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Performance of an Ideal Quantum Receiver of a Coherent Signal of Random Phase

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ABSTRACT

An ideal quantum receiver is to detect a coherent narrowband optical signal in the presence of thermal background radiation. Curves are given both of the average probability of error in a binary communication system transmitting 0's (blanks) and 1's (pulses) with equal probabilities, and of the probability of detection for various fixed values of the false-alarm probability.

In analyzing a proposed optical communication or radar receiver it is useful to compare its performance with that of an ideal receiver limited only by background radiation and by the quantum nature of the signals to be detected. In this correspondence, we present performance curves for a receiver of coherent optical signals of random phase in the presence of thermal background radiation of absolute temperature T. The time of arrival and the form of the envelope of each signal are assumed known.

The signal is postulated to be a coherent narrowband pulse such as would be emitted by an ideal laser. The ideal receiver consists of a cavity with an aperture directed toward the source of the signal. The cavity is initially empty. During the period when the signal is expected to arrive, the aperture is opened to admit it and then closed. An observer makes the best possible measurements on the field in the cavity for the purpose of deciding whether it contains a component due to a signal or not. It has been shown that the optimum procedure for detecting a signal of this kind is equivalent to counting the number m of photons -- or measuring the energy -- in a composite mode that is in effect matched to the signal.[1] If m exceeds a certain decision level μ, the observer decides that a signal is present. The resulting performance is the same as though the signal occupied a single mode of the receiver.

Let S be the average number of photons in the signal; $S = E_s/h\nu$, where E_s is the energy in the signal, h is Planck's constant, and ν is the carrier frequency of the signal. Let N be the average number of noise photons per mode of the receiver. For thermal radiation of absolute temperature T,

$$N = [\exp(h\nu/kT) - 1]^{-1}, \tag{1}$$

where k is Boltzmann's constant.

In the absence of a signal (hypothesis H_0) the probability that m photons are counted is

$$P_{0m} = (1 - v)\, v^m, \quad v = N/(N + 1)\,. \tag{2}$$

When the signal is present (hypothesis H_1), the probability that m photons are counted is given by the Laguerre distribution[2]

$$P_{1m} = (1 - v)\, e^{-(1-v)S}\, v^m\, L_m[- (1 - v)^2\, S/v] \tag{3}$$

where $L_m(x)$ is the m-th Laguerre polynomial.

We analyze first a binary communication system sending messages coded into 0's and 1's. A 1 is transmitted by dispatching a laser pulse, a 0 by dispatching nothing. We suppose 0's and 1's to be equally probable. The optimum receiver compares the likelihood ratio P_{1m}/P_{0m} with the decision level 1, deciding that a 1 was sent (choosing H_1) whenever $P_{1m}/P_{0m} > 1$. The decision level μ is therefore the smallest integer for which

$$L_{\mu}\,[-(1-v)^2\,S/v\,] \geq e^{(1-v)S}.$$

The probability of an error of the first kind (choosing hypothesis H_1 when H_0 is true) is

$$Q_0 = \sum_{\mu+1}^{\infty} P_{0m} = v^{\mu+1}. \tag{4}$$

The probability of an error of the second kind (choosing H_0 when H_1 is true) is

$$Q_1 = \sum_{m=0}^{\mu} P_{1m}. \tag{5}$$

The average probability of error is

$$P_e = \frac{1}{2}\,(Q_0 + Q_1). \tag{6}$$

It is plotted in Fig. 1 versus the average number S of signal photons, and in Fig. 2 versus the signal-to-noise ratio S/N, for various values of N. In Table 1 we give the values of the ratio $h\nu/kT$ corresponding to the average numbers N of noise photons used in the graphs.

TABLE 1

N	0.01	0.03	0.05	0.1	0.2	0.3	0.4
$h\nu/kT$	4.61	3.54	3.04	2.40	1.792	1.466	1.253
N	0.5	1	2	3	5	7	9
$h\nu/kT$	1.099	0.693	0.406	0.288	0.1823	0.1334	0.1054

When N is large, the error probabilities are nearly the same as for detecting a classical narrowband signal of random phase in white Gaussian noise,

$$Q_0 = \exp(-\beta^2/2), \tag{7}$$

$$Q_1 = 1 - Q(\alpha, \beta) = \int_0^\beta x \exp[-\tfrac{1}{2}(x^2 + \alpha^2)]\, I_0(\alpha x)\, dx, \tag{8}$$

where

$$\beta^2/2 = \mu/N, \; \alpha^2/2 = S/N \;, \tag{9}$$

and $Q(\alpha, \beta)$ is Marcum's Q-function. For this binary communication system the value of β is the solution of the equation

$$e^{-\alpha^2/2} \; I_0(\alpha\beta) = 1, \tag{10}$$

and the average probability of error is[3]

$$P_e = \tfrac{1}{2} Q(\beta, \alpha) . \tag{11}$$

It has been plotted as the curve marked "$N = \infty$" in Fig. 2.

In a laser radar the receiver will be based on the Neyman-Pearson criterion and will maximize the detection probability $Q_d = 1 - Q_1$ for a fixed false-alarm probability Q_0. In order to attain exactly a pre-assigned value of Q_0, the decisions must be randomized.
For all numbers m of photons less than a certain number μ, hypothesis H_0 is chosen; for all $m > \mu$, H_1 is chosen. Whenever exactly μ photons are counted, hypothesis H_1 is chosen with probability f, H_0 with probability $(1 - f)$.

