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A STUDY OF HUMAN OPERATOR PERFORMANCE USING REGRESSION ANALYSIS

by August L. Burgett

Prepared by
UNIVERSITY OF MICHIGAN
Ann Arbor, Mich.

for

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ABSTRACT

A STUDY OF HUMAN OPERATOR PERFORMANCE USING REGRESSION ANALYSIS

August Llewellyn Burgett

The purpose of this research is twofold: the development of a parameter identification technique which can be used with intervals of data which are on the order of 20-seconds and shorter in length and secondly, the use of this technique in a study of various aspects of human operator performance in low order compensatory control tasks.

As developed here, the parameter identification technique is a modification of the classical statistical regression analysis, modified in the sense that integrals of continuous functions of time are used to obtain the desired parameter estimates instead of sums of discrete data samples. In addition to the integral formulation, a technique proposed by A. I. Rubin is used to implicitly invert the correlation matrix which is part of any regression analysis formulation. This makes the entire technique amenable to implementation on an analog computer. In the present research the parameters being estimated appear as elements of a dynamical system. The estimates of the system parameter values are obtained by first constructing a model of the system for which the parameter values are known. The system parameter estimates are then obtained by combining the known model parameter values with estimates, obtained with the regression analysis technique, of the difference between the corresponding model and system parameters.

The regression analysis technique is used to analyze the performance of human operators in low order compensatory manual control systems. This study is based on two experiments in which the subjects controlled single and double integrator dynamical systems with an input which was low frequency noise. In modeling the human

operator system, the "crossover model" proposed by D. T. McRuer is used. This model expresses the entire forward-loop of the compensatory control system as a series of operators: a gain K, a time-delay τ and a single integration. The study takes the form of obtaining estimates of the parameters K and τ for twenty-five 20-second intervals for each day of testing. From an analysis of these parameter values, inferences are made about the performance of the human operators.

A novel approach which is taken in the analysis of the parameter values is to divide the variance of both K and τ, based on 20-second data intervals, into a within-subject component and a between-subject component for each day of testing. On the basis of the components of variance for K and τ the following characteristics of the human operator when controlling low order dynamical systems are inferred.

(1) The human operator adopts a more consistent "signal processing path" as he learns the tracking task.

(2) The subjects are more uniform in control strategy for the double integrator system than for the single integrator system.

(3) The variance of τ is a more sensitive indicator of learning than is the average value of either K or τ.

(4) There appears to be an inherent variability in the human operator gain on which training has little effect.

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LIST OF SYMBOLS

$\underline{b}$	Parameter estimate vector
b_i	Element of $\underline{b}$
$\underline{b}_e$	Vector of best parameter estimates $\underline{b}_e = R^{-1}\underline{v}$
b_{ie}	Element of $\underline{b}_e$
$\underline{b}_\infty$	Vector of best estimates based on an infinite data interval
$b_{i\infty}$	Element of $\underline{b}_\infty$
$c(t)$	Output of control device
$\underline{\tilde{c}}$	Vector of system parameter values
$\underline{\tilde{c}}_e$	Best estimate of $\underline{\tilde{c}}$
$e(t)$	Model error, $e(t) = \theta(t) - u_0(t)$
$E[x]$	Statistical mean of x
$G(p)$	Open-loop transfer operator
$\overline{G}_i$	Subpopulation sample mean
$\overline{G}$	Population sample mean
$h(t, \underline{c})$	Closed-loop weighting function
$H(p, \underline{c})$	Closed-loop transfer operator
J	Cost function
k	Constant
K	Crossover model gain
$\tilde{K}_e$	Estimate of human operator gain
MS_B	Unbiased estimate of between-subject variance
MS_{Total}	Unbiased estimate of total variance
MS_W	Unbiased estimate of within-subject variance

$n(t)$	Human operator remnant term referred to output of human operator
p	Differential operator
PM	Power match
$r(t)$	Human operator remnant referred to system output
R	$R = \frac{1}{T}\int_0^T \underline{u}\,\underline{u}^{\#} dt$
T	Data interval
$u_0(t)$	Model output
$\underline{u}(t)$	Vector of parameter influence coefficients
$u_i(t)$	Element of $\underline{u}(t)$
$\underline{v}$	$\underline{v} = \frac{1}{T}\int_0^T e(t)\underline{u}(t)dt$
$Y_C(p)$	Transfer operator for controlled element
$Y_P(p)$	Random input describing function for human operator
α	Auxiliary parameter, $\alpha = \frac{2}{\tau}$
$\epsilon(t)$	System error, $\epsilon(t) = \psi(t) - \theta(t)$
$\theta(t)$	Human operator system output
$\theta_0(t)$	Output of time-invariant linear system
ρ	Normalized cross-correlation
σ_x^2	Statistical variance of x
σ_B^2	Between-subject variance
σ_{Total}	Total variance, $\sigma^2_{Total} = \sigma^2_B + \sigma^2_W$
σ_W^2	Within-subject variance
τ	Crossover model time-delay
$\tilde{\tau}_e$	Estimate of human operator time-delay
$\phi_x(t)$	Statistical autocorrelation of x

$\phi_{xy}(t)$	Statistical cross-correlation of x and y
$\Phi_{x}(j\omega)$	Power spectral density of x
$\Phi_{xy}(j\omega)$	Cross-spectral density of x and y
$\psi(t)$	Input signal
ω	Frequency, rad/sec
ω_c	Cut-off frequency of input signal
$(\cdot)^{\#}$	Vector or matrix transpose
$(\tilde{\cdot})$	Parameter value of human operator or other system being studied
$(\hat{\cdot})$	Parameter value of model

CHAPTER 1

INTRODUCTION

During the last twenty-five years there has been a large amount of research directed toward describing various aspects of human operator tracking behavior. There are two rather distinct though not unrelated motivations for this research. The psychologist or physiologist is interested in human tracking behavior as one part of the overall study of human perceptual-motor performance. The control engineer is interested in human tracking behavior because there are many situations today where the human operator is an integral part of a complex system. One current example is a manned spacecraft, which cannot be designed without knowledge of the capabilities of the different components of this machine.

1.1 Compensatory Tracking Tests

One method that is used extensively for studying human operator behavior is the compensatory tracking test. The basic block diagram for a compensatory tracking test is shown in Fig. 1.1-1. As can be seen from this figure, the compensatory tracking test has several aspects which are similar to manual control tasks such as driving an automobile or flying an airplane.

The primary objective of the human operator (subject) in a compensatory tracking test is to perform in such a manner that the system output $\theta(t)$ follows the system input $\psi(t)$ as closely as possible. In a great many cases the input is in the form of a continuously varying random-appearing signal. To realize physcially this control situation, a signal that is proportional to the instantaneous difference between system input and output, $\epsilon(t)$, is presented to the subject. The subject, in many mechanizations of such tests today, sees the error as the displacement of a line or dot from the center of an oscilloscope screen.

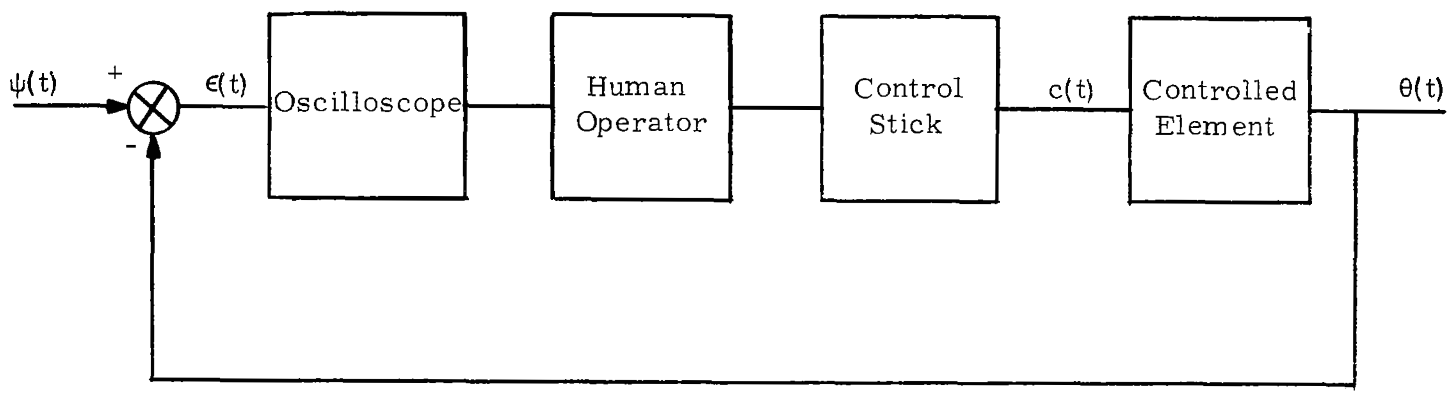

Figure 1.1-1 Block Diagram of Compensatory Control System

Thus the objective of the subject can be restated in terms of the error signal. The objective of the subject is to perform in such a manner as to keep the indication of the system error as near the center of the oscilloscope screen as possible. To accomplish this the subject is provided with a control device. Commonly this device is some form of control stick which the subject manipulates with his arm or hand. As is seen in Fig. 1.1-1, the manipulation of the control stick produces a signal, c(t), which is used as the forcing function or input signal for the controlled element. A more complete description of the actual test situation analyzed in this report is given in Section 5.1.

1.2 Human Operator Modeling

One method of analyzing compensatory tracking test data is to attempt to match the human operator behavior with a mathematical model. There are two basic approaches to developing a model to use in the analysis of test data. One method involves qualitative or psychological models. The second method uses quantitative or engineering models.

Psychological models are characterized by an attempt to model the fine or micro-structure of the human operator. In the qualitative approach algorithms are proposed for individual elements such as muscles, joints, neural pathways and other components of the neuromuscular system [30]*. Other aspects of psychological models are the inclusion of the human capacity to remember and predict [24] and also the inclusion of the capability for adapting to changing situations [9, 30]. Although this type of model will in theory account for many aspects of the human operator behavior, they are difficult to apply due to the sometimes non-quantitative description that the models give. In the models which do give a quantitative description there is inevitably a large number of undetermined parameters. Determining these parameter values is a difficult job.

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*Bracketed numbers are references given at the end of the report.

On the other hand, engineering models are based on an attempt to describe the macroscopic performance of the human operator. In this regard little effort is directed toward associating the various components of the model with corresponding physiological elements of the human operator. The particular models that have been proposed are almost as many in number as the number of investigators [8, 35]. Of the many quantitative models that have been proposed, one of the most popular today is the random input describing function model [16].

In summary, there are two basic approaches that can be taken to extend the knowledge of human operator performance. One approach would be to begin with a psychological model and attempt to manipulate the various model components in such a way that a given piece of compensatory test data is matched. The results of such a method are often less than satisfactory due in large part to the difficulties of psychological models mentioned previously. The second approach that can be taken is to start with an engineering model such as a describing function model and determine as much as possible about the "black box" which is being modeled. This approach does not give explicit information about the physiology of the human operator but it does provide a means for making inferences about the human operator. For the reasons just mentioned, the research reported here utilizes an engineering model which is based on a random input describing-function model.

1.3 Random Input Describing Function

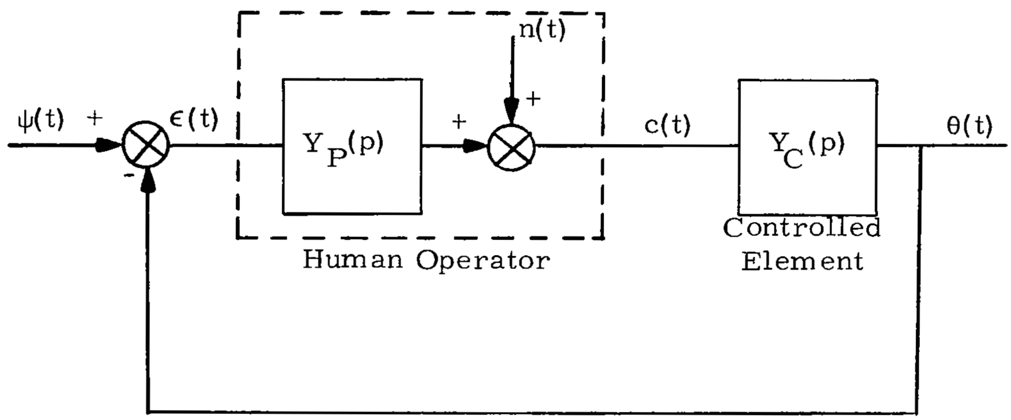
The random input describing function is an extension of the more common sinusoidal describing function for nonlinear systems [39]. The main difference in the description of a system characterized by the two types of describing function is the input. The sinusoidal describing function is applicable to a system which has a periodic input and a periodic response. The random input describing function, as the name implies, is applied to a system which has a random signal as the input. A common type of random input is described in Chapter 5. Both types of describing function are strictly defined only for time-invariant nonlinear systems. For this type of system, the random input describing function represents the "best", in the mean square error sense, linear transfer function for the nonlinear element involved [16].

The use of a random input describing function to describe the human operator is expressed well in the comment by Elkind [8]. "The essential idea of the describing function approach is that the dynamic characteristics of the human pilot, which are non-linear, noisy, and time-varying, can be represented by a linear operator $Y_P(p)$ (the describing function) and a remnant noise n(t), added to the output of $Y_P(p)$." A representation of the random input describing function model of the human operator is shown in Fig. 1.3-1. For this type of model, the random input describing function is

$$Y_P(j\omega) = \frac{\Phi_{\psi c}(j\omega)}{\Phi_{\psi \epsilon}(j\omega)} \qquad 1.3\text{-}1$$

where $\Phi_{\psi c}(j\omega)$ is the cross-spectral density of the input signal and the human operator output signal and $\Phi_{\psi \epsilon}(j\omega)$ is the cross-spectral density of the input signal and the system error signal. In practice, $Y_P(j\omega)$ is most easily evaluated by computing the two cross-spectral densities from experimental data and performing the division indicated. As mentioned above, $Y_P(p)$ is strictly defined only for stationary systems, while in practice the definition given in Eq. 1.3-1 is applied formally even when there is strong evidence that the remnant term is due in part to time-variations of the human operator.

As is seen from Eq. 1.3-1, $Y_P(j\omega)$ is a complex function of frequency. Thus to be completely specified, the gain and phase or real and imaginary parts of $Y_P(j\omega)$ must be specified for all frequencies.



$$Y_P(j\omega) = \frac{\Phi_{\psi c}(j\omega)}{\Phi_{\psi \epsilon}(j\omega)}$$

$$\Phi_{\psi n}(j\omega) = 0$$

Figure 1.3-1 Block Diagram for Describing Function of Human Operator

In practice, however, it is more convenient to assume an a priori mathematical form for $Y_P(j\omega)$ which approximately fits the experimentally-determined frequency function given by Eq. 1.3-1. In such a procedure the coefficients that are part of the assumed $Y_P(j\omega)$ are undetermined and must be estimated to give the complete form for each individual case tested. Of the many mathematical forms that have been proposed [27, 28, 35, 37] one that is useful and currently popular is the "crossover model" proposed by McRuer, et al. [28]. It was found by these authors that a consistent expression could be obtained if the human operator describing function was combined with the transfer function for the controlled element. Thus the crossover model is expressed as follows:

$$Y_P(p)Y_C(p) = \frac{Ke^{-\tau p}}{p} \qquad 1.3\text{-}2$$

The crossover model has been shown to fit experimental data quite well for such controlled elements as $Y_P(p) = 1/p$, $1/p^2$ and $1/(p-2)$ [22, 28]. In addition to fitting the experimental data well, the crossover model is characterized by only two parameters, the gain, K, and the time-delay, τ. For these reasons the crossover model is very useful when automatic parameter identification techniques are utilized.

1.4 Remnant

The remnant, $n(t)$, as used in this report, represents the portion of the human operator's output, $c(t)$, which is not linearly correlated with the system input, $\psi(t)$. Under the assumption that the remnant has zero statistical mean, the fact that the remnant and input are uncorrelated is expressed by the cross-spectral density of the input and the remnant being equal to zero, i.e.,

$$\Phi_{\psi n}(j\omega) = 0 \qquad 1.4\text{-}1$$

This formulation of the remnant is actually a computational artifact since there is strong evidence that the remnant is composed of terms due to such human operator characteristics as nonlinearity and time-variation as well as a certain amount of additive noise [27].

Several studies of the time-varying and nonlinear aspects of the human operator have been performed [3]. These studies can be interpreted as efforts to account for some of the remnant signal which is part of the describing function characterization. Rather than present an extensive review of these studies, a short discussion of a few pertinent studies is given here.

Two approaches that have been taken in studies of human operator time-variation are discussed here. One approach that has been taken is to represent the human operator by a time-varying weighting function. Estimates are then obtained for this time-varying weighting function. Elkind [7] has applied a regression analysis technique to this problem and obtained a piece-wise constant representation of the weighting function. Wierwille and Gagné [41] have generalized this approach to a method which gives a continuously varying estimate of the time-varying weighting function. Both of these methods give a good qualitative representation of the human operator time-variation. However, a time-varying weighting function is not an easily interpretable description of time-variation.

Another description which is a restricted case of the time-varying weighting function is to represent the human operator by a time-varying differential equation. This is equivalent to an a priori specification of the form of the weighting function. This approach has been taken by McDonnell [29]. The particular method used by McDonnell was to assume that the human operator could be represented by a modified crossover model which had a time-varying gain in place of the constant gain given by Eq. 1.3-2. McDonnell's results [29] suggest that this is a reasonable representation of the human operator. Having estimates of time-varying parameters of the human operator provides a more interpretable representation than does a time-varying weighting function.

One approach to the study of nonlinearities of the human operator has been proposed by Weirwille and Gagné [42]. This method is a further extension of the method discussed in the preceding paragraph. The method makes use of predetermined nonlinearities which are operated in parallel with the human operator closed-loop system. This writer feels that the method is inappropriate for analyzing human operator nonlinearities, due to the fact that any human operator nonlinearities appear in the forward-loop of the closed-loop system. Thus any nonlinearity which matched the closed-loop response would be very difficult to interpret in terms of a forward-loop nonlinearity. A proposed method for circumventing this problem is discussed in Chapter 6.

A less general approach to analyzing human operator nonlinearity has been taken by Smith [34] and Young and Meiry [40]. The results presented in both papers indicate that for certain tasks the human operator utilizes a saturating or bang-bang type of response. Although this effect is not readily apparent in all control situations, these results give a basis for assuming that a portion of the human operator remnant is due to some type of nonlinearity.

1.5 Description of the Research

In the research reported here certain aspects of human operator performance in compensatory tracking tests are analyzed. In the two compensatory tracking experiments analyzed the subject was presented with a random input. The distinguishing feature of the experiments was the controlled element, one having $Y_C(p) = 5/p$ and the other having $Y_C(p) = 5/p^2$. In the analysis of the data, values of gain and time-delay of the crossover model are obtained for 20-second intervals using a computational technique based on regression analysis. A major analysis technique employed in this work is to obtain estimates of the within-subject and the between-subject variance of both the gain and the time-delay of the human operator for each day of testing. The estimates of the variance components are then used to make inferences about such characteristics of the human operator as sources of remnant, uniformity of human operators for the controlled elements used and the effect of training on human operator signal processing.

The contents of this report are divided as follows. The application of regression analysis to system identification is developed in Chapter 2. Chapter 3 discusses the use of regression analysis in determining estimates of gain and time-delay for the crossover model. Various sources of error in the regression analysis technique are analyzed in Chapter 4 with particular emphasis on application to the crossover model. Chapter 5 deals with the analysis and results of the compensatory tracking task experiments that were conducted. Chapter 6 is the concluding chapter in which a review of results is presented along with suggested areas of additional research. Several appendices are included which present details of computer implementation as well as theoretical and experimental details.

CHAPTER 2

PARAMETER ESTIMATION BASED ON LINEAR REGRESSION ANALYSIS

2.1 Simple Linear Regression Analysis

Consider two random variables, X_0 and X_1, which have a continuous joint density function, $p(x_0, x_1)$. For the analysis that will be discussed here it is convenient to consider X_1 as an independent variable and X_0 as a dependent variable. The conditional density function for X_0 is then expressed by

$$p(x_0/x_1) = \frac{p(x_0, x_1)}{p(x_1)} \qquad 2.1\text{-}1$$

where $p(x_1)$ is the marginal density function for X_1 [4]. Since the conditional density function for X_0 is dependent on the value of X_1, all conditional moments of X_0, and in particular the first moment or mean of X_0, will be dependent on the value of X_1. Thus the expression for the conditional mean of X_0, $\bar{x}_0$, is

$$\bar{x}_0 = \int_{-\infty}^{\infty} x_0\, p(x_0/x_1)\, dx_0 \qquad 2.1\text{-}2$$

The relation between $\bar{x}_0$ and the value of X_1 can be written as

$$\bar{x}_0 = f(x_1) \qquad 2.1\text{-}3$$

where $f(x_1)$ represents the regression of X_0 on X_1.

Regression analysis is used here to obtain a "best estimate", in the least square sense [2], of the function $f(x_1)$ from samples of experimental data. With no restrictions on the form of $f(x_1)$ this is a problem in the calculus of variations. Rather than treating the general problem we will consider here only linear functions, $f(x_1) = \beta_1 + \beta_2(x_1 - \bar{x}_1)$. If the form of $f(x_1)$ is restricted to this class of functions, Eq. 2.1-3 can be written as:

$$\bar{x}_0 = \beta_1 + \beta_2(x_1 - \bar{x}_1) \qquad 2.1\text{-}4$$

With this restriction the problem of finding a "best estimate" is reduced to a problem of ordinary calculus. In this situation it is only necessary to obtain estimates for the parameters β_1 and β_2 in Eq. 2.1-4. When $f(x_1)$ is restricted as in Eq. 2.1-4 to the class of linear functions, this is known as a problem in linear regression analysis.

In applying regression analysis to obtain estimates of the parameters, N observations of the variables X_0 and X_1 are made from experimental data. For each observation, an estimate of X_0, y_0, is obtained from the expression:

$$y_0 = b_1 + b_2(x_1 - \bar{x}_{1S}) \qquad 2.1\text{-}5$$

where b_1 is an estimate of β_1 and b_2 is an estimate of β_2. $\bar{x}_{1S}$ is the sample mean of X_1 which is defined by:

$$\bar{x}_{1S} = \frac{1}{N}\sum_{i=1}^{N} x_{1i} \qquad 2.1\text{-}6$$

The best estimates, in the mean square sense, of the parameters β_1 and β_2 are obtained by minimizing a cost function, J. This cost function is a measure of the errors between the observed values of X_0 and the estimated values from Eq. 2.1-5. The cost function, J, is

$$J = \frac{1}{2N}\sum_{i=1}^{N} (x_{0i} - y_{0i})^2 \qquad 2.1\text{-}7$$

To obtain the best estimate of the parameters, the gradient of J is set equal to zero. This yields the equations;

$$\frac{\partial J}{\partial b_1} = -\frac{1}{N}\sum_{i=1}^{N} [x_{0i} - (b_1 + b_2(x_{1i} - \bar{x}_{1S}))] = 0 \qquad 2.1\text{-}8$$

$$\frac{\partial J}{\partial b_2} = \frac{1}{N}\sum_{i=1}^{N}\{-(x_{1i}-\bar{x}_{1S})[x_{0i}-(b_1+b_2(x_{1i}-\bar{x}_{1S}))]\} = 0 \qquad 2.1\text{-}9$$

Rearranging Eqs. 2 1-8 and 2 1-9 gives:

$$\frac{1}{N}\sum_{i=1}^{N} x_{0i} - b_1 - \frac{b_2}{N}\sum_{i=1}^{N}(x_{1i} - \bar{x}_{1S}) = 0 \qquad 2.1\text{-}10$$

$$\frac{1}{N}\sum_{i=1}^{N} x_{0i}(x_{1i} - \bar{x}_{1S}) - \frac{b_1}{N}\sum_{i=1}^{N}(x_{1i} - \bar{x}_{1S}) - \frac{b_2}{N}\sum_{i=1}^{N}(x_{1i} - \bar{x}_{1S})^2 = 0 \qquad 2.1\text{-}11$$

Note from the definition of the sample mean given in Eq. 2.1-6 that:

$$\frac{1}{N}\sum_{i=1}^{N}(x_{1i} - \bar{x}_{1S}) = 0 \qquad 2.1\text{-}12$$

Combining Eqs 2.1-10 through 2.1-12 yields the best estimates of β_1 and β_2 which are denoted respectively by b_{1e} and b_{2e}.

$$b_{1e} = \frac{1}{N}\sum_{i=1}^{N} x_{0i} \qquad 2.1\text{-}13$$

$$b_{2e} = \frac{\frac{1}{N}\sum_{i=1}^{N} x_{0i}(x_{1i} - \bar{x}_{1S})}{\frac{1}{N}\sum_{i=1}^{N}(x_{1i} - \bar{x}_{1S})^2} \qquad 2.1\text{-}14$$

By adopting definitions of sample mean similar to that given in Eq. 2.1-6, Eqs. 2.1-13 and 2.1-14 can be written in terms of the various sample means as:

$$b_{1e} = \bar{x}_{0S} \qquad 2.1\text{-}15$$

$$b_{2e} = \frac{\overline{(x_0 x_1)}_S - \bar{x}_{0S}\bar{x}_{1S}}{\overline{(x_1^2)}_S - (\bar{x}_{1S})^2} \qquad 2.1\text{-}16$$

It can be shown [2] that the best estimates, b_{1e} and b_{2e}, as defined by Eqs. 2.1-15 and 2.1-16 are unbiased estimates of β_1 and β_2. Also it can be shown [13] that if X_0 and X_1 are jointly gaussian random variables, then the estimates defined by Eqs. 2.1-15 and 2.1-16 are maximum likelihood estimates of β_1 and β_2.

2.2 An Example

As an application of the theory presented in Section 2.1 consider the system given in Fig. 2.2-1. This is a simple linear system which has a random signal, $X_1(t)$, as input. The observable system response, $X_0(t)$, is a combination of the actual system response and an additive noise signal $r(t)$. The additive noise term could be due to many sources, just one of which is measurement uncertainties. From Fig. 2.2-1 it can be seen that $X_0(t)$ and $X_1(t)$ are related by:

$$X_0(t) = \beta X_1(t) + r(t) \qquad 2.2\text{-}1$$

As a simplifying assumption, let both $X_1(t)$ and $r(t)$ have zero mean.

