, J.H., and Hodge, P.G., Elasticity and Plasticity. John Wiley & Sons, New York, 1958.

[137] Gol'denblatt, I.I., Nekotoryye voprosy mekhaniki deformiruyemykh sred (Certain Questions of Mechanics of Deforming Media), Gostekhizdat, Moscow, 1955.

[138] Hill, R., The Mathematical Theory of Plasticity. Oxford, at the Clarendon Press, 1950; Russian translation: R. Hill, Matematicheskaya teoriya plastichnosti, Gostekhizdat, Moscow, 1956.

[139] Hodge,Ph.C., A General Theory of Piecewise Linear Plasticity Based on Maximum Shear. Journ. Mech. and Phys. of Solids, Vol.5, No.4, 1957.

[140] Ishlinskiy, A.Yu., General Theory of Plasticity with Linear Reinforcement. Ukr. matem. zhurn. (Ukrainian Math. Journ.), Vol.6, 314, 1954.

[141] Kachanov, L.M., Osnovy teorii plastichnosti (Bases of Plasticity Theory), Gostekhizdat, Moscow, 1956.

[142] Kachanov, L.M., Teoriya polzuchesti (Theory of Creep), Fizmatgiz, Moscow, 1960.

[143] Kodashevich, Yu.I., and Novozhilov, V.V., Theory of Plasticity Taking the Baushinger Effect Into Consideration. Dokl. Akad. Nauk (Proc. of Acad. Sci.), Vol.117, No.4, 1957.

[144] Koiter, W.T., Stress-strain Relations, Uniqueness and Variational Theorems for Elastic-Plastic Materials with a Sing lar Field Surface. Quart. Appl. Math., No.3, pp. 350-354, 1953.

[145] Koiter, W.T., General Theorems for Elastic-Plastic Solids. Chapt. IV. Progress in Solid Mechanics, vol.I. Edited by J. Sheddon and R. Hill, North Holland Publishing Co., 1960.

[146] Mieses, R., Mechanics of Solid Bodies in Plastically Deformable State (in German). Gott. Nachr., Math.-Phys. Klasse, 582, 1913.

[147] Mieses, R., Mechanics of Plastic Form Modification of Crystals (in German). ZAMM, Vol.8, No.3, pp. 161-185, 1928.

[148] Namestnikov, V., and Rabotnov, Yu.N., On the Hypothesis of Equation of State During Creep. Zhurn. Prikl. Matem. i Tekhn. Fiz. (Journal of Applied Mathematics and Technical Physics), No.3, pp. 101-102, 1961.

[149] Prager, W., The Effect of Deformation on the Flow State of Viscous-Plastic Bodies. ZAMM, No. 1-2, pp. 76-80, 1935.

[150] Prager, W., Method of Analyzing Stress and Strain in Work-Hardening Plastic Solids. Journ. Appl. Mech., Vol.23, p. 493, 1956.

[151] Prager, V., Problemy teorii plastichnosti (Problems of Theory of Plasticity), Fizmatgiz, Moscow, 1958.

[152] Prager, V., Neizotermicheskoye plasticheskoye deformirovaniye (Nonisothermal Plastic Deformation). Collec. of translations from foreign periodical literature. Mekhanika (Mechanics), FLP (Foreign Lit. Press), No.5, 1959.

[153] Rabotnov, Yu.N., Model Illustrating Certain Properties of a Hardening Plastic Body. Prikl. Matem. i Mekh., Vol.23, No.1, 1959.

[154] Rabotnov, Yu.N., Modelling of Plasticity. Zhurn. Prikl. Matem. i Tekhn. Fiz., No.2, pp. 89-95, 1961.

[155] Sanders, T.L., Plastic Stress-Strain Relations Based on Linear Loading Functions. Proc. of Second U.S. National Congress of Appl. Mech., pp. 455-460, 1954.

[156] Taylor, G.J., and Quinney, H., The Plastic Distortion of Metals. Phil. Trans. Roy. Soc., Vol.230, pp. 323-362, London, 1932.

[157] Ziegler, H., Methods of Plasticity Theory Applied in Snow Mechanics (in German). Zeitschrift für angewandte Math. und Phys. (ZAMP) (Journ. of Applied Math. and Phys.), Vol. 14, No.6, pp. 713-737, 1963.