The false-alarm probability is now

$$\begin{aligned} Q_0 &= f(1-v)\, v^{\mu} + v^{\mu+1}, \\ v &= N/(N+1), \quad 0 < f \le 1, \end{aligned} \tag{12}$$

from which μ and f can be calculated for given values of Q_0 and N; μ is the greatest integer in the quotient $(\ln Q_0/\ln v)$. The probability of detection is now

$$Q_d = 1 - \sum_{m=0}^{\mu} P_{1m} + f\, P_{1\mu} \tag{13}$$

It has been plotted in Figs. 3-5 versus the average number S of signal photons and in Figs. 6-8 versus the signal-to-noise ratio S/N, for various values of N and for $Q_0 = 10^{-2}$, 10^{-4}, and 10^{-6}. The curves marked "$N = \infty$"

in Figs. 6-8 represent the detection probability for a classical receiver of a narrowband signal of unknown phase,[4]

$$Q_d = Q(\alpha, \beta) \ , \ \alpha = (2S/N)^{1/2} \ , \tag{14}$$

where β is determined by the false-alarm probability,

$$Q_0 = \exp(-\beta^2/2) \ . \tag{15}$$

In a previous paper[5] the probability of detection was erroneously stated to be given as in (14) for all values of N, but with $\alpha = [\,2S/(N+\frac{1}{2})\,]^{\frac{1}{2}}$. The detection statistic was regarded there as the modulus of a complex quantity, the mode amplitude, whose real and imaginary parts are independent Gaussian random variables. Those parts represent non-commuting observables, however, and their joint probability density function, from which the distribution of the modulus was derived, cannot legitimately be specified. The squared modulus is in fact proportional to the quantized variable m, whose probability distributions are given by (2) and (3) under the two hypotheses.

FOOTNOTES

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1. C. W. Helstrom, "Fundamental limitations on the detectability of electromagnetic signals," Int. J. Theor. Phys. vol. 1, pp. 37-50; May, 1968.
2. G. Lachs, "Theoretical aspects of mixtures of thermal and coherent radiation," Phys. Rev. vol. 138, pp. B 1012-B 1016; May 24, 1965.
3. C. W. Helstrom, Statistical Theory of Signal Detection, Pergamon Press, Oxford; 2nd ed., 1968, pp. 192-193.
4. Ibid., pp. 166-171.
5. C. W. Helstrom, "Quantum Limitations on the Detection of Coherent and Incoherent Signals," Trans. IEEE, vol. IT-11, pp. 482-490; October, 1965.

FIGURE CAPTIONS

FIG. 1: Average probability P_e of error versus average number S of signal photons.

FIG. 2: Average probability P_e of error versus signal-to-noise ratio S/N.

FIG. 3: Probability Q_d of detection versus average number S of signal photons; $Q_0 = 10^{-2}$.

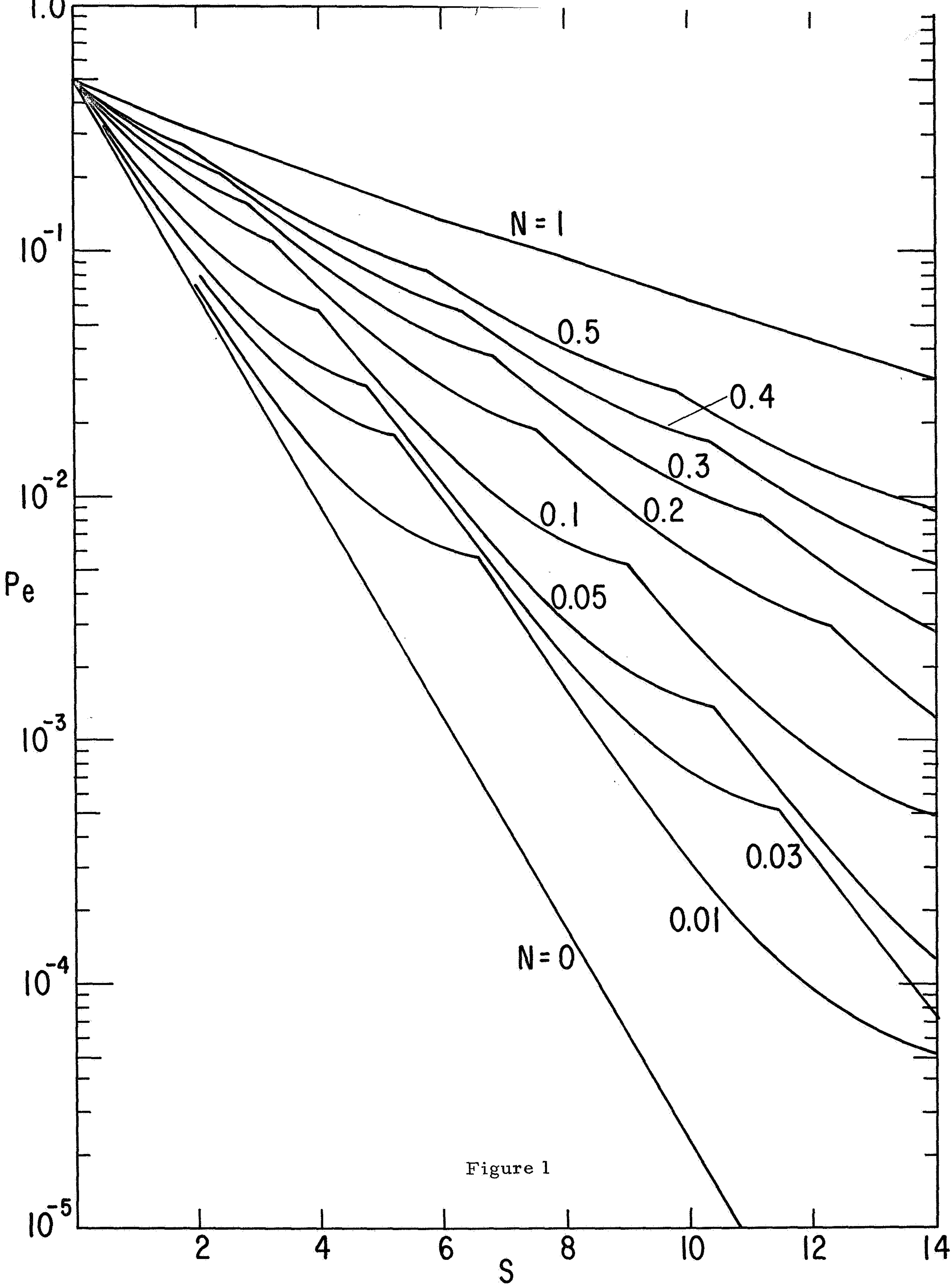
FIG. 4: Probability Q_d of detection versus average number S of signal photons; $Q_0 = 10^{-4}$.