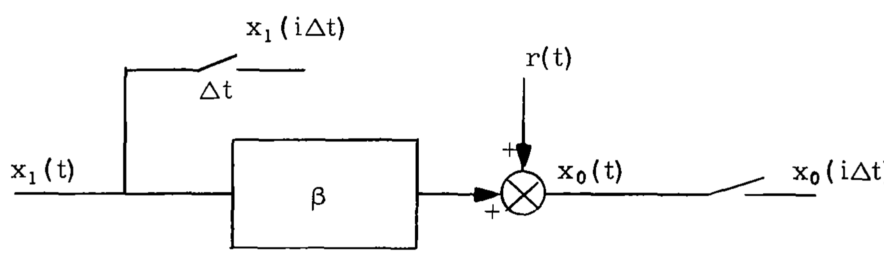


Figure 2.2-1 A Linear Regression Analysis Example

The switch symbols shown in Fig. 2.2-1 represent sampling devices. Thus, instead of using the entire time histories of $X_0(t)$ and $X_1(t)$, samples of these functions are used in the calculations. Although it is not critical to the discussion of this section let us assume that the samples are taken periodically in time. The sample period, as shown in Fig. 2.2-1 is Δt seconds. For consistency with Section 2.1, define the sampled values of $X_0(t)$ by the following notation:

$$x_0(i\Delta t) = x_{0i} \qquad 2.2\text{-}2$$

A similar notation is used for sampled values of $X_1(t)$ and $r(t)$. Since the means of both $X_1(t)$ and $r(t)$ have been assumed to be zero, let us further assume that the number of samples taken, N, is large enough that the sample means of X_0 and X_1 are essentially zero, i.e.,

$$\bar{x}_{0S} = \frac{1}{N}\sum_{i=1}^{N} x_{0i} \approx 0 \qquad 2.2\text{-}3$$

$$\bar{x}_{1S} = \frac{1}{N}\sum_{i=1}^{N} x_{1i} \approx 0 \qquad 2.2\text{-}4$$

From Fig. 2.2-1 it might be expected that the conditional mean of X_0 would be a linear function of values of X_1. Thus in this situation restricting the function $f(x_1)$ of Eq. 2.1-3 to the class of linear functions is a valid step. With this restriction the assumed form for the conditional mean of X_0 is:

$$\bar{x}_0 = \beta x_1 \qquad 2.2\text{-}5$$

The cost function defined by Eq. 2.1-7 is then:

$$J = \frac{1}{2N}\sum_{i=1}^{N}\left\{(x_{0i} - b x_{1i})^2\right\} \qquad 2.2\text{-}6$$

where b is the estimate of β. Since the cost function, J, is a function of only one parameter in this case, the partial derivatives of Eqs. 2.1-8 and 2.1-9 are replaced by the total derivative giving:

$$\frac{dJ}{db} = \frac{1}{N}\sum_{i=1}^{N}(x_{0i} - bx_{1i})(-x_{1i}) = 0 \qquad 2.2\text{-}7$$

Rearranging Eq. 2.2-7 gives the expression for the best estimate of β.

$$b_e = \frac{\frac{1}{N}\sum_{i=1}^{N} x_{0i}x_{1i}}{\frac{1}{N}\sum_{i=1}^{N}(x_{1i})^2} \qquad 2.2\text{-}8$$

From Fig. 2.2-1 it is seen that the value of x_{0i} can be expressed as the sum of two components.

$$x_{0i} = \beta x_{1i} + r_i \qquad 2.2\text{-}9$$

Substituting the expression from Eq. 2.2-9 into Eq. 2.2-8 yields the expression for the best estimate of the system parameter β.

$$b_e = \beta + \frac{\frac{1}{N}\sum_{i=1}^{N} x_{1i} r_i}{\frac{1}{N}\sum_{i=1}^{N} (x_{1i})^2} \qquad 2.2\text{-}10$$

Note that if $X_1(t)$ and $r(t)$ are uncorrelated then b_e is an unbiased estimate of the system parameter β. A more complete discussion of the effect of additive noise is given in Chapter 4.

2.3 Integral Formulation of Regression Analysis

As will be seen in Section 2.5 it is often convenient to apply analog computer techniques to the solution of regression analysis problems. To facilitate the application of analog techniques, consider integral representations for the summations given in Section 2.2. The Euler approximation [18] for the integral of a general variable $y(t)$ is:

$$\frac{1}{N}\sum_{i=1}^{N} y_i \approx \frac{1}{T}\int_0^T y(t)\,dt \qquad 2.3\text{-}1$$

where,

$$T = N\Delta t; \quad y_i = y(i\Delta t)$$

Thus the cost function, J, given by Eq. 2.2-6 can be considered as the Euler approximation of a corresponding integral cost function, J_I.

$$J = \frac{1}{2N}\sum_{i=1}^{N} [x_{0i} - b x_{1i}]^2 \approx J_I = \frac{1}{2T}\int_0^T [x_0(t) - b x_1(t)]^2\,dt \qquad 2.3\text{-}2$$

The gradient of J_I is

$$\frac{dJ_I}{db} = \frac{1}{T}\int_0^T [-x_1(t)][x_0(t) - b x_1(t)]dt = 0$$

and the value of b that minimizes J_I is given by:

$$b_{eI} = \beta + \frac{\frac{1}{T}\int_0^T r(t)x_1(t)dt}{\frac{1}{T}\int_0^T [x_1(t)]^2 dt} \qquad 2.3\text{-}3$$

Comparing the terms in Eqs. 2.2-10 and 2.3-3, it is seen that the terms in Eq. 2.2-10 represent the Euler approximations of the corresponding terms in Eq. 2.3-3. It is seen then that the integral formulation gives results which are comparable with the results using the classical summation type of cost function. Thus for the type of problem in which analog computer methods may be used, the formulation using integrals can directly replace the formulation using summations.

2.4 Multiple Linear Regression Analysis

Consider the extension of the concepts discussed in Sections 1, 2 and 3 of this chapter to the case where there is one dependent variable, X_0, and L independent variables, $X_1, X_2, \ldots, X_L$. In this case the conditional density function for X_0 is expressed by

$$p(x_0/x_1, x_2, \ldots, x_L) = \frac{p(x_0, x_1, \ldots, x_L)}{p(x_1, x_2, \ldots, x_L)} \qquad 2.4\text{-}1$$

where $p(x_1, x_2, \ldots, x_L)$ is the marginal joint density function for $X_1, X_2, \ldots, X_L$. The conditional mean of X_0, $\bar{x}_0$, is then a function of the values of $X_1, X_2, \ldots, X_L$ and can be expressed by

$$\bar{x}_0 = f(x_1, x_2, \ldots, x_L) \qquad 2.4\text{-}2$$

As was done in Section 2.1 let us consider here only the linear regression analysis problem. Then for this restricted problem Eq. 2.4-2 is written as:

$$\bar{x}_0 = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_L x_L \qquad 2.4\text{-}3$$

To simplify the following analysis, let us introduce the following matrix notation. Let $\underline{x}$ and $\underline{\beta}$ be L dimensional column vectors which have elements $x_1, x_2, \ldots, x_L$ and $\beta_1, \beta_2, \ldots, \beta_L$ respectively. Then Eq. 2.4-3 can be rewritten in vector notation as

$$\bar{x}_0 = \underline{\beta}^{\#}\underline{x} \qquad 2.4\text{-}4$$

where the superscript # indicates matrix or vector transpose [1].

The application of regression analysis discussed here makes use of the integral formulation presented in Section 2.3. Therefore, the cost function, J, that is to be minimized is expressed as a time integral of data for the interval $0 \leqq t \leqq T$.

$$J = \frac{1}{2T}\int_0^T (x_0 - \underline{b}^{\#}\underline{x})^2\, dt \qquad 2.4\text{-}5$$

The subscript I as used in Eq. 2.3-2 to indicate an integral formulation of the performance index J will be omitted for the remainder of the report since only integral formulations are used. In Eq. 2.4-5 the vector $\underline{b}$ represents an estimate of the vector $\underline{\beta}$. The best estimate of $\underline{\beta}$ for this interval of data is obtained by setting the gradient of J equal to zero.

$$\frac{\partial J}{\partial \underline{b}} = \frac{\partial}{\partial \underline{b}}\left\{\frac{1}{2T}\int_0^T (x_0 - \underline{b}^{\#}\underline{x})^2\, dt\right\} = 0 \qquad 2.4\text{-}6$$

Performing the partial differentiation indicated in Eq. 2.4-6 yields

$$\frac{\partial J}{\partial \underline{b}} = R\underline{b} - \underline{v} = 0 \qquad 2.4\text{-}7$$

where

$$R = \frac{1}{T}\int_0^T \underline{x}\,\underline{x}^{\#}\, dt \qquad 2.4\text{-}8$$

$$\underline{v} = \frac{1}{T}\int_0^T x_0\,\underline{x}\, dt \qquad 2.4\text{-}9$$

Under the assumption that the inverse of R exists, the expression for the best estimate, $\underline{b}_e$, is

$$\underline{b}_e = R^{-1}\underline{v} \qquad 2.4\text{-}10$$

The value of $\underline{b}_e$ obtained from Eq. 2.4-10 is the best estimate, in the mean square sense, of the vector of regression coefficients, $\underline{\beta}$, based on the T seconds of data used. The effect of additive noise, such as was illustrated in Section 2.3, and other factors are discussed in Chapter 4.

2.5 Implicit Matrix Inversion

In Section 2.4 it is seen that an L × L matrix, R, must be inverted to obtain the parameter estimate. If an analog computer is being used to perform the necessary operations, matrix inversion is a difficult and equipment-consuming process. Similarly, if a digital computer is used to solve Eq. 2.4-7, matrix inversion on the digital computer can be a time consuming process. In problems such as the one illustrated in Section 2.2 it is convenient to use an analog computer, which implies using an integral formulation as suggested in Section 2.3. However, the integral formulation presents the problem of solving Eq. 2.4-7 on the analog computer.

Rubin [31,32] has suggested a means for eliminating the necessity for matrix inversion when an analog computer is used to solve Eq. 2.4-7. The operation expressed by Eq. 2.4-10 can be thought of as one method of adjusting the value of $\underline{b}$ subsequent to the interval (0, T). This is theoretically an instantaneous adjustment of the value of $\underline{b}$. In actual practice, however, using either an analog or digital computer, the inversion of the matrix R requires a small but nonzero amount of time. The method proposed by Rubin also requires a small amount of time to adjust the value of $\underline{b}$ but in this case the matrix R is not explicitly inverted. Instead of the algebraic adjustment procedure suggested by Eq. 2.4-10, the adjustment of the value of $\underline{b}$ is controlled by a differential equation. The appropriate differential equation is obtained by equating the time derivative $\underline{b}'$ and the negative gradient of J

$$\underline{b}' = -k \frac{\partial J}{\partial \underline{b}} = -k[R\,\underline{b} - \underline{v}] \qquad 2.5\text{-}1$$

where k is an arbitrary positive constant. The prime notation, $(\cdot)'$, is used to denote differentiation with respect to time, but with time restricted to values subsequent to the interval (0, T). Rearranging Eq. 2.5-1 yields the differential equation which regulates the adjustment of $\underline{b}$

$$\underline{b}' + k R \underline{b} = k \underline{v} \qquad 2.5\text{-}2$$

Consider now some properties of the matrix R. If it is assumed that

$$\frac{1}{T}\int_0^T (x_i)^2 dt \neq 0 \quad i = 1, 2, \ldots, L$$

which will be the case in any practical situation, then it can be shown as follows that R is a nonnegative definite matrix. The expression for Λ, the diagonal matrix of eigenvalues of R, is

$$\Lambda = P^{-1} R P \qquad 2.5\text{-}3$$

where P is the matrix of eigenvectors of R. Since R is real and symmetric, P is an orthogonal matrix and

$$P^{-1} = P^{\#}$$

Now:

A necessary and sufficient condition that R be nonnegative definite is that all the eigenvalues of R be nonnegative [1].

To show that all eigenvalues of R are nonnegative, consider the form of Λ.

$$\Lambda = P^{\#}\left[\frac{1}{T}\int_0^T \underline{x}\,\underline{x}^{\#} dt\right] P \qquad 2.5\text{-}5$$

and since P is a constant matrix,

$$\Lambda = \frac{1}{T}\int_0^T P^{\#}\underline{x}\,x^{\#}\;P\,dt$$

Now define

$$\underline{z}(t) = P^{\#}\,x(t) \qquad 2.5\text{-}6$$

then

$$\Lambda = \frac{1}{T}\int_0^T \underline{z}\,\underline{z}^{\#}\,dt \qquad 2.5\text{-}7$$

Equating the diagonal terms of both sides of Eq. 2.5-7 yields

$$\lambda_i = \frac{1}{T}\int_0^T (z_i)^2\,dt \geqq 0 \qquad i = 1, 2, \ldots, L \qquad 2.5\text{-}8$$

Therefore all eigenvalues are nonnegative and R is a nonnegative definite matrix.

In many situations, especially for the matrix R used in the data analysis of Chapter 5, the eigenvalues will be strictly positive. For this case let us denote the smallest eigenvalue of the matrix R by λ_{min} Since Eq. 2.5-2 represents a linear differential equation with a constant input, the response will be within 1% of the steady-state solution in approximately five times the longest time-constant, i.e.,

$$\underline{b}' \rightarrow 0 \qquad 2.5\text{-}9$$

for

$$t > 5\left(\frac{1}{k\lambda_{min}}\right)$$

where t is measured from the end of the interval (0, T).

As $\underline{b}' \rightarrow 0$ it is seen from Eq. 2.5-2 that:

$$k R \underline{b} \rightarrow k \underline{v} \qquad 2.5\text{-}10$$

or

$$\underline{b} \rightarrow \underline{b}_e = R^{-1} \underline{v} \qquad 2.5\text{-}11$$

Thus by using the differential equation approach to adjusting the value of $\underline{b}$, the awkward step of matrix inversion using the analog computer can be circumvented.

By combining the integral formulation presented in Section 2.3 and the implicit matrix inversion discussed in this section, it is seen that the regression analysis problem is amenable to solution using an analog computer.

2.6 Parameter Identification in Linear Dynamic Systems

In this section we will treat the problem of parameter identification in linear dynamic systems. The system is represented by either the weighting function $h(t, \underline{c})$ or the corresponding transfer operator $H(p, \underline{c})$ [23], as shown in Fig. 2.6-1, where $\underline{c}$ is an L dimensional parameter vector. The system under study has parameter value $\underline{c} = \tilde{\underline{c}}$ and output $\theta_0(t)$. In addition it is necessary to construct a model of this system which has parameter value $\underline{c} = \hat{\underline{c}}$ and output $u_0(t)$. Regression analysis is used to obtain an estimate of the difference between $\tilde{\underline{c}}$ and $\hat{\underline{c}}$ which is then combined with $\hat{\underline{c}}$ to obtain the estimate of $\tilde{\underline{c}}$.

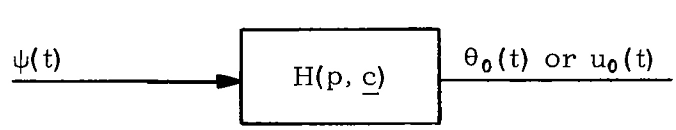


Figure 2.6-1 Linear System Representation

Before discussing the parameter identification problem, let us

consider some aspects of the system response. The response of **the** system and model for a given input can be expressed by convolution integrals as,

$$\theta_0(t) = \int_{-\infty}^{\infty} h(t-\mu, \underline{\tilde{c}})\, \psi(\mu)\, d\mu \qquad 2.6\text{-}1$$

$$u_0(t) = \int_{-\infty}^{\infty} h(t-\mu, \underline{\hat{c}})\, \psi(\mu)\, d\mu \qquad 2.6\text{-}2$$

or in operator notation,

$$\theta_0(t) = H(p, \underline{\tilde{c}})\, \psi(t) \qquad 2.6\text{-}3$$

$$u_0(t) = H(p, \underline{\hat{c}})\, \psi(t) \qquad 2.6\text{-}4$$

The two time histories can be symbolized as in Fig. 2.6-2.

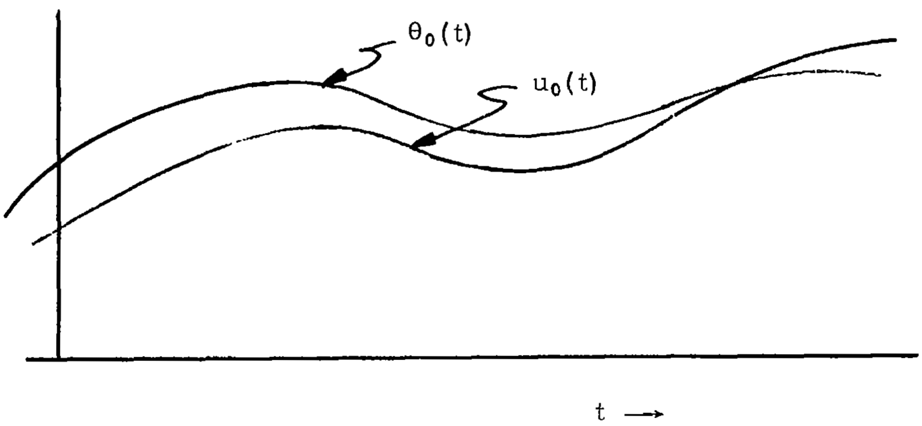


Figure 2.6-2 Symbolization of System Response

In a region of the parameter space where the system represented by $H(p, \underline{\tilde{c}})$ is stable, typical weighting functions for $\underline{c} = \underline{\tilde{c}}$ can be expanded in a **Taylor** series about $\underline{c} = \underline{\hat{c}}$.

$$h(t, \underline{\tilde{c}}) = h(t, \underline{\hat{c}}) + \left.\frac{\partial h(t, \underline{c})}{\partial \underline{c}}\right|_{\underline{\hat{c}}}^{\#} \Delta \underline{c} + \tfrac{1}{2}\left\{\Delta \underline{c}^{\#} \left.\frac{\partial^2 h(t, \underline{c})}{\partial \underline{c}\,\partial \underline{c}}\right|_{\underline{\hat{c}}} \Delta \underline{c}\right\} + \cdots \qquad 2.6\text{-}5$$

where

$$\Delta \underline{c} = \underline{\tilde{c}} - \underline{\hat{c}} \qquad 2.6\text{-}6$$

Using this expansion of $h(t, \underline{\tilde{c}})$, the corresponding response is

$$\theta_0(t) = u_0(t) + \underline{u}^{\#}(t, \underline{\hat{c}})\Delta\underline{c} + \tfrac{1}{2}\{\Delta\underline{c}^{\#} V(t, \underline{\hat{c}})\Delta\underline{c}\} + \cdots \qquad 2.6\text{-}7$$

where the elements of the vector $\underline{u}(t, \underline{\hat{c}})$ are convolutions

$$u_i(t, \underline{\hat{c}}) = \int_{-\infty}^{\infty} \left.\frac{\partial h(t-\mu, \underline{c})}{\partial c_i}\right|_{\underline{\hat{c}}} \psi(\mu)\, d\mu \qquad i = 1, 2, \ldots, L \qquad 2.6\text{-}8$$

and where the elements of the matrix $V(t, \underline{\hat{c}})$ are convolutions

$$v_{ij}(t, \underline{\hat{c}}) = \int_{-\infty}^{\infty} \left.\frac{\partial^2 h(t-\mu, \underline{c})}{\partial c_i\, \partial c_j}\right|_{\underline{\hat{c}}} \psi(\mu)\, d\mu \qquad \begin{matrix} i = 1, 2, \ldots, L \\ j = 1, 2, \ldots, L \end{matrix} \qquad 2.6\text{-}9$$

If, in a sufficiently small region about $\theta_0(t)$ defined by $|\theta_0(t) - u_0(t)| \leqq \delta$, $\delta > 0$, the nonlinear terms of Eq. 2.6-7 are negligible, then the linear approximation

$$\theta_0(t) \approx u_0(t) + \underline{u}^{\#}(t, \underline{\hat{c}})\Delta\underline{c} \qquad 2.6\text{-}10$$

is valid. In the region specified by δ, the $|\Delta c_i|$ must satisfy

$$|\Delta c_i| = |\tilde{c}_i - \hat{c}_i| \leqq \frac{\delta}{L\psi_{max} B} \qquad i = 1, 2, \ldots, L \qquad 2.6\text{-}11$$

where

$$\int_{-\infty}^{\infty} \left| \left.\frac{\partial h(t, \underline{c})}{\partial c_i}\right|_{\underline{\hat{c}}} \right| dt \leqq B \qquad i = 1, 2, \ldots, L \qquad 2.6\text{-}12$$

and

$$|\psi(t)| \leqq \psi_{max} \quad \text{for all } t \qquad 2.6\text{-}13$$

The transfer operator representation for the expression of Eq. 2.6-10 is:

$$\theta_0(t) = H(p, \underline{\tilde{c}})\psi(t) \approx H(p, \underline{\hat{c}})\psi(t) + \left[\frac{\partial H(p, \underline{c})}{\partial \underline{c}} \bigg|_{\underline{\hat{c}}} \psi(t) \right]^{\#} \Delta \underline{c} \qquad 2.6\text{-}14$$

From Eq. 2.6-14 it is seen that when Eq. 2.6-11 is satisfied, $\theta_0(t)$ can be approximated by the system shown in Fig. 2.6-3.

The functions $u_i(t, \underline{\hat{c}})$ are the parameter influence coefficients [38] for the system which represents the sensitivity of the system response to small changes in the system parameters. Other authors [22, 38] have obtained the parameter influence coefficients for a system by differentiating the differential equation of the system with respect to the parameters in question.

In the analysis above it is shown that the system response for one set of parameter values, $\underline{c} = \underline{\tilde{c}}$, can be obtained approximately by combining the model response for a second set of parameter values, $\underline{c} = \underline{\hat{c}}$, and the corresponding parameter influence coefficients. Consider now the reverse problem of determining the value of $\underline{\tilde{c}}$ when the value of $\underline{\hat{c}}$ and the time functions $\theta_0(t)$, $u_0(t)$ and $\underline{u}(t, \underline{\hat{c}})$ are all known. One method of obtaining an estimate of the value of $\underline{\tilde{c}}$ is to apply the regression analysis technique discussed in Sections 2.4 and 2.5. To be a valid calculation, the value of $\underline{\tilde{c}}$ must be such that Expression 2.6-11 is satisfied.

Before discussing the application of regression analysis to the problem of estimating $\underline{\tilde{c}}$, let us define the variable $e_1(t)$ as:

$$e_1(t) = \theta_0(t) - u_0(t) \qquad 2.6\text{-}15$$

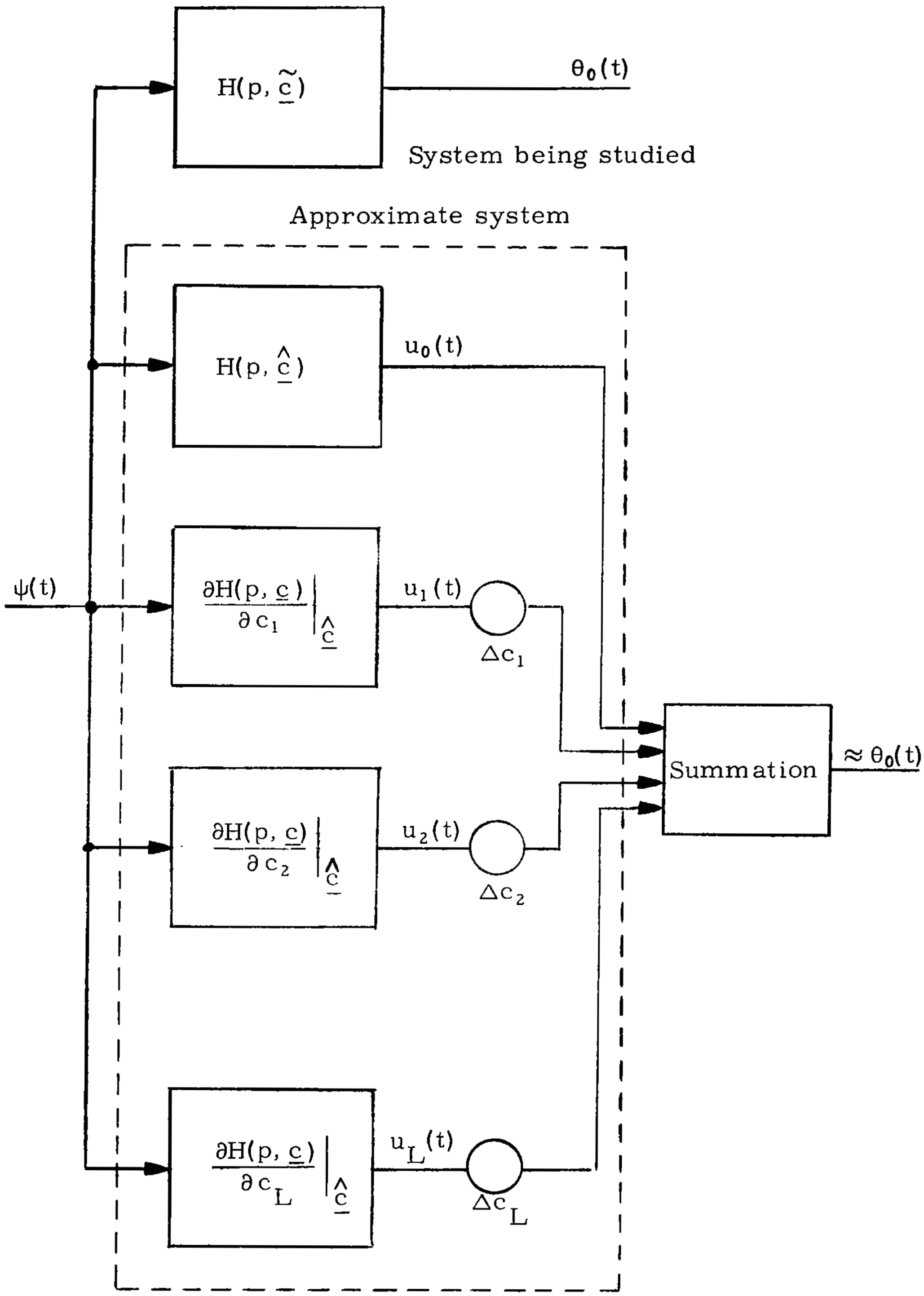


Figure 2.6-3 Generation of Approximation to System Response

From Eq. 2.6-10, this can be rewritten as:

$$e_1(t) \approx u^{\#}(t, \underline{\hat{c}})\Delta\underline{c} \qquad 2.6\text{-}16$$

Rather than consider identification based on $e_1(t)$ let us consider a system output $\theta(t)$ which is the sum of $\theta_0(t)$ and the effects of such factors as additive noise. Due to the randomness of $\theta(t)$ it is more appropriate to consider the conditional mean of e(t) for a given input, where

$$e(t) = \theta(t) - u_0(t) \qquad 2.6\text{-}17$$

and the conditional mean is:

$$\overline{e}(t) \approx \underline{u}^{\#}(t, \underline{\hat{c}})\Delta\underline{c} \qquad 2.6\text{-}18$$

Using this expression for $\overline{e}(t)$ it is desired to obtain estimates of the parameter vector $\Delta\underline{c}$ which can be combined with the known value of $\underline{\hat{c}}$ to obtain an estimate of $\underline{\tilde{c}}$. To obtain an estimate of $\Delta\underline{c}$ a cost function similar to that of Eq. 2.4-5 is defined as

$$J = \frac{1}{2T}\int_0^T [e(t) - \underline{u}^{\#}(t, \underline{\hat{c}})\underline{b}]^2 dt \qquad 2.6\text{-}19$$

where $\underline{b}$ is an estimate of $\Delta\underline{c}$ and data is available for the interval (0, T). As in Section 2.4, the best estimate of $\Delta\underline{c}$, in the mean square sense, is obtained by setting the gradient of J equal to zero, i.e.,

$$\frac{\partial J}{\partial \underline{b}} = \frac{1}{T}\int_0^T [-\underline{u}(t, \underline{\hat{c}})][e(t) - \underline{u}^{\#}(t, \underline{\hat{c}})\underline{b}]dt = 0 \qquad 2.6\text{-}20$$

Rearranging Eq. 2.6-20 gives the best estimate of $\Delta\underline{c}$ to be:

$$\underline{b}_e = R^{-1}(\underline{\hat{c}})\, \underline{v}(\underline{\hat{c}}) \qquad 2.6\text{-}21$$

where

$$R(\hat{\underline{c}}) = \frac{1}{T}\int_0^T \underline{u}(t, \hat{\underline{c}})\underline{u}^{\#}(t, \hat{\underline{c}})dt \qquad 2.6\text{-}22$$

$$\underline{v}(\hat{\underline{c}}) = \frac{1}{T}\int_0^T e(t)\underline{u}(t, \hat{\underline{c}})dt \qquad 2.6\text{-}23$$

Under the assumption that the inverse of $R(\hat{\underline{c}})$ exists, the method of Section 2-5 may be used to implicitly perform the matrix inversion. Having the best estimate of $\Delta\underline{c}$ defined by Eq. 2.6-21, an estimate of $\tilde{\underline{c}}$ can be obtained from:

$$\tilde{\underline{c}}_e = \hat{\underline{c}} + \underline{b}_e \qquad 2.6\text{-}24$$

2.7 Iterative Considerations

In most practical situations, the value of $\tilde{\underline{c}}$ is not known well enough to guarantee that Eq. 2.6-11 will be satisfied initially. In such situations an iterative identification technique is necessary to obtain good estimates of $\tilde{\underline{c}}$. One technique, based on the method of Section 2.6, is to initially guess at the value of $\tilde{\underline{c}}$ and set $\hat{\underline{c}}$ equal to this guess. Then apply the method of Section 2.6 to obtain an estimate, $\tilde{\underline{c}}_e$. The data then can be rerun with a new value of $\hat{\underline{c}}$ equal to the last value of $\tilde{\underline{c}}_e$, i.e.,

$$\hat{\underline{c}}_{i+1} = \tilde{\underline{c}}_{ei} \qquad 2.7\text{-}1$$

or

$$\hat{\underline{c}}_{i+1} = \hat{\underline{c}}_i + \underline{b}_{ei} \qquad 2.7\text{-}2$$

This method is known as the "Gauss-Newton" iteration technique [13]. Unfortunately this method is not guaranteed to be convergent for arbitrarily large values of $\Delta\underline{c}$. However, the method is known to be quadratically convergent in some small neighborhood of $\tilde{\underline{c}}$ [13].

