THE APPLICATION OF SIGNAL DETECTION THEORY TO OPTICS

PROGRESS REPORT

NASA GRANT 05-009-079

December 15, 1968

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ABSTRACT

Progress in research on the application of signal detection theory to optics is described. The quantum-mechanical threshold detector for an incoherent object observed against a background of thermal light has been derived and its performance analyzed. Curves of detection probability versus signal strength for coherent and incoherent optical signals in thermal noise are presented. A method for calculating cumulative probability from a moment-generating function is proposed. The restoration of degraded images is treated as a problem in statistical estimation theory.

I. Detection of Incoherent Objects

When viewing a scene, an optical instrument such as a telescope performs two functions, deciding whether objects are present in the scene and estimating parameters, such as location and brightness, of the objects it discovers. The performance of the instrument with respect to the first of these tasks can be measured by the probability that it will detect a certain test object as a function of the size and contrast of the object, for a fixed probability of a false alarm (saying the object is present when it is not). It is useful to compare this detection probability with the maximum probability of detecting the object by any instrument subject to the same conditions of background radiation and admitting light through an aperture of the same area.

A natural object normally emits or reflects incoherent light over a spectral range whose width W is much greater than the reciprocal T^{-1} of the time during which it is observed. In the neighborhood of the object, the emitted or reflected light, under most conditions of radiation or illumination, possesses negligible first-order coherence. The light is propagated for a great distance to the aperture of the viewing instrument, which usually subtends such a small solid angle from the object that the object light possesses some considerable degree of first-order spatial coherence over the aperture. Mixed with the object light is background light of a thermal or quasi-thermal variety, which can be considered a type of interfering noise. The frequency spectrum and angular distribution of this background light are much broader than those of the light from the object.

The detection of the object can be treated as a problem of hypothesis testing. The task of the optical instrument is viewed as one of choosing between two hypotheses, (H_0) that only background light is entering the instrument and (H_1) that in addition the light contains a component coming from the object. One seeks that design which will permit correct choice of hypothesis H_1 (a detection) with maximum probability Q_d, for a fixed probability Q_0 of choosing H_1 when H_0 is true (a false alarm).[1]

In a previous paper[2] the detection of an incoherently radiating object in the presence of background radiation was treated under the classical assumption that the electromagnetic field at the aperture of the instrument is completely measurable. The maximum detection probability was found to depend on an equivalent signal-to-noise ratio

$$D = (E/N)\ (TW)^{-1/2}\ \mathcal{F}, \tag{1}$$

where E is the total radiant energy received from the object during the observation interval $(0, T)$, N is the spatio-temporal spectral density of the background, W is the bandwidth of the object light, and $\mathcal{F}$ is a spatial factor that equals 1 when the object light possess full first-order coherence at the aperture, but which decreases to 0 with the degree of first-order coherence. If $\mathcal{T}$ is the effective temperature of the background light, $N = K\mathcal{T}$, where K is Boltzmann's constant.

A second paper[3] treated the detection of an incoherent object whose light has passed through a turbulent medium before reaching the aperture. A similar dependence of the detectability on the degrees of spatial and temporal coherence of the object light was discovered.

Much of our research under this grant during the past few months has been devoted to extended this theory to cover quantum-limited detection. The results are described in a paper, "Detection of Incoherent Objects by a Quantum-Limited Optical System", attached to this report.

In this study we no longer assumed that the field at the aperture is classically measurable. Instead, it is subject to the laws of quantum mechanics, which limit the extent to which it can be measured and require that the hypothesis-testing problem be attacked by the methods of quantum detection theory.[4]

Again the detectability of the object was found to depend on the number $M' = TW/\mathcal{F}^2$ of effectively independent spatio-temporal degrees of freedom of the light from the object as received at the aperture during the observation interval $(0, T)$. An important parameter is the product $\mathcal{N}M'$, where

$$\mathcal{N} = (e^{h\nu/K\mathcal{T}} - 1)^{-1} \qquad (2)$$

is the mean number of thermal photons per mode of the field, h is Planck's constant and ν is the central angular frequency of the object light.

For an object radiating light with a rectangular spectrum of width W, the detectability of the object was found to be governed by a cumulative Poisson distribution with mean $\mathcal{N}M' + N_s$, where $N_s = E/h\nu$ is the mean total number of photons received by the object. In general, $\mathcal{N} << 1$, but $M' >> 1$, so that $\mathcal{N}M'$ may be of the order of 1. As $\mathcal{N}$ increases, the governing distribution becomes approximately Gaussian, and detectability depends on a signal-to-noise ratio

$$D = N_s \left[M'\mathcal{N}(\mathcal{N}+1) \right]^{-1/2} \tag{3}$$

When $K\mathcal{T} >> h\nu$ (the classical limit), $\mathcal{N} >> 1$ and this signal-to-noise ratio becomes equal to the one in Eq. (1) derived before.[2]

In Figs. 1-3, we have plotted the probability Q_d of detection versus the mean number N_s of signal photons for three values of the false-alarm probability Q_0. In Figs. 4-6 we plotted Q_d versus the signal-to-noise ratio D for the same false-alarm probabilities. In these figures the curves are indexed by the mean total number $\mathcal{N}M'$ of thermal photons.

When the object spectrum is not rectangular, the distribution governing detectability is no longer Poisson. For a Lorentz spectrum, more typical of naturally radiating objects, the moment-generating function of the distribution was worked out. Numerical methods will be required for evaluating the false-alarm and detection probabilities.

II. Detection of Coherent Optical Signals

In laser radar and communication systems the transmitted pulses possess a high degree of coherence. The performance of such systems depends on how well such pulses can be detected in the presence of background radiation. The maximum probability of detecting a coherent optical pulse, for a fixed false-alarm probability, was calculated as a function of the signal energy and the background level. Details and results are given in the attached paper, "Performance of an Ideal Quantum Receiver of a Coherent Signal of Random Phase."

The optimum detector of such a coherent signal of random phase in effect filters it by creating a field mode matched to the signal field itself. It then counts the number of photons in that matched mode when it is exposed to the incident light, a number that has a Laguerre distribution when the signal is present and an exponential distribution when it does not.[5]

III. Evaluation of Detection Probabilities

An optical or radio-frequency receiver makes its decisions whether a signal is present by comparing with a fixed decision level x_o the value of a certain quantity, or "statistic", x, which it generates. The false-alarm probability Q_0 is the probability that x exceeds the level x_o when no signal is present; the detection probability is the probability that $x > x_o$ when the signal is present.

In many cases it is difficult to calculate these probabilities. The most that can easily be done is to determine the moment-generating function (m. g. f.) of x, which is the average value of e^{sx} as a function of s. (For $s = i\omega$ this is the familiar characteristic function.) What is needed is a method of calculating false-alarm and detection probabilities from the m. g. f.'s of x under the two hypotheses H_0 and H_1.

Such a method, based on Laguerre functions, has been worked out and tested with some simple distributions. Details are given in a paper, "Approximate Calculation of Cumulative Probability from a Moment-Generating Function," attached to this report. The method is to be used in calculating the detectability of an incoherent object having a Lorentz spectrum, a problem mentioned at the end of Section I.

An alternative method is also being tried. It involves expanding a rectangular pulse in a series of orthonormal functions composed of exponential functions. Computational difficulties have so far prevented our approximating the rectangular pulse closely enough to permit accurate calculation of detection probabilities.