FIG. 5: Probability Q_d of detection versus average number S of signal photons; $Q_0 = 10^{-6}$.

FIG. 6: Probability Q_d of detection versus signal-to-noise ratio S/N; $Q_0 = 10^{-2}$.

FIG. 7: Probability Q_d of detection versus signal-to-noise ratio S/N; $Q_0 = 10^{-4}$.

FIG. 8: Probability Q_d of detection versus signal-to-noise ratio S/N; $Q_0 = 10^{-6}$.


1.0
10^-1
10^-2
10^-3
10^-4
10^-5
Pe
N = 1
0.5
0.4
0.3
0.2
0.1
0.05
0.03
0.01
N = 0
2
4
6
8
10
12
14
S
Figure 1

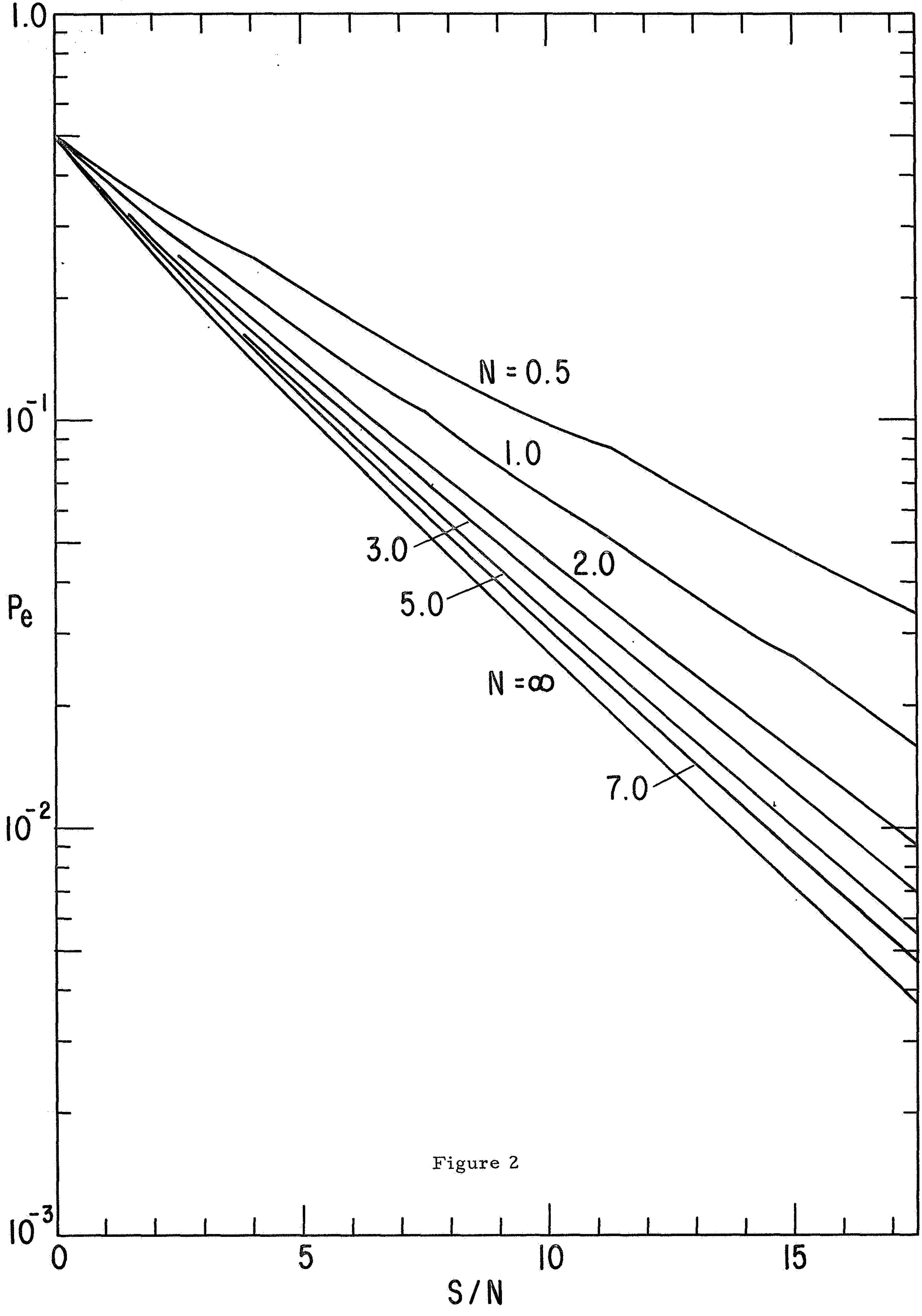


Figure 2

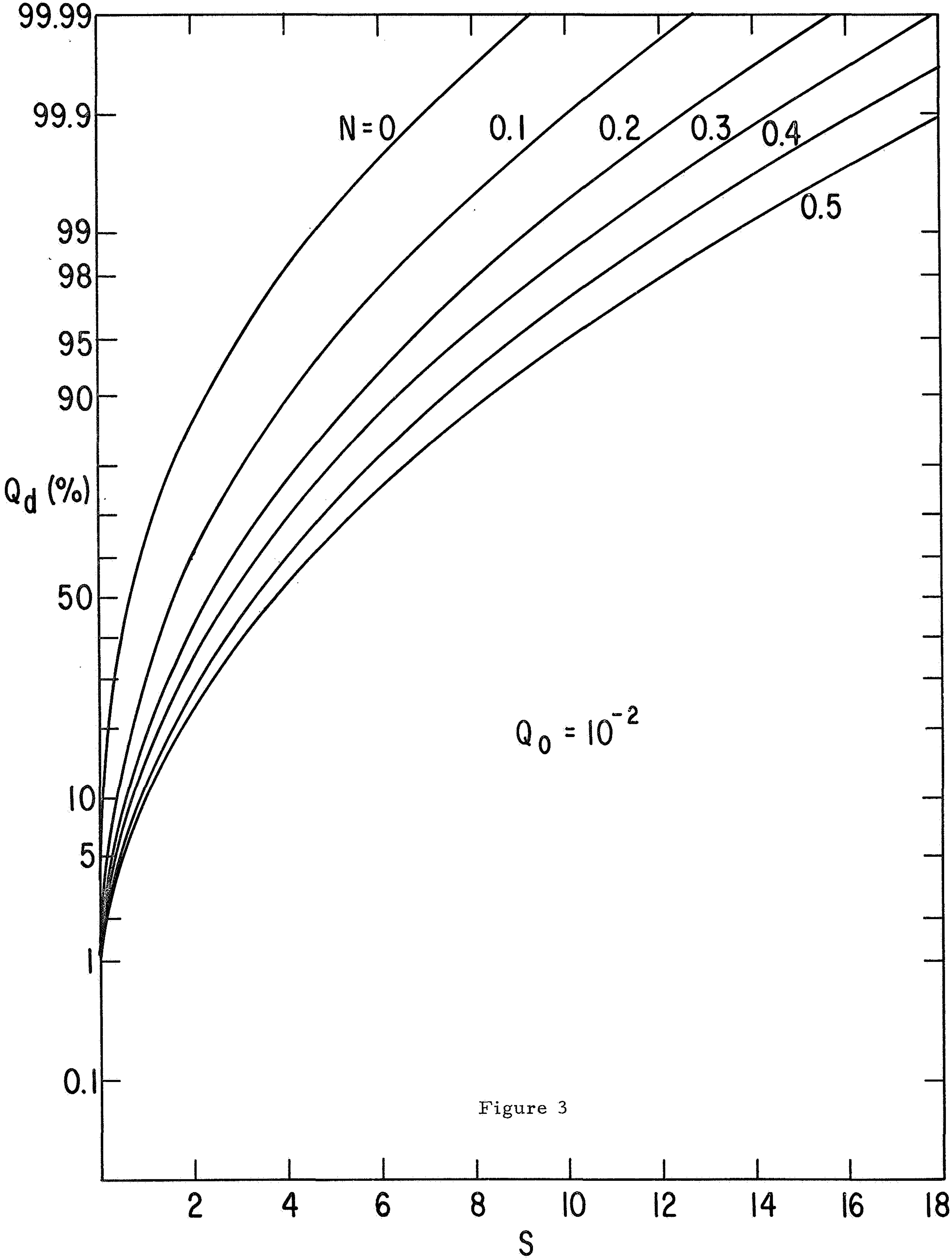

99.99
99.9
99
98
95
90
50
10
5
1
0.1
Qd (%)
N=0
0.1
0.2
0.3
0.4
0.5
Q0 = 10-2
2
4
6
8
10
12
14
16
18
S