To apply this iterative technique to the data from a given interval, (0, T), the time histories of $\psi(t)$ and $\theta_0(t)$ must be recorded or stored in some manner for use during each iteration. One method for accomplishing this is to record the time histories of $\psi(t)$ and $\theta_0(t)$ on magnetic tape which then can be replayed for each iteration. An objection to this approach is the fact that each iteration requires T seconds of actual time. If many iterations are required this can be a time consuming process. However, due to equipment limitations, this is often the method that must be used. As is discussed in Chapter 5, this is the method of iteration that was used in analysing the experimental data from the human operator tests.

If a hybrid computer is available, the iteration process can be accomplished with a great reduction in time. A block diagram for implementing the iterative regression analysis method on a hybrid computer is given in Fig. 2.7-1. In this implementation, the first estimate of $\underline{\tilde{c}}$ is obtained from an on-line computation of $R(\hat{\underline{c}})$ and $\underline{v}(\hat{\underline{c}})$. All subsequent computations of $R(\hat{\underline{c}})$ and $\underline{v}(\hat{\underline{c}})$ are obtained from a "fast-time" solution of Eqs. 2.6-22 and 2.6-23. This "fast-time" computation is made possible by being able to reproduce $\psi(t)$ and $\theta_0(t)$ from the digital storage at a much faster rate than the sampling rate of the analog to digital converter. Analog to digital conversion rates of 10^5 samples per second of a single variable are possible with available equipment [36]. It has been found [22] that conversion rates as low as 20 samples per second are adequate for signals of the type considered in Chapter 5. Using a "fast-time" scale which is 10^3 times real-time and an interval length of the order of twenty seconds it is possible to obtain ten iterations in less than one second of actual time. Thus the time required for data reduction in the analysis of Chapter 5 could be reduced considerably with the use of a hybrid computer.

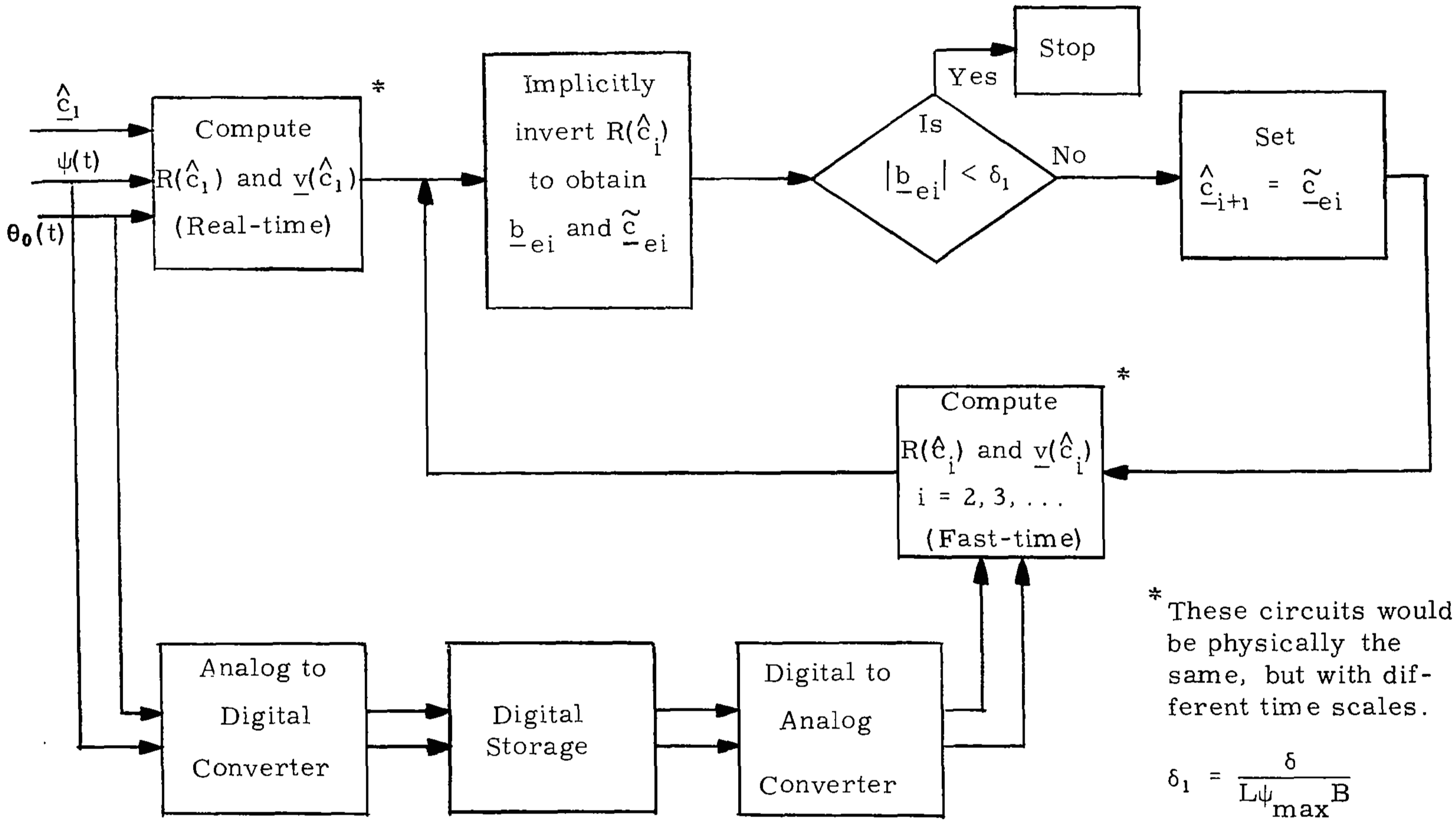


Figure 2.7-1 Hybrid Computer Parameter Identification Block Diagram

CHAPTER 3

APPLICATION OF REGRESSION ANALYSIS TO THE CROSSOVER MODEL

3.1 The Model

As discussed in Chapter 1, the first choice in determining a model of the human operator is between a psychological model and an engineering model. Once this decision has been made it is necessary to choose the actual model to be used based on the particular aspects of the experiment that is to be performed. For the type of study proposed here, i.e., inference of gross characteristics of the human operator, it was decided to use an engineering model for analysis of the experimental data.

Of the many engineering models that have been proposed, the crossover model proposed by McRuer, et al. [28] has advantages which are not shared by other models. The transfer operator which characterizes the crossover model is given in Eq. 1.3-2 and is repeated here.

$$Y_C(p)Y_P(p) = \frac{Ke^{-\tau p}}{p} \qquad 3.1\text{-}1$$

The fundamental strong point of the crossover model is the good fit to experimental data in the frequency domain for controlled elements such as $Y_C(p) = 1/p$ and $Y_C(p) = 1/p^2$. The fit is especially good in the frequency range where the open-loop system, including the human operator, has unity gain. Although other models such as the extended crossover model [28] fit the experimental frequency data better for the entire frequency spectrum, McRuer, et al. [28] point out that the crossover model is "adequate to describe key trends in the crossover region." Coupled with the good match to frequency domain data is the important point that the model includes only two parameters, open-loop gain and time-delay. Thus in any parameter identification technique, the necessary computational equipment is minimized. In addition, Jackson [22] has shown "...that the crossover model output is most sensitive to parameter changes in the same (frequency) region it most accurately describes human response."

Due to difficulties in implementing a pure time-delay, especially when the length of delay is not known a priori, an approximation of pure time-delay is used. The form used is a first order Padé approximation [39]. Using this approximation the transfer operator for a pure time-delay of τ seconds can be approximated as

$$e^{-\tau p} \approx \frac{\frac{2}{\tau} - p}{\frac{2}{\tau} + p} \qquad 3.1\text{-}2$$

The frequency domain phase shift characteristics for both a pure time-delay and the Padé approximation are shown in Fig. 3.1-1. From this figure it is seen that the Padé approximation has a phase shift which agrees quite well with the phase shift of the pure time-delay for low values of frequency. Also note that both the pure time-delay and the approximation have unity gain for all frequencies. Since the dominant frequencies present in the experimental data are of the order of 2 rad/sec and τ is the order of 0.25 sec, it is felt that the Padé approximation used is sufficiently good for analyzing the experimental data ($\tau\omega = 0.5$).

With this approximation the actual model used is a modified crossover model, modified in the sense that the pure time-delay is represented by the Padé approximation. Thus the model used to analyze experimental data has a transfer operator,

$$Y_C(p)Y_P(p) = \frac{K(\frac{2}{\tau} - p)}{p(\frac{2}{\tau} + p)} \approx \frac{Ke^{-\tau p}}{p} \qquad 3.1\text{-}3$$

As is discussed in Chapter 5, the data for each experimental trial is divided into successive intervals. Estimates of the two model parameters, K and τ, are then obtained for each interval. The

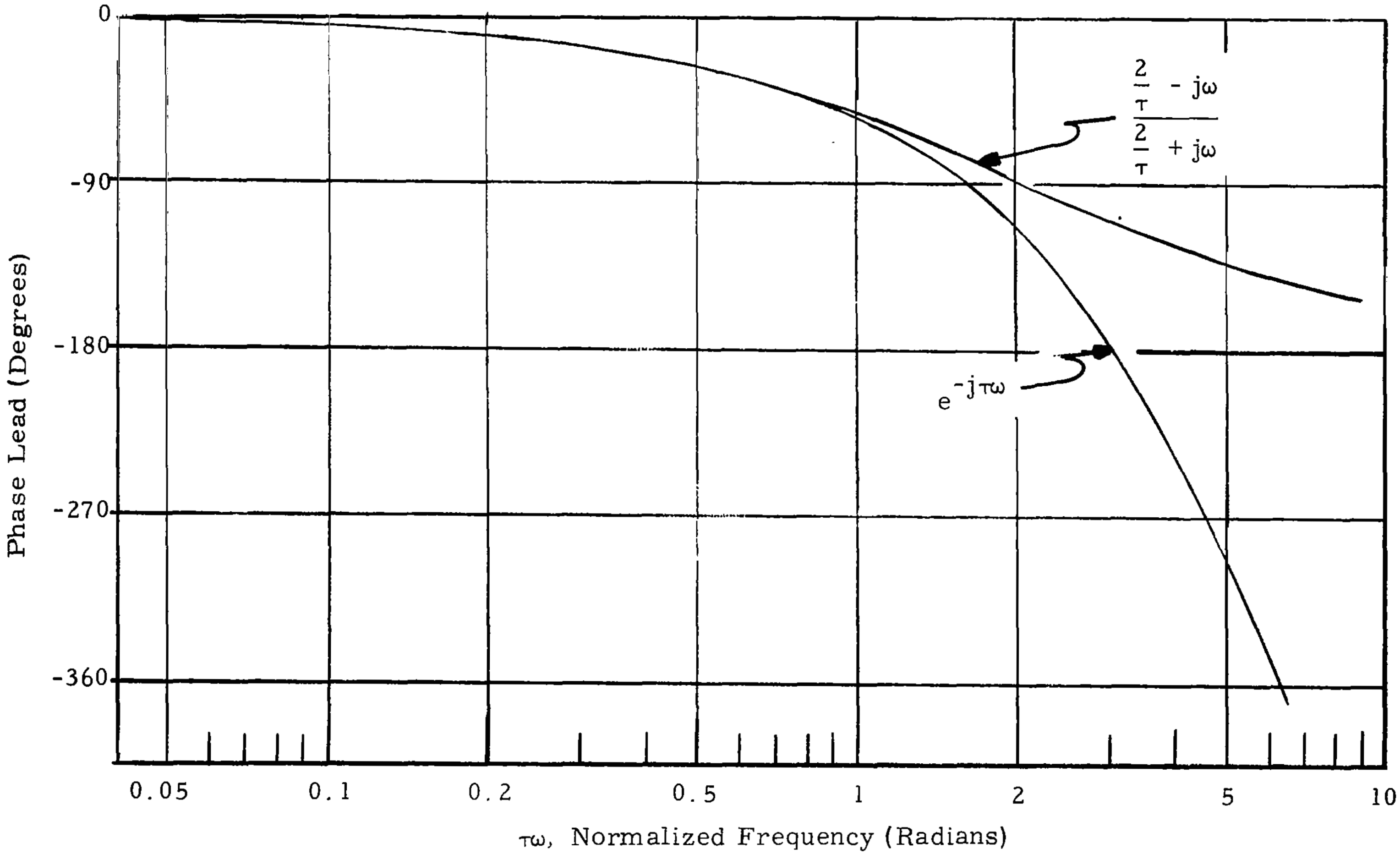


Figure 3.1-1 Comparison of Phase Characteristics of Pure Time-Delay and First Order Padé Approximation

remainder of this chapter is devoted to developing the circuits, and various aspects of these circuits, which are used in the analysis of the experimental data. Block diagrams for the circuits are presented in this chapter while the corresponding analog computer circuit diagrams are presented in Appendix A.

3.2 Comparison of Equation Error and Response Error Techniques

Two approaches that are commonly taken in parameter identification problems are the equation error approach and the response error approach [20]. The two methods are compared here and reasons for the use of the response error technique in the present data analysis are developed.

For the purpose of this discussion, consider the example of a closed-loop system of the form shown in Fig. 3.2-1.

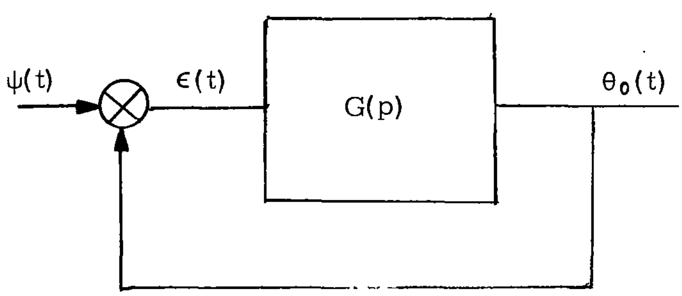


Figure 3.2-1 Closed-Loop Linear System Representation

For simplicity let us assume that the open-loop transfer operator, $G(p)$, is given by

$$G(p) = \frac{1}{\frac{1}{K}p + 1} \qquad 3.2\text{-}1$$

Then the closed-loop transfer operator is:

$$H(p) = \frac{G(p)}{1 + G(p)} = \frac{K}{p + 2K} \qquad 3.2\text{-}2$$

Let the value of K in the system be $K = \tilde{K}$. Then the system output can be represented by

$$\dot{\theta}_0 + \tilde{K}\theta_0 = \tilde{K}\epsilon \qquad 3.2\text{-}3$$

The equation error, $e_1(t)$, for this system is defined by

$$e_1(t) = \dot{\theta}_0 + b\theta_0 - b\epsilon \qquad 3.2\text{-}4$$

where b is an estimate of $\tilde{K}$. The best estimate of $\tilde{K}$ is then the value of b which minimizes an appropriate norm of $e_1(t)$. A block diagram which would be used in the application of the equation error method is given in Fig. 3.2-2.

When regression analysis is used to obtain the parameter estimates, the norm or cost function used is the integral squared error. Thus the cost function, J_1, for the equation error method is

$$J_1 = \frac{1}{2T}\int_0^T [e_1(t)]^2 dt \qquad 3.2\text{-}5$$

The best estimate of $\tilde{K}$, in the mean square sense, is obtained by setting $\frac{\partial J_1}{\partial b}$ equal to zero.

$$\frac{\partial J_1}{\partial b} = \frac{1}{T}\int_0^T (\theta_0 - \epsilon)[\dot{\theta}_0 + b\,\theta_0 - b\epsilon]dt = 0 \qquad 3.2\text{-}6$$

For this example, the best estimate, b_e, is

$$b_e = -\frac{\frac{1}{T}\int_0^T \dot{\theta}_0(\theta_0 - \epsilon)dt}{\frac{1}{T}\int_0^T (\theta_0 - \epsilon)^2 dt} \qquad 3.2\text{-}7$$

A distinct advantage of this formulation over that of Section 2-6 is that the estimate of $\tilde{K}$ is obtained with no requirement that a

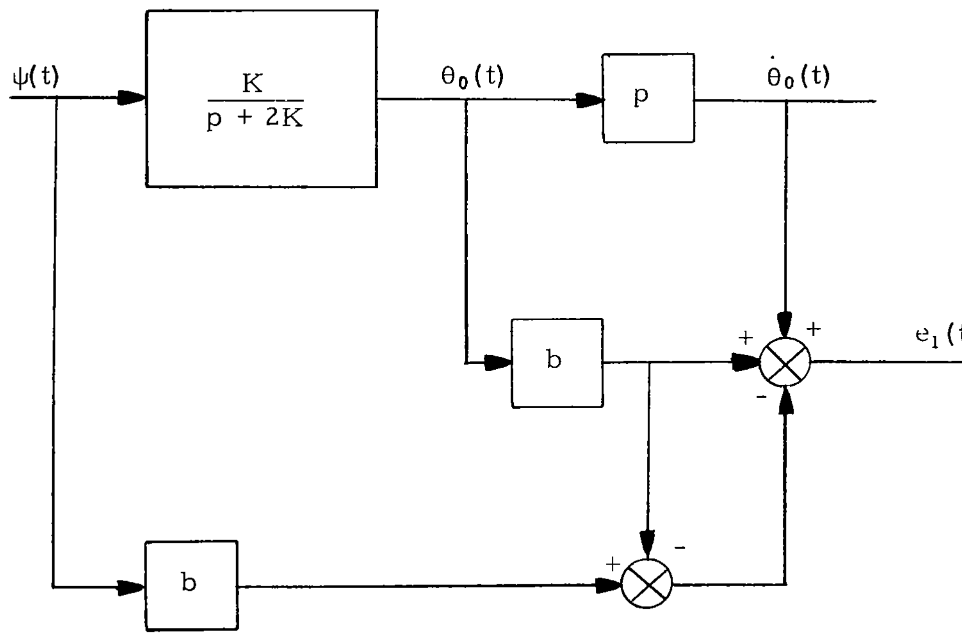


Figure 3.2-2 Block Diagram for Equation Error Parameter Identification

corresponding model parameter be sufficiently close to $\tilde{K}$. Thus this method does not require an iterative computational procedure. Variations of the equation error formulation have been used to determine such system parameters as aircraft dynamic stability derivatives from flight data [32, 33], and servomechanism linear and nonlinear terms [19, 25].

In addition to the advantage of not requiring an iterative computational procedure there are two rather serious disadvantages. The first disadvantage is the fact that all state variables must be measured to obtain the parameter estimate given by Eq. 3.2-7. In systems of high order, the requirement that all state variables be measurable can be almost impossible to meet. One method for circumventing this problem has been proposed by Kohr [25]. This method involves passing a given signal through a "state variable filter" from which not only the desired signal but also all necessary derivatives of the signal can be obtained, at least approximately.

A second disadvantage of the equation error formulation involves the presence of additive noise such as was discussed in Section 2.2. In such situations it has been shown [7] that the parameter estimates are statistically biased due to the noise. This effect can be quite pronounced when large amounts of noise are present. In fact, this effect was the main reason for not using the equation error formulation in the analysis of the human operator data in Chapter 5.

In applying the response error method, a computer model of the process shown in Fig. 3.2-1 must be physically constructed. The response of this model, which has the same input as the system being a analysed, is denoted by z(t). The response error, $e_2(t)$, is then defined by:

$$e_2(t) = \theta_0(t) - z(t) \qquad 3.2\text{-}8$$

A block diagram for use in applying the response error technique is given in Fig. 2.6-3. In this method, as in the equation error method, the best estimates of the model parameters are obtained by minimizing an appropriate norm of $e_2(t)$.

In a general formulation of the response error method, the variables $u_i(t)$ discussed in connection with Fig. 2.6-3 do not have to be sensitivity coefficients. As Elkind [7] has formulated the problem, the functions $u_i(t)$ correspond to outputs from filters which are orthogonal to each other. In this type of formulation the parameter estimates are combined with the individual filter weighting functions to obtain an estimate of the unknown system weighting function. Thus in the example given the regression analysis parameter estimates would be used to obtain an estimate of the closed-loop weighting function:

$$h(t) = \tilde{K} e^{-2\tilde{K}t} \qquad 3.2\text{-}8$$

or the closed-loop transfer operator

$$H(p) = \frac{\tilde{K}}{p + 2\tilde{K}} \qquad 3.2\text{-}9$$

Although this formulation does not require iteration to obtain the parameter estimates, the method does not give a direct estimate of the parameter $\tilde{K}$. To obtain estimates of the unknown system parameters, the formulation discussed in Section 2.6 must be used. This formulation in general requires an iteration process to obtain the best estimate of the desired parameters.

A major advantage of either formulation of the response error technique is the fact that the parameter estimates obtained are statistically unbiased in the presence of additive noise. This is an important characteristic in situations such as human operator analysis where large amounts of equivalent additive noise are present.

In this research the choice between using the equation error approach or one of the response error formulations was based on two factors. One factor was the desire to obtain estimates of the actual system parameters rather than an estimate of the system transfer operator or weighting function. The second factor was the desire to have statistically unbiased estimates of the parameters in the presence of additive noise. The only formulation that satisfies both factors is the method described in Section 2.6 Unfortunately, this is also the only one of the three methods which requires an iterative computational procedure.

3.3 Regression Analysis Applied to the Crossover Model

In this section the parameter influence coefficient equations and other related expressions discussed in Sections 2.5 and 2.6 will be developed for the approximate crossover model. The expression of Eq. 3.1-3 is the open-loop transfer function for this model. The corresponding closed-loop transfer operator for the model is

$$H(p, K, \tau) = \frac{Y_C Y_P}{1 + Y_C Y_P} = \frac{K(\frac{2}{\tau} - p)}{p^2 + (\frac{2}{\tau} - K)p + \frac{2K}{\tau}} \qquad 3.3\text{-}1$$

Note that the time-delay parameter, τ, appears in Eq. 3.3-1 as the denominator of an equivalent parameter, α,

$$\alpha = \frac{2}{\tau} \qquad 3.3\text{-}2$$

If the parameter influence coefficient for τ is obtained, τ will always appear as a denominator which requires many division circuits in the implementation. Although division circuits are completely valid and practical to implement they are in general not as convenient to use as are circuits which perform multiplication. For this reason, the use of α eliminates the need for division circuits in the implementation. The use of α in analyzing the data of Chapter 5 requires the transformation of each value of α that is obtained back into a corresponding value of τ. The equations for the crossover model and the corresponding closed-loop transfer operator in terms of α are given respectively by Eqs. 3.3-3 and 3.3-4.

$$Y_P(p)Y_C(p) = \frac{K(\alpha - p)}{p(\alpha + p)} \qquad 3.3\text{-}3$$

$$H(p,K,\alpha) = \frac{K(\alpha - p)}{p^2 + (\alpha - K)p + \alpha K} \qquad 3.3\text{-}4$$

As described in Section 2.6, the parameter influence coefficients are obtained by the use of filters which have transfer operators which are $\frac{\partial H}{\partial K}$ and $\frac{\partial H}{\partial \alpha}$. The expressions for these transfer operators are

$$\frac{\partial H}{\partial K} = \frac{-p^3 + \alpha^2 p}{[p^2 + (\alpha - K)p + \alpha K]^2} \qquad 3.3\text{-}5$$

$$\frac{\partial H}{\partial \alpha} = \frac{2Kp^2}{[p^2 + (\alpha - K)p + \alpha K]^2} \qquad 3.3\text{-}6$$

Thus the expressions for $u_1(t)$, the parameter influence coefficient for K, and $u_2(t)$, the parameter influence coefficient for α, are

$$u_1(t) = \frac{-p^3 + \alpha^2 p}{[p^2 + (\alpha - K)p + \alpha K]^2}\,\psi(t) \qquad 3.3\text{-}7$$

$$u_2(t) = \frac{2Kp^2}{[p^2 + (\alpha - K)p + \alpha K]^2}\,\psi(t) \qquad 3.3\text{-}8$$

Having these expressions, the system used to analyze the experimental data is shown in Fig. 3.3-1. In Fig. 3.3-1, the parameter values of the human operator system are denoted by a tilde (~) and the parameter values in the model are denoted by a caret (^). The differences between the two values are denoted as

$$\Delta K = \tilde{K} - \hat{K} \qquad 3.3\text{-}9$$

$$\Delta \alpha = \tilde{\alpha} - \hat{\alpha} \qquad 3.3\text{-}10$$

The expressions for the parameter influence coefficients given by Eqs. 3.3-7 and 3.3-8 can be rearranged to give:

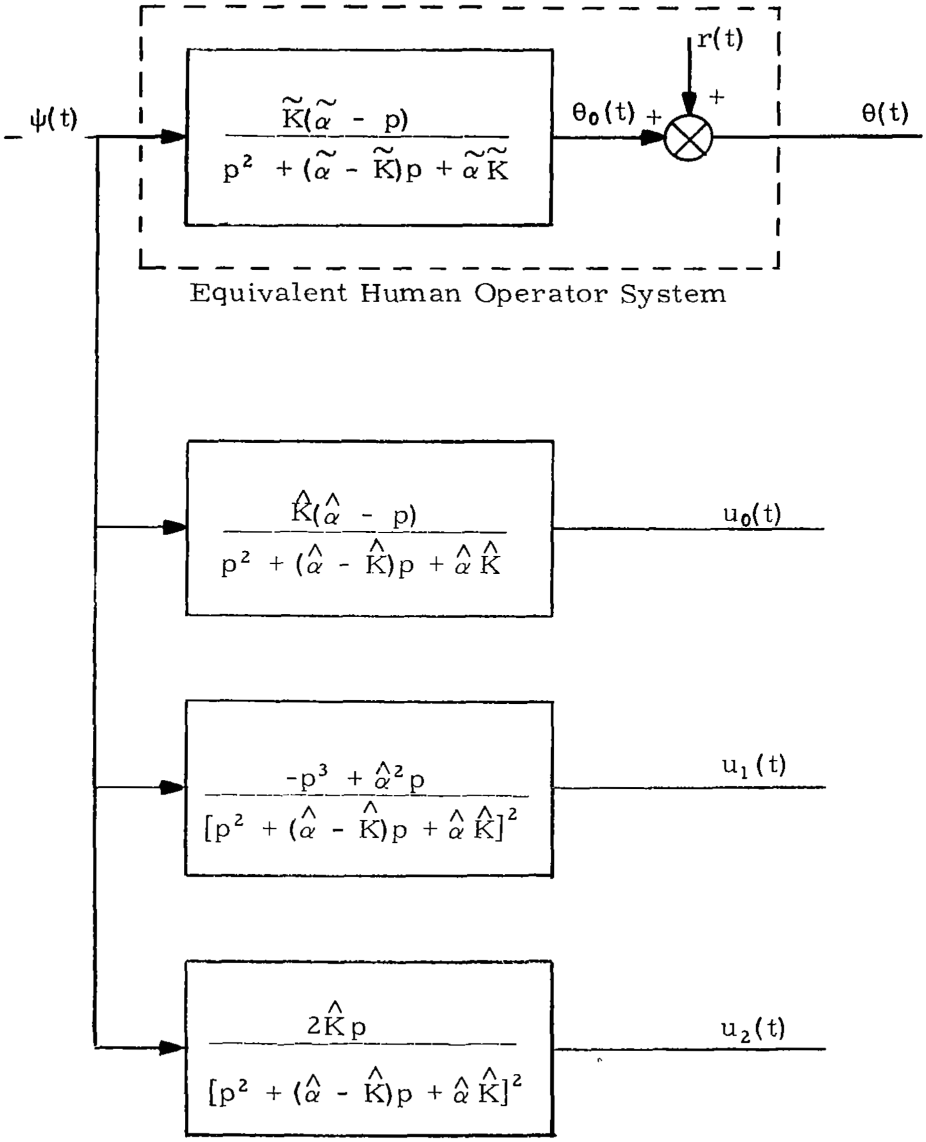


Figure 3.3-1 Block Diagram of Crossover Model Analysis System

$$u_1(t) = \frac{p(p+\alpha)}{K[p^2 + (\alpha - K)p + \alpha K]} u_0(t) \qquad 3.3\text{-}11$$

$$u_2(t) = \frac{2p^2}{(\alpha - p)[p^2 + (\alpha - K)p + \alpha K]} u_0(t) \qquad 3.3\text{-}12$$

A block diagram for implementing Eqs. 3.3-4, 3.3-11 and 3.3-12 is given in Fig. 3.3-2.