[158] Ziegler, H., Thermodynamic Considerations in Continuum Mechanics. The 1964 Minta Martin Lecture, MIT.

VII. PHYSICALLY NONHOLONOMOUS MEDIA. HYPOELASTICITY.

[159] Bernstein, B., Hypo-elasticity and Elasticity. Arch. Rational Mech. Anal., Vol.6, No.2, pp. 89-104, 1960.

[160] Bernstein, B., and Ericksen, J.L., Work Function in Hypo-elasticity. Arch. Rational Mech. Anal., Vol.1, No.5, pp. 396-409, 1958.

[161] Noll, W., On the Continuity of the Solid and Fluid States. Journ. Rational Mech. Anal., Vol.4, pp. 3-81, 1955.

[162] Sedov, L.I., and Eglit, M.E., Construction of Nonholonomous Models of Continuous Media with Allowance for Finiteness of Deformations and Certain Physico-Chemical Effects. Dokl. Akad. Nauk, Vol.142, No.1, pp. 54-57, 1962.

[163] Truesdell, C., Hypoelasticity. Journ. Rational Mech. Anal., Vol.4, pp. 83-133, 1955.

[164] Truesdell, C., Hypo-elasticity Shear. Journ. Appl. Phys., Vol. 27, pp. 441-447, 1956.

VIII. MODELS OF LIQUID MEDIA WITH COMPLICATED PROPERTIES

[165] Anzelius, A., On the Motion of Anisotropic Fluids (in German). The Annals of Univ. of Uppsala, 1931.

[166] Bergen, J.T., Messersmith, D.C., and Rivlin, R.S., Stress Relaxation for Biaxial Deformation of Filled High-Polymers. Journ. Appl. Polymer Science, Vol.3,Issue No.8, pp. 153-167, 1960.

[167] Bland, D.R., The Theory of Linear Visco-elasticity. Pergamon Press, N.Y. 1960.

[168] Coleman, B.D., and Noll, W., On the Thermostatics of Continuous Media. Arch. Rational Mech. Anal., Vol.4, 97, 1959.

[169] Ericksen, J.L., Theory of Anisotropic Fluids. Trans. Soc. Rheol.,Vol.4, pp. 29-39, 1960.

[170] Ericksen, J.L., Anisotropic Fluids. Arch. Rational Mech. Anal. 4, pp. 231-237, 1960.

[171] Ericksen, J.L., Conservation Laws for Liquid Crystals. Trans. Soc. Rheol., Vol.5, pp. 23-34, 1961.

[172] Ericksen, J.L., Transversely Isotropic Fluids. Kolloid-Zeitschrift (Colloid Journal), Vol.173, No.2, pp. 117-122, 1960.

[173] Ericksen, J.L., A Vorticity Effect in Anisotropic Fluids. Journ. Polym. Sci., Vol.47, pp. 327-333,1960.

[174] Ericksen, J.L., Recoil of Orientable Fluids. Presented at 7th Congress on Theoretical and Applied Mechanics. Bombay, 1961.

[175] Ericksen, J.L., Kinematics of Macromolecules. Arch. Rational Mech.Anal., Vol.9, No.1, pp. 1-8, 1962.

[176] Ericksen, J.L., Hydrostatic Theory of Liquid Crystals. Arch. Rational Mech. Anal., Vol.9, No.5, pp. 371-378, 1962.

[177] Ericksen, J.L., Nilpotent Energies in Liquid Crystal Theory. Arch. Rational Mech. Anal., Vol.10, pp. 189-196, 1962.

[178] Ericksen, J.L., Singular Surfaces in Anisotropic Fluids. Int. Journ. Engng. Sci., Vol.1, pp. 157-171, 1963. Pergamon Press. Printed in Great Britain.

[179] Frank, F.C., On the Theory of Liquid Crystals. Discussions Faraday Soc., Vol.25, pp. 19-28, 1958.

[180] Hand, G.L., A Theory of Dilute Suspensions. Arch. Rational Mech. Anal., Vol. 7, No.1, pp. 81-86, 1961.

[181] Hand, G.L., A Theory of Anisotropic Fluids. Fluid Mech., Vol.13, Part 1, pp. 33-46, 1962.

[182] Shizuo, Hayashi, Theory of Viscoelasticity in Temporarily Crosslinked Polymers, (1), Treatment in Single Relaxation Time. Journ. Phys. Soc. Japan, Vol.18, No. 1, pp. 131-138, 1963.