Figure 3

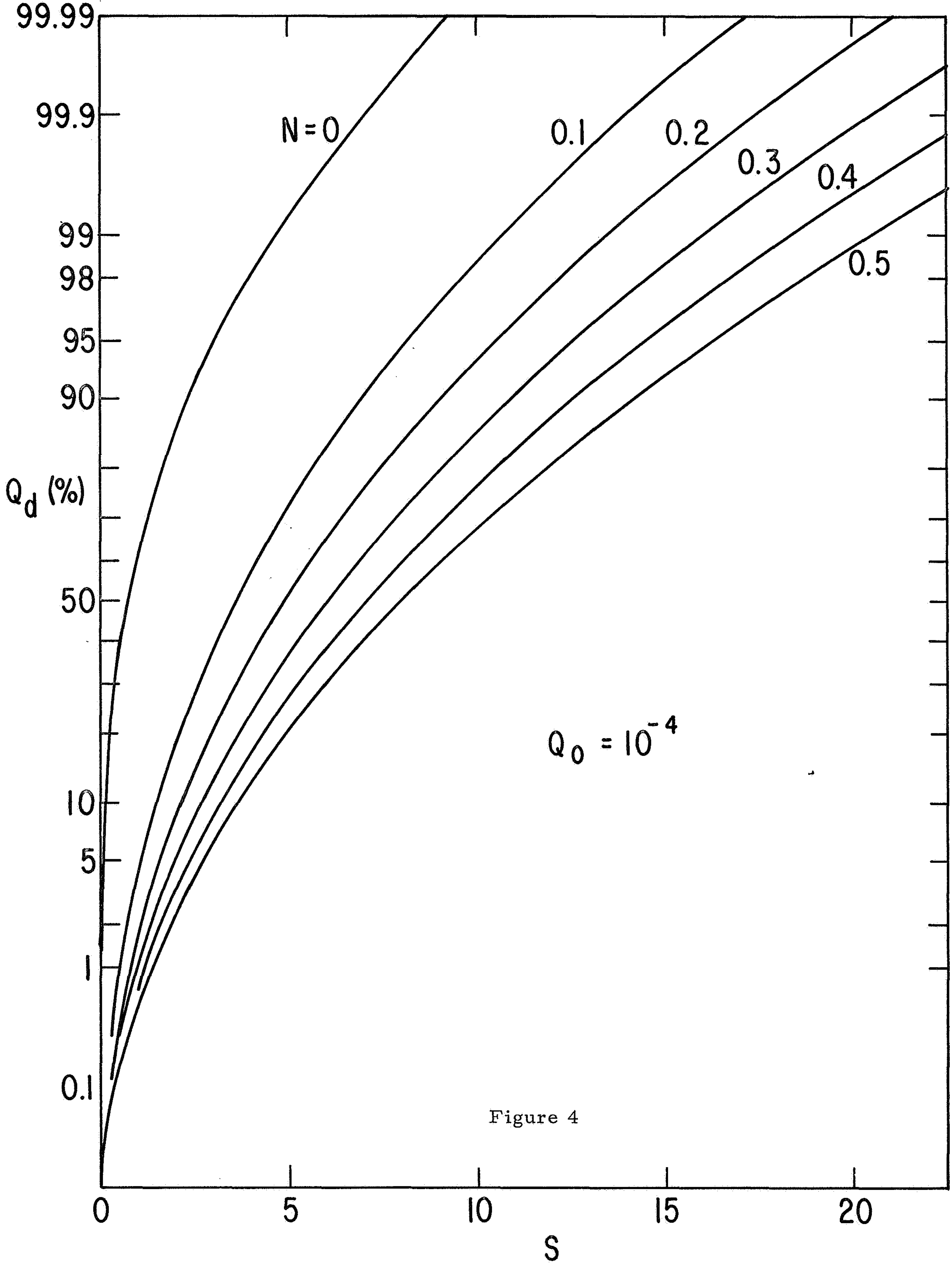