As a check on setting up the computer circuits, a dynamic check was performed on the computer circuits. This dynamic check involves comparing the computer response for a sinusoidal input with the calculated response for the same input. This dynamic check is also presented in Appendix A.

3.4 Estimation of Parameters

In applying regression analysis to the estimation of crossover model parameters, the cost function used is

$$J = \frac{1}{2T}\int_0^T [\theta(t) - u_0(t) - b_1 u_1(t) - b_2 u_2(t)]^2 dt \qquad 3.4\text{-}1$$

where b_1 is an estimate of ΔK and b_2 is an estimate of $\Delta\alpha$. As in Section 2.4, the best parameter estimates, i.e., values of b_1 and b_2 which minimize J, are obtained by setting the gradient of J equal to zero.

$$\frac{\partial J}{\partial b_1} = \frac{1}{T}\int_0^T [-u_1(t)][e(t) - b_1 u_1(t) - b_2 u_2(t)]dt = 0 \qquad 3.4\text{-}2$$

$$\frac{\partial J}{\partial b_2} = \frac{1}{T}\int_0^T [-u_2(t)][e(t) - b_1 u_1(t) - b_2 u_2(t)]dt = 0 \qquad 3.4\text{-}3$$

where

$$e(t) = \theta(t) - u_0(t) \qquad 3.4\text{-}4$$

Rearranging Eqs. 3.4-2 and 3.4-3 yields the best estimate:

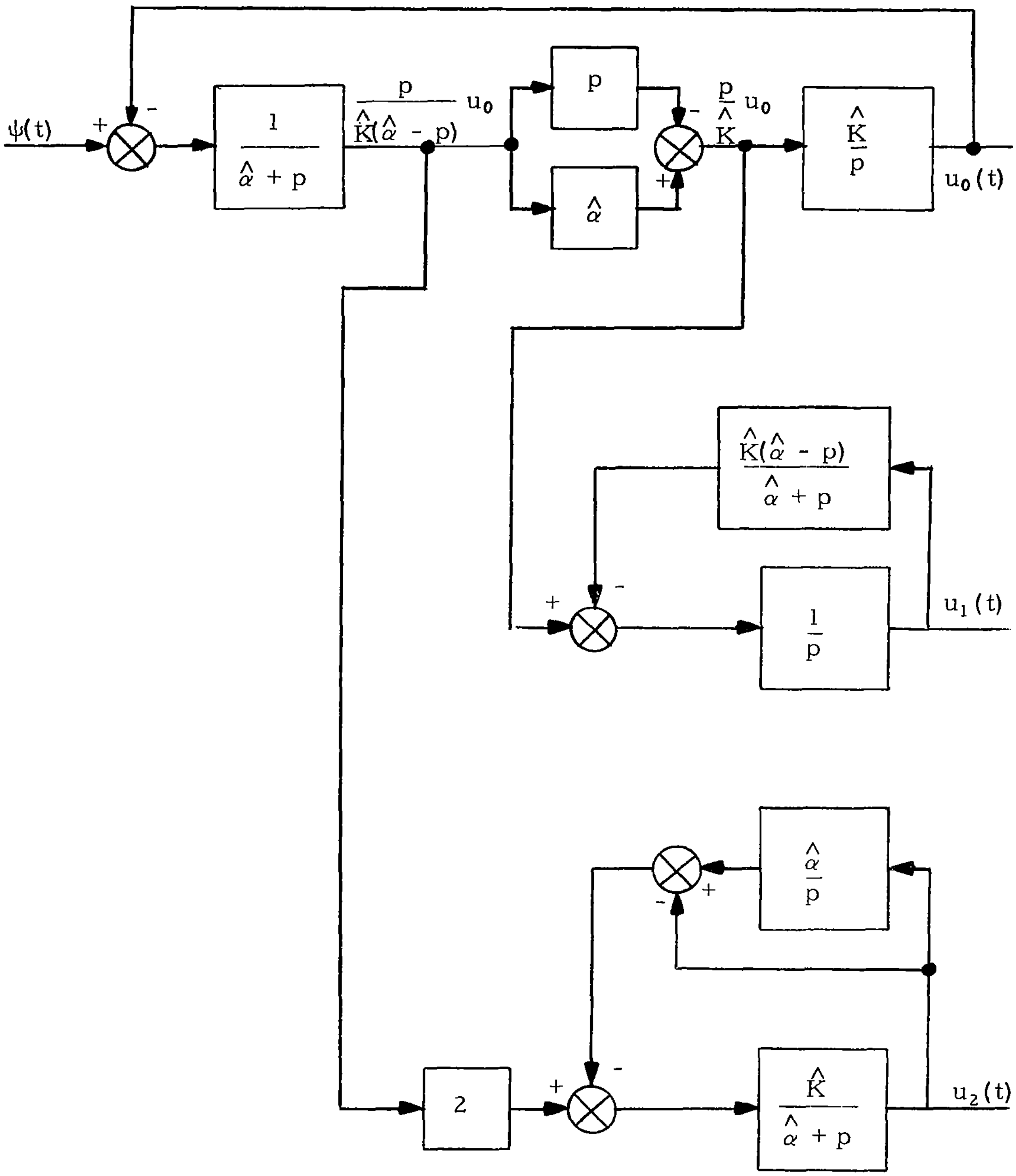


Figure 3.3-2 Block Diagram for Crossover Model Implementation

$$\underline{b}_e = R^{-1} \underline{v} \qquad 3.4\text{-}5$$

where $\underline{b}_e$ is a two dimensional vector with elements b_{1e} and b_{2e}, R is a 2 × 2 matrix,

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \qquad 3.4\text{-}6$$

with elements,

$$r_{ij} = \frac{1}{T} \int_0^T u_i(t)\, u_j(t)\, dt \qquad 3.4\text{-}7$$

$$i = 1, 2$$
$$j = 1, 2$$

and $\underline{v}$ is a vector with elements,

$$v_i = \frac{1}{T} \int_0^T u_i(t)\, e(t)\, dt \qquad i = 1, 2 \qquad 3.4\text{-}8$$

The block diagram for evaluating the elements of R and $\underline{v}$ is given in Fig. 3.4-1.

In the actual analysis, successive T-second intervals of data are analyzed. Thus in obtaining the elements of R and $\underline{v}$ it is necessary to begin calculating a set of values for a second interval immediately after the termination of a given interval. This capability is provided by using two sets of integrators with the proper set being automatically addressed for each interval.

As described in Section 2.5, R is inverted implicitly by solving

$$\underline{b}' + k R \underline{b} = k \underline{v} \qquad 3.4\text{-}9$$

The block diagram for the implicit matrix inversion calculation is shown in Figure 3.4-2. A typical response for this computation is shown in Figure 3.4-3. As seen in this figure, the implicit matrix inversion requires no more than 50 milliseconds. With a state-of-the-art

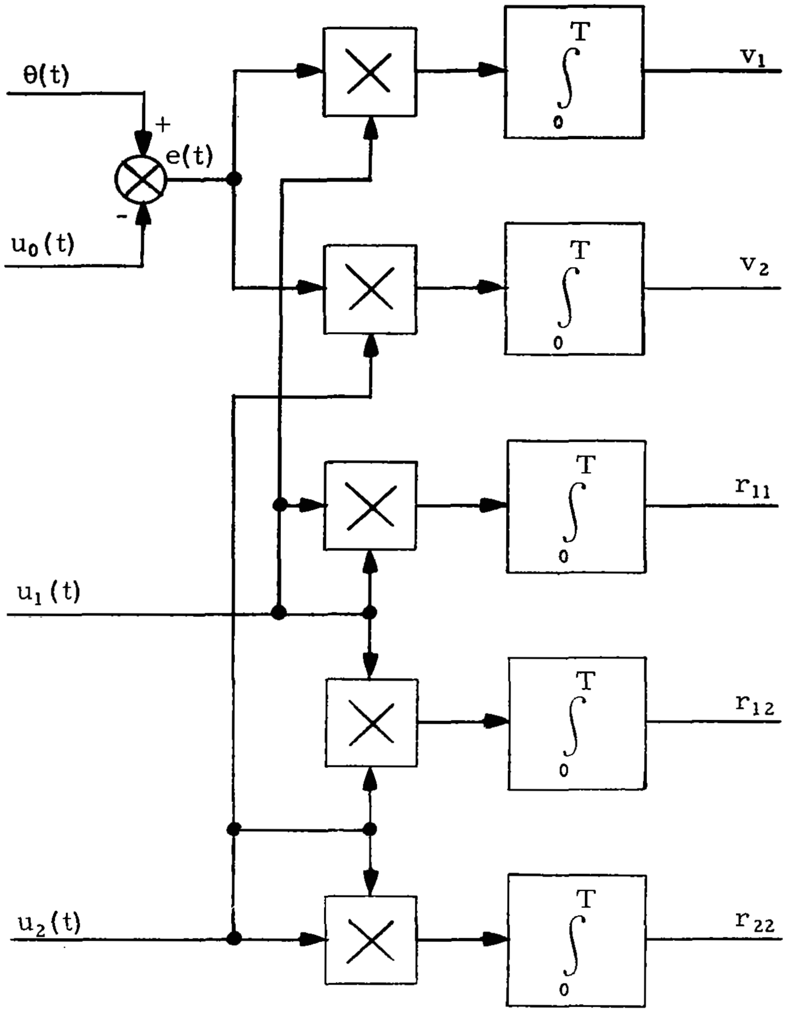
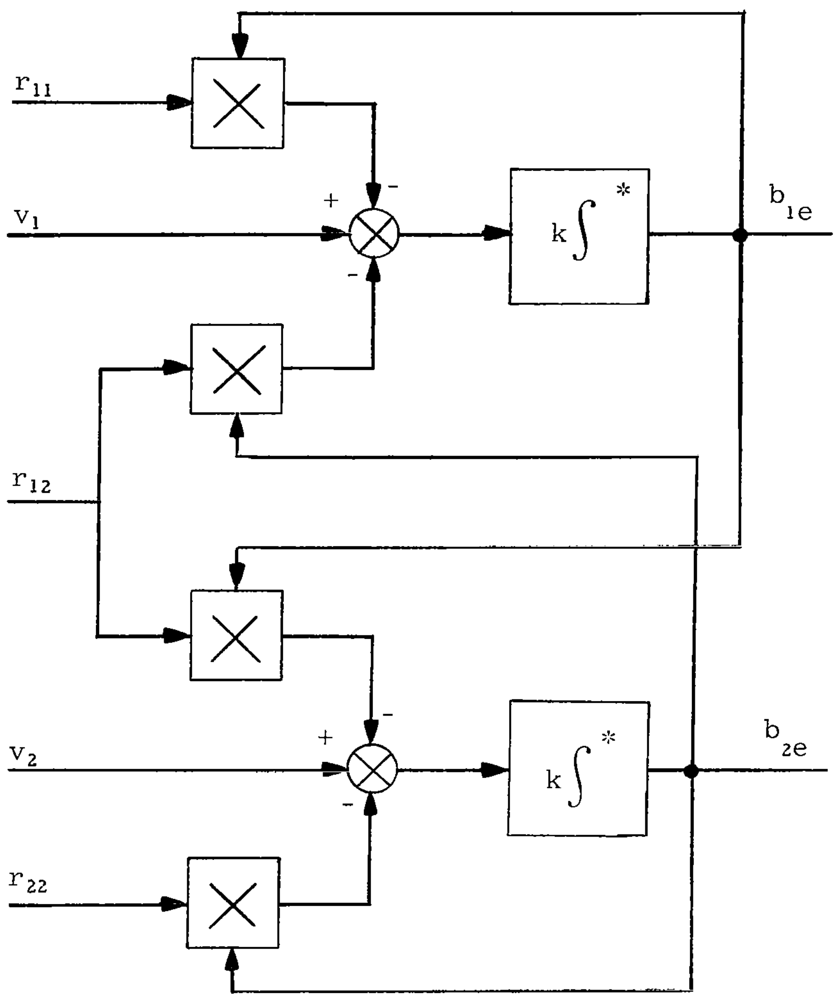


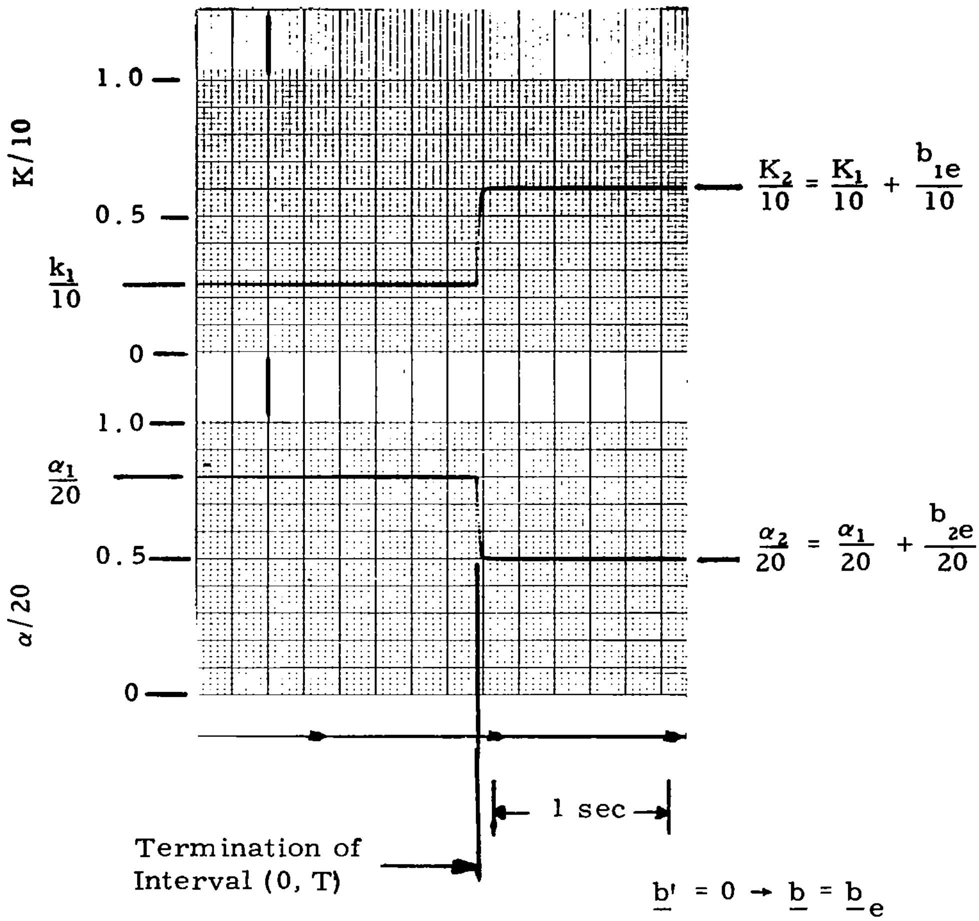
Figure 3.4-1 Block Diagram for Evaluating R and $\underline{v}$



*This integration occurs subsequent to the computation of R and $\underline{v}$.

Figure 3.4-2 Block Diagram for Solving

$$\underline{b}' + kR\underline{b} = k\underline{v}$$



Note that these time histories include such effects as recorder dynamics and thus represent a least upper bound on the response time.

Figure 3.4-3 Typical Response of:

$$\underline{b}' + kR\underline{b} = k\underline{v}$$

high speed computer this time could be reduced by at least a factor of 10.

As in Section 2.6, the estimates of K and α are obtained from

$$\tilde{K}_e = \hat{K} + b_{1e} \qquad 3.4\text{-}10$$

and

$$\tilde{\alpha}_e = \hat{\alpha} + b_{2e} \qquad 3.4\text{-}11$$

3.5 Modifications to Parameter Estimation Computations

In the interest of computing accuracy and simplicity of mechanization, two modifications were made in the computation of R and $\underline{v}$ discussed in Sections 3.3 and 3.4.

The first modification is based on the analysis of infinite data intervals presented in Appendix B. In this appendix it is shown that

$$\lim_{T\to\infty} r_{12} = \lim_{T\to\infty} \left\{ \frac{1}{T} \int_0^T u_1(t)u_2(t)\,dt \right\} = 0 \qquad 3.5\text{-}1$$

If Eq. 3.5-1 were approximately satisfied for sufficiently short finite data intervals, then r_{12} could be eliminated from the computations.

To determine the relative effect of r_{12} for short data intervals the normalized covariance, ρ, of $u_1(t)$ and $u_2(t)$ is considered.

$$\rho = \frac{r_{12}}{[r_{11} r_{22}]^{\frac{1}{2}}} \qquad 3.5\text{-}2$$

The matrix R rewritten in terms of ρ is

$$R = \begin{bmatrix} r_{11} & \rho\sqrt{r_{11} r_{22}} \\ \rho\sqrt{r_{11} r_{22}} & r_{22} \end{bmatrix} \qquad 3.5\text{-}3$$

and R^{-1} is

$$R^{-1} = \frac{1}{(1-\rho^2)} \begin{bmatrix} \frac{1}{r_{11}} & -\frac{\rho}{\sqrt{r_{11} r_{22}}} \\ -\frac{\rho}{\sqrt{r_{11} r_{22}}} & \frac{1}{r_{22}} \end{bmatrix} \qquad 3.5\text{-}4$$

Then from Eq. 3.4-5, the expression for the best parameter estimate, $\underline{b}_e$, is

$$\underline{b}_e = \frac{1}{(1-\rho^2)} \begin{bmatrix} \frac{1}{r_{11}} & -\frac{\rho}{\sqrt{r_{11} r_{22}}} \\ -\frac{\rho}{\sqrt{r_{11} r_{22}}} & \frac{1}{r_{22}} \end{bmatrix} \underline{v} \qquad 3.5\text{-}5$$

From Eq. 3.5-5, it is seen that if ρ is small compared to unity, then the cross-correlation (off-diagonal terms) of R can be neglected.

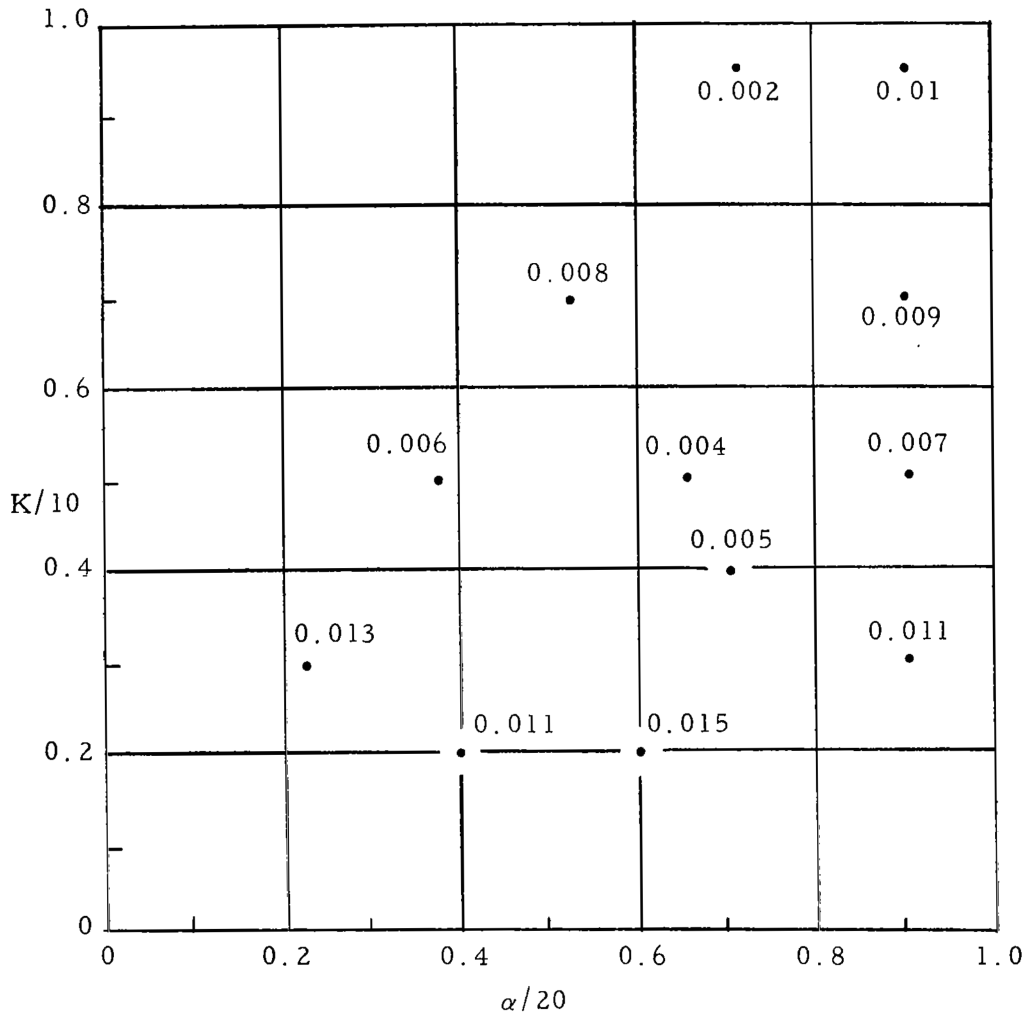
The results of an empirical survey of the value of ρ for different sets of parameter values are presented in Fig. 3.5-1. These data are for an interval 20 seconds in length. Shorter intervals produced larger values of ρ. From Fig. 3.5-1 it is seen that for T = 20 seconds the value of $|\rho|$ is less than 0.015 for all of the values of K and α used. Thus neglecting the cross-correlation terms is a valid step.

The second modification was made to compensate for the effect on $u_2(t)$ of variations in the model parameter values. The variation in the low frequency gain of $u_1(t)$ and $u_2(t)$ as a function of the value of K and α is seen by rewriting Eqs. 3.3-7 and 3.3-8 as:

$$u_1(t) = \frac{1}{K^2} \left\{ \frac{p^3/\alpha^2 + p}{\left[\frac{p^2}{\alpha K} + \frac{(\alpha - K)}{\alpha K} p + 1\right]^2} \right\} \psi(t) \qquad 3.5\text{-}6$$

$$u_2(t) = \frac{2}{K\alpha^2} \left\{ \frac{p^2}{\left[\frac{p^2}{\alpha K} + \frac{(\alpha - K)}{\alpha K} p + 1\right]^2} \right\} \psi(t) \qquad 3.5\text{-}7$$

In the experiments discussed in Chapter 5, the input, $\psi(t)$, is a random signal with a cut-off frequency of 2 rad/sec. Also, typical values of K



Numbers indicate value of $|\rho|$

$$\rho = \frac{\frac{1}{T}\int_0^T u_1 u_2 \, dt}{\left[\left(\frac{1}{T}\int_0^T u_1^2 \, dt\right)\left(\frac{1}{T}\int_0^T u_2^2 \, dt\right)\right]^{\frac{1}{2}}}$$

T = 20 seconds

Figure 3.5-1 A Survey of $|\rho|$ as a Function of K and α

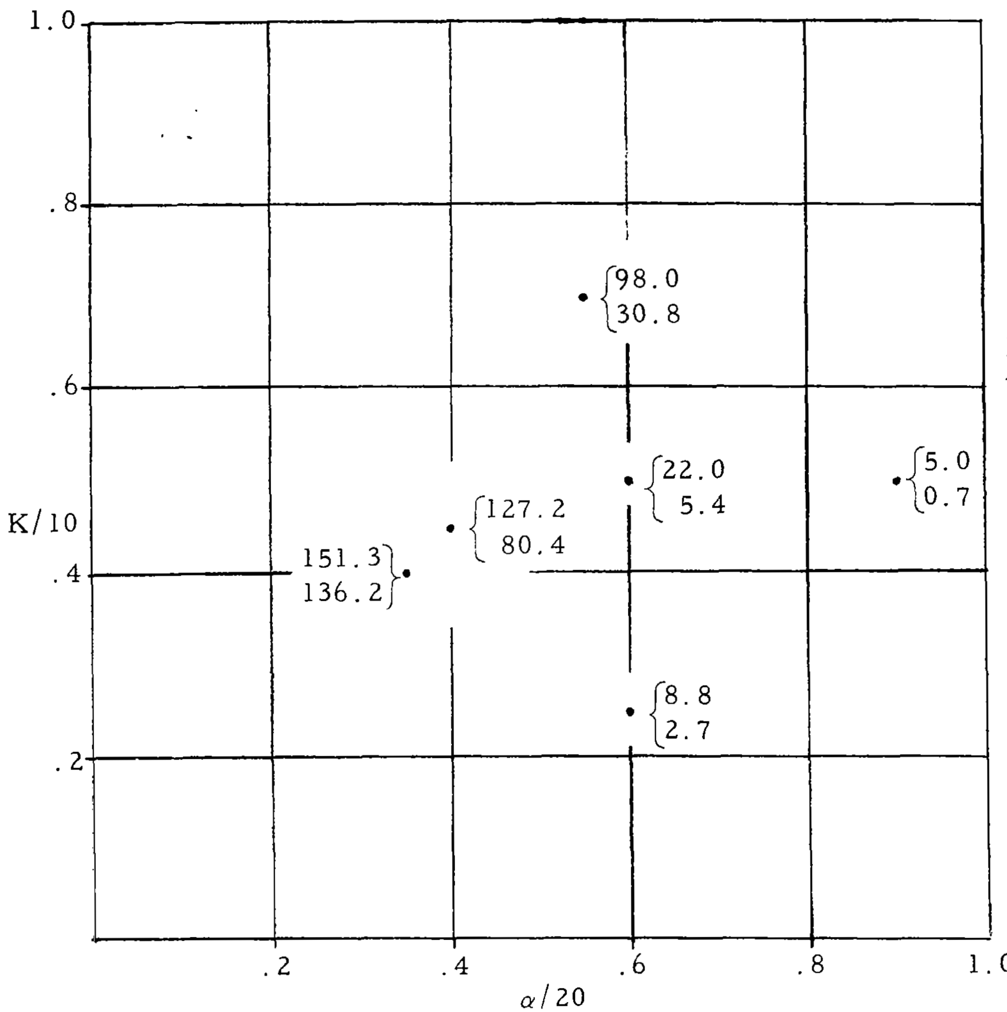
and α are 4 and 7 respectively. For these conditions the $\frac{p^2}{\alpha K}$ and $\frac{(\alpha - K)}{\alpha K}$ p terms in the denominator of Eq. 3.5-7 are small enough that for a given input the magnitude of $u_2(t)$ is, at least to a first approximation, inversely proportional to α^2.

The comparison of empirical values of both $\frac{1}{T}\int_0^T [u_2(t)]^2 dt$ and $\frac{\hat{\alpha}^2}{T}\int_0^T [u_2(t)]^2 dt$ given in Fig. 3.5-2 substantiates this hypothesis. From this figure it is seen that more uniform values are obtained for $\frac{\hat{\alpha}^2}{T}\int_0^T [u_2(t)]^2 dt$ than for $\frac{1}{T}\int_0^T [u_2(t)]^2 dt$ over the range of values of α considered. If the fixed scale is used, the values of $\frac{1}{T}\int_0^T [u_2(t)]^2 dt$ for large α would introduce analog-computer errors in the parameter calculation. Thus the equations were modified to be based on the automatically-scaled $\frac{\alpha^2}{T}\int_0^T [u_2(t)]^2 dt$. The modified expressions involving $u_2(t)$ are:

$$r_{22} = \frac{\hat{\alpha}^2}{T}\int_0^T [u_2(t)]^2 dt \qquad 3.5\text{-}8$$

$$v_2 = \frac{\hat{\alpha}^2}{T}\int_0^T [u_2(t)e(t)]dt \qquad 3.5\text{-}9$$

The modified analysis block diagrams are given in Figs. 3.5-3, 3.5-4 and 3.5-5. These block diagrams reflect the changes to Figs. 3.3-2, 3.3-3 and 3.3-4 due to deleting the cross-correlation terms in R and also due to the automatic scaling of $u_2(t)$.