[183] Noll, W., On the Continuity of Solid and Fluid States. Journ. Rational Mech. Anal., Vol.4, pp. 3-81, 1955.

[184] Noll, W., A Mathematical Theory of the Mechanical Behavior of Continuous Media. Arch. Rational Mech. Anal., Vol.2, pp. 198-226, 1958.

[185] Oldroyd, J.G., On the Formulations of Rheological Equations of State. Proc. Roy. Soc. (A), Vol.200, No.1063, pp. 523-541, 1950.

[186] Oldroyd, J.G., A Rational Formulation of the Equations of Plastic Flow for a Binham Solid. Proc. Cambridge Phil. Soc., Vol.43, pp. 100-105, 1947.

[187] Oseen, C.W., The Theory of Liquid Crystals. Trans. Faraday Soc., Vol.29, pp. 883-889, 1933.

[188] Reiner, M., The Theory of Non-Newtonian Liquids. Physics, Vol.5, No.11, November 1934.

[189] Reiner, M., A Mathematical Theory of Dilatancy. Amer. Journ. Math., Vol.67, 350, 1945.

[190] Reiner, M., Rheology, in: S. Flügge, Handbuch der Physik (Physics Manual), Vol. VI, Springer-Verlag, Berlin 1958.

[191] Rivlin, R.S., The Hydrodynamics of Non-Newtonian Fluids, Part 1, Proc. Roy. Soc., A193, 260, 1948.

[192] Rivlin, R.S., Some Flow Properties of Concentrated High-Polymer Solutions. Proc. Roy. Soc. A200, pp. 168-178, 1950.

[193] Rivlin, R.S., Some Reflections on Non-linear Viscoelastic Fluids, 83-93, (in French). Phenomena of Relaxation in Nonlinear Rheology. International Colloquia of the National Center for Scientific Research 98 (Paris, 27 June - 2 July 1960); Publications of the National Center for Scientific Research, 15, Quai Anatole-France-Paris (VIIe), 1961.

[194] Rivlin, R.S., and Ericksen, J.L., Stress-Deformation Relations for Isotropic Material. Journ. Rational Mech. Anal. (JRMA), Vol.4, No.2, pp. 323-425, 1955.

[195] Wehrli, C. and Ziegler, H., Some Material Equations Compatible with the Principle of the Greatest Dissipative Yield. ZAMP, Vol.13, No.4, pp. 372-393, 1962.

[196] Eglit, M.E., One Generalization of Model of an Ideal Compressible Fluid. Prikl. Matem. i Mekh., Vol.29, No.2, 1965.

IX. MODELS OF A LIQUID WITH BUBBLES

[197] Ackeret, J., Research in the Field of Engineering (in German). Edition A, 1, 63, 1930.

[198] Kogarko, B.S., Concerning One Model of a Cavitating Liquid. Dokl. Akad. Nauk (Proc. of Acad. Sci.), Vol.137, No.6, pp. 1331-1333, 1961.

[199] Epshteyn, L.A., On the Possibilities of a Theoretical Study of Cavitation As of the Motion of a Special Type of Compressible Fluid. Dokl. Akad. Nauk, Dept. of Tech. Sci., Vol.49, No.6, 1945.

[200] Epshteyn, L.A., Cavitation and the Possibility of Its Theoretical Study As of a Supersonic Flow of Hypothetical Fluid. Trudy TSAGI (Trans. of N.Ye. Zhukovskiy Central Aerohydrodynamic Institute), No.584, 1946.

X. RAREFIED GASES

The Boltzmann Equation

[201] Boltzmann, L., Lectures on the Theory of Gases; L. Boltzmann, Reports of Vienna. Acad., Vol.56, 275, 1872 (No.1, p. 345, No. 33).

Equations of Moments

[202] Maxwell, I.C., Scientific Papers, 1890 (General Equations of Conservation for an infinite series of macroscopic values (Infinite system of equations).

[203] Grad, H., Comm. Pure Appl. Math., Vol.2, 331, 1949 (20 and 13 equations respectively for 20 and 13 macroscopic values).

Derivation of Macroscopic Equations by Expansion in Respect to Small Parameter

[204] Hilbert, D., Basic Features of a General Theory of Linear Integral Equations. Vienna, 1924.