IV. Image Restoration as an Estimation Problem

When a scene is viewed through the atmosphere, turbulence causes distortion that impedes identification of features in the scene. Background light further degrades any image that can be formed, and if the light from the scene has passed through apertures and lenses, diffraction and aberration introduce additional distortion. An important problem in optics is to discover, by measuring the light at some plane in the receiving optical instrument, the nature of the original scene. It is often called "image restoration."

Image restoration is really a matter of estimating the radiance of the object plane in a composite optical system made up of that plane at one end, the intervening medium, the aperture of the observing instrument, any lenses and stops it may contain, and -- at the other end -- an image plane. How accurately that radiance can be estimated depends on the amount of corrupting background radiation, on the turbulence of the medium, and on the aperture size and other characteristics of the optical instrument. In particular, one would like to know how accurate an estimate can be obtained by any instrument in which the light is taken in through an aperture of given shape and size. To attack this problem, one must draw upon statistical estimation theory. Here we shall describe the progress that has been made toward a solution.

1. The Optical Field.

We use the same notation and make the same assumptions as in Reference 2. In particular, the radiance distribution of the object plane is $B(\underset{\sim}{u})$, where $\underset{\sim}{u}$ is a 2-vector of coordinates in that plane. For simplicity we consider a quasimonochromatically radiating object, and we let the point-spread function between object and aperture planes --see Fig. 7 -- be $S(\underset{\sim}{r}, \underset{\sim}{u})$. In the absence of any turbulence,

$$S(\underset{\sim}{r}, \underset{\sim}{u}) = \frac{ik}{2\pi R} \exp\left(ikR + \frac{ik}{2R} \left| \underset{\sim}{r} - \underset{\sim}{u} \right|^2\right), \qquad (1.1)$$

where k is the propagation constant of the object light and R is the distance between object and aperture planes.

Let $\Psi_s(\underset{\sim}{u}, z; t)$ be the scalar light field at a point $(\underset{\sim}{u}, z)$; then

$$\Psi_s(\underset{\sim}{r}, 0; t) = \int_O S(\underset{\sim}{r}, \underset{\sim}{u}) \Psi_s(\underset{\sim}{u}, R; t)\, d^2\underset{\sim}{u} \qquad (1.2)$$

is the field at the aperture z = 0 in terms of that at the object plane z = R. Here "O" indicates that the integral is carried out over the object plane.

The object plane is assumed to radiate completely incoherently, and $\Psi_s(\underset{\sim}{u}, R; t)$ is a circular complex Gaussian random process of mean 0 and autocovariance functions

$$\frac{1}{2} \langle \Psi_s(\underset{\sim}{r}_1, R; t_1) \Psi_s^*(\underset{\sim}{r}_2, R; t_2)\rangle =$$

$$\pi k^{-2} B(\underset{\sim}{r}_1)\, \delta(\underset{\sim}{r}_1 - \underset{\sim}{r}_2)\, \chi(t_1 - t_2), \qquad (1.3)$$

$$\langle \Psi_s(\underset{\sim}{r}_1, R; t_1) \Psi_s(\underset{\sim}{r}_2, R; t_2)\rangle = 0 .$$

Here $\chi(\tau)$ is the temporal autocovariance function of the object light,

$$\chi(\tau) = \int_{-\infty}^{\infty} X(\omega)\, e^{i\omega\tau}\, d\omega/2\pi, \qquad (1.4)$$

where $X(\omega)$ is the temporal spectral density of the object light, with angular frequency ω measured from the central frequency $\Omega = kc$ of the object spectrum. It is so normalized that $\chi(0) = 1$.

After propagation from object plane to aperture, the field of the object light is as given by Eq. (1.2), and its autocovariance function is

$$\varphi_s(\underset{\sim}{r}_1, t_1; \underset{\sim}{r}_2, t_2) = \frac{1}{2} \langle \Psi_s(\underset{\sim}{r}_1, 0; t_1) \Psi_s^*(\underset{\sim}{r}_2, 0; t_2) \rangle$$

$$= \varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2) \chi(t_1 - t_2), \tag{1.5}$$

where

$$\varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2) = \pi k^{-2} \int_O S(\underset{\sim}{r}_1, \underset{\sim}{u}) S^*(\underset{\sim}{r}_2, \underset{\sim}{u}) B(\underset{\sim}{u}) d^2\underset{\sim}{u}. \tag{1.6}$$

If turbulence is present, the right-hand side of Eq. (1.6) must be further averaged with respect to its ensemble of configurations. For simplicity we assume in the sequel that there is no turbulence.

The field $\Psi_s(\underset{\sim}{r}, 0; t)$ at the aperture is corrupted by background radiation whose field $\Psi_n(\underset{\sim}{r}, 0; t)$ is circular complex, Gaussian, and -- we postulate -- spatially and temporally white. Its autocovariance function is

$$\frac{1}{2} \langle \Psi_n(\underset{\sim}{r}_1, 0; t_1) \Psi_n^*(\underset{\sim}{r}_2, 0; t_2) \rangle = \varphi_n(\underset{\sim}{r}_1, t_1; \underset{\sim}{r}_2, t_2) =$$

$$N\delta(\underset{\sim}{r}_1 - \underset{\sim}{r}_2) \delta(t_1 - t_2). \tag{1.7}$$

The net observed field is

$$\Psi(\underset{\sim}{r}, t) = \Psi_s(\underset{\sim}{r}, 0; t) + \Psi_n(\underset{\sim}{r}, 0; t); \tag{1.8}$$

its autocovariance is the sum

$$\frac{1}{2} \langle \Psi(\underset{\sim}{r}_1, t_1) \, \Psi^*(\underset{\sim}{r}_2, t_2) \rangle = \varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2) \, \chi(t_1 - t_2) + \varphi_n(\underset{\sim}{r}_1, t_1, \underset{\sim}{r}_2, t_2). \tag{1.9}$$

This total autocovariance function depends through Eq. (1.6) on the radiance distribution $B(\underset{\sim}{u})$. The problem is to estimate $B(\underset{\sim}{u})$ as accurately as possible from measurements of the field $\Psi(\underset{\sim}{r}, t)$ over the aperture A during a fixed observation interval $(0, T)$. The problem is equivalent to estimating parameters of the covariance matrix of a Gaussian random process on the basis of measurements of the process.

2. Maximum Likelihood Estimation*

The radiance distribution $B(\underset{\sim}{u})$ is to be estimated by the method of maximum likelihood. The likelihood ratio for detecting an object having the radiance distribution $B(\underset{\sim}{u})$ is written down in terms of the field $\Psi(\underset{\sim}{r}, t)$ at the aperture A during $(0, T)$. The maximum-likelihood estimate $B(\underset{\sim}{u})$ is that radiance function for which the likehood ratio takes on its greatest value.

In order to calculate the likelihood ratio we sample the field $\Psi(\underset{\sim}{r}, t)$ by expanding it in a series of functions $\eta_n(\underset{\sim}{r}, t)$ orthonormal over A and $(0, T)$,

$$\iint_{A\,0}^{T} \eta_n(\underset{\sim}{r}, t) \, \eta_m^*(\underset{\sim}{r}, t) \, d^2\underset{\sim}{r} \, dt = \delta_{nm}. \tag{2.1}$$

*The calculations in this section were carried out by Mr. Y. M. Hong.