Each number is the avarage of ten 20 second runs.

Lower number = $50\int_0^T u_2^2\,dt$; gain of $u_2(t)$ fixed.

Upper number = $1.388\alpha^2\int_0^T u_2^2\,dt$ gain of $u_2(t)$ variable.

Figure 3.5-2 Comparison of Values of $\int_0^T u_2^2\,dt$ for Fixed Gain and Variable Gain on $u_2(t)$

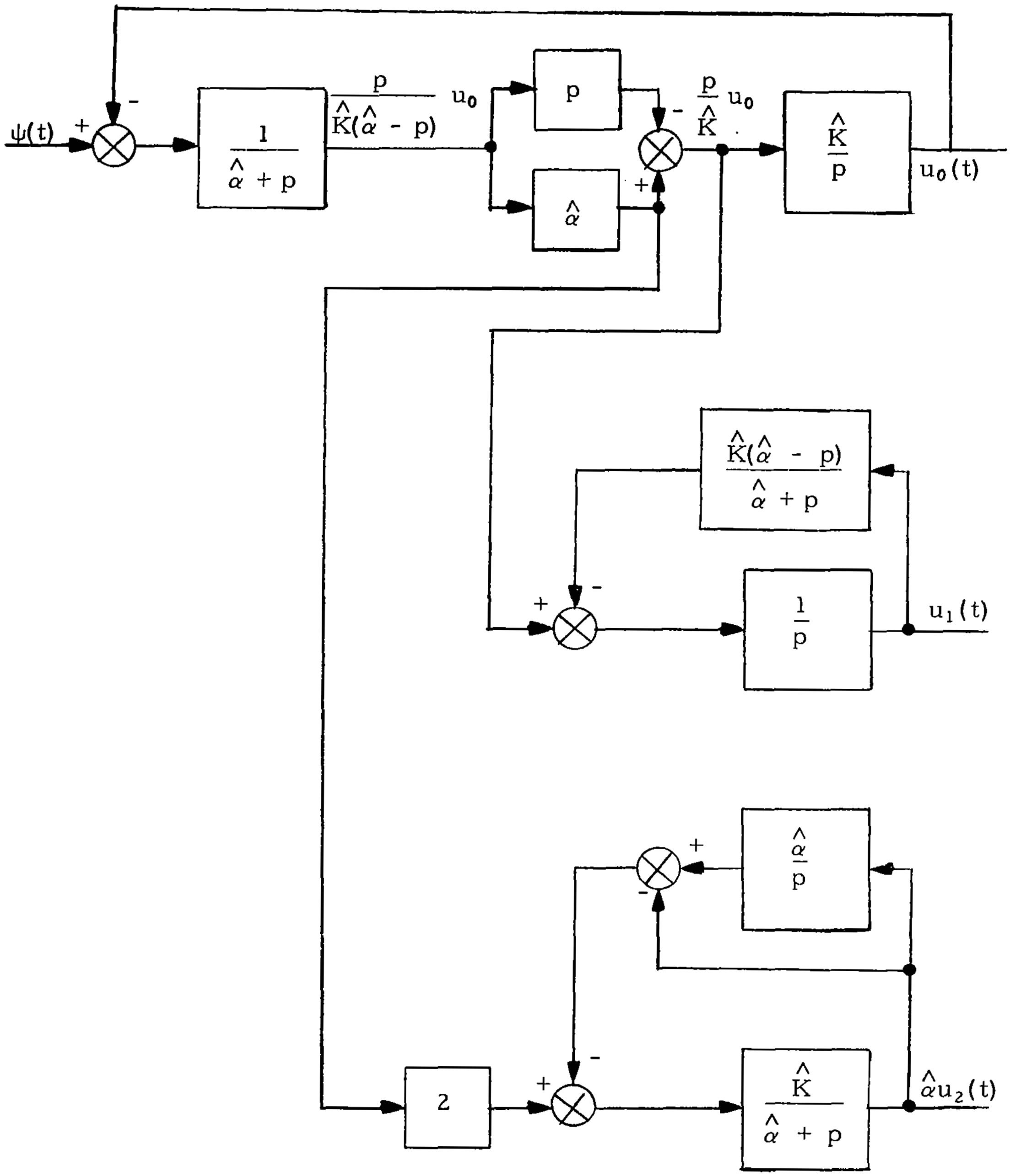


Figure 3.5-3 Modified Block Diagram for Crossover Model Implementation

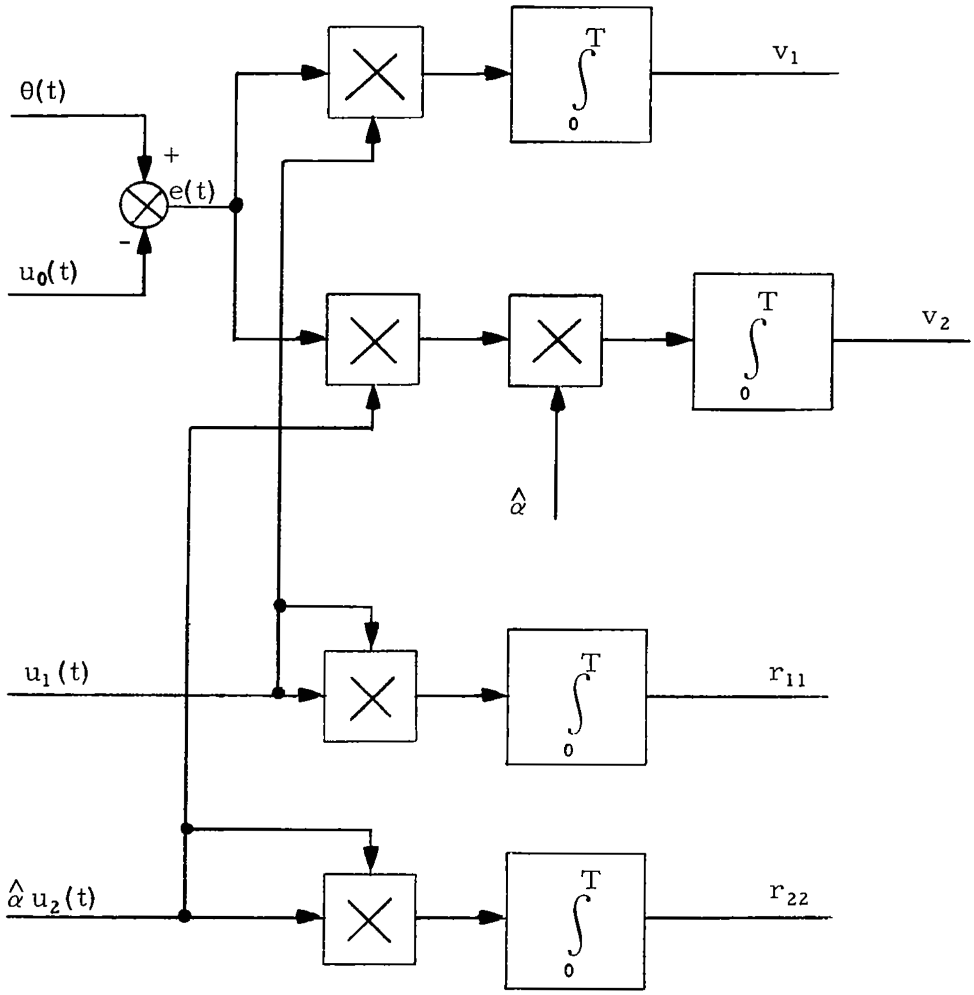


Figure 3.5-4 Modified Block Diagram for Evaluating R and $\underline{v}$

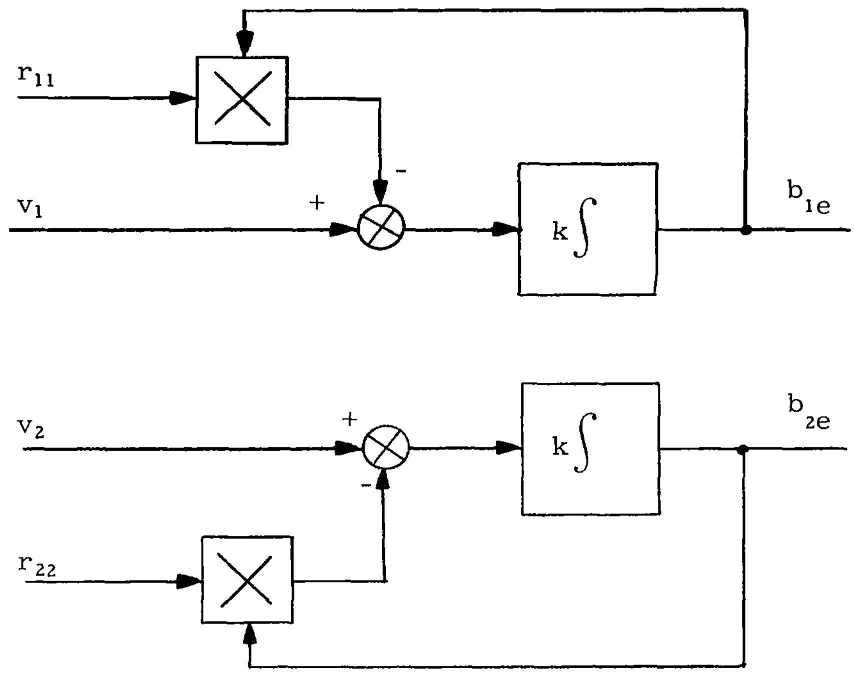


Figure 3.5-5 Modified Block Diagram for Solving

$$\underline{b}' + kR\underline{b} = k\underline{v}$$

CHAPTER 4

SOURCES OF ERROR IN APPLICATION OF REGRESSION ANALYSIS TO THE CROSSOVER MODEL

As with any computational technique, there are several sources of error when using the regression analysis technique. In this chapter four different sources of error are discussed. To obtain the maximum benefit, the analysis is based on a known system which is in the form of the crossover model. Except for the discussion of Sections 4.4 and 4.5 it is assumed that no external noise is present.

4.1 Analysis Using an Infinite Interval of Data

In the initial phases of this work it was thought that the non-iterative on-line technique discussed in Section 3.6 might provide a reasonably accurate computational technique. This was based on two assumptions. The first assumption was that a good estimate of the system parameter values would be available prior to the actual analysis. The second assumption was that the variation of the human operator parameter values from one interval to the next would be small. This would then imply that the linear approximation given by Eq. 2.6-14 is valid.

To determine the magnitude of error that is produced by the non-iterative technique, parameter estimates were obtained using a known model in place of the human operator system. Thus the analysis discussed here is based on the system shown in Fig. 3.3-1 with a simulated human operator. To eliminate as many sources of error as possible in these calculations it was decided to obtain the results analytically based on an infinite interval of data. The results obtained for the infinite interval are also useful as a reference in Section 4.3.

Let us define the estimate obtained from an infinite data interval as $\underline{b}_{\infty}$. Then

$$\underline{b}_\infty = \lim_{T \to \infty} [R^{-1} \underline{v}] \qquad 4.1\text{-}1$$

The expressions for the estimated values of $\tilde{K}$ and $\tilde{\alpha}$ are then

$$\tilde{K}_\infty = \hat{K} + b_{\infty 1} \qquad 4.1\text{-}2$$

$$\tilde{\alpha}_\infty = \hat{\alpha} + b_{\infty 2} \qquad 4.1\text{-}3$$

Rather than directly evaluate the expressions of Eq. 4.1-1 in the time domain, an indirect frequency domain method is used. This method is based on the input, $\psi(t)$, being an ergodic stationary random process which then allows the infinite integrals in the time domain to be replaced by corresponding integrals in the frequency domain. This frequency domain method is developed in Appendix B along with the necessary expressions for determining $\underline{b}_\infty$. Also in Appendix B is a copy of the digital computer program that is used to numerically evaluate these expressions.

Parameter estimates are obtained for three different sets of system parameter values:

Condition 1, $\tilde{K} = 5.5$, $\tilde{\alpha} = 12$
Condition 2, $\tilde{K} = 4.5$, $\tilde{\alpha} = 15$
Condition 3, $\tilde{K} = 3.0$, $\tilde{\alpha} = 8$

For each set of system parameter values, sixteen different sets of model parameter values are used. The model parameter values correspond to

$$\frac{1}{10} \Delta K = \frac{1}{10} (\tilde{K} - \hat{K}) = \pm 0.01,\ \pm 0.025,\ \pm 0.05,\ \pm 0.1$$

$$\frac{1}{20} \Delta \alpha = \frac{1}{20} (\tilde{\alpha} - \hat{\alpha}) = \pm 0.01,\ \pm 0.025,\ \pm 0.05,\ \pm 0.1$$

The relation between the system and model parameter values is shown in Fig. 4.1-1. Due to scaling requirements of the analog computer,

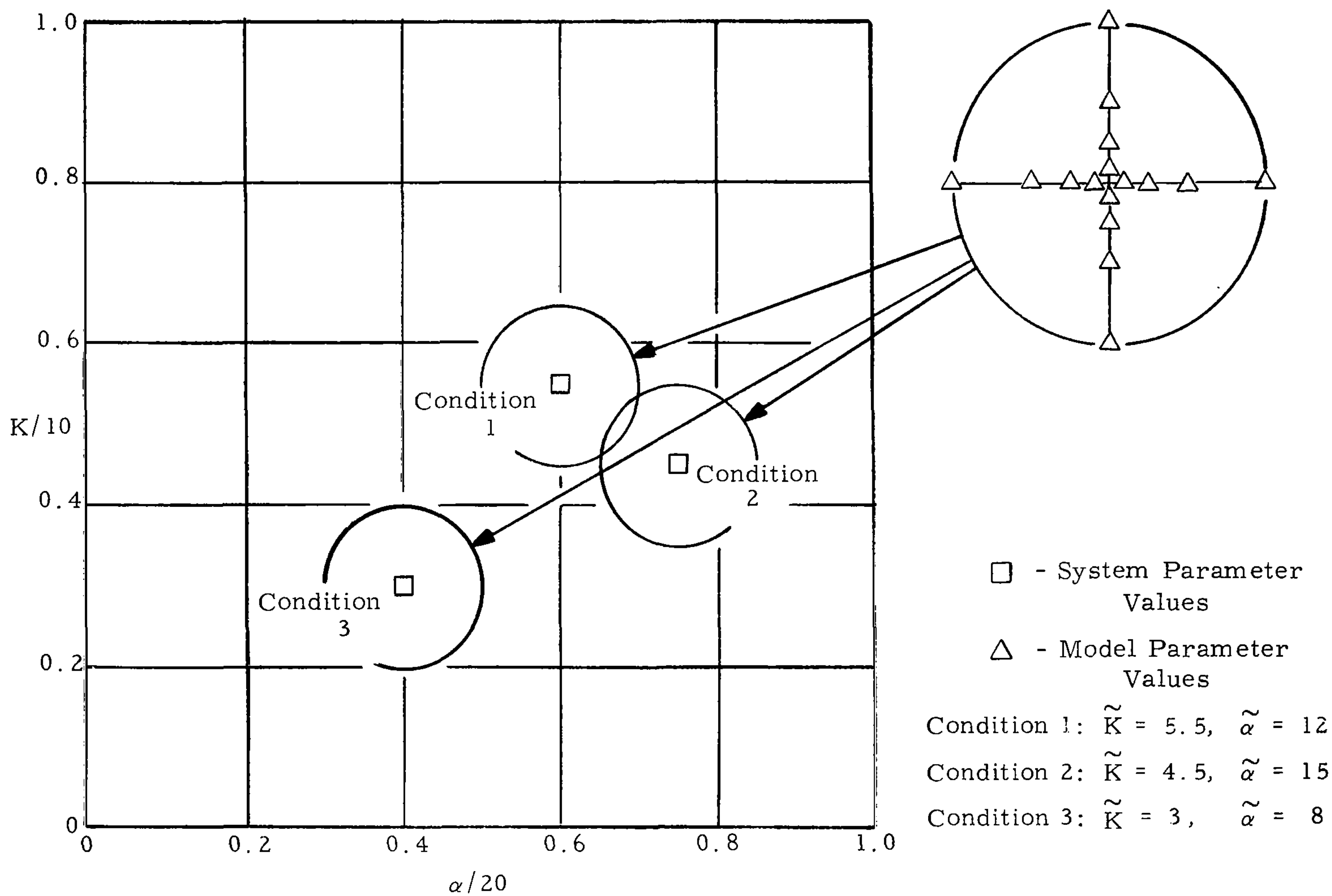


Figure 4.1-1 Relation Between System and Model Parameter Values

$b_{1e}/10$ and $b_{2e}/20$ are computed rather than b_{1e} and b_{2e}. For this reason, the data presented in Sections 4.1, 4.2 and 4.3 are based on $b_{1e}/10$ and $b_{2e}/20$.

The results of the numerical calculations are listed in Table 4.1-1 and are also presented in Figs. 4.1-2 and 4.1-3. The results in Fig. 4.1-2 are for the case where $\hat{K} = \tilde{K}$. The results in Fig. 4.1-3 are for the case where $\hat{\alpha} = \tilde{\alpha}$.

One observation that can be made from Figs. 4.1-2 and 4.1-3 is that for a given difference and especially for large differences between the system and model parameter values, the relative error in the estimation of $\tilde{\alpha}$ is invariably larger than the relative error in the estimation of $\tilde{K}$. The magnitude of the errors involved indicates that if a noniterative on-line procedure is used, one might expect better accuracy in the estimates of $\tilde{K}$ than in the estimates of $\tilde{\alpha}$.

Another application of the data presented in Figs. 4.1-2 and 4.1-3 is in connection with Eq. 2.6-11. In this equation an expression is given for the maximum difference between system and model for which linearization is valid. Rather than analytically obtain the necessary maximum values that are a part of this expression, the empirical results of Figs. 4.1-2 and 4.1-3 can be used. The results presented in these figures indicate that if $\hat{K}$ and $\hat{\alpha}$ are within 8% of $\tilde{K}$ and $\tilde{\alpha}$ respectively, the error in $\tilde{K}_\infty$ is less than 0.5% and the error in $\tilde{\alpha}_\infty$ is less than 1%. Similarly, if $\hat{K}$ and $\hat{\alpha}$ are within 15% of $\tilde{K}$ and $\tilde{\alpha}$ respectively, the error in $\tilde{K}_\infty$ is less than 1% and the error in $\tilde{\alpha}_\infty$ is less than 4%.

4.2 Convergence of Iterative Regression Analysis

In Section 4.1 it is seen that for large differences between the system and model parameter values, the first estimate obtained by regression analysis will not be accurate which indicates that an iterative technique is required. It was also noted in Section 2.7 that the regression analysis converges in an arbitrarily small neighborhood of the

TABLE 4.1-1

Parameter Estimates Based on an Infinite Interval of Data

Condition 1: $\frac{\tilde{K}}{10} = 0.55, \quad \frac{\tilde{\alpha}}{20} = 0.60$

$\hat{K}/10$	$b_{1\infty}/10$	$\tilde{K}_{\infty}/10$	$\hat{\alpha}/20$	$b_{2\infty}/20$	$\tilde{\alpha}_{\infty}/20$
0.650	-0.0980	0.552	0.600	0.0216	.622
* 0.600	-0.0504	0.550	0.600	0.0063	.606
0.575	-0.0252	0.550	0.600	0.0017	.602
0.560	-0.0100	0.550	0.600	0.0003	.600
0.540	0.0100	0.550	0.600	0.0003	.600
0.525	0.0246	0.550	0.600	0.0019	.602
* 0.500	0.0479	0.548	0.600	0.0082	.608
0.450	0.0891	0.539	0.600	0.0368	.637
0.550	-0.0066	0.543	0.500	0.0696	.570
* 0.550	-0.0014	0.549	0.550	0.0426	.593
0.550	-0.0003	0.550	0.575	0.0232	.598
0.550	-0.0000	0.550	0.590	0.0097	.600
0.550	-0.0000	0.550	0.610	-0.0102	.600
0.550	-0.0002	0.550	0.625	-0.0267	.598
* 0.550	-0.0009	0.549	0.650	-0.0569	.593
0.550	-0.0030	0.547	0.700	-0.1271	.573

The asterisks (*) denote conditions that are studied in Section 4.3.

TABLE 4.1-1 (cont.)

Condition 2: $\frac{\tilde{K}}{10} = 0.45$, $\frac{\tilde{\alpha}}{20} = 0.75$

$\hat{K}/10$	$b_{1\infty}/10$	$\tilde{K}_\infty/10$	$\hat{\alpha}/20$	$b_{2\infty}/20$	$\tilde{\alpha}_\infty/20$
* 0.550	-0.1089	0.441	0.750	0.0373	0.787
0.500	-0.0527	0.447	0.750	0.0110	0.761
0.475	-0.0257	0.449	0.750	0.0030	0.753
0.460	-0.0101	0.450	0.750	0.0005	0.750
0.440	0.0099	0.450	0.750	0.0005	0.750
0.425	0.0241	0.449	0.750	0.0035	0.753
0.400	0.0464	0.446	0.750	0.0153	0.765
* 0.350	0.0839	0.434	0.750	0.0729	0.823
* 0.450	-0.0008	0.449	0.650	0.0813	0.731
0.450	-0.0002	0.450	0.700	0.0453	0.745
0.450	-0.0000	0.450	0.725	0.0238	0.749
0.450	-0.0000	0.450	0.740	0.0098	0.750
0.450	-0.0000	0.450	0.760	-0.0102	0.750
0.450	-0.0000	0.450	0.775	-0.0261	0.749
0.450	-0.0001	0.450	0.800	-0.0545	0.746
* 0.450	-0.0004	0.450	0.850	-0.1181	0.732

Condition 3: $\frac{\tilde{K}}{10} = 0.30$, $\frac{\tilde{\alpha}}{20} = 0.40$

$\hat{K}/10$	$b_{1\infty}/10$	$\tilde{K}_\infty/10$	$\hat{\alpha}/20$	$b_{2\infty}/20$	$\tilde{\alpha}_\infty/20$
0.400	-0.0944	0.306	0.400	0.0342	0.434
* 0.350	-0.0505	0.300	0.400	0.0114	0.411
0.325	-0.0254	0.300	0.400	0.0032	0.403
0.310	-0.0101	0.300	0.400	0.0005	0.400
0.290	0.0099	0.300	0.400	0.0006	0.401
0.275	0.0242	0.299	0.400	0.0042	0.404
* 0.250	0.0459	0.296	0.400	0.0190	0.419
0.200	0.0780	0.278	0.400	0.0988	0.499
0.300	-0.0073	0.293	0.300	0.0593	0.359
* 0.300	-0.0015	0.299	0.350	0.0401	0.390
0.300	-0.0003	0.300	0.375	0.0225	0.398
0.300	-0.0000	0.300	0.390	0.0096	0.400
0.300	-0.0000	0.300	0.410	-0.0104	0.400
0.300	-0.0003	0.300	0.425	-0.0273	0.398
* 0.300	-0.0009	0.299	0.450	-0.0593	0.391
0.300	-0.0027	0.297	0.500	-0.1366	0.363

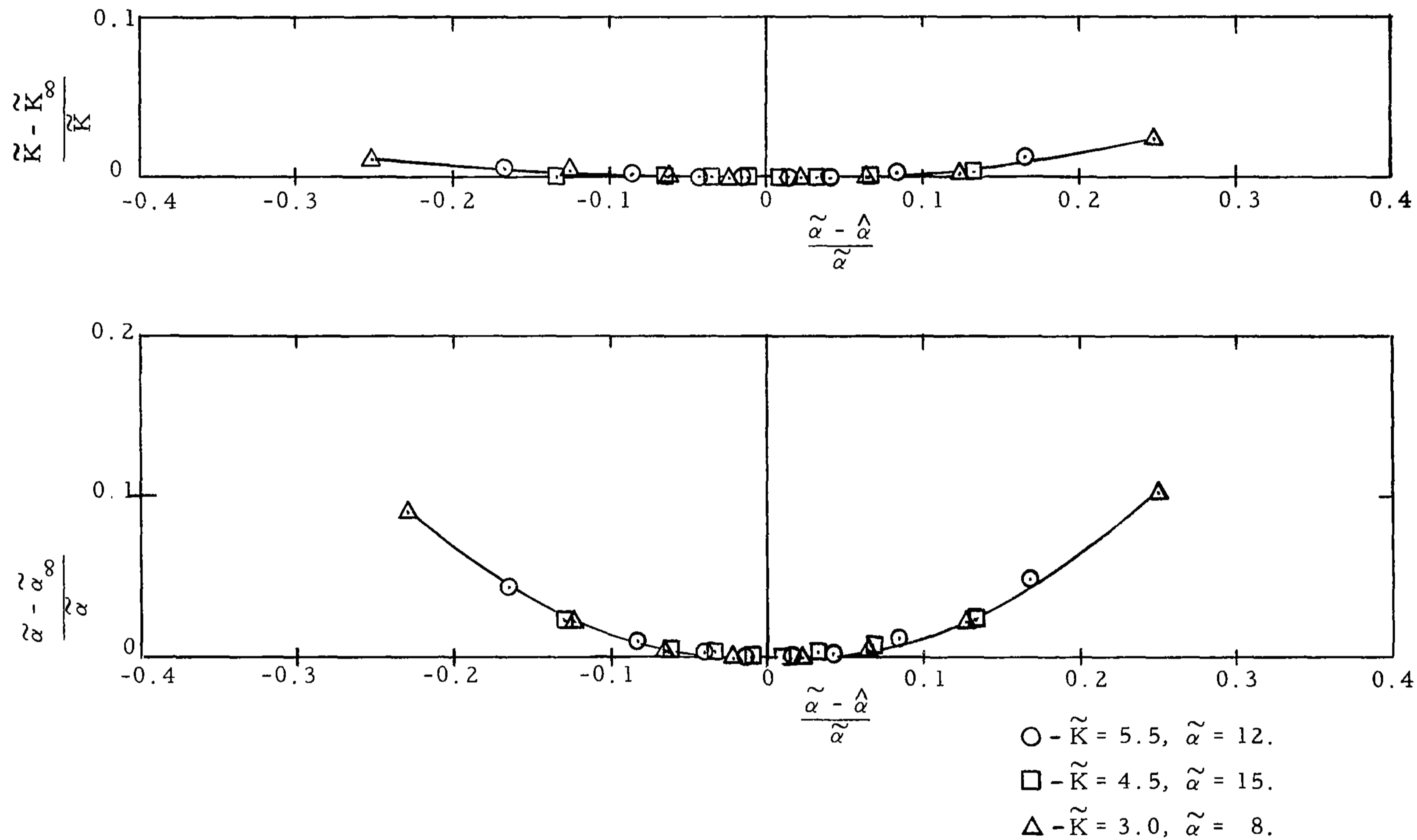


Figure 4.1-2 Results of Infinite Data Interval Analysis, $\hat{K} = \tilde{K}$

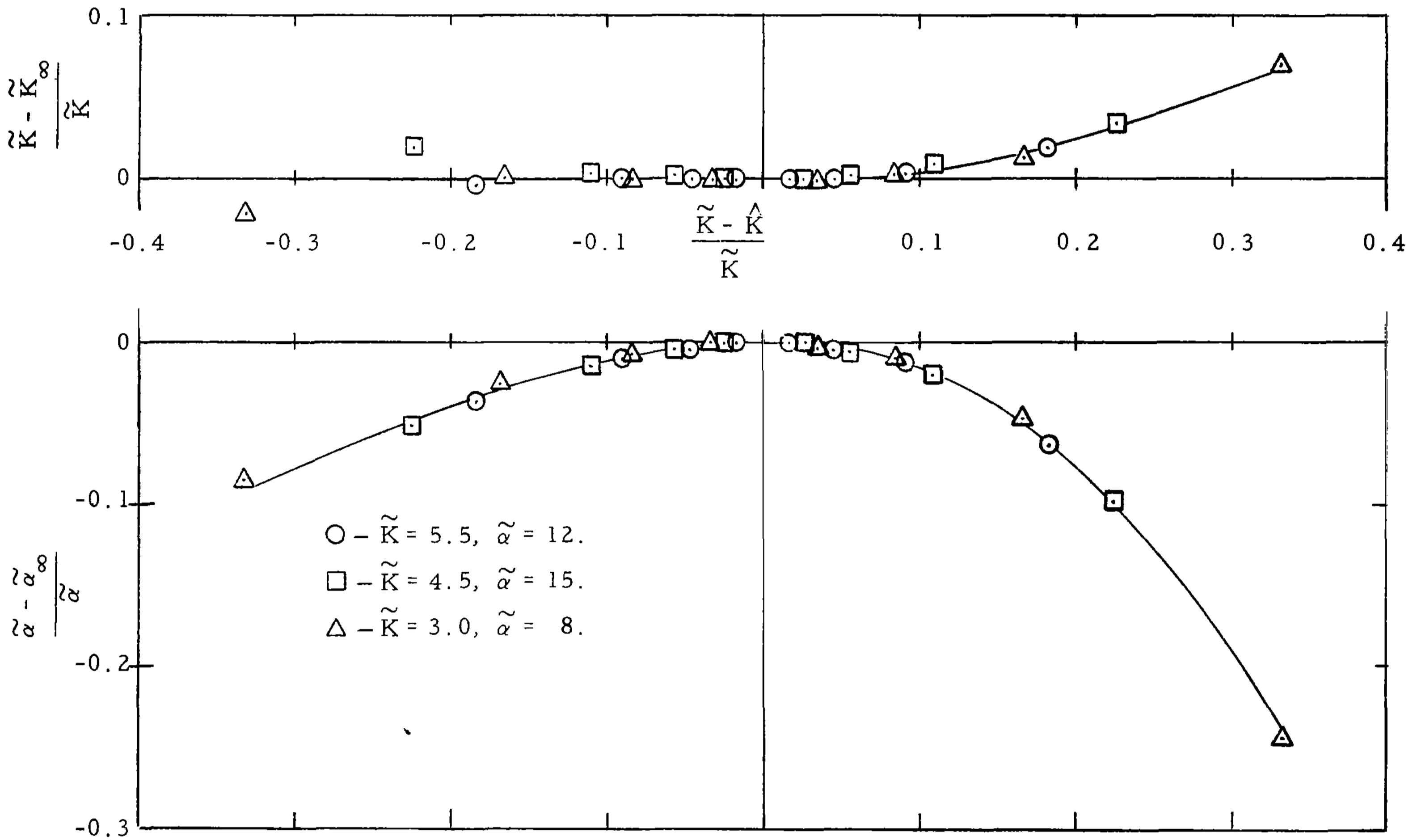


Figure 4.1-3 Results of Infinite Data Interval Analysis, $\hat{\alpha} = \tilde{\alpha}$

system parameter values, $\tilde{K}$ and $\tilde{\alpha}$. In general this convergence is not guaranteed in a global domain. To test the convergence of this technique, the human operator system was replaced by a known system as described in Section 4.1. A typical result of the convergence study is given in Fig. 4.2-1. From these results it is seen that for quite large initial differences between the system and model parameter values, the regression analysis technique not only converges but takes less than ten iterations to converge to the proper value.