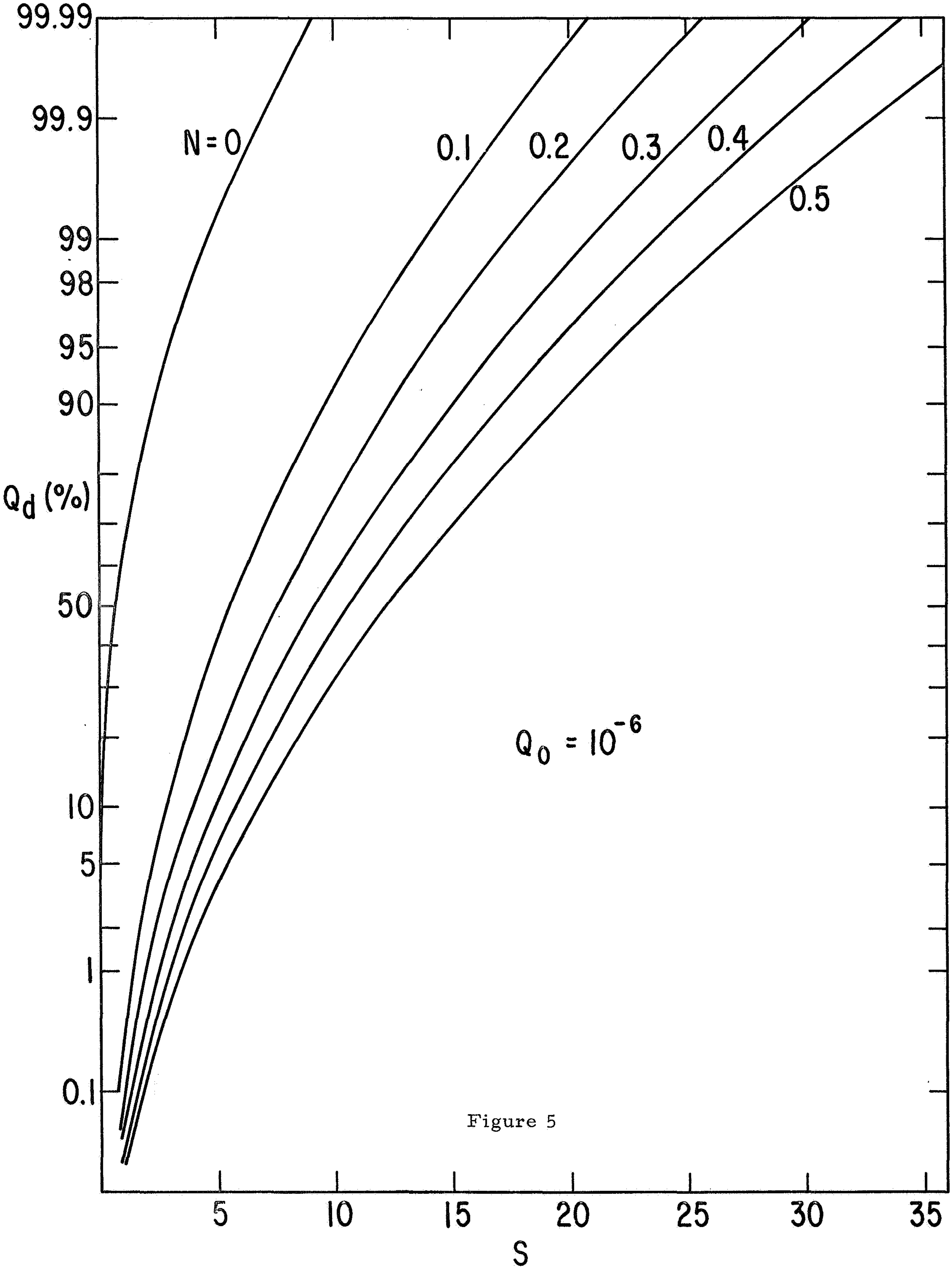
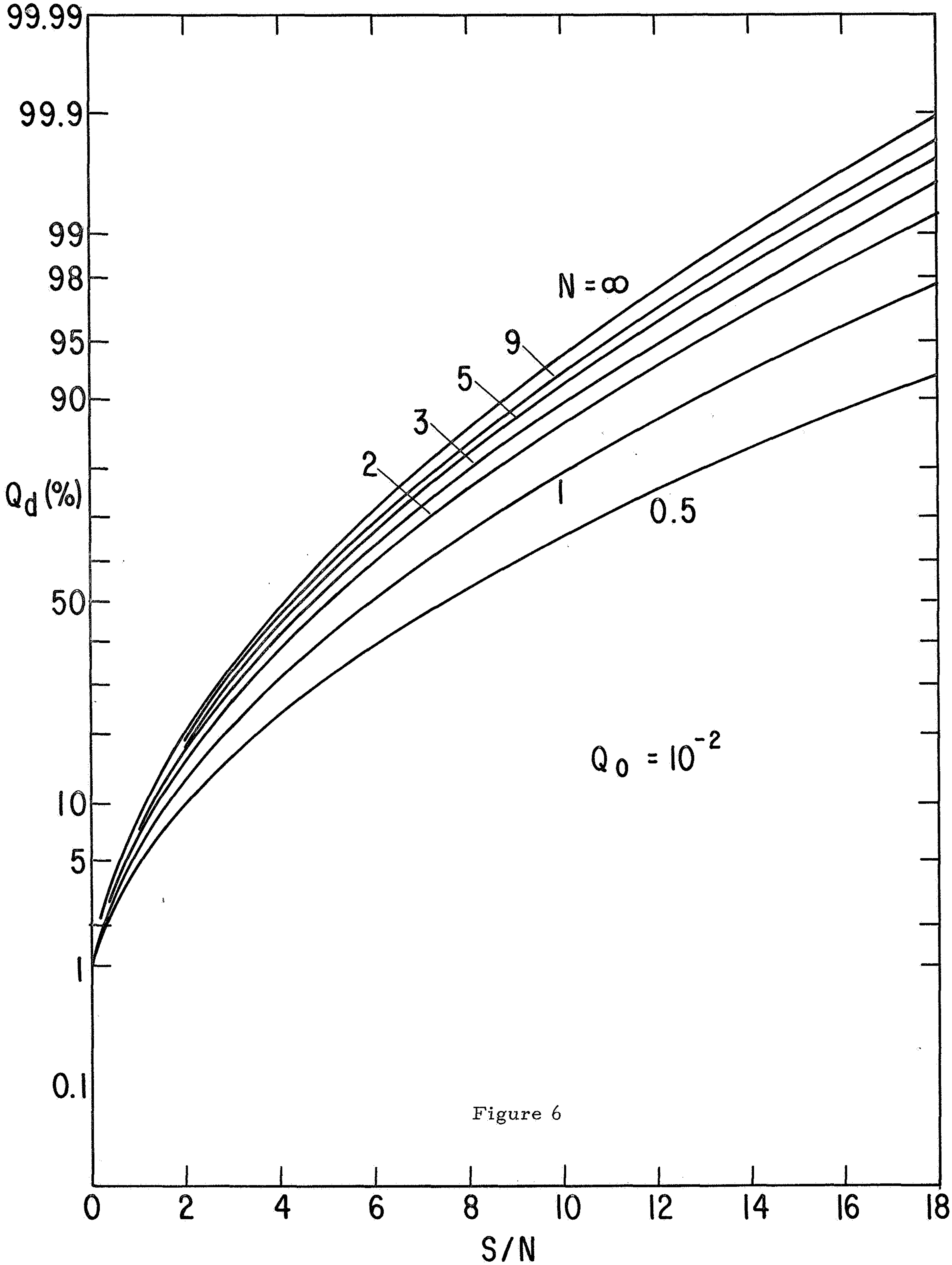