The coefficients of the expansion are

$$\Psi_n = \int_A \int_0^T \eta_n^*(\underset{\sim}{r}, t)\, \Psi(\underset{\sim}{r}, t)\, d^2 \underset{\sim}{r}, \tag{2.2}$$

and the field is

$$\Psi(\underset{\sim}{r}, t) = \sum_n \Psi_n \eta_n (\underset{\sim}{r}, t) \ . \tag{2.3}$$

We arrange the coefficients in a column vector $\underset{\sim}{\Psi}$, whose Hermitian-transpose row vector $\underset{\sim}{\Psi}^+$ is

$$(\Psi_1^*, \Psi_2^*, \ldots \Psi_n^*, \ldots)$$

The coefficients are circular complex Gaussian random variables with covariance matrix

$$\underset{\sim}{\varphi}_1 = \frac{1}{2} \langle \underset{\sim}{\Psi} \underset{\sim}{\Psi}^* \rangle = \underset{\sim}{\varphi}_s + \underset{\sim}{\varphi}_b , \tag{2.4}$$

where $\underset{\sim}{\varphi}_s$ is the covariance matrix of the samples of the object field and

$$\underset{\sim}{\varphi}_b = N\, \underset{\sim}{I} , \tag{2.5}$$

with $\underset{\sim}{I}$ the identity matrix, is the covariance matrix of the samples of the background field.

The joint probability density function (p. d. f.) of the samples Ψ_n is

$$p_1(\underset{\sim}{\Psi}) = M \left| \det \underset{\sim}{\varphi}_1 \right|^{-1} \exp \left(-\frac{1}{2} \underset{\sim}{\Psi}^+ \underset{\sim}{\varphi}_1^{-1} \underset{\sim}{\Psi}\right) \tag{2.6}$$

where M is a normalization constant. If there were no object present, their joint p. d. f. would be

$$p_0 (\underset{\sim}{\Psi}) = M \left|\det \underset{\sim}{\varphi}_n \right|^{-1} \exp \left(- \frac{1}{2} \underset{\sim}{\Psi}^+ \underset{\sim}{\varphi}_n^{-1} \underset{\sim}{\Psi}\right), \tag{2.7}$$

which does not depend on the radiance distribution $B(\underset{\sim}{u})$. Maximizing $p_1(\underset{\sim}{\Psi})$ with respect to $B(\underset{\sim}{u})$ is therefore the same as maximizing the likelihood ratio

$$\Lambda(\underset{\sim}{\Psi}) = \frac{p_1(\underset{\sim}{\Psi})}{p_0(\underset{\sim}{\Psi})} = \left| \det \underset{\sim}{\varphi}_n \underset{\sim}{\varphi}_1^{-1} \right| \exp\left[\frac{1}{2} \underset{\sim}{\Psi}^*(\underset{\sim}{\varphi}_n^{-1} - \underset{\sim}{\varphi}_1^{-1}) \underset{\sim}{\Psi}\right], \tag{2.8}$$

or its logarithm. The reason for introducing $p_0(\underset{\sim}{\Psi})$ is that it is easier to take the logarithm of the likelihood ratio $\Lambda(\underset{\sim}{\Psi})$ to the limit to an infinite number of samples than to do the same to the p.d.f. $p_1(\underset{\sim}{\Psi})$ alone. We shall, however, postpone this passage to the limit.

A natural source has a bandwidth W so great that for ordinary observation intervals $(0, T)$, the product TW is very large, $TW \gg 1$. As discussed in reference 2, the object light can then be thought of as composed of so large a number of effectively independent degrees of freedom that the signal-to-noise ratio for each is very small. As a result, the logarithm $\ell n\, \Lambda(\underset{\sim}{\Psi})$ of the likelihood ratio can be expanded in a series, the so-called "threshold expansion".

We write

$$\begin{aligned}
\underset{\sim}{\varphi}_1^{-1} &= (\underset{\sim}{\varphi}_n + \underset{\sim}{\varphi}_s)^{-1} = \underset{\sim}{\varphi}_n^{-1}(\underset{\sim}{I} + \underset{\sim}{\varphi}_s \underset{\sim}{\varphi}_n^{-1})^{-1} \\
&= \underset{\sim}{\varphi}_n^{-1} - \underset{\sim}{\varphi}_n^{-1} \underset{\sim}{\varphi}_s \underset{\sim}{\varphi}_n^{-1} + \underset{\sim}{\varphi}_n^{-1} \underset{\sim}{\varphi}_s \underset{\sim}{\varphi}_n^{-1} \underset{\sim}{\varphi}_s \underset{\sim}{\varphi}_n^{-1} - \ldots \\
&= N^{-1}\underset{\sim}{I} - N^{-2}\underset{\sim}{\varphi}_s + N^{-3}\underset{\sim}{\varphi}_s^2 - \ldots
\end{aligned} \tag{2.9}$$

by virtue of Eq. (2.5). Similarly

$$\ln \det (\underset{\sim}{\varphi}_n \underset{\sim}{\varphi}_1^{-1}) = - \mathrm{Tr} \ln (\underset{\sim}{\varphi}_n^{-1} \underset{\sim}{\varphi}_1)$$

$$= -\mathrm{Tr} \ln (\underset{\sim}{I} + \underset{\sim}{\varphi}_n^{-1} \underset{\sim}{\varphi}_s) =$$

$$- \mathrm{Tr} (\underset{\sim}{\varphi}_n^{-1} \underset{\sim}{\varphi}_s - \frac{1}{2} \underset{\sim}{\varphi}_n^{-1} \underset{\sim}{\varphi}_s \underset{\sim}{\varphi}_n^{-1} \underset{\sim}{\varphi}_s + \ldots)$$

$$= - N^{-1} \mathrm{Tr} (\underset{\sim}{\varphi}_s - \frac{1}{2} N^{-1} \underset{\sim}{\varphi}_s^{\,2} + \ldots) \tag{2.10}$$

where "Tr" stands for the trace of the matrix following it. As a result, the logarithm of the likelihood ratio is

$$U = \ln \Lambda (\underset{\sim}{\Psi}) = \frac{1}{2} N^{-2} \underset{\sim}{\Psi}^{+} (\underset{\sim}{\varphi}_s - N^{-1} \underset{\sim}{\varphi}_s^{\,2} + \ldots) \underset{\sim}{\Psi}$$

$$- N^{-1} \mathrm{Tr} \underset{\sim}{\varphi}_s + \frac{1}{2} N^{-2} \mathrm{Tr} \underset{\sim}{\varphi}_s^{\,2} - \ldots \tag{2.11}$$

When this expression is converted back to a spatio-temporal representation,[2] the sums involved in the matrix products and traces go into integrals over A and $(0, T)$, and the result is

$$U = \frac{1}{2} N^{-2} \int_A \int_A d^2 \underset{\sim}{r}_1 \, d^2 \underset{\sim}{r}_2 \int_0^T \int_0^T dt_1 \, dt_2 \, \Psi^*(\underset{\sim}{r}_1, t_1) \times$$

$$\Big[\varphi_s (\underset{\sim}{r}_1, \underset{\sim}{r}_2) \, \chi (t_1 - t_2) - \frac{1}{N} \int_A d^2 \underset{\sim}{r}_3 \int_0^T dt_3 \, \varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_3) \, \chi (t_1 - t_3) \times$$

$$\varphi_s (\underset{\sim}{r}_3, \underset{\sim}{r}_2) \, \chi (t_3 - t_2) + \ldots \Big] \Psi (\underset{\sim}{r}_2, t_2)$$

$$-N^{-1} \int_A \int_0^T \varphi (\underset{\sim}{r}, \underset{\sim}{r}) \;\; \chi (0) \, d^2 \underset{\sim}{r} \, d t$$