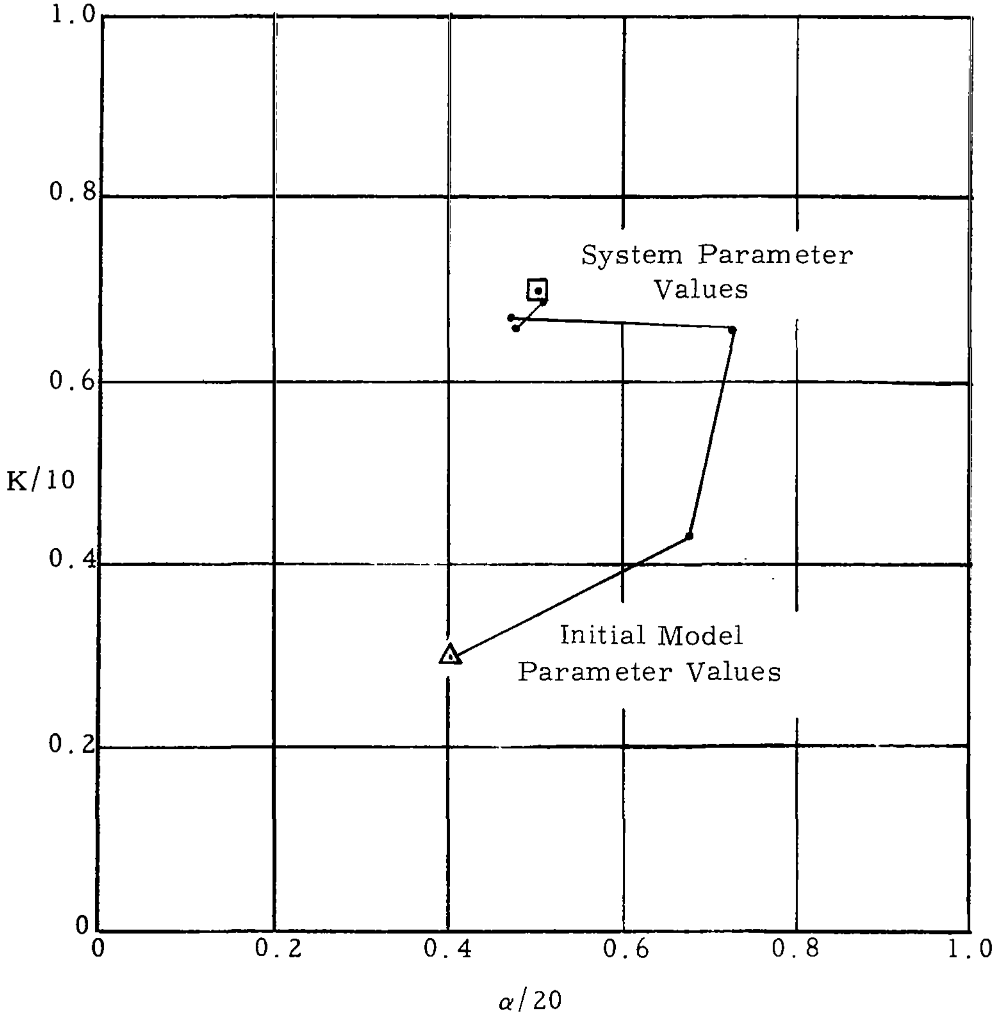
Although the conditions of Figs. 4.1-2 and 4.1-3, namely either $\hat{K} = \tilde{K}$ or $\hat{\alpha} = \tilde{\alpha}$, are not met exactly, the results of these figures can be generally compared with Fig. 4.2-1. For instance, the first iteration where $\frac{\tilde{K} - \hat{K}}{\tilde{K}} = 0.57$ and $\frac{\tilde{\alpha} - \hat{\alpha}}{\tilde{\alpha}} = 0.20$ yields an estimate $\tilde{K}_e$ which is 39% in error and an estimate $\tilde{\alpha}_e$ which is 35% in error. These results are outside the range considered in Section 4.1 but agree in general with Fig. 4.1-3 where $\hat{\alpha} = \tilde{\alpha}$. Similarly for the fifth iteration where $\frac{\tilde{K} - \hat{K}}{\tilde{K}} = 0.057$ and $\frac{\tilde{\alpha} - \hat{\alpha}}{\tilde{\alpha}} = 0.05$, the estimate $\tilde{K}_e$ is 1.4% in error and the estimate $\tilde{\alpha}_e$ is 1% in error which is predictable from Figs. 4.1-2 and 4.1-3.

4.3 Effect of Finite Data Interval

A source of error that is related to the material discussed in Section 4.1 is the effect of a finite interval of data. It has been shown by Gilbert [14] that a statistical error in the value of the elements of R and $\underline{v}$ results when a finite interval of data is used. As in Section 4.1 let us consider an input which is a stationary ergodic random process. As discussed in Appendix C, it also is necessary to require that the input signal be a gaussian process.

In Section 4.1 the parameter estimate, $\underline{b}_e$, for an infinite data interval was defined as

$$\underline{b}_\infty = \lim_{T\to\infty} \underline{b}_e = \lim_{T\to\infty} [R^{-1} \underline{v}] \qquad 4.3\text{-}1$$



Each dot ● represents an intermediate estimate of the system parameters. In the noniterative on-line implementation, each such set of values is used as the model parameter values during the subsequent interval.

Note that six iterations were required for this set of system and model parameter values.

Figure 4.2-1 Convergence of Regression Analysis Technique

The effect of using finite intervals of data can be considered as a perturbation, $\underline{\delta}$, from $\underline{b}_{\infty}$. Thus,

$$\underline{b}_e = \underline{b}_{\infty} + \underline{\delta} \qquad 4.3\text{-}2$$

It is shown in Appendix C that

$$E[\underline{\delta}] = 0 \qquad 4.3\text{-}3$$

Also in Appendix C, expressions are obtained for the upper bound of the variance of the elements of $\underline{\delta}$. For the crossover model, the upper bounds on the variances are given by

$$\sigma^2_{\delta_1} \leqq \frac{\pi}{T}\left\{\frac{\int_0^{\infty}\{\Phi_{u_1}(j\omega)\,\Phi_e(j\omega) + \mathrm{Re}[\Phi^2_{eu_1}(j\omega)]\}\,d\omega}{[\int_0^{\infty}\Phi_{u_1}(j\omega)\,d\omega]^2}\right\} \qquad 4.3\text{-}4$$

and

$$\sigma^2_{\delta_2} \leqq \frac{\pi}{T}\left\{\frac{\int_0^{\infty}\{\Phi_{u_2}(j\omega)\,\Phi_e(j\omega) + \mathrm{Re}[\Phi^2_{eu_2}(j\omega)]\}\,d\omega}{[\int_0^{\infty}\Phi_{u_2}(j\omega)\,d\omega]^2}\right\} \qquad 4.3\text{-}5$$

The integrals of Eqs. 4.3-4 and 4.3-5 were evaluated numerically using a digital computer program which is given in Appendix C. The twelve sets of parameter values for which the variance upper bounds were calculated are denoted by an asterisk (*) in Table 4.1-1. The upper bounds are calculated for two values of T: 5 seconds and 20 . seconds.

Rather than tabulate the variance upper bounds, the upper bound on the standard deviation of δ_1 and δ_2, which have the same dimensionality as K and α respectively, are presented in Table 4.3-1. It is seen from Table 4.3-1 that the effect of using finite lengths of data is quite small, even for lengths of data as short as five seconds. Table 4.3-1 shows that for a five second interval, estimates of b_{1e} and b_{2e} will, in general, vary not more than ten per cent from the value obtained from

TABLE 4.3-1

Upper Bound on Standard Deviation of Parameter Estimates, b_{1e} and b_{2e}, Due to Finite Data Intervals

				T = 20 seconds		T = 5 seconds	
$\tilde{K}/10$	$\hat{K}/10$	$\tilde{\alpha}/20$	$\hat{\alpha}/20$	$b_{1e}/\tilde{K}$	$b_{2e}/\tilde{\alpha}$	$b_{1e}/\tilde{K}$	$b_{2e}/\tilde{\alpha}$
0.55	0.60	0.60	0.60	0.0170	0.0025	0.0340	0.0050
0.55	0.50	0.60	0.60	0.0168	0.0315	0.0336	0.0630
0.55	0.55	0.60	0.55	0.0008	0.0155	0.0016	0.0310
0.55	0.55	0.60	0.65	0.0005	0.0190	0.0010	0.0380
0.45	0.55	0.75	0.75	0.0470	0.0102	0.0940	0.0204
0.45	0.35	0.75	0.75	0.0440	0.0222	0.0880	0.0444
0.45	0.45	0.75	0.65	0.0006	0.0218	0.0011	0.0436
0.45	0.45	0.75	0.85	0.0003	0.0302	0.0005	0.0604
0.30	0.35	0.40	0.40	0.0406	0.0082	0.0812	0.0165
0.30	0.25	0.40	0.40	0.0404	0.0140	0.0808	0.0280
0.30	0.30	0.40	0.35	0.0017	0.0278	0.0035	0.0556
0.30	0.30	0.40	0.45	0.0010	0.0375	0.0019	0.0750

an infinite interval of data. This is not entirely unexpected for the following reason. Although the effect of finite averaging time on the values of the individual components of R and $\underline{v}$ may be large, the value of $\underline{b}_e$ is determined by a ratio of these elements. The error effects in the elements of R and $\underline{v}$ then essentially cancel giving a rather small random error in the parameter estimates, $\underline{b}_e$.

In addition to the digital computer numerical analysis discussed here and in Appendix C, a series of trials was run on the analog computer using a simulated human operator. Each trial was for a different set of system and model parameter values. During each trial, parameter estimates were obtained for each of twenty-five 20 second intervals. The parameter estimate data obtained from this experiment was used to compute estimates of the variance of $b_{1e}/10$ and $b_{2e}/20$. The upper bound on the variances computed with the digital program and the sample variances obtained from the analog data are compared in Table 4.3-2.

From Table 4.3-2 it is seen that in general the upper bounds obtained from the numerical analysis agree quite well with the analog data. However, it should be pointed out that the variances computed from the analog data reflect not only the effect of using a finite data interval but also any random errors in the computer mechanization. This would seem to be verified by the two cases where the sample variance is larger than the upper bound computed with the digital program. Note however that the variance of the parameter estimates due to finite data intervals is extremely small for both of these cases.

4.4 Effect of Additive Noise

As discussed in Section 1.3, the human operator control system can be characterized by an equivalent "black-box" system. Such a "black-box" system contains an equivalent human operator which has an output consisting of the response of a linear time-invariant system plus noise which is uncorrelated with the input signal. The noise can be redefined to be a signal, r(t), which is added outside of the closed-loop system [27].

TABLE 4.3-2

Comparison of Theoretical and Empirical Data on Statistical Characteristics of Parameter Estimates, b_{1e} and b_{2e}

$b_{1e}/10$ Data, T = 20 seconds

$\tilde{K}/10$	$\hat{K}/10$	$\tilde{\alpha}/20$	$\hat{\alpha}/20$	$b_{1\infty}/10$	Empirical Average	Variance Upper Bound	Empirical Variance Estimate
0.45	0.55	0.75	0.75	-0.1089	-0.1096	4.52E-04*	3.64E-06
0.45	0.35	0.75	0.75	0.0839	0.0813	3.99E-04	1.76E-06
0.45	0.45	0.75	0.65	-0.0008	-0.0014	6.36E-08	1.02E-07
0.45	0.45	0.75	0.85	-0.0004	-0.0012	1.44E-08	1.26E-07

$b_{2e}/20$ Data, T = 20 seconds

$\tilde{K}/10$	$\hat{K}/10$	$\tilde{\alpha}/20$	$\hat{\alpha}/20$	$b_{2\infty}/10$	Empirical Average	Variance Upper Bound	Empirical Variance Estimate
0.45	0.55	0.75	0.75	0.0373	0.0385	5.90E-05	1.51E-05
0.45	0.35	0.75	0.75	0.0729	0.0738	2.76E-04	4.49E-05
0.45	0.45	0.75	0.65	0.0813	0.0834	2.68E-04	1.00E-06
0.45	0.45	0.75	0.85	-0.1181	-0.1176	5.17E-04	3.02E-06

*E-04 = 10^{-4}

In the notation of Fig. 1.3-1, n(t) and r(t) are related by:

$$r(t) = \frac{Y_C(p)}{1 + Y_C(p)Y_P(p)} n(t) \qquad 4.4\text{-}1$$

The equivalent system is shown in Fig. 3.3-1 along with the corresponding model and parameter influence coefficients.

As has been noted by other authors [7, 20], the noise signal will cause a statistical error in the estimates of the parameters of the linear time-invariant part of the equivalent system. Elkind, et al. [7] have performed an analysis that is similar to the analysis presented in this section. However, the use of a sampled-data system in the above reference resulted in an analysis method the details of which are substantially different from that presented here.

Consider the system given in Fig. 3.3-1. It is assumed that the additive noise, r(t), has zero mean and is statistically independent of the input signal, $\psi(t)$. To restrict the sources of error in this analysis to r(t), it is assumed that the model is identical to the closed-loop portion of the system. For the case of the crossover model, this implies that $\hat{K} = \tilde{K}$ and $\hat{\alpha} = \tilde{\alpha}$. Then in Fig. 3.3-1,

$$u_0(t) = \theta_0(t) \qquad 4.4\text{-}2$$

If the equations developed in Section 3.4 are applied, it is seen that

$$e(t) = \theta(t) - u_0(t) = r(t) \qquad 4.4\text{-}3$$

and the expression for the best parameter estimate is given by

$$\underline{b}_e = R^{-1} \frac{1}{T} \int_0^T r(t)\, \underline{u}(t)\, dt \qquad 4.4\text{-}4$$

Let us define the variable $\underline{z}(t)$ as

$$\underline{z}(t) = R^{-1} \underline{u}(t) \qquad 4.4\text{-}5$$

Then the expression for $\underline{b}_e$ is

$$\underline{b}_e = \frac{1}{T} \int_0 r(t)\, \underline{z}(t)\, dt \qquad 4.4\text{-}6$$

Consider now the statistical effect of r(t). Since the effect of r(t) is desired for any given interval of data and both $\psi(t)$ and r(t) are considered to be random processes, it is necessary to consider the conditional effect of r(t). However since $\psi(t)$ and r(t) are assumed to be statistically independent, identical results are obtained if r(t) is assumed to be random and ψ(t) is assumed to be deterministic. Then the expected value of $\underline{b}_e$ is obtained as follows.

$$E[\underline{b}_e] = E\left\{\frac{1}{T}\int_0^T r(t)\,\underline{z}(t)\,dt\right\} \qquad 4.4\text{-}7$$

$$= \frac{1}{T}\int_0^T E[r(t)]\,\underline{z}(t)\,dt \qquad 4.4\text{-}8$$

Since r(t) is assumed to have zero mean,

$$E[\underline{b}_e] = 0 \qquad 4.4\text{-}9$$

Thus $\underline{b}_e$ is an unbiased estimate in the presence of additive noise. Now consider the variance of $\underline{b}_e$. Note that:

$$\underline{b}_e\underline{b}_e^{\#} = \frac{1}{T}\int_0^T r(t_1)\,\underline{z}(t_1)\,dt_1\Big\}\left\{\frac{1}{T}\int_0^T r(t_2)\,\underline{z}^{\#}(t_2)\,dt_2\right\} \qquad 4.4\text{-}10$$

Then the covariance matrix for $\underline{b}_e$ is given by:

$$E[\underline{b}_e\underline{b}_e^{\#}] = E\left\{\frac{1}{T^2}\int_0^T\int_0^T r(t_1)\,r(t_2)\,\underline{z}(t_1)\,\underline{z}^{\#}(t_2)\,dt_1\,dt_2\right\} \qquad 4.4\text{-}11$$

Under the additional assumption that r(t) is a stationary process, Eq. 4.4-11 can be rewritten in terms of the autocorrelation function of r(t) as:

$$E[\underline{b}_e\underline{b}_e^{\#}] = \frac{1}{T^2}\int_0^T\int_0^T \phi_r(\sigma)\,\underline{z}(t_1)\,\underline{z}^{\#}(t_2)\,dt_1\,dt_2 \qquad 4.4\text{-}12$$

where $\phi_r(\sigma)$ is the autocorrelation of r(t) and

$$\sigma = t_2 - t_1$$

Without further assumptions on the statistical properties of r(t), Eq. 4.4-12 cannot be further simplified. However, it has been shown [28] that the equivalent noise for the human operator has a flat power spectral density over a wide range of frequencies. In this case, it is not unreasonable to approximate r(t) by white noise with a spectral density given by

$$\Phi_r(j\omega) = N \qquad 4.4\text{-}13$$

The corresponding autocorrelation for r(t) is then,

$$\phi_r(\sigma) = N\delta(\sigma) \qquad 4.4\text{-}14$$

where $\delta(\sigma)$ is the Kronecker delta function. Substituting this expression for the autocorrelation into Eq. 4.4-12 yields

$$E[\underline{b}_e \underline{b}_e^{\#}] = \frac{N}{T^2} \int_0^T \underline{z}(t_1)\,\underline{z}^{\#}(t_1)\,dt_1 \qquad 4.4\text{-}15$$

Referring back to the definition of $\underline{z}(t)$ given by Eq. 4.4-5 suggests rewriting Eq. 4.4-15 as

$$E[\underline{b}_e \underline{b}_e^{\#}] = \frac{N}{T^2} R^{-1} \left\{ \int_0^T \underline{u}(t)\, \underline{u}^{\#}(t)\,dt \right\} [R^{-1}]^{\#} \qquad 4.4\text{-}16$$

Note that

$$R = \frac{1}{T} \int_0^T \underline{u}(t)\, \underline{u}^{\#}(t)\,dt$$

is a symmetric matrix. Therefore

$$R^{-1} = [R^{-1}]^{\#}$$

and the expression for the covariance of $\underline{b}_e$ becomes

$$E[\underline{b}_e \underline{b}_e^{\#}] = \frac{N}{T} R^{-1} \qquad 4.4\text{-}17$$

It has been shown in Section 3.5 that if T is sufficiently long, then

$$R \approx \begin{bmatrix} r_{11} & 0 \\ 0 & r_{22} \end{bmatrix} \qquad 4.4\text{-}18$$

In the case where the approximation of Eq. 4.4-18 is valid, the variances of the elements of $\underline{b}_e$ are

$$\sigma^2_{b_{1e}} = \frac{N}{r_{11} T} \qquad 4.4\text{-}19$$

$$\sigma^2_{b_{2e}} = \frac{N}{r_{22} T} \qquad 4.4\text{-}20$$

Two conclusions can be drawn from Eq. 4.4-17. For a given interval of data, the variance of the parameter estimates is directly proportional to the magnitude of the noise spectral density. Secondly, the variance of the parameter estimates is inversely proportional to the length of the data interval. Thus to achieve a given variance of the estimates, the length of the data interval must be increased as the amount of equivalent noise in the human operator system increases.

4.5 Effect of Model Initial Conditions

The discussion in this section is meant to be qualitative in nature rather than quantitative as in the preceding sections of this chapter. To that end, the crossover model is not considered per se but rather a general linear time-invariant system is considered. For completeness it is assumed that the system output is corrupted by additive noise. A block diagram for the system and corresponding model is given in Fig. 4.5-1.

The system responses can be written as

$$\begin{aligned}
\theta(t) &= r(t) + \theta_{0c}(t) + \theta_{0q}(t) \\
u_0(t) &= u_{0c}(t) + u_{0q}(t) \\
u_1(t) &= u_{1c}(t) + u_{1q}(t) \\
&\vdots \\
u_L(t) &= u_{Lc}(t) + u_{Lq}(t)
\end{aligned}$$

where the subscript c denotes the transient solution and the

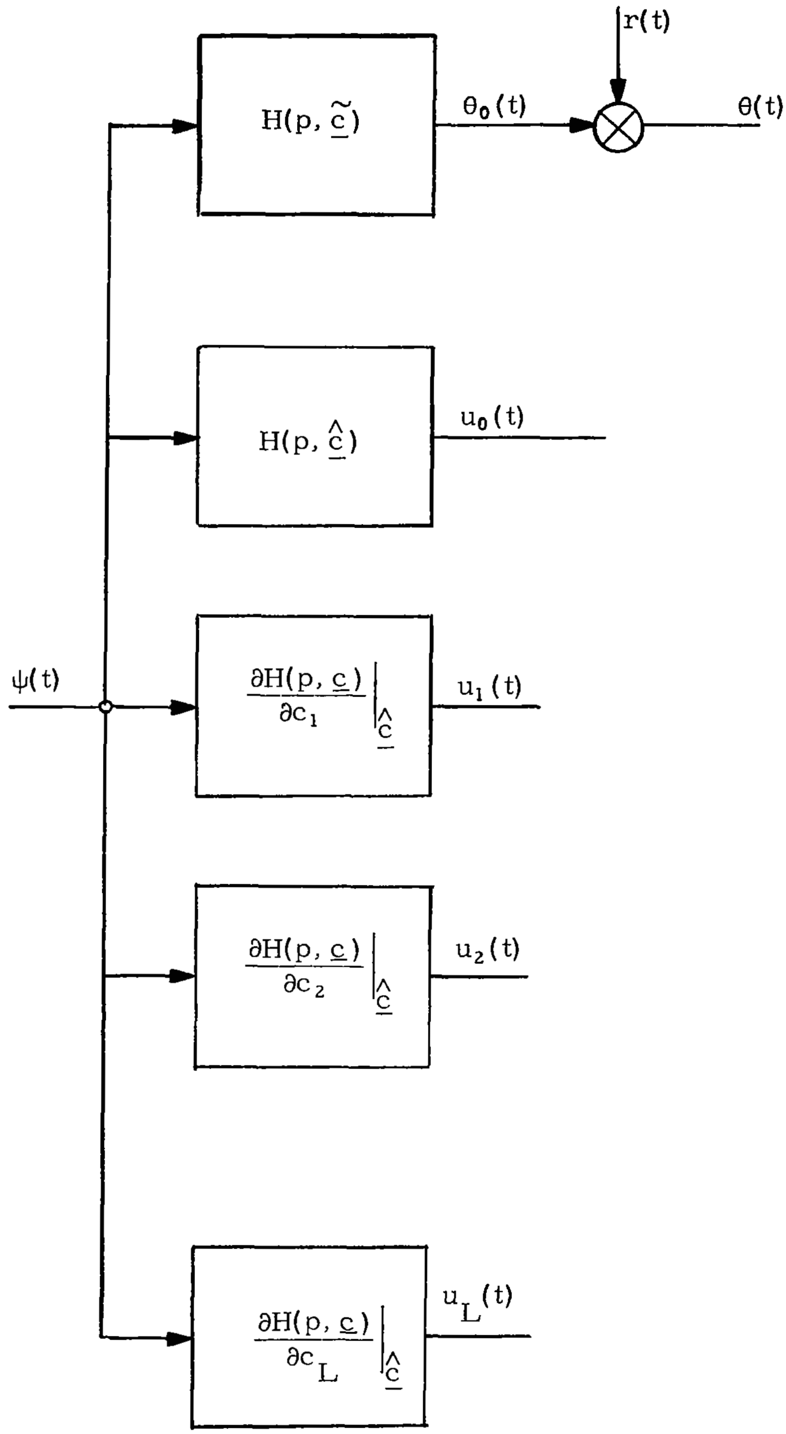


Figure 4.5-1 Block Diagram of General Linear System

subscript q denotes the steady-state solution [23].

If the system represented by $H(p, \tilde{\underline{c}})$ is performing in a steady-state manner, then $\theta_{0c}(t) = 0$ and

$$\theta(t) = r(t) + \theta_{0q}(t)$$

To obtain correct results from the parameter estimation calculations, the elements of R and $\underline{v}$ should be obtained for the condition where the model is also operating in the steady-state. Thus for any interval of data, the transient solution of $u_0(t)$ and the $u_i(t)$ should be zero, or,

$$\begin{aligned} u_0(t) &= u_{0q}(t) \\ u_1(t) &= u_{1q}(t) \\ &\vdots \\ u_L(t) &= u_{Lq}(t) \end{aligned}$$

However, in general, the parameter values of the model will change at the beginning of each data interval. This change of parameter values will introduce a transient effect which is characterized by $u_{0c}(t)$ and the $u_{ic}(t)$ being non-zero during the initial moments of each interval. This effect is symbolized in Fig. 4.5-2. The presence of the transient terms will cause the values of R and $\underline{v}$ to be in error.

