Approximate Calculation of Cumulative Probability from a Moment-Generating Function

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ABSTRACT

A numerical method is presented for calculating the cumulative distribution of a positive random variable from its moment-generating function. It involves an expansion of the rectangular function in Laguerre functions. As examples, the cumulative exponential and cumulative Poisson probability functions are approximated.

A common problem is the calculation of the cumulative probability distribution

$$Q(x) = \int_0^x p(y)\, dy, \quad 0 < x < \infty, \tag{1}$$

of a positive random variable y of which one knows only the moment-generating function (m. g. f.),

$$\mu(s) = \underset{\sim}{E}(e^{ys}) = \int_0^\infty e^{ys}\, p(y)\, dy, \tag{2}$$

where p(y) is the probability density function (p. d. f.) of y.

In signal detection theory, for instance, y is related to the likelihood ratio, and 1-Q(x) is the false-alarm or detection probability for a decision level x. Often the m. g. f. can be worked out rather easily, but it is impossible to determine p(y) from $\mu(s)$ analytically by, for instance, taking the inverse Laplace transform of $\mu(-s)$ or the inverse Fourier transform of $\mu(i\omega)$.

A technique for calculating Q(x) numerically can be derived by writing (2) as

$$Q(x) = \int_0^\infty R(y/x)\, p(y)\, dy, \tag{3}$$

where R(t) is the rectangular function

$$R(t) = 1, \ 0 < t < 1; \ R(t) = 0, \ t > 1. \tag{4}$$

One expands R(t) in a series of Laguerre functions,[1]

$$R(t) = e^{-kt/2} \sum_{m=0}^{\infty} a_m L_m(kt), \tag{5}$$

where[2]

$$a_m = k \int_0^1 e^{-kt/2} L_m(kt)\, dt =$$

$$2e^{-k/2} \left[L_{m-1}(k) - L_m(k) \right] - a_{m-1}. \tag{6}$$

The series in (5) is to be truncated at a finite number M of terms.

The cumulative distribution is

$$Q(x) = \sum_{m=0}^{\infty} a_m C_m(x), \tag{7}$$

where the coefficients

$$C_m(x) = \int_0^{\infty} e^{-ky/2x} L_m(ky/x)\, p(y)\, dy \tag{8}$$

can be expressed in terms of $\mu(-k/2x)$ and its derivatives. In particular,

$$C_0(x) = \mu(-k/2x), \tag{9}$$

and by using the fomula[3]

$$L_m(t) = (-1)^m t^m/m! - \sum_{r=1}^{m} (-1)^r \binom{m}{r} L_{m-r}(t), \tag{10}$$

a recurrence relation for $C_m(x)$ is easily obtained.

$$C_m(x) = 2^m (m!)^{-1} \left\{ s^m \frac{d^m}{ds^m} [\mu(s)] \right\} \Bigg|_{s=-k/2x} - \sum_{r=1}^{m} (-1)^r \binom{m}{r} C_{m-r}(x), \tag{11}$$

which facilitates numerical computation.

The method was tried out with two very different distributions,

the exponential,

$$p(y) = e^{-y}, \quad y > 0; \; p(y) = 0, \quad y < 0, \tag{12}$$

whose m. g. f. is

$$\mu(s) = (s-1)^{-1}, \quad R\ell\, s < 1, \tag{13}$$

and the Poisson,

$$p(y) = e^{-\lambda} \sum_{n=0}^{\infty} \lambda^n \delta(y-n)/n! \tag{14}$$

whose m. g. f. is

$$\mu(s) = \exp[\lambda(e^s - 1)] \tag{15}$$

First it was necessary to determine the best value of the scale parameter k when M terms are used. This was done by hunting the value of k that yielded the minimum mean-square error

$$\mathcal{E} = 1 - \sum_{m=0}^{M-1} a_m^2 \qquad (16)$$

in fitting the truncated version of (5) to R(t). For M = 20, we found that k = 43 gives a mean-square error $\mathcal{E} = 0.01567$. The coefficients a_m are listed in Table 1.

For the exponential p. d. f. we list in Table 2 the percentage error in Q(x) for $0 < Q(x) < 1/2$ and the percentage error in $1 - Q(x)$ for $1/2 < Q(x) < 1$. The relative error decreases with increasing x.

For the Poisson distribution we evaluated Q(x) by the approximation method for values of x halfway between the integers and compared the results with the Poisson distribution summed from y = 0 to the greatest integer in x. Table 3 lists the percentage errors in Q(x) for $0 < Q(x) < 1/2$ and in $1 - Q(x)$ for $1/2 < Q(x) < 1$.

The accuracy is greatest near the mean and poorest in the tails of the Poisson distribution, and this can be expected in most applications. There exist other approximation methods best suited for the tails of a distribution. For large x the inverse Laplace transform of $\mu(-s)$ can be approximated by the method of steepest descents.[4] For x near 0, an approximation to Q(x) can be obtained from the asymptotic behavior[5] of $\mu(-s)$ for large s. The method described here fills the gap.

An alternative method is the Edgeworth series, but it has an asymptotic character that restricts its usefulness.[6] There is an optimum number of terms in the Edgeworth series, and if more are used, the accuracy decreases markedly. Numerical Fourier transformation of $\mu(i\omega)$, followed by numerical integration of the p. d. f. $p(y)$, might be used in some cases, but would hardly be suitable for a discrete random variable like the Poisson-distributed one of our second example.

TABLE 1

Coefficients of Laguerre Expansion

m	0	1	2	3	4	5	6	7	8
a_m	2	-2	2	-2	1.9999	-1.9992	1.9956	-1.9800	1.9261

m	9	10	11	12	13	14
a_m	-1.7766	1.4460	-0.87710	0.15695	0.41416	-0.49039

m	15	16	17	18	19
a_m	0.071648	0.33527	-0.22910	-0.18569	0.24351

TABLE 2

Exponential Distribution

x	0.1	0.3	0.5	1.0	1.5	2.0
Error (%)	0.743	0.561	0.422	-0.351	-0.336	-0.285
x	3.0	4.0	5.0	6.0	8.0	10.0
Error (%)	-0.170	-0.0878	-0.0408	-0.0174	-0.00239	-0.000224

TABLE 3

Poisson Distribution

x	6	8	10	12	14	16
Q	0.00763	0.0374	0.1185	0.2676	0.4656	0.6641
Error (%)	-59.6	-17.6	-2.82	-0.133	0.00493	0.135

x	18	20	22	24	26	28	30
Q	0.8195	0.9170	0.9673	0.9888	0.99669	0.99914	0.99980
Error (%)	0.241	0.735	1.199	1.379	-0.415	-20.2	72.2

FOOTNOTES

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1. J. W. Head and W. P. Wilson, "Laguerre Functions: Tables and Properties," Proceedings of the Institute of Electrical Engineers, vol. 103C, pp. 428-436; June, 1956.
2. See eq. (38), reference 1, p. 434.
3. See eq. (67), reference 1, p. 435.
4. G. Doetsch, Handbuch der Laplace-Transformation, Birkhäuser Verlag, Basel and Stuttgart, 1955, vol. 2, ch. 3, § 5, pp. 83-88.
5. Ibid, vol. 2, ch. 3, § 1, pp. 45-50 and § 7, pp. 92-94.
6. T. C. Fry, Probability and Its Engineering Uses, D. Van Nostrand Co., Princeton, 2nd ed., 1965, p. 262.