$$+ \frac{1}{2} N^{-2} \int_A \int_A d^2 \underset{\sim}{r}_1 \, d^2 \underset{\sim}{r}_2 \int_0^T \int_0^T dt_1 \, dt_2 \, |\varphi_s (\underset{\sim}{r}_1, \underset{\sim}{r}_2)|^2 \, |\chi (t_1 - t_2)|^2$$

$$- + \ldots \tag{2.12}$$

After substituting from Eq. (1.6), we obtain for the logarithmic likelihood ratio

$$U = \int_O M(\underset{\sim}{u})\, B(\underset{\sim}{u})\, d^2 \underset{\sim}{u}$$

$$- \frac{1}{2} \int_O \int_O L\,(\underset{\sim}{u}, \underset{\sim}{v})\, B(\underset{\sim}{u})\, B(\underset{\sim}{u})\, d^2 \underset{\sim}{u}\, d^2 \underset{\sim}{v} + \ldots \tag{2.13}$$

where

$$M(\underset{\sim}{u}) = \frac{1}{2} \pi k^{-2} N^{-2} \int_0^T \int_0^T \mu^*(\underset{\sim}{u}, t_1)\, \chi\, (t_1 - t_2) \times$$

$$\mu(\underset{\sim}{u}, t_2)\, dt_1\, dt_2 - (AT/4\pi N R^2)\,, \tag{2.14}$$

with

$$\mu\,(\underset{\sim}{u}, t) = \int_A \Psi\,(\underset{\sim}{r}, t)\; S^*(\underset{\sim}{r}, \underset{\sim}{u})\, d^2\, \underset{\sim}{r}, \tag{2.15}$$

and

$$L\,(\underset{\sim}{u}, \underset{\sim}{v}) = \frac{1}{2} \pi^2 N^{-3} k^{-4}\, V\,(\underset{\sim}{v}, \underset{\sim}{u}) \times$$

$$\int_0^T \int_0^T \mu^*(\underset{\sim}{u}, t_1)\, Y\,(t_1, t_2)\, \mu\,(\underset{\sim}{v}, t_2)\, dt_1\, dt_2$$

$$- \frac{1}{2} \pi^2 N^{-2} k^{-4}\, (T/W)\, |V(\underset{\sim}{u}, \underset{\sim}{v})|^2, \tag{2.16}$$

with

$$V\,(\underset{\sim}{u}, \underset{\sim}{v}) = \int_A S(\underset{\sim}{r}, \underset{\sim}{u})\, S^*(\underset{\sim}{r}, \underset{\sim}{v})\, d^2\, \underset{\sim}{r} \tag{2.17}$$

and

$$Y\,(t_1, t_2) = \int_0^T \chi\,(t_1 - t_3)\, \chi(t_3 - t_2)\, dt_3$$

$$\cong \int_{-\infty}^{\infty} \left[X\,(\omega)\right]^2 \exp\,[i\omega(t_1 - t_2)]\, d\omega\, /2\pi \tag{2.18}$$

Here the bandwidth W has been defined as[2]

$$W = |\chi(0)|^2 \Big/ \int_{-\infty}^{\infty} |\chi(\tau)|^2 \, d\tau =$$

$$\left| \int_{-\infty}^{\infty} X(\omega) \, d\omega / 2\pi \right|^2 \Big/ \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega / 2\pi , \qquad (2.19)$$

and the assumption $TW >> 1$ has been used.

If we cut off the series in Eq. (2.13) for the logarithmic likelihood ratio after the second term, the function $B(\underset{\sim}{u})$ that maximizes U is the solution of the integral equation

$$M(\underset{\sim}{u}) = \int_O L(\underset{\sim}{u}, \underset{\sim}{v}) \, B(\underset{\sim}{v}) \, d^2 \underset{\sim}{v} . \qquad (2.20)$$

The data appear in the function $M(\underset{\sim}{u})$ and the kernel $L(\underset{\sim}{u}, \underset{\sim}{v})$, both of which depend on the field $\Psi(\underset{\sim}{r}, t)$ at the aperture. When $TW >> 1$, the terms neglected in Eq. (2.13) will be insignificant.

It was shown in reference 2, Section 5, how the function $M(\underset{\sim}{u})$ could be generated. The aperture is provided with a lens focusing the object plane on to a rectifying surface, and the light passing into the aperture is filtered by a frequency filter whose transfer function, measured with respect to the central frequency $\Omega = kc$, is proportional to $|X(\omega)|^{1/2}$, with an arbitrary phase factor. If the surface has a quadratic characteristic, the response at a point corresponding to $\underset{\sim}{u}$ will be proportional to the first term of $M(\underset{\sim}{u})$ in Eq. (2.14). The second term is a known constant. If a similar method could be found for generating the kernel $L(\underset{\sim}{u}, \underset{\sim}{v})$, given by Eq. (2.16), it would be unnecessary to measure the field $\Psi(\underset{\sim}{r}, t)$ itself.

The integral equation (2.20) differs from the usual one for image restoration in that the kernel $L(\underset{\sim}{u}, \underset{\sim}{v})$ depends on the data -- the field $\Psi(\underset{\sim}{r}, t)$ -- and is hence random. Its mean value is

$$\underset{\sim}{E}[L(\underset{\sim}{u}, \underset{\sim}{v})] = \pi^2 N^{-2} k^{-4} (T/W) |V(\underset{\sim}{u}, \underset{\sim}{v})|^2 + L_1(\underset{\sim}{u}, \underset{\sim}{v}), \qquad (2.21)$$

where $L_1(\underset{\sim}{u}, \underset{\sim}{v})$ depends on the object radiance $B(\underset{\sim}{u})$ and is much smaller, when $TW >> 1$, than the first term. If we neglect $L_1(\underset{\sim}{u}, \underset{\sim}{v})$, the integral equation (2.13) becomes

$$\int_0^T\int_0^T \mu^*(\underset{\sim}{u}, t_1)\, \chi(t_1 - t_2)\, \mu(\underset{\sim}{u}, t_2)\, dt_1\, dt_2$$

$$- (k^2 ATN/2\pi^2 R^2) =$$

$$2\pi k^{-2} (T/W) \int_O |V(\underset{\sim}{u}, \underset{\sim}{v})|^2 B(\underset{\sim}{v})\, d^2\underset{\sim}{v} \,. \qquad (2.22)$$

The expected value of the left-hand side of this equation, which now is the only term depending on the input field $\Psi(\underset{\sim}{r}, t)$, is equal to the right-hand side.

If we remember that the left-hand side of Eq. (2.22) is obtained by frequency-filtering the incoming light and passing it through lenses that focus the object plane on a rectifying, or flux-measuring, surface, we realize that Eq. (2.22) corresponds to the usual integral equation for image restoration. The kernel $|V(\underset{\sim}{u}, \underset{\sim}{v})|^2$ is proportional to the incoherent point-spread function for this system. All the difficulties of solving that integral equation affect this one as well. The original integral equation, eq. (2.13), on the other hand, has a modified kernel that depends on the input. Whether it can be solved in a way that avoids the difficulties of solving Eq. (2.22) remains to be seen.

3. Solution of the Integral Equation for Image Restoration

As mentioned in Section IV.2, image restoration generally involves solving an integral equation that can be written as

$$J\,(\underset{\sim}{x}) = \int S\,(\underset{\sim}{x} - \underset{\sim}{x}')\, J_0(\underset{\sim}{x}')\, d^2\underset{\sim}{x}' \,, \tag{3.1}$$

where $J(\underset{\sim}{x})$ is the observed illuminance in the image plane of some optical system, $J_0(\underset{\sim}{x})$ is the illuminance of the "true" or "geometrical" image that would be seen if there were no distortions due to turbulence, diffraction, or aberrations, and $S(\underset{\sim}{x})$ is the point-spread function of the optical system, assumed isoplanatic. From measurements of $J(\underset{\sim}{x})$ one would like to determine $J_0(\underset{\sim}{x})$.