The effect of this error term is difficult to determine in any given situation due to $\psi(t)$ being a random process. This effect is most easily compensated for by starting the calculation of the elements of R and $\underline{v}$ at some time after the change of value has been made in the model parameters. For any stable system, which the human operator is, the model transient solutions will decay to zero. Thus by delaying the calculation of the elements of R and $\underline{v}$ a sufficiently long time, the effect of the change of parameter value can be minimized. Typical values of K and α are 4 and 7, respectively, which corresponds to a time-constant for the model of about 0.6 seconds. In the implementation of both techniques a two second delay or approximately three time-constants

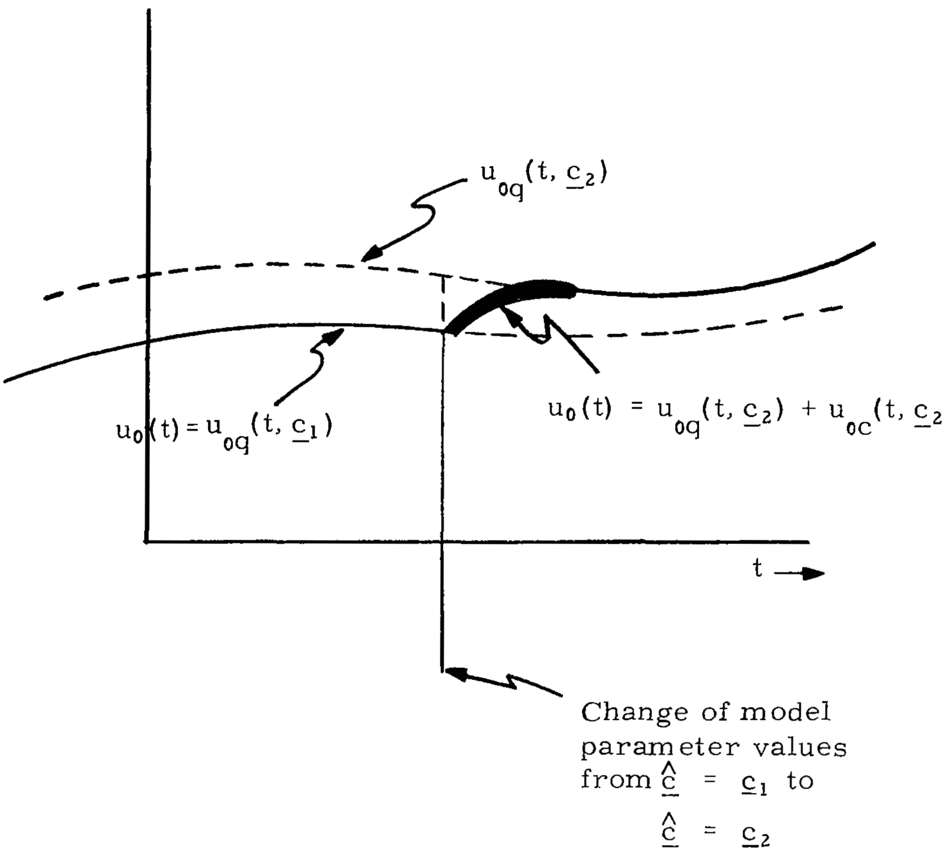


Figure 4.5-2 Symbolization of Effect of Parameter Change on $u_0(t)$

was included between the time that the model parameter values were changed and the start of the calculation of R and v.

CHAPTER 5

HUMAN OPERATOR PERFORMANCE TESTS AND RESULTS

The regression analysis technique described in Chapters 2 and 3 has been applied to data from two compensatory tracking experiments. These experiments and the analysis of the data are discussed in this chapter.

5.1 Description of the Compensatory Tracking Experiments

The data that are analyzed in this chapter are the results of experiments performed by Jackson and are described in detail elsewhere [22]. For completeness of this report, however, the major aspects of the experiments are presented here.

The general arrangement of the experimental set-up is shown in the block diagram of Fig. 1.1-1. The oscilloscope used was a 5-inch Fairchild x-y indicator with a P-31 phosphor coating. The oscilloscope display was in the form of a dot which moved horizontally with respect to a vertical cursor located in the center of the screen. The displacement of the dot from the center was proportional to the system error. The face of the oscilloscope was located approximately 28 inches from the eyes of the subject.

The subject was seated in a straight backed chair with his right arm on the control stick. The stick is of the side arm type, i.e., the subject's elbow joint was constrained to a fixed angle of about 90 degrees. This type of control stick constrains the arm motion of the subject to rotation at the shoulder joint using such upper torso muscles as the subscapularis and infraspinatus [21]. The control stick incorporates a light spring to provide an indication of the center position and has essentially no damping. All subjects were right handed males with no known physical abnormalities.

The experiments had two distinguishing characteristics, controlled element and subjects involved. The transfer operator of the controlled element for the first experiment was $Y_C(p) = 5/p$ while the transfer operator of the controlled element for the second experiment was $Y_C(p) = 5/p^2$. Each experiment had a separate group of three subjects who took part.

In both experiments the subjects were tested for a total of ten days. Within each day, each subject completed five two-minute trials at each of three input cut-off frequencies for a total of 15 trials each day. The blocks of 5 trials for a given cut-off frequency were randomly ordered on each day of testing.

The input signal for these experiments was pseudo-random noise which had an approximately gaussian amplitude distribution [15]. This signal was produced by passing a binary sequence from a pseudo-random noise generator through an analog filter [17] with a transfer operator of the form $\frac{1}{(p/\omega_c + 1)^3}$. Input cut-off frequencies of 1, 2 and 4 radians per second were obtained by using the appropriate value of ω_c. Note that the data for a cut-off frequency of 2 radians per second are the only data analyzed and discussed in this report.

During each experimental trial the input signal and the output signal were recorded on separate channels of a four channel magnetic tape. These recorded signals were then replayed in the process of analyzing the experimental data.

5.2 Iterative Parameter Identification Technique

The regression analysis method of parameter identification described in Chapters 2 and 3 was used to analyze the experimental data. In applying this regression analysis technique, each two minute trial was divided into five non-overlapping 20 second subintervals. During each 20 second interval, the best value of the parameters $K/10$ and $\alpha/20$ were obtained using the iteration procedure. This normalization for K and α was chosen because the parameter ranges are typically $0.2 \leqq K/10 \leqq 1.0$ and $0.15 \leqq \alpha/20 \leqq 0.75$. The criterion used to terminate the iterative process was

$$\left|(\tilde{K}_e/10)_{nm} - (\hat{K}/10)_{nm}\right| \leqq 0.015 \quad n = 1, 2, \ldots, 5 \qquad 5.2\text{-}1$$

and

$$\left|(\tilde{\alpha}_e/20)_{nm} - (\hat{\alpha}/20)_{nm}\right| \leqq 0.015 \quad n = 1, 2, \ldots, 5 \qquad 5.2\text{-}2$$

where n denotes the subinterval within each two minute trial and m denotes the number of the iteration. The best estimates of the human operator parameters K and τ are given in Appendix E. Note that $\tau = 2/\alpha$. In applying the iterative procedure, the initial values of $\hat{K}/10$ and $\hat{\alpha}/20$ for all five subintervals were set equal to corresponding parameter values obtained by Jackson [22] using a different identification technique. With these initial conditions, $m \leqq 6$ was sufficient for all trials analyzed and in a large number of trials $m \leqq 2$ was sufficient to satisfy Eqs. 5.2-1 and 5.2-2.

As mentioned previously, only the 2 radian per second cut-off-frequency data were analyzed during this investigation. For the single integrator controlled element the second, sixth and tenth days of testing were analyzed. For the double integrator controlled element the third, seventh and ninth days of testing were analyzed. All indications are that the intermediate days of testing have results which are consistent with results for those days that were analyzed.

5.3 Analysis of Iterated Parameter Values

One method of analyzing the parameter values obtained was to study the time histories of the parameters. It was thought that a subject might follow some consistent trend in the variation of gain and time-delay during a trial or during a single day of testing. This type of consistency would become apparent from a visual examination of the parameter time histories. Typical parameter time histories for one day of testing are shown in Fig. 5.3-1. It is apparent from the time histories such as that shown in Fig. 5.3-1 that the subjects did not have any consistent trends in gain or time-delay within a single day of testing.

A second method of analyzing the parameter values was to consider the two parameters, K and τ, as independent random variables. With this point of view the distributions of the parameters might well give some insight into subject behavior.

Rather than study the entire distribution of each of the parameters it was decided to study the mean and variance of each distribution. Since the mean and variance of a random variable are theoretical parameters which are not measurable, it is necessary to obtain estimates of these quantities from the empirical data. On a given day of testing the sample average for either parameter value for a given subject is represented by

$$\overline{G}_i = \frac{1}{25} \sum_{j=1}^{25} x_{ij} \qquad i = 1, 2, 3 \qquad 5.3\text{-}1$$

In Appendix D, it is shown that $\overline{G}_i$ is an unbiased estimate of the true mean value, μ_i. In Eq. 5.3-1, the x_{ij} represent samples of either K or τ.

The sample values of K and τ were obtained on each day of testing, using the regression analysis technique to obtain the best estimate of the parameters for each of twenty-five 20 second intervals of data for each subject. This gives a total of 75 estimates of both K and τ for each day of testing. The parameter average values are presented in Fig. 5.3-2 for the case $Y_C(p) = 5/p$ and in Fig. 5.3-3 for the case $Y_C(p) = 5/p^2$.

Two major characteristics of the average parameter values are apparent from Figs. 5.3-2 and 5.3-3, namely:

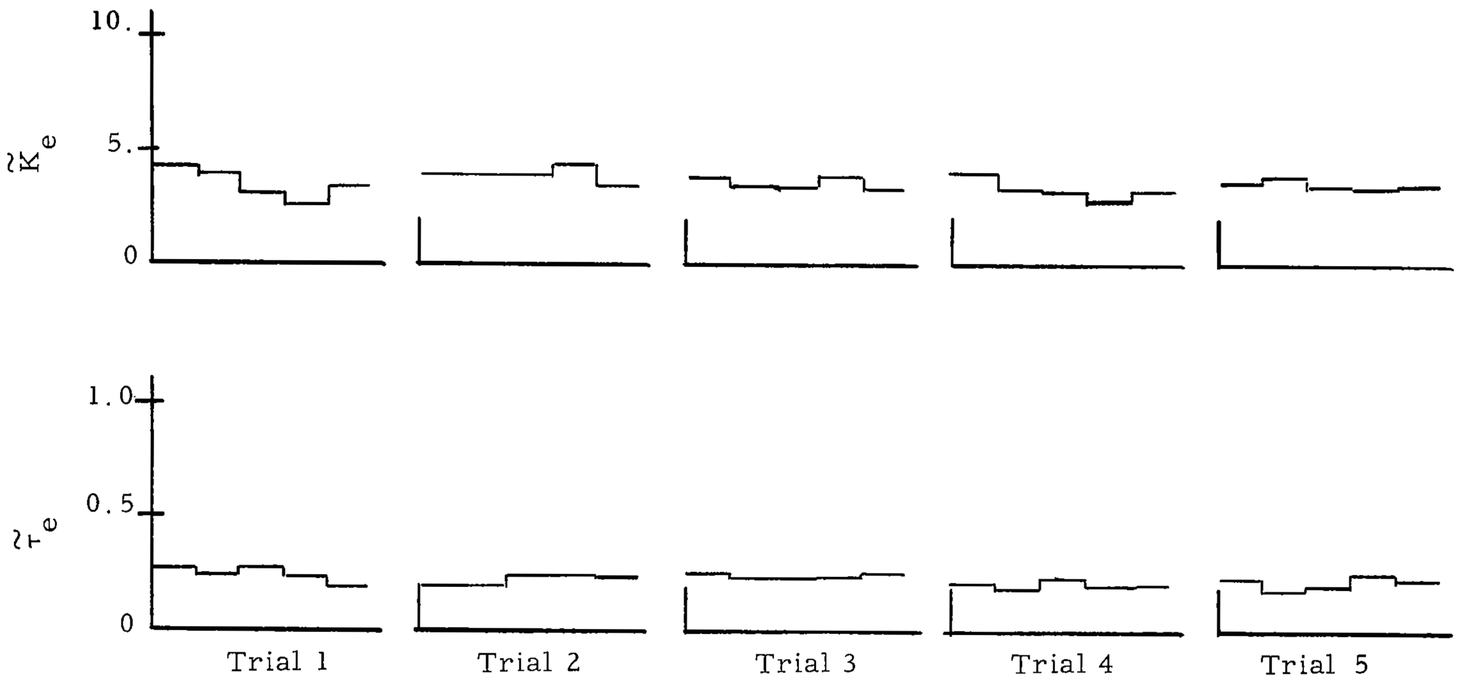


Figure 5.3-1 Iterative Parameter Estimation Time History, $Y_C(p) = 5/p$, Subject 3, Day 6

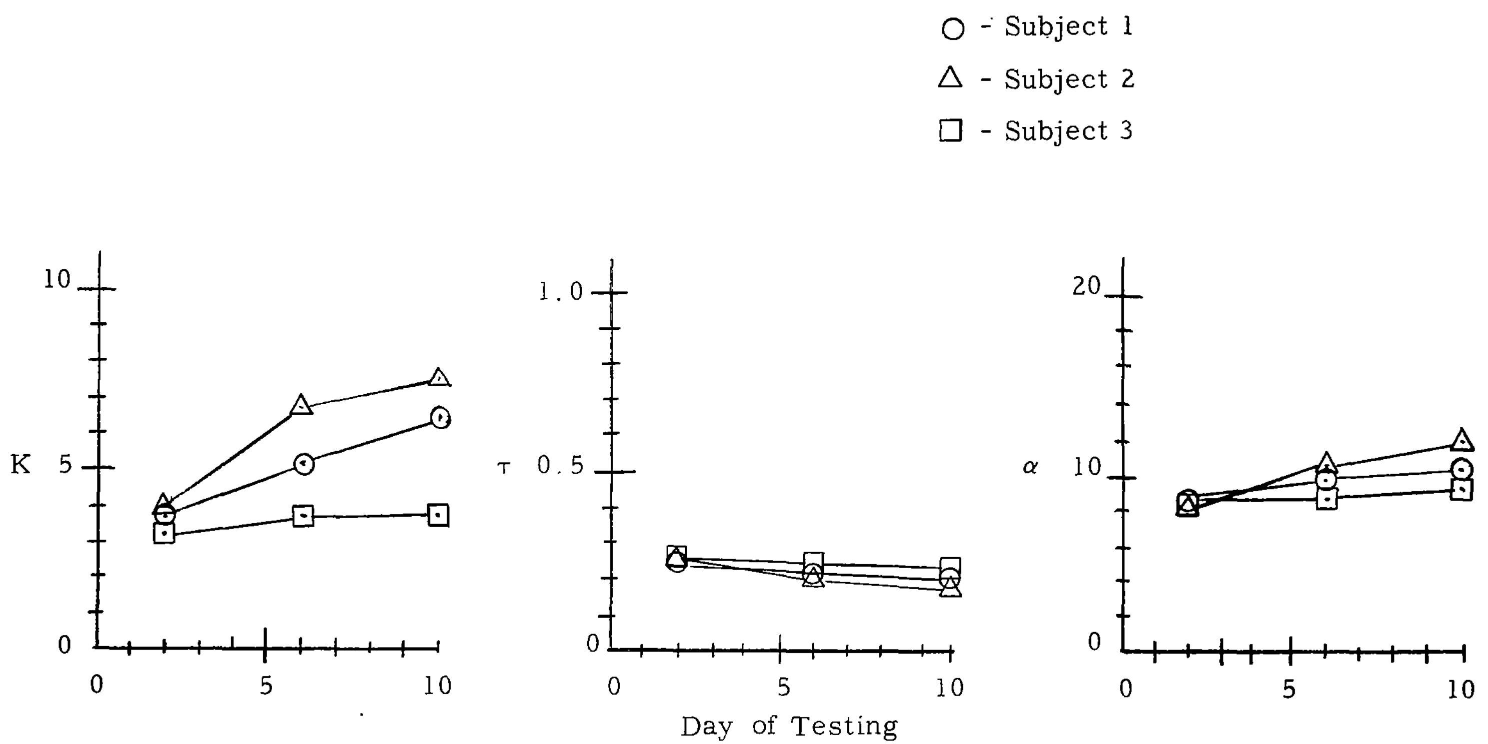


Figure 5.3-2 Average Parameter Values, $Y_C(p) = 5/p$

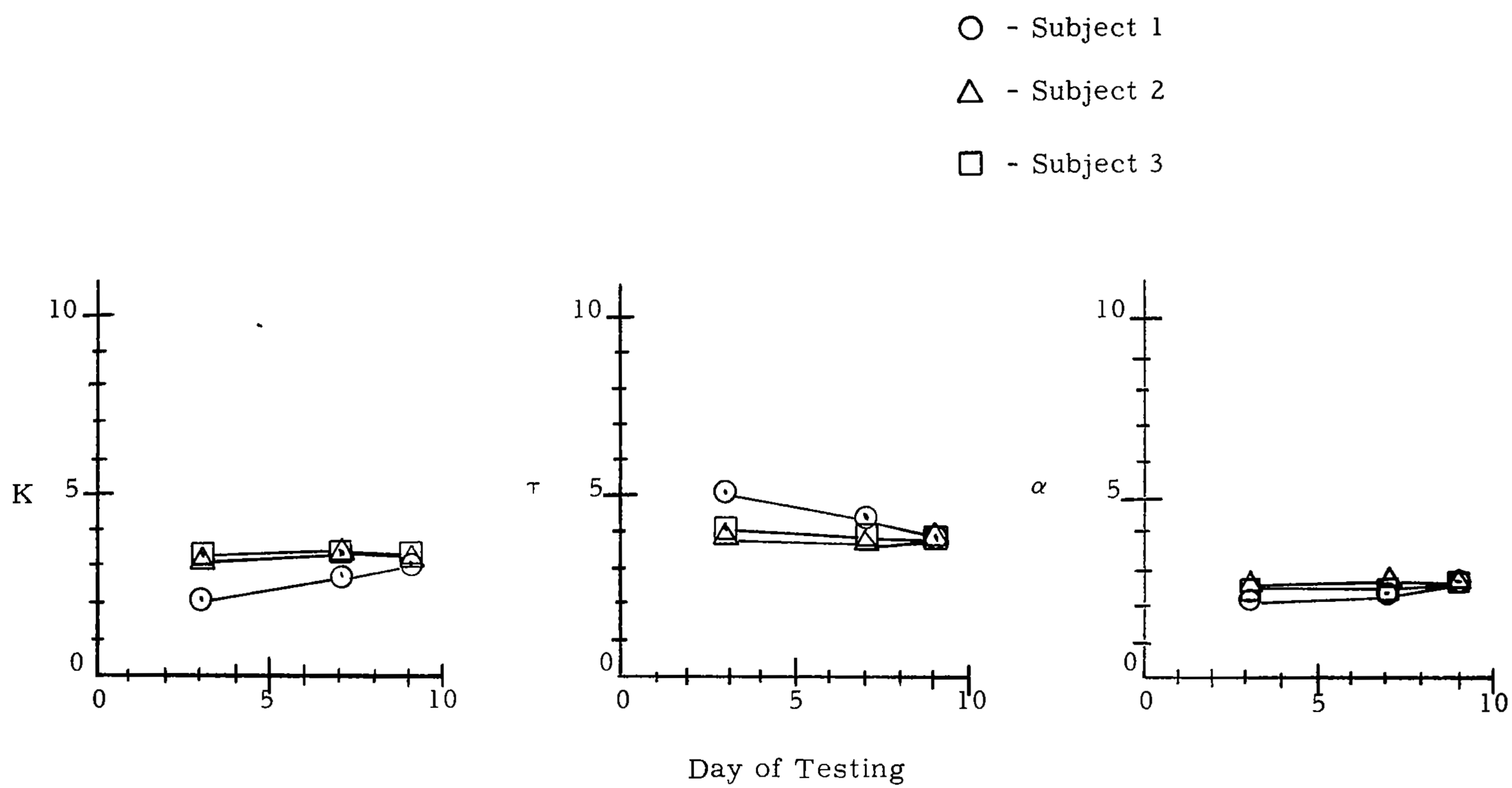


Figure 5.3-3 Average Parameter Values, $Y_C(p) = 5/p^2$

(1) The daily average value of K increases with learning.

(2) The daily average value of τ decreases with learning.

The results of Figs. 5.3-2 and 5.3-3 agree completely with those presented by Jackson [22]. This is entirely expected since the experimental data is the same as that analyzed by Jackson using a different parameter identification method. In addition to the average parameter data, Jackson also presented error score data which shows error scores that decrease with learning. Thus one interpretation of the characteristics of the average parameter values mentioned above is that in the process of consciously attempting to improve his error score the subject increases his gain, K, and shortens his time-delay τ.

This interpretation is born out by the following analysis. The spectral density of the system error signal can be written as:

$$\Phi_\epsilon(j\omega) = \left|\frac{1}{1+Y_C(j\omega)Y_P(j\omega)}\right|^2 \Phi_\psi(j\omega) + \left|\frac{Y_C(j\omega)}{1+Y_C(j\omega)Y_P(j\omega)}\right| \Phi_n(j\omega) \tag{5.3-2}$$

where the various signals are those given in Fig. 1.3-1 for the equivalent human operator. Let us take the simple case of $Y_C(p) = \frac{1}{p}$ and assume that the crossover model gives a sufficiently good representation of the system. Then,

$$Y_C(j\omega)Y_P(j\omega) = \frac{Ke^{-j\omega\tau}}{j\omega} \tag{5.3-3}$$

and

$$\Phi_\epsilon(j\omega) = \left\{\frac{\omega^2}{K^2[1 - \frac{2\omega}{K}\sin\omega\tau + \frac{\omega^2}{K^2}]}\right\}\Phi_\psi(j\omega)$$

$$+ \left\{\frac{1}{K^2[1 - \frac{2\omega}{K}\sin\omega\tau + \frac{\omega^2}{K^2}]}\right\}\Phi_n(j\omega) \tag{5.3-4}$$

The reasoning followed in this analysis is that if the spectral density of the error signal, $\Phi_\epsilon(j\omega)$, is small, then the error signal itself is in general small. Now consider the following cases.

(1) τ fixed: Inspection of Eq. 5.3-4 shows that for this case, $\Phi_\epsilon(j\omega)$ decreases as K increases.

(2) K fixed: Again inspection of Eq. 5.3-4 shows that for small values of τ the denominator of both terms increases for decreasing value of τ. Thus $\Phi_\epsilon(j\omega)$ decreases as τ decreases.

Thus it is seen that increasing the value of K and decreasing the value of τ corresponds to decreasing the magnitude of $\epsilon(t)$. A third mechanism for reducing $\Phi_\epsilon(j\omega)$ is to reduce the remnant signal, n(t). Note also that $\Phi_n(j\omega)$ is not necessarily independent of the value of K and τ.

In the analysis of the variance of the parameters, K and τ, the following approach is taken. The total of three subjects on any given day of testing is considered as a source of a population, A_K, of values of the random variable K and also as a source of a population, A_τ, of values of the random variable τ. Within the total population, either A_K or A_τ, there are three subpopulations, A_{K1}, A_{K2}, etc., each representing parameter values for one of the three individual subjects. In Appendix D, it is shown that the total variance of either K or τ is given by

$$\sigma^2 = \sigma_W^2 + \sigma_B^2 \qquad 5.3\text{-}5$$

For the approach outlined above, the first component of the total variance which is the within-subject variance, σ_W^2, is given by

$$\sigma_W^2 = \frac{1}{3}\sum^{3} \sigma_i^2 \qquad 5.3\text{-}6$$

The σ_i^2 represents the variance of the parameter value within each of the three individual subjects. The second component of the total variance is the between-subject variance, σ_B^2, and is given by

$$\sigma_B^2 = \frac{1}{3} \sum_{i=1}^{3} (\mu - \mu_i)^2 \qquad 5.3\text{-}7$$

The μ_i represent the average parameter value for each individual subject and μ represents the average parameter value for the total of three subjects. Thus,

$$\mu = \frac{1}{3} \sum_{i=1}^{3} \mu_i \qquad 5.3\text{-}8$$

To study the components of variance of K and τ given by Eq. 5.3-5, unbiased estimates of the elements of this equation are obtained from the empirical data. In Appendix D, it is shown that the following are unbiased estimates,

$$E[MS_{Total}] = \sigma^2 \qquad 5.3\text{-}9$$

$$E[MS_W] = \sigma_W^2 \qquad 5.3\text{-}10$$

$$E[MS_B] = \sigma_B^2 \qquad 5.3\text{-}11$$

where

$$MS_W = \frac{1}{72} \left[\sum_{i=1}^{3} \sum_{j=1}^{25} (x_{ij} - \overline{G}_i)^2 \right] \qquad 5.3\text{-}12$$

$$MS_B = \frac{1}{3} \left[\sum_{i=1}^{3} (\overline{G} - \overline{G}_i)^2 \right] - \frac{2}{75} \left[MS_W \right] \qquad 5.3\text{-}13$$

$$MS_{Total} = \frac{1}{74} \left[\sum_{i=1}^{3} \sum_{j=1}^{25} (x_{ij} - \overline{G}_i)^2 \right] - \frac{1}{74} \left[MS_B \right] \qquad 5.3\text{-}14$$

In Eqs. 5.3-12 through 5.3-14, x_{ij} represents a sample of either K or τ, and

$$\overline{G} = \frac{1}{3} \sum_{i=1}^{3} \overline{G}_i \qquad 5.3\text{-}15$$

The total variance and the components of the total variance were calculated for both K, τ and the auxiliary variable α for the days of testing given in Section 5.2. The results of these calculations are presented in Figs. 5.3-4 and 5.3-5.

Before proceeding further, let us define parameter time-variation. It has been shown [26] that small variations of gain and time-delay can be represented by an equivalent additive noise term. Thus the problem of separating the remnant term into components due to parameter time-variation and due to motor or additive noise is indeterminate. Also, Wierwille and Gagné have pointed out [41] that if no constraint is placed on the rate of variation of the parameters or gains that "...instead of having the time-varying gains follow the changes in the human operator's dynamics, the gains simply track the (output) signal itself." Thus one arbitrary method for partitioning the remnant term would be to attribute low frequency components to parameter time-variation and high frequency components to motor noise. The distinction between low and high frequency is also a question which each experimenter must decide. As implemented here, the parameters K and τ are restricted to frequencies on the order of one cycle per minute and lower. This restriction is imposed by taking the best parameter values for successive 20-second intervals.