99.99
99.9
99
98
95
90
50
10
5
1
0.1
Q_d (%)
N=0
0.1
0.2
0.3
0.4
0.5
Q_0 = 10^-4
0
5
10
15
20
S

Figure 4



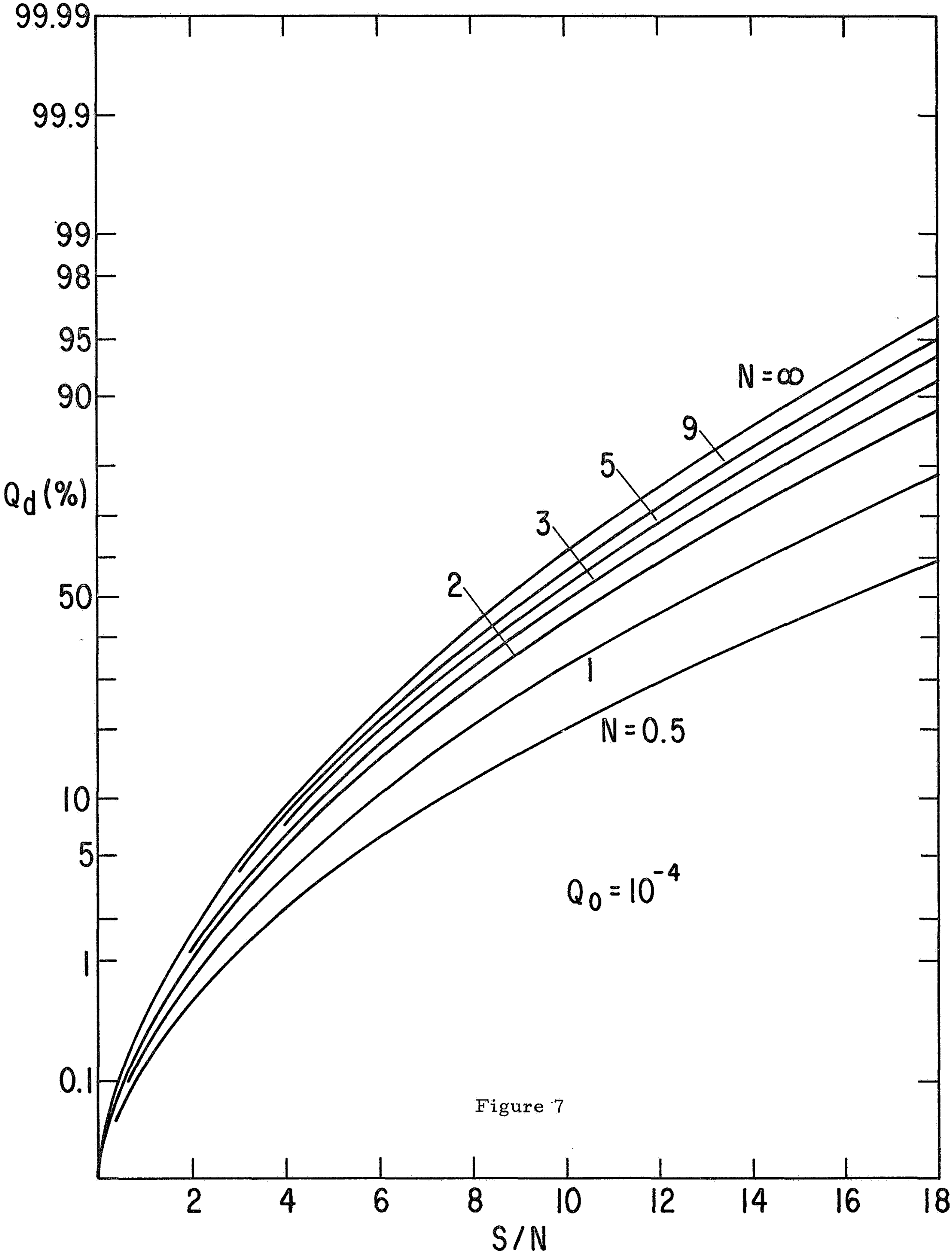
99.99
99.9
99
98
95
90
Q_d (%)
50
10
5
1
0.1
N = 0
0.1
0.2
0.3
0.4
0.5
$Q_0 = 10^{-6}$
5
10
15
20
25
30
35
S