Actually, these measurements are subject to random errors, which can be represented as a spatial noise $N(\underset{\sim}{x})$, and Eq. (3.1) should be written

$$J(\underset{\sim}{x}) = \int S(\underset{\sim}{x} - \underset{\sim}{x}')\, J_0(\underset{\sim}{x}')\, d^2\underset{\sim}{x}' + N(\underset{\sim}{x})\,. \tag{3.2}$$

Since the noise $N(\underset{\sim}{x})$ is unknown, the integral equation cannot be solved for $J_0(\underset{\sim}{x})$ exactly. Conventional methods, such as Fourier transformation, that would apply to Eq. (3.1), actually amplify the noise, which usually overwhelms the solution one is looking for. Since similar integral equations arise in many branches of physics, such as nuclear and optical spectroscopy, methods for solving them when the data -- here $J(\underset{\sim}{x})$ -- are corrupted by noise and experimental error are of great interest.

Since the noise $N(\underset{\sim}{x})$ is a random process, the best one can do is to estimate the solution $J_0(\underset{\sim}{x})$. Estimation in the least-squares sense has been suggested for image restoration and other applications of the integral equation (3.2).[6] Under the supervision of the principal investigator, Charles Rino, a NASA trainee, is studying numerical methods for estimating the solution of such an integral equation when the data are provided only at a finite number of discrete values of $\underset{\sim}{x}$. In particular, he has been studying image restoration for bandlimited spread functions S(x) in one-dimension, with particular attention to the use of expansions in prolate spheroidal wave functions. He has shown that with continuous data, if the noise is bandlimited as well, the data can be extended from the finite interval to an infinite one, whereupon the same minimum mean-square error can be attained as when the original data are given over an infinite interval. Details are given in his paper, "Bandlimited Image Restoration by Linear Mean-Square Estimation," attached to this report.

V. Conclusion

The performance of the quantum threshold detector of an incoherent object in the presence of thermal radiation shows a distinct dependence on the form of the object spectrum. Numerical calculations of detection probability are needed to determine the significance of this dependence. Methods of carrying them out will be investigated.

The quantum threshold detector, defined in terms of maximizing a certain signal-to-noise ratio, is in some sense an approximation to the optimum detector in the limit of small signal-to-noise ratio and large time-bandwidth product, but the relation is not so clear as in conventional detection theory. This point will receive further study.

Work on estimation theory in connection with image restoration will continue, and additional topics described in our proposal[8] for renewal of this grant will be pursued as time permits.

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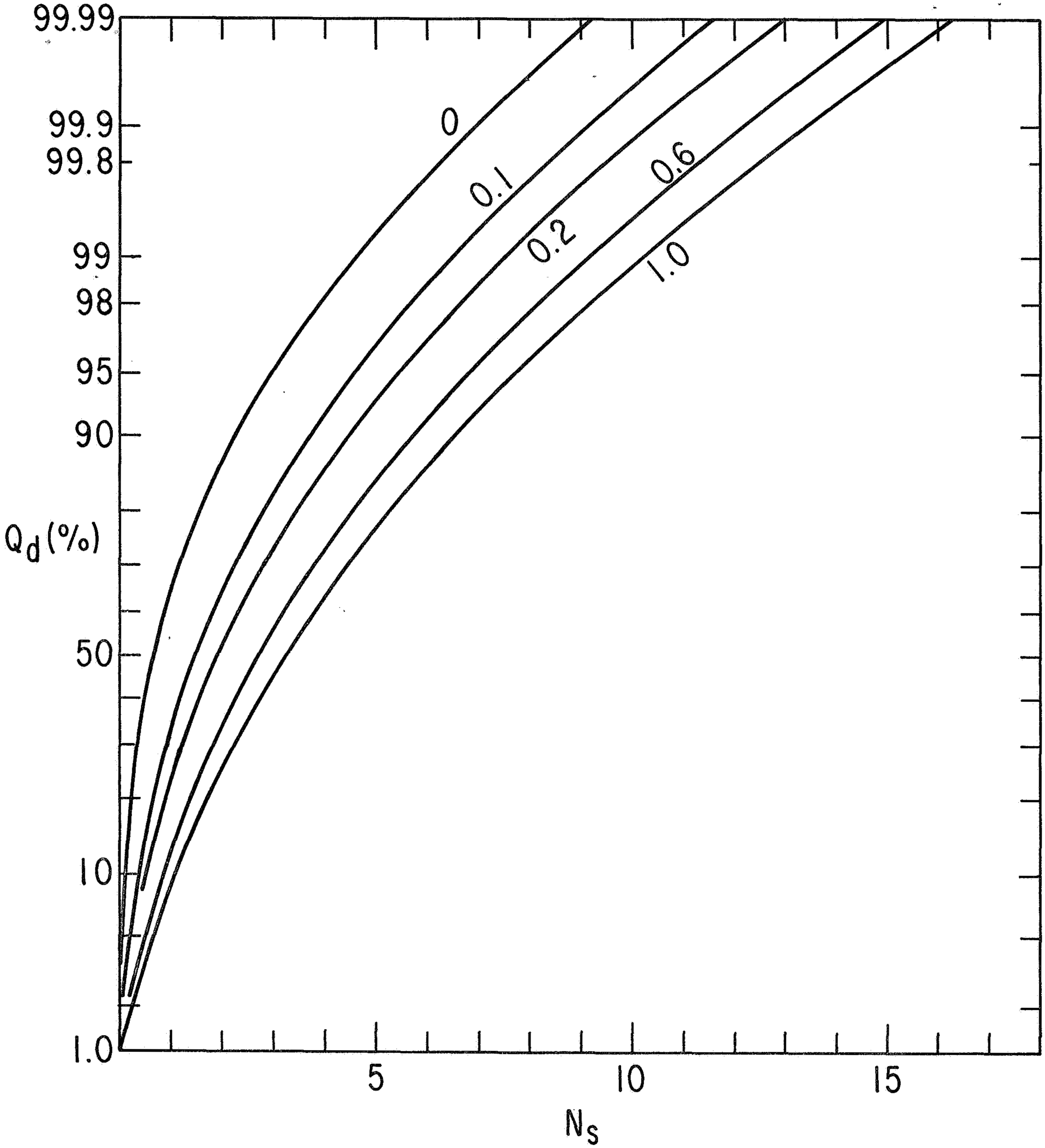

Fig. 1. Probability Q_d of detection versus mean number N_s of signal photons, for various values of $\mathcal{N}M'$. $Q_o = 10^{-2}$.