[205] Chapman, S., Phil. Trans. Roy. Soc. Lond. A217, 115, 1917.

[206] Enskog, D., Kinetic Theory of Processes in Moderately Thinned Gases (in German). Uppsala, 1917.

[207] Barnett, D., Proc. Lond. Math. Soc., Vol.40, 382, 1935.

Derivation of Boltzmann Equation from the Liouville Equation

[208] Bogolyubov, N.N., Kinetic Equations. ZHETF (JETP), Vol.16, No.3, 691,1946.

[209] Kirkwood, J.G., Journ. Chem. Phys., Vol.15, 72, 1947.

XI. KINETIC MODELS OF PLASMA. KINETIC EQUATIONS OF PLASMA WITH COLLISIONS

[210] Vlasov, A.A., Concerning the Vibrational Properties of an Electron Gas. ZHETF (JETP), Vol. 8, 291, 1938.

Term of Collisions in Kinetic Equations of Plasma

[211] Landau, D., Kinetic Equation in Case of Coulomb Interaction. ZHETF, Vol.7, 203, 1936.

[212] Balescu, R., Irreversible Processes in Ionized Plasma. Phys. of Fluids, Vol.3, 52, 1960.

[213] Lenard, On Bogolyubov Kinetic Equation for a Spatially Homogeneous Plasma. Ann. of Phys., Vol.10, 390, 1960.

Array of Equations for Plasma and Its Breaking

[214] Bogolyubov, N.N., Dynamic Problems in Statistical Physics. Gostekhizdat, Moscow-Leningrad, 1946.

Derivation of Hydrodynamic Plasma Equations

[215] Braginskiy, S.I., Phenomena in a Fully Ionized Two-Temperature Plasma. ZHETF, Vol.33, No.2(8), 1957 (the hydrodynamic equations were derived by the Chapman-Ensky method).

[216] Zhdanov, V.M., Phenomena of Transport in a Partially Ionized Gas. Prikl. Matem. i Mekh., Vol.26, No.3, 280, 1962 (the Gred method is utilized in the report).

Hydrodynamic Models of Plasma for Plasma Without Collisions, Situated in an Intensive Magnetic Field

[217] Chew, G.F.,Goldberger, M.L., and Low, F.E., The Boltzmann Equation and the One-Fluid Hydromagnetic Equation in the Absence of Particle Collisions. Proc. Roy. Soc., A236, 112, London 1956.

[218] Chandrasekhar, S., Kaufman, A., and Watson, K., The Properties of Ionized Gas of Low Density in a Magnetic Field. Ann. of Phys., Vol.2, 435, 1957.

XII' RELATIVISTIC MECHANICAL MODELS OF CONTINUOUS MEDIA. CONSIDERATION OF THE ELECTRODYNAMIC EFFECTS

[219] Boa-Teh-Chu, Thermodynamics of Electrically Conducting Fluids. Phys. Fluids, Vol.2, No.5, pp. 473-484, 1959.

[220] Zel'manov, A.L., Application of Comoving Coordinates in Nonrelativistic Mechanics. Dokl. Akad. Nauk, Vol.61, pp. 993-996, 1948.

[221] Zel'manov, A.L., Chronometric Invariants and Comoving Coordinates in the General Theory of Relativity. Dokl. Akad. Nauk, Vol. 107, No.6, 1956.

[222] Zel'manov, A.L., On Question of Deformation of Comoving Space in the Einsteinian Theory of Gravitation. Dokl. Akad. Nauk, Vol.135, No.6, pp. 1367-1370, 1960.

[223] Zommerfel'd, A., Elektrodinamika (Electrodynamics), FLP, Moscow, 1958.

[224] Landau, L.D., and Lifshits, Ye.M., Teoriya polya (Theory of a Field), Fizmatgiz, Moscow, 1962.

[225] Lorentz, H.A., Einstein, A., and Minkowski, H., The Principle of Relativity (in German). 4th Ed., Leipzig, 1962.

[226] Moller, C., Kgl. Danske Vidensk Selk., Mat. fys. Medd. (Danish), Vol.31, No.14, 1959; Russian translation published in collec.: "Most Recent Problems in Gravitation", FLP, Moscow, 1961.

[227] Pauli, V., Teoriya otnositel'nosti (Theory of Relativity), Gostekhizdat, Moscow, 1947.