Figure 5



99.99
99.9
99
98
95
90
Q_d (%)
50
10
5
1
0.1
$N = \infty$
9
5
3
2
1
0.5
$Q_0 = 10^{-2}$
0
2
4
6
8
10
12
14
16
18
S/N

Figure 6



99.99
99.9
99
98
95
90
50
10
5
1
0.1
Qd (%)
N = ∞
9
5
3
2
1
N = 0.5
Q0 = 10-4
2
4
6
8
10
12
14
16
18
S/N

Figure 7

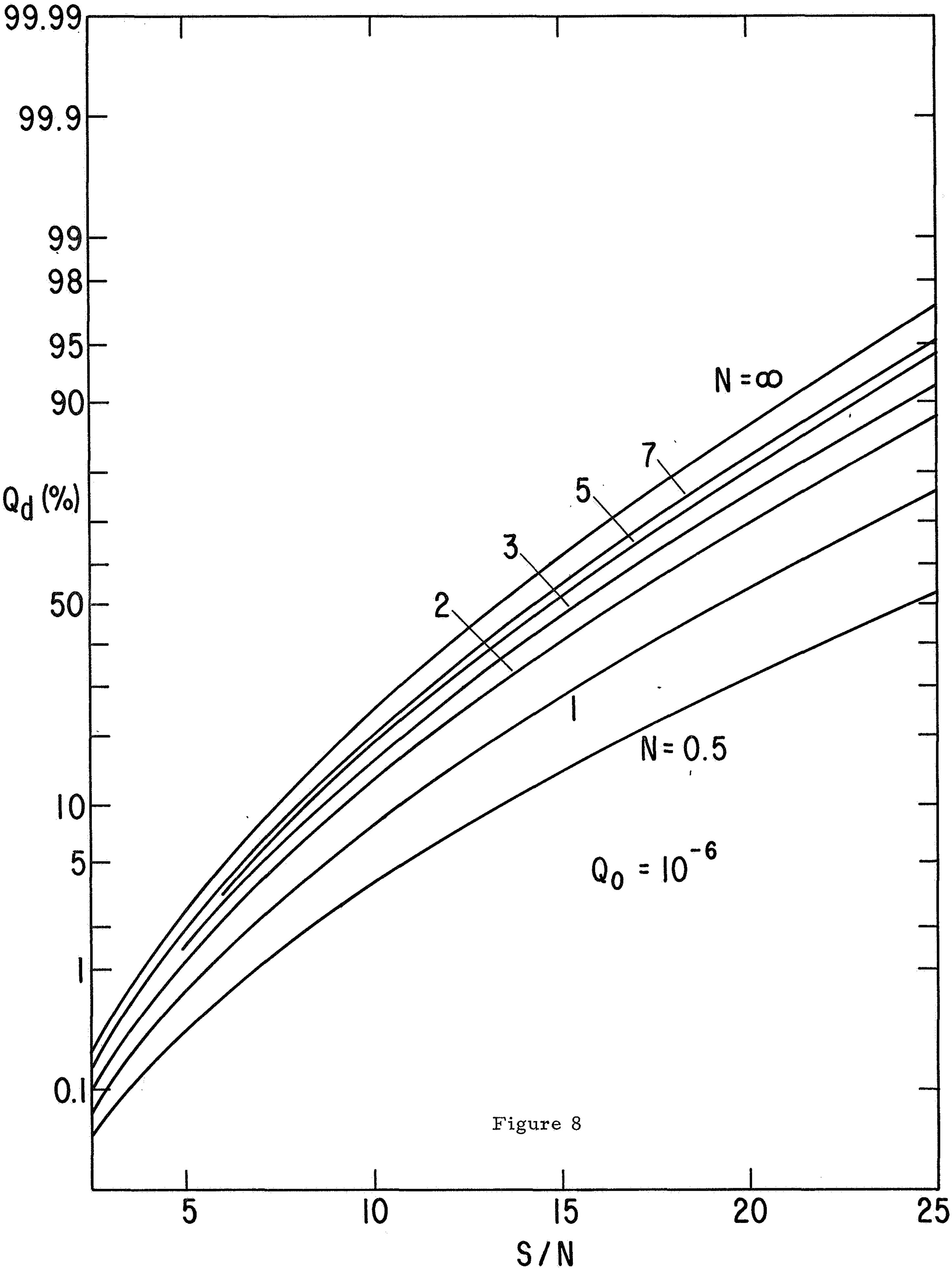

99.99
99.9
99
98
95
90
50
10
5
1
0.1
Qd (%)
N = ∞
7
5
3
2
1
N = 0.5
Q0 = 10^-6
5
10
15
20
25
S / N

Figure 8