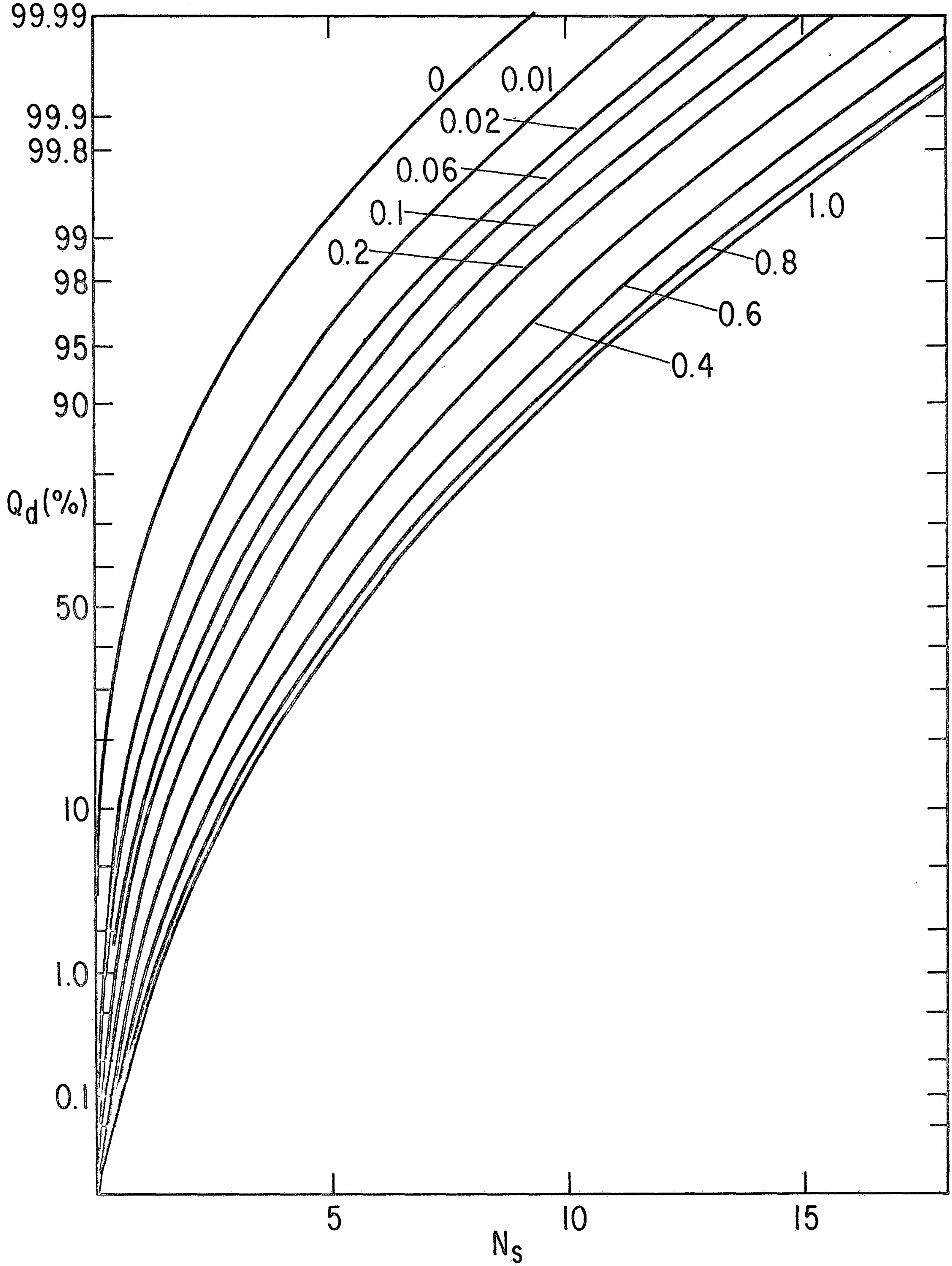

Fig. 2. Probability Q_d of detection versus mean number N_s of signal photons, for various values of $\mathcal{N}M'$. $Q = 10^{-4}$.

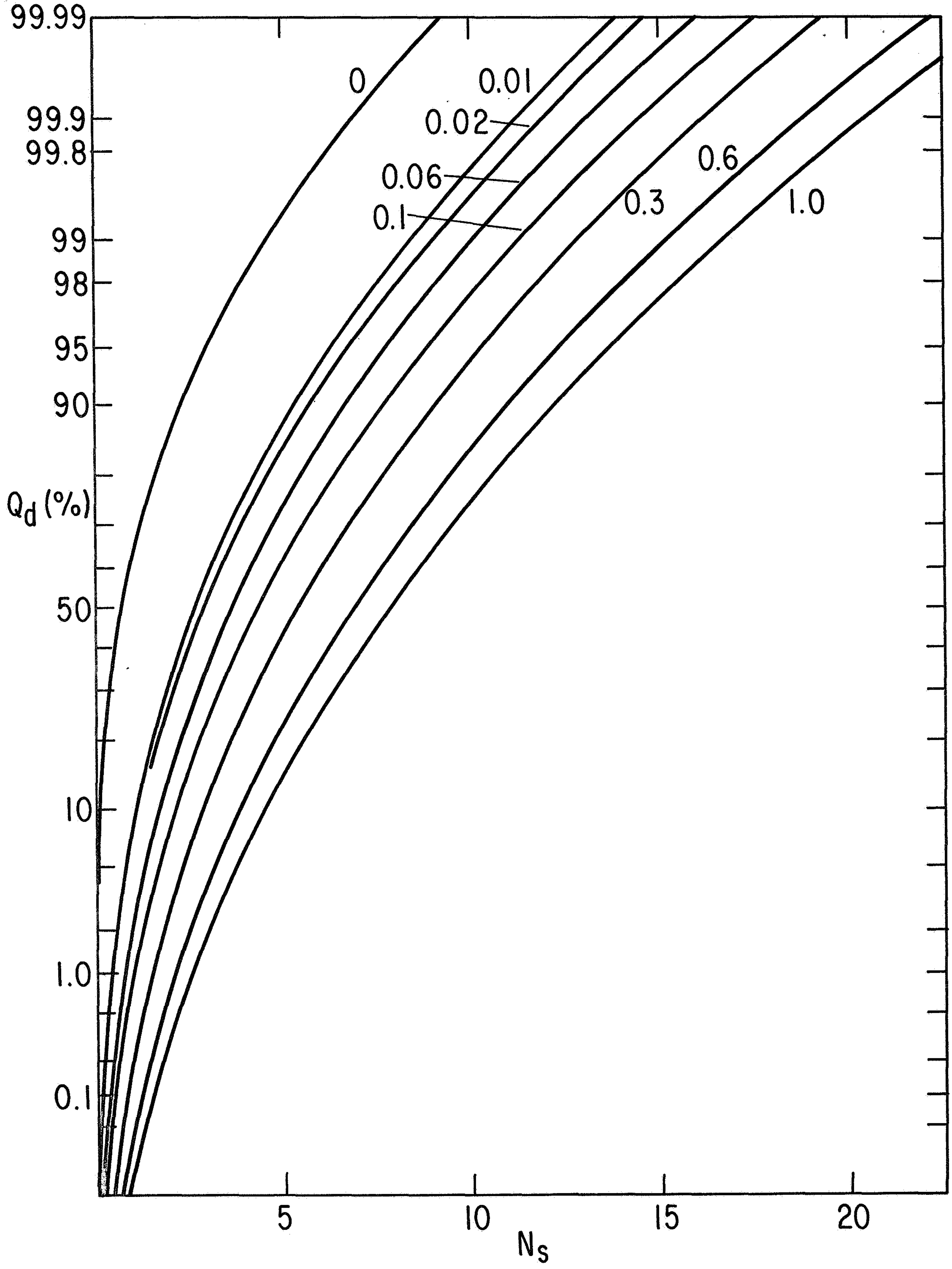

Fig. 3. Probability Q_d of detection versus mean number N_s of signal photons, for various values of $\mathcal{N}M'$. $Q_o = 10^{-6}$.