[228] Rayner, C.B., Proc. Roy. Soc. London, 272, 44, 1963.

[229] Rosen, N., Phys. Rev., 54, 1947.

[230] Sedov, L.I., On the Ponderomotive Forces of Interaction of an Electromagnetic Field and of an Accelerated Moving Material Continuum with Allowance for Finiteness of Deformations. PMM, No.1, 1965.

[231] Schöpf, H.G., Concerning an Approximate Solution of the Einsteinian Field Equations Which Is Represented by a Source-Free Gravitational Field (in German). Ann. der Physik (Annals of Physics), No.7, Ser. 9, No. 11-12, pp. 65-78, 1961.

[232] Schöpf, H.G., On the Relativity Theory of Dielectrics (in German). Ann. der Phys., Vol.7, Ser. 9, No. 5-6, pp. 301-312, 1962.

[233] Schöpf, H.G., Moving Anisotropic Dielectrics in Relativistic Visibility. Ann. der Phys., Vol.7, Ser. 13, No.1-2, pp. 41-52, 1964.

[234] Schöpf, H.G., General Relativistic Principles of Continuum Mechanics (in German). Ann. der Phys., Vol. 7, Ser. 12, No. 7-8, pp. 377-395, 1963.

[235] Schöpf, H.G., Clebsch-Transformations in the General Theory of Relativity (in German). Acta Phys. Hung., Vol.7, No.1-2, pp. 41-56, 1964.

[236] Synge, I.L., Relativity, the General Theory. North Holland Publishing Co., 116, Amsterdam, 1960.

[237] Tiersten, H.F., Coupled Magnetomechanical Equations for Magnetically Saturated Insulators. Journ. of Math. Phys., Vol.5, No.9, 1964.

[238] Tonnela, M.A., Fundamentals of Electromagnetism and Theory of Relativity. FLP (Foreign Literature Press), 143, 1962.

[239] Winogradzki, J., On the Metrical Pulse-Energy Tensor and the Noether Theorem (in French). Cahier de Physique, No. 67, pp.1-5, 1956.

[240] Winogradzki, J., Invariance and Conservation in the Field Theory; Application of the General Theory to the Study of a Spinor Field (in French). Cahier de Physique, No.101, pp. 17-26, 1959.

[241] Zorski, H., General Solutions of the Conservation Equation in Curved Spaces. Bull. Acad. Polonaise Sci., Techn. Sci. Ser., Vol.7, No. 10, pp. 567-571, 1959.

[242] Zorski, H., Variational Principle for Compatibility Equations. Bull. Acad. Polonaise Sci., Techn. Sci. Ser., Vol.7, No. 7-8, pp. 445-446, 1959.

[243] Zorski, H., Projective Formulation of the Continuum Mechanics Equations (Part 1), Arch. Mech. Stosowaney, 5/6, Vol.12, pp. 617-647, 1960.

XIII. RELATIVISTIC HYDROMECHANICS

[244] Landau, L.D., Izvestiya Akad. Nauk, ser. fiz. (Bull. of Acad. Sci., Physics Series), Vol.17, 51, 1953.

[245] Khalatnikov, I.M., Certain Questions of Relativistic Hydrodynamics. ZHETF, Vol.27, 529, 1954.

[246] Landau, L.D., and Lifshits, Ye.M., Mekhanika sploshnykh sred (Mechanics of Continuous Media), Gostekhizdat, Moscow, 1954.

[247] Shikin, I.S., Contribution to General Theory of Stationary Movements in Relativistic Hydrodynamics. Dokl. Akad. Nauk (DAN), Vol.142, 296, 1962.

[248] Shikin, I.S., On the Interpretation of Nonstationary Relativistic Hydrodynamics in a Minkowski Space. DAN, Vol. 142, 564, 1962.

XIV. MODELS WITH NONEQUILIBRIUM PROCESSES

[249] Coleman, B.D., and Truesdell, On the Reciprocal Relations of Onsager. Journal of Chemical Physics, Vol.33, No.1, pp. 28-31, 1960.

[250] Denbing, K., Thermodynamics of Irreversible Processes. FLP, Moscow, 1954.

[251] de Donder, Th., Affinity (in French). Paris, 1927.

[252] Hirschfelder, D., Curtis, C., and Byrd, R., Molecular Theory of Liquids and Gases. FLP, Moscow, 1961.