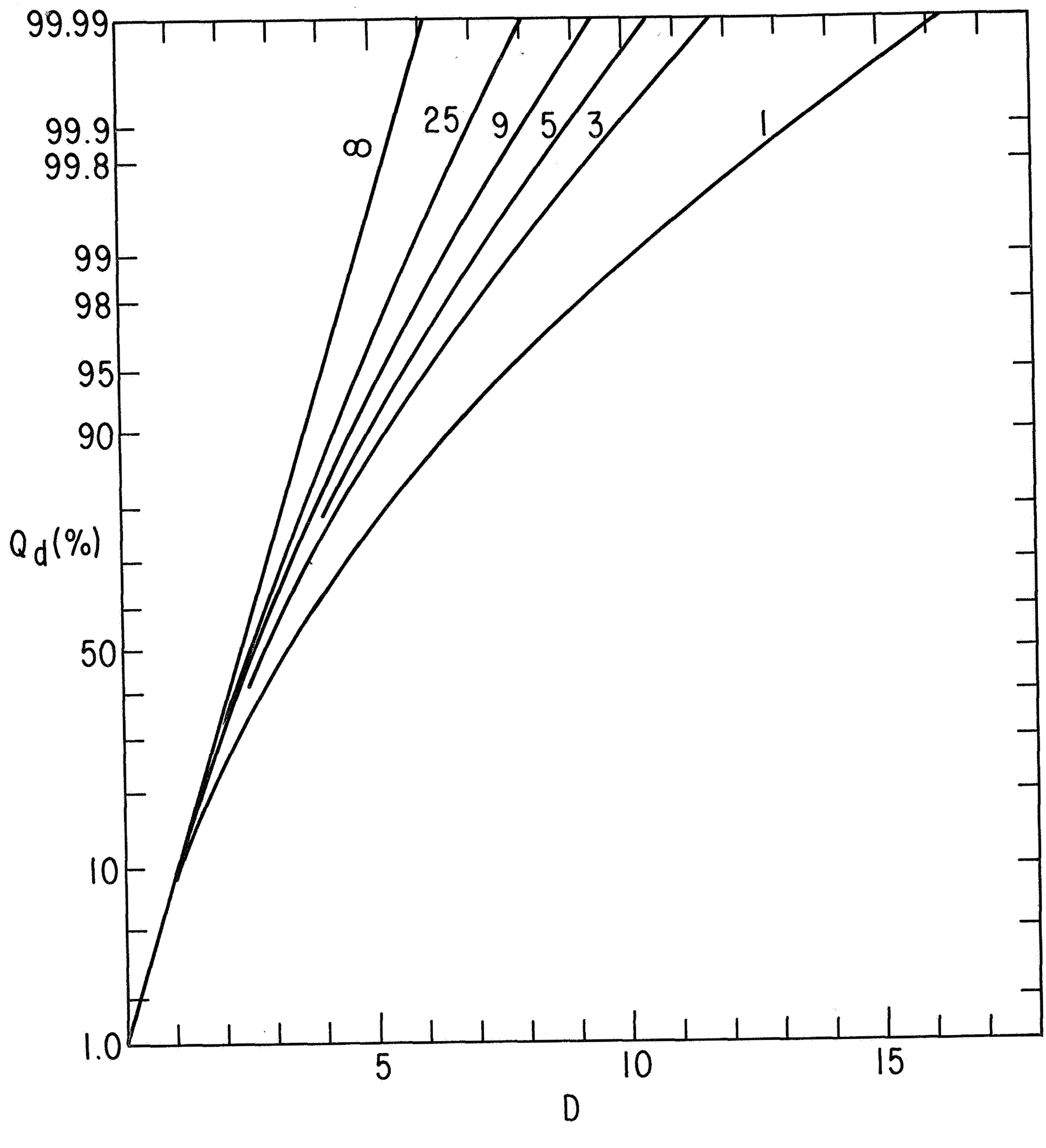

Fig. 4. Probability Q_d of detection versus signal-to-noise ratio $N_s/\sqrt{\mathcal{N}M'}$ for various values of $\mathcal{N}M'$. $Q_o = 10^{-2}$.

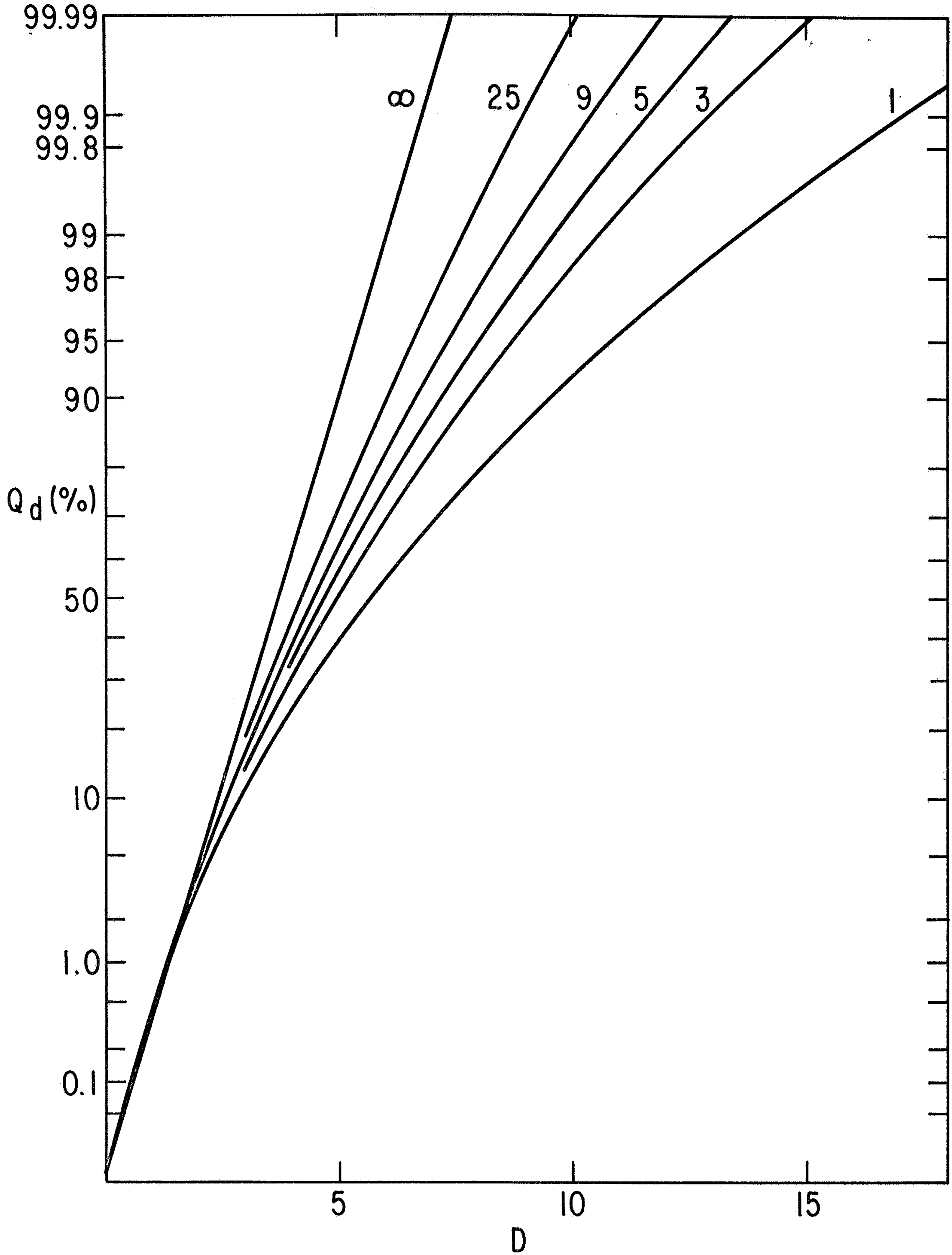

Fig. 5. Probability Q_d of detection versus signal-to-noise ratio $N_s/\sqrt{\mathcal{N}M'}$ for various values of $\mathcal{N}M'$. $Q_o = 10^{-4}$.

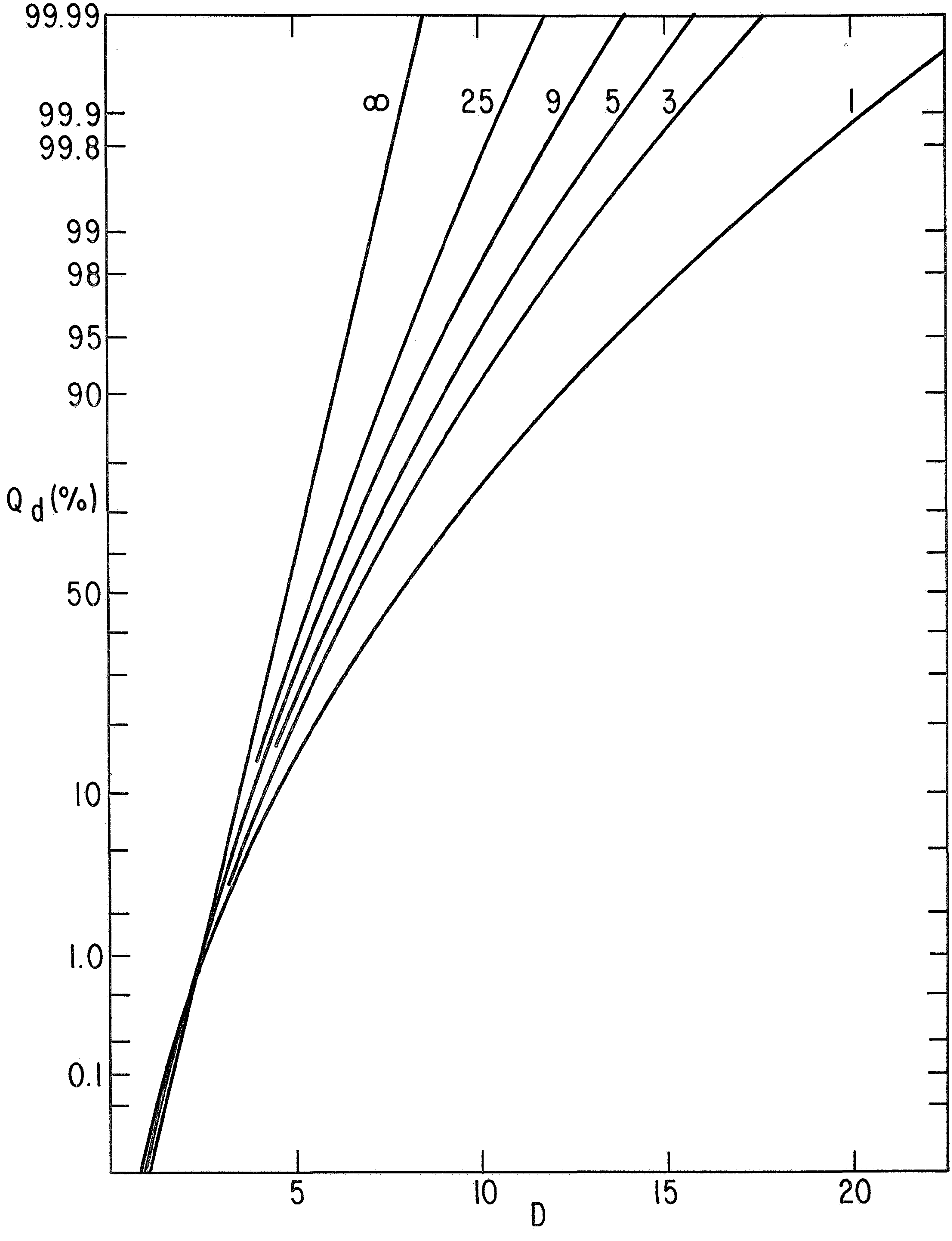

Fig. 6. Probability Q_d of detection versus signal-to-noise ratio $N_s/\sqrt{\mathcal{N}M'}$ for various values of $\mathcal{N}M'$. $Q_o = 10^{-6}$.

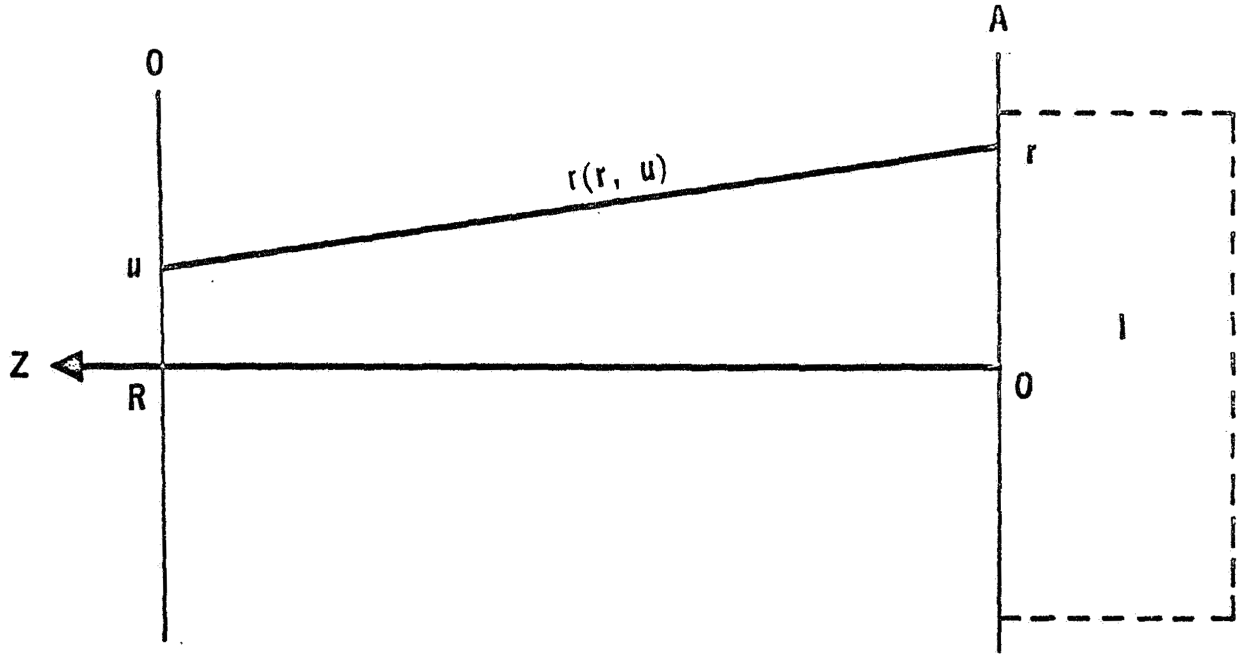

Fig. 7. Geometrical configuration of object plane O and aperture plane A.
Light from the object and background falls on plane A from the left.
I is an optical instrument for processing the field ψ $(\underline{r},t)$ on plane A.