[253] de Groot, S., Thermodynamics of Irreversible Processes. Gostekhizdat, Moscow, 1955; see also: S.R. de Groot, Thermodynamics of Irreversible Processes. Amsterdam, 1961.

[254] de Groot, S., and Mazur, P., Non-equilibrium Thermodynamics. Peace Publishing House, Moscow 1964; see also: S.R. deGroot, P. Mazur (same title), North Holland Publishing Co., Amsterdam, 1962.

[255] Glansdorff, P., Formulations of Hydrodynamic Equations of Ionized Gas (in French). Combustion and Propulsions, Pergamon Press, 1963.

[256] Glansdorff, P. and Prigogin, I., On the Differential Properties in the Production of Entropy. Phys., Vol.20, pp. 773-780, 1954.

[257] Glansdorff, P. and Prigogin, I., On a General Evolution Criterion in Macroscopic Physics. Science Dept. of Univ. of Brussels, Brussels, Belgium.

[258] Glansdorff, P., Prigogin, I., and Hays, D.F., Variational Properties of Viscous Liquid of a Nonuniform Temperature. Phys. Fluids, Vol.5, No.2, pp. 144-149, 1962.

[259] Kluitenberg, G.A., Relativistic Thermodynamics of Irreversible Processes. Leyden, 1954.

[260] Kluitenber, G.A., Thermodynamical Theory of Elasticity and Plasticity. Phys., Vol.28, pp. 217-232, 1962.

[261] Kluitenberg, G.A., A Note of the Thermodynamics of Maxwell Bodies, Kelvin Bodies (Voight Bodies) and Fluids. Phys., Vol.28, pp. 561-568, 1962.

[262] Kluitenberg, G.A., On Rheology and Thermodynamics of Irreversible Processes. Phys., Vol.28, pp. 1173-1183, 1962.

[263] Kluitenberg, G.A., On the Thermodynamics of Viscosity and Plasticity. Phys. Vol. 29, pp. 633-652, 1963.

[264] Landau, L.D., and Lifshits, Ye.M., Theoretical Physics; Statistical Physics. Gostekhizdat, Moscow-Leningrad, 1951; Electrodynamics of Continuous Media. Fizmatgiz, Moscow 1959. Mekhanika sploshnykh sred (Mechanics of Continuous Media), Gostekhizdat, Moscow, 1953.

[265] Onsager, L., Reciprocal Relations in Irreversible Processes. Phys. Rev., Vol. 37, 1, 405, 1931; Pys. Rev., Vol. 38, 2265, 1931; see: H.B.G. Casimir, On Onsager's Principle of Microscopic Reversibility. Rev. Mod. Phys., Vol.17, 343, 1945.

[266] Oshima, N., General Equations of Motion of Composite Fluids. Proc. 11th Japan Nat. Congr. Appl. Mech., 1961 (1961, Osaka), pp. 139-147, Tokyo, 1962.

[267] Prigogin, I., Introduction to Thermodynamics of Irreversible Processes. FLP, Moscow 1960; see also: I. Prigogin (same title, in French), Liege, Desoer, 1947; I. Prigogin, same title (in English), Springfield, 1955.

[268] Thermodynamics of Irreversible Processes. FLP, Moscow 1962; refer to: same title (Italian), Proc. of Enrico Fermi International School of Physics, Bologna, 1960.

[269] Uhlhorn, U., Onsager's Reciprocal Relations for Non-linear Systems. Arkiv för Fysik, Vol.7, No.20, Iss. 3-4, pp. 361-368 (year not given).

[270] Ziegler, H., An Attempt to Generalize Onsager's Principle and Its Significance for Rheological Problems. ZAMP, Vol.96, 748, 1958.

[271] Ziegler, H., Two Extremal Principles of Irreversible Thermodynamics (in German) Ing. Arch., Vol. 30, 410, 1961.

[272] Ziegler, H., The Statistical Bases of Irreversible Thermodynamics (in German). Ing. Arch., Vol.31, No.5, pp. 317-342, 1962.

[273] Ziegler, H., Concerning a Principle of the Greatest Specific Energy Production and Its Significance for Rheology (in German). Rheologica Acta, Vol. 2, No.3, pp. 230-235, 1962.

[274] Ziegler, H., etc. identical to Ref. [272]

END