Jackson has shown [22] that the human operator remnant is larger for the case of the double integrator controlled element than for the single integrator controlled element. In addition, it has been postulated in the literature [27, 28] that this increased remnant is due to, among other sources, a more pronounced time variability of the human operator in the first case. If this is true, then the within-subject variance of the parameters should be appreciably larger for the double integrator controlled element than for the single integrator controlled element. The hypothesis of a larger within-subject variance is not substantiated by Figs. 5.3-4 and 5.3-5. Although the results for the time-delay, τ, indicate a larger within-subject variance for the double integrator controlled element, reference to Eq. 3.3-4 shows that this does not directly account for a larger remnant. Thus the results indicate that the increased remnant for the case of the double integrator controlled

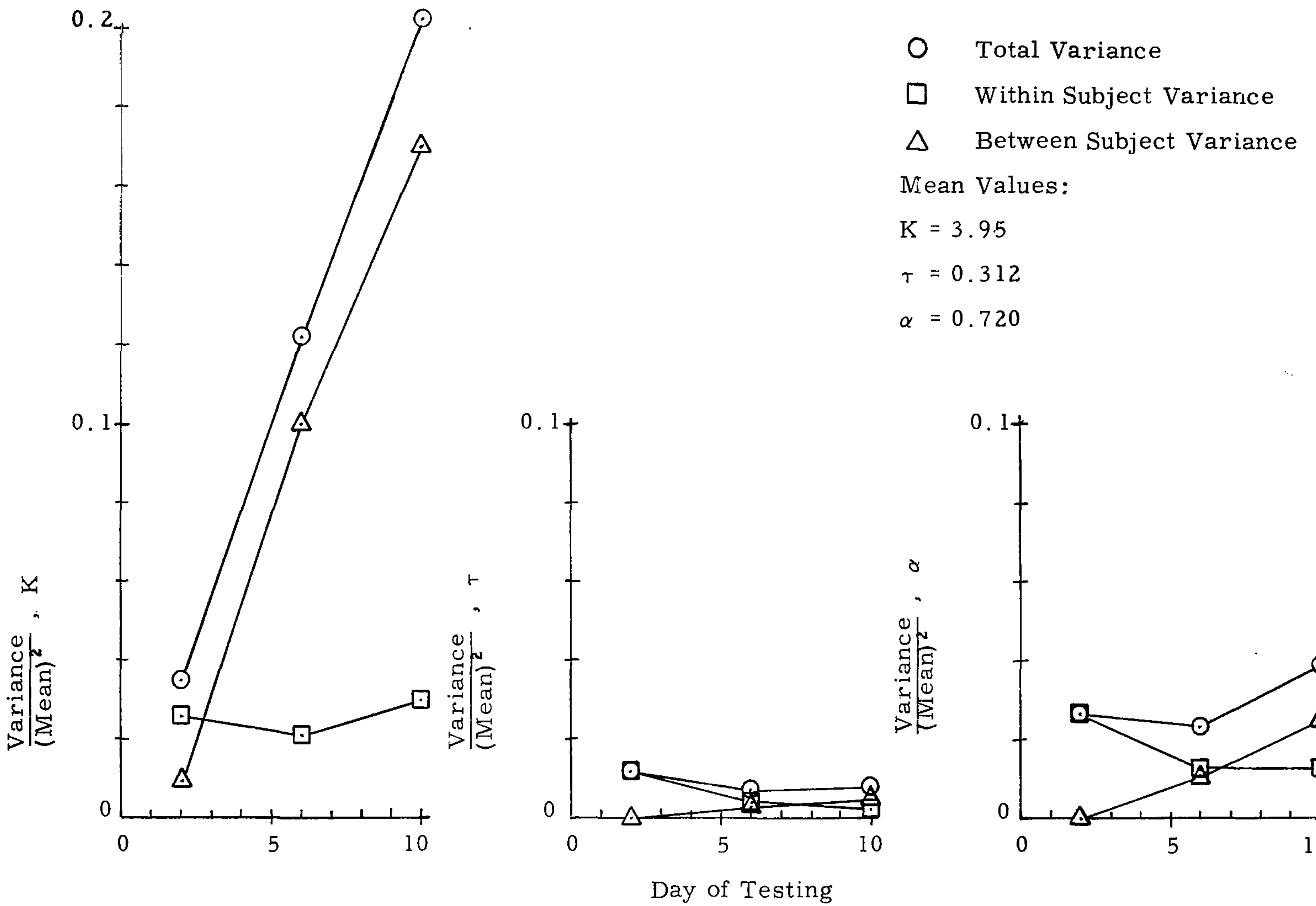


Figure 5.3-4 Components of Parameter Variance, $Y_C(p) = 5/p$

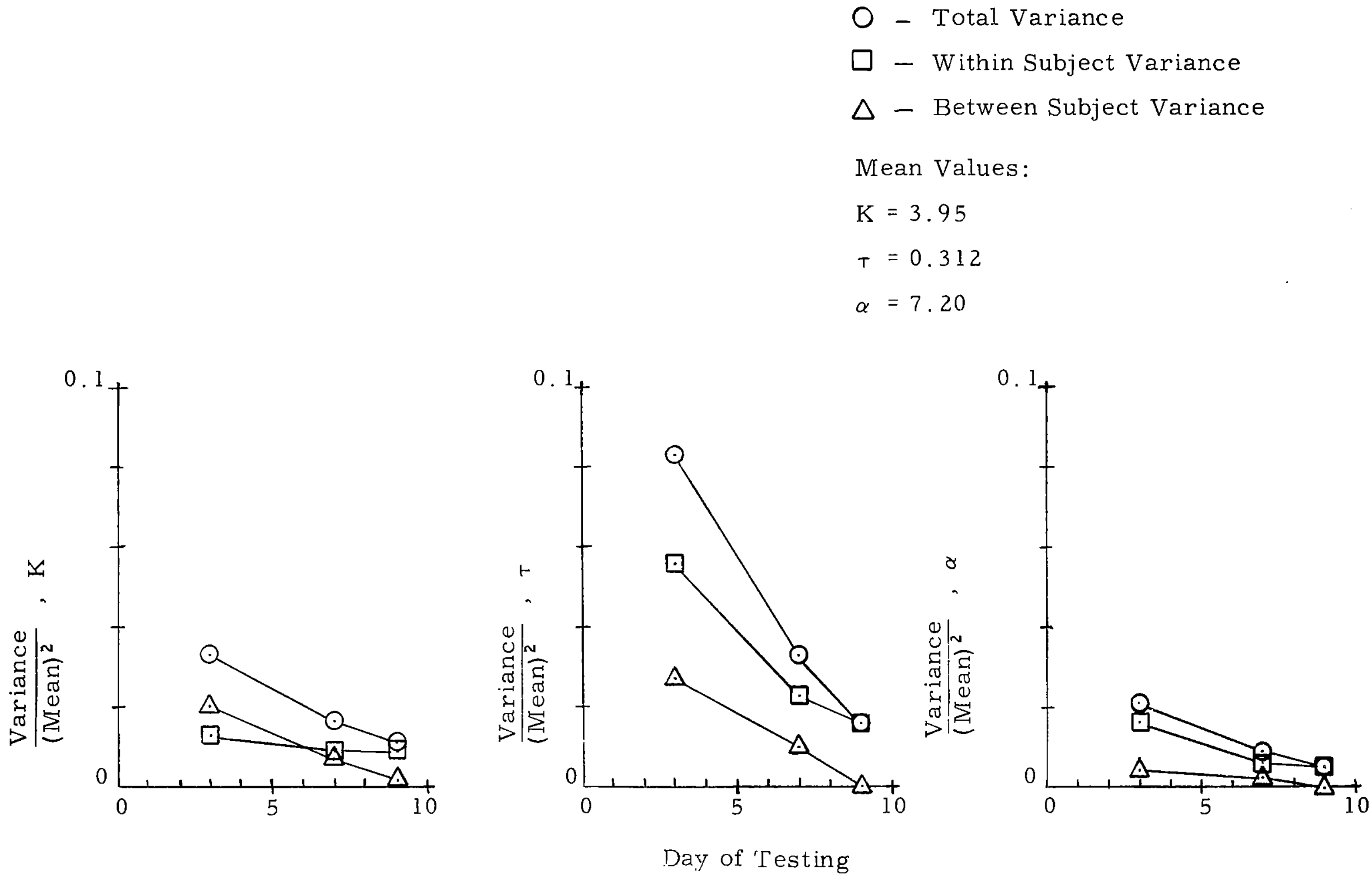


Figure 5.3-5 Components of Parameter Variance, $Y_C(p) = 5/p^2$

element is not due to increased time variation of the human operator. By implication then, this suggests that the increased remnant is due to such sources as more pronounced nonlinearity of the human operator.

Another observation that can be made is that the within-subject variance for K shows very little change as the subjects learn while the within-subject variance for the time-delay shows a marked decrease with learning. This is a significant finding which has not been reported in the literature previously. The fact that the within-subject variance of the gain, K, is essentially constant for all days of testing indicates that there is an inherent variability in the gain on which training has little effect. On the other hand, the decrease in within-subject variance for the time-delay indicates that the variability of τ is a characteristic of the human operator which is very dependent on the amount of training.

One explanation of the relationship between variability of τ and training is the following. In Figs. 5.3-2 and 5.3-3 it was pointed out that the subject increases his average gain and decreases his average time-delay as he learns to perform the compensatory tracking task. These learning trends were associated with a conscious effort on the part of the subject to reduce the system error. The total results then indicate that in the process of learning, the human operator not only reduces the average value of his time-delay by consciously trying to do a better job of tracking, but also subconsciously adopts a more consistent signal processing mechanism. One analogy that has been suggested [10] for the mental operations inherent in the learning process is a modern electronic data-processing system. Using such an analogy, the signal processing mechanism mentioned above would correspond to the computer program used in the performance of the tracking task. This program would consist of many subroutines which can be changed or modified. The large initial within-subject variance of τ would correspond to the subject experimenting with a wide variety of subroutines. Then as the subject learns he would reduce the variety of subroutines that he tries as well as modifying the complete program to make it more efficient. In experiments of a different nature, learning to roll cigars, Crossman [5] has arrived at a similar description: "The writer has taken the basic premise that a learner faced by a new task tries out various methods, retains the more successful ones and rejects the less successful ones."

Along this same line, it is seen from Figs. 5.3-4 and 5.3-5 that the within-subject variance of the time-delay is appreciably larger for the double integrator controlled element than for the single integrator case. This in all likelihood is due to the increased difficulty of the double integrator case. More important than the relative magnitudes is the noticeable decrease of the within-subject variance in Fig. 5.3-5 between the seventh and the ninth day of testing. This indicates that the subjects have not completely learned the task by the ninth day of testing. The average parameter values presented in Figs. 5.3-2 and 5.3-3 do not show as readily this apparent incompletion of learning. Thus the results suggest that the variance of a human operator's time-delay is a more sensitive criterion of learning than is the mean value of the time-delay.

The similarity between the time-delay within-subject variance curves for $Y_C(p) = 5/p$ and $Y_C(p) = 5/p^2$ is not apparent from Figs. 5.3-4 and 5.3-5. However in Fig. 5.3-6 where the same data are plotted on a logarithmic scale, it is seen that the curves are strikingly similar except for magnitude. From Fig. 5.3-6 then, it can be concluded that the effect of training on the time-delay variance is similar for both controlled elements.

Another observation that can be made from the data presented in Figs. 5.3-4 and 5.3-5 deals with the between-subject variance of gain and time-delay. It is seen that on the final day of testing the between-subject variance for both parameters is much smaller for the double integrator case than for the single integrator case. This agrees with the finding of McRuer, et al. [28] that the more difficult task constrains the subjects to behave in a uniform manner. Also, for the single

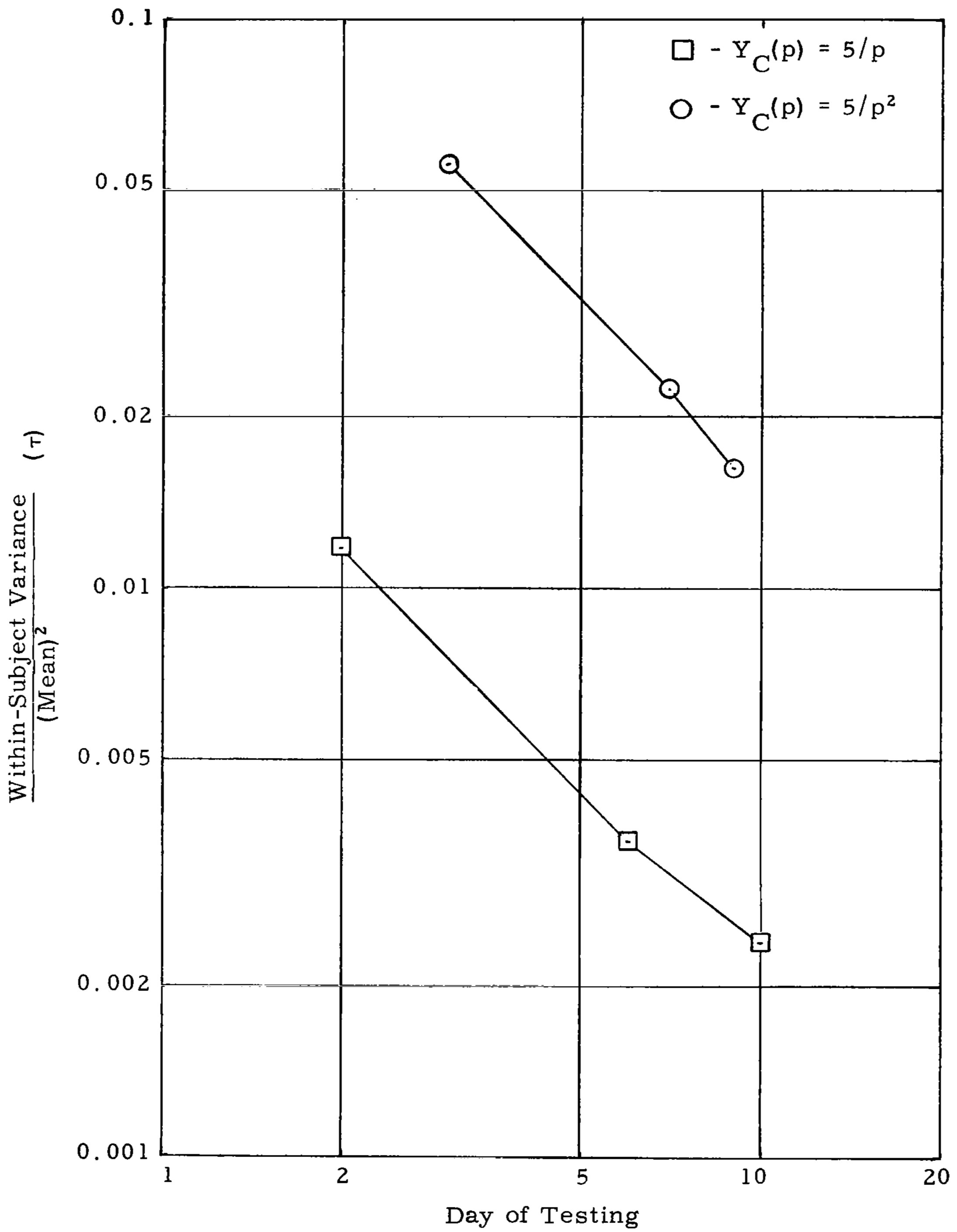


Figure 5.3-6 Within-Subject Variance of τ

integrator controlled element the between-subject variance for the human operator gain is much more pronounced than for the time-delay. This indicates that for the more easily controlled case, the human operator gain is a better indicator of individuality than is time-delay.

5.4 Power Match Considerations

A performance measure has been suggested which indicates the percentage of the human operator system output power that is accounted for by the model being used. This performance measure is called the power match, PM, [37], and for these experiments is

$$PM = 1 - \frac{\int_0^{120} e^2(t)\,dt}{\int_0^{120} \theta^2(t)\,dt} \qquad 5.4\text{-}1$$

During the analysis performed on the experimental data, two values of power match were computed for each trial. The two values of power match correspond to two different sets of parameter values that are used in the model during the calculation. One value of power match was obtained using the average of the five parameter values for each trial. In this calculation the model parameters were fixed at the average value for the entire trial. A second value of power match was obtained for each trial using the best parameter values for each of the five 20 second intervals within the trial. In this calculation the model parameters were set automatically at the best value during each of the 20 second intervals during the trial.

The values of power match for each of the five trials within a given day of testing were averaged together to give a single value of power match for each subject for each day of testing. These values of power match for the two sets of parameter values are presented in Figs. 5.4-1 and 5.4-2. The data presented in these figures indicate a small but consistent improvement in power match when the best parameter values are used for each 20 second interval over the power match obtained when the average parameter values are used during the

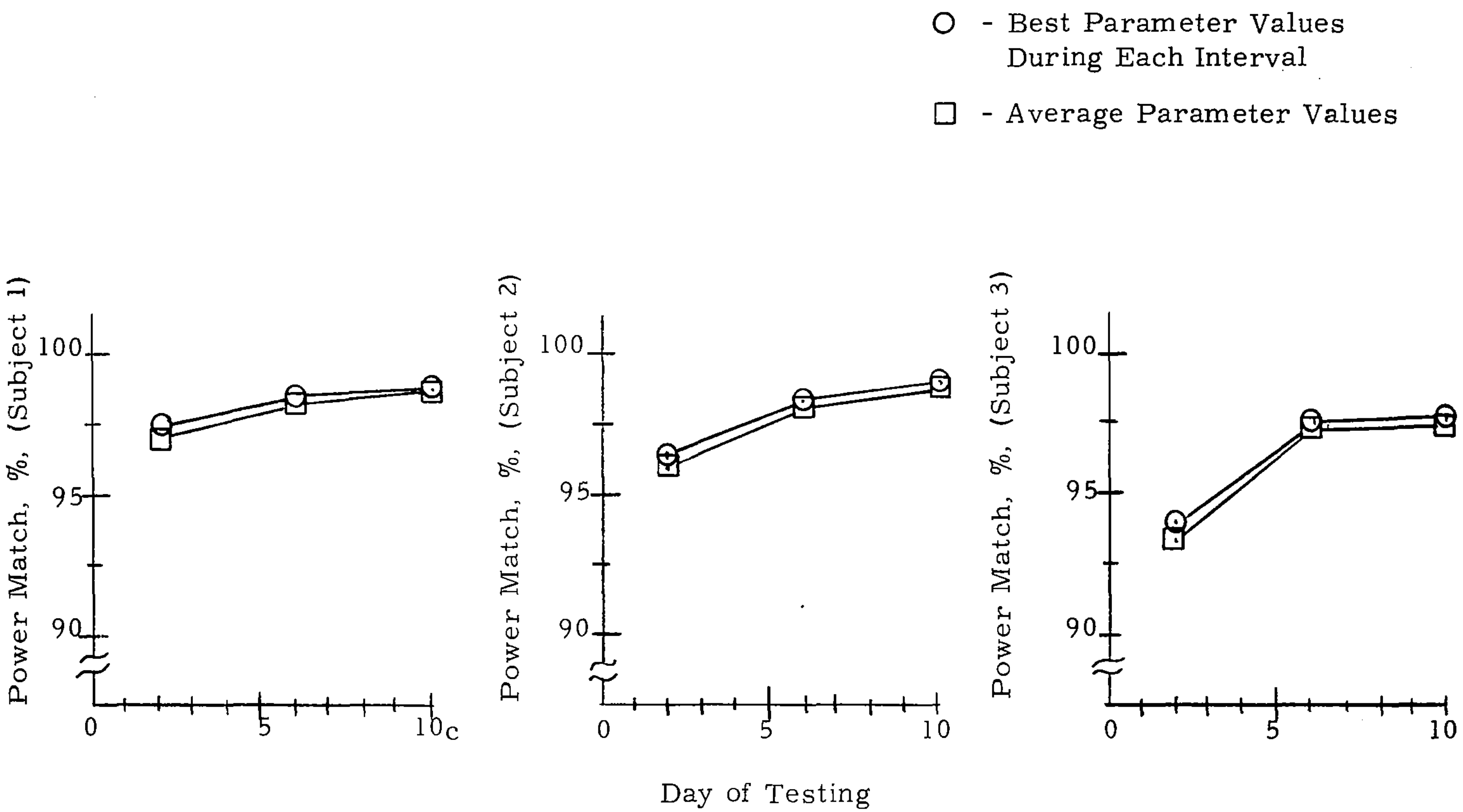


Figure 5.4-1 Power Match Values, $Y_C(p) = 5/p$

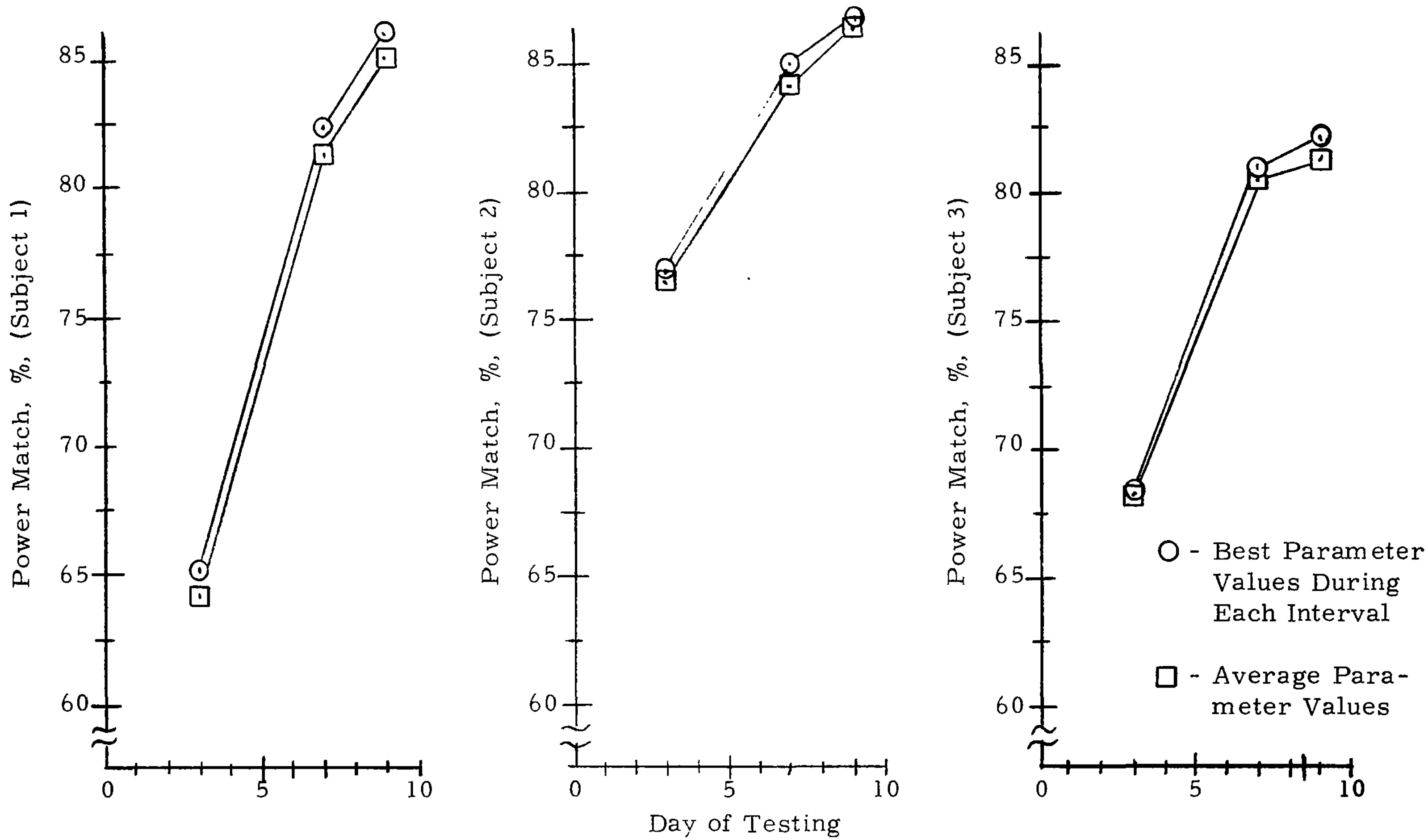


Figure 5.4-2 Power Match Values, $Y_C(p) = 5/p^2$

entire trial. Jackson [22] has shown that the crossover model as formulated in Eq. 3.3-4 is very close to the best linear constant coefficient model for the human operator controlling the first and second order controlled elements used here. Thus the power match obtained using the crossover model gives a good indication of the amount of remnant power present in the human operator system output. The small improvement in the power match when the best parameter values are used for each interval is then another indication that human operator time variation accounts for only a small part of the remnant.

5.5 Noniterative On-Line Parameter Identification

It has been shown [13] that the iterative regression analysis described in Chapter 2 converges quadratically near the optimum values of the parameters. It is therefore conceivable that good approximations of the system parameters could be obtained without iterations.

Because the iterations are costly and time consuming, for comparison purposes a noniterative on-line regression analysis was applied to obtain estimates of the crossover model parameters for the same test conditions analyzed in Section 5.3. In this application the model parameters were initially set at values which were known to be good estimates of the average parameter values for the given trial. These known parameter values were used for the first interval of the trial. At the beginning of each subsequent interval within the trial the model parameter values were updated to the best estimate from the preceding interval, i.e.,

$$\hat{K}_{i+1} = \tilde{K}_{ei} \qquad 5.5\text{-}1$$

$$\hat{\alpha}_{i+1} = \tilde{\alpha}_{ei} \qquad 5.5\text{-}2$$

where i = 1, 2, . . . , 5, represents the interval within a trial.

Some typical time histories of the parameters obtained by this method are shown in Fig. 5.5-1 along with the corresponding system input and output functions. The error E_K, E_τ, or E_α, between the

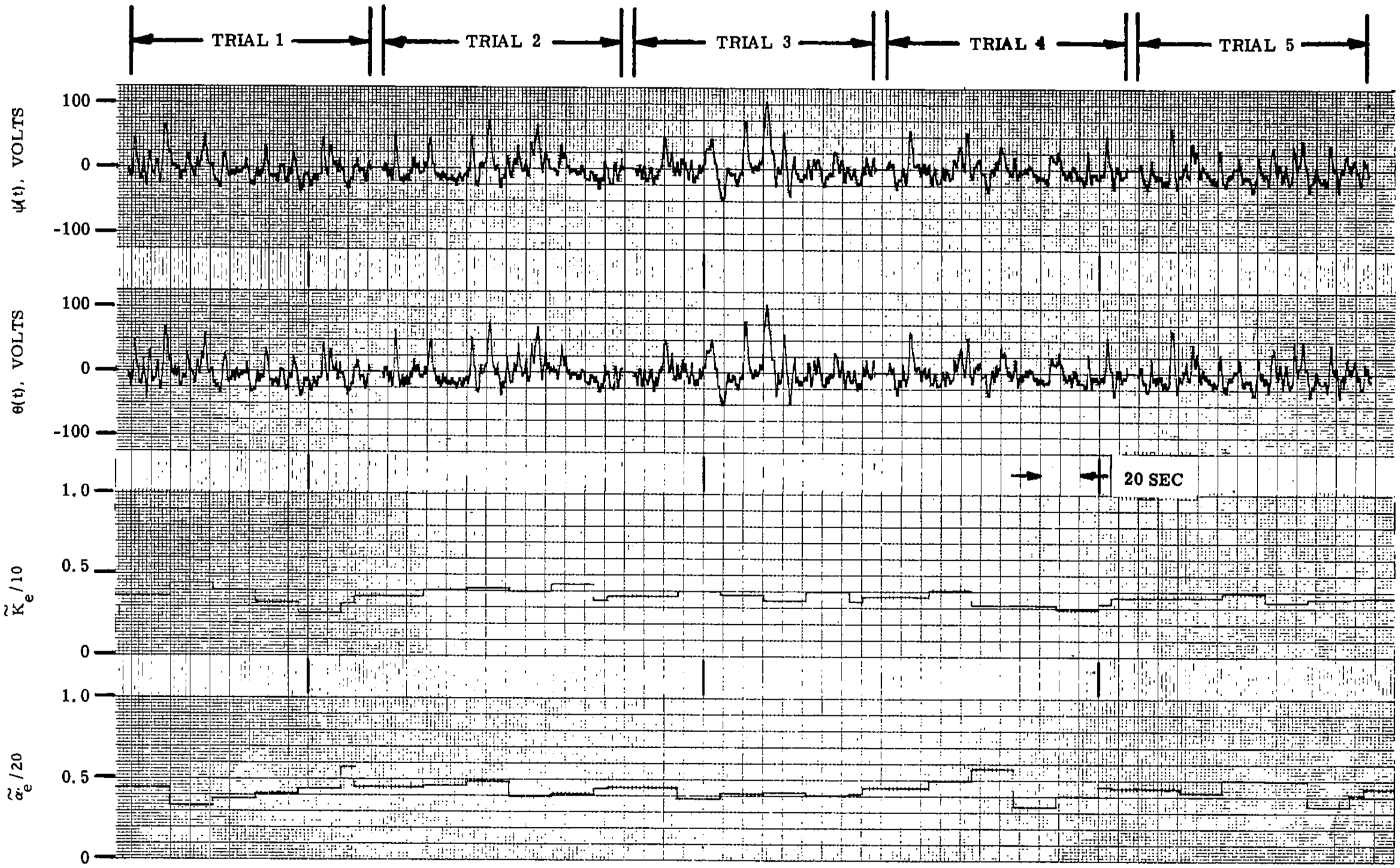


Figure 5.5-1 Noniterative On-Line Parameter Estimation Time History

$Y_C(p) = 5/p$, Subject 3, Day 6

parameter values obtained by noniterative on-line technique and the values obtained by the iterative technique described in Section 5.2 was computed for each 20 second interval that was analyzed. This error is defined by

$$E = 100 \frac{(\text{Iterative Parameter Value})-(\text{On-line Parameter Value})}{\text{Iterative Parameter Value}} \qquad 5.5\text{-}3$$

The average $|E|$ was computed for each day of testing. These data are presented in Figs. 5.5-2 and 5.5-3. It can be seen from these data that except for the early days of testing, the average $|E|$ of all parameters is less than 10% although much larger errors are not uncommon. Thus if an investigator should have a rather noncritical situation where errors in the parameter values of 10% can be tolerated the noniterative on-line technique would be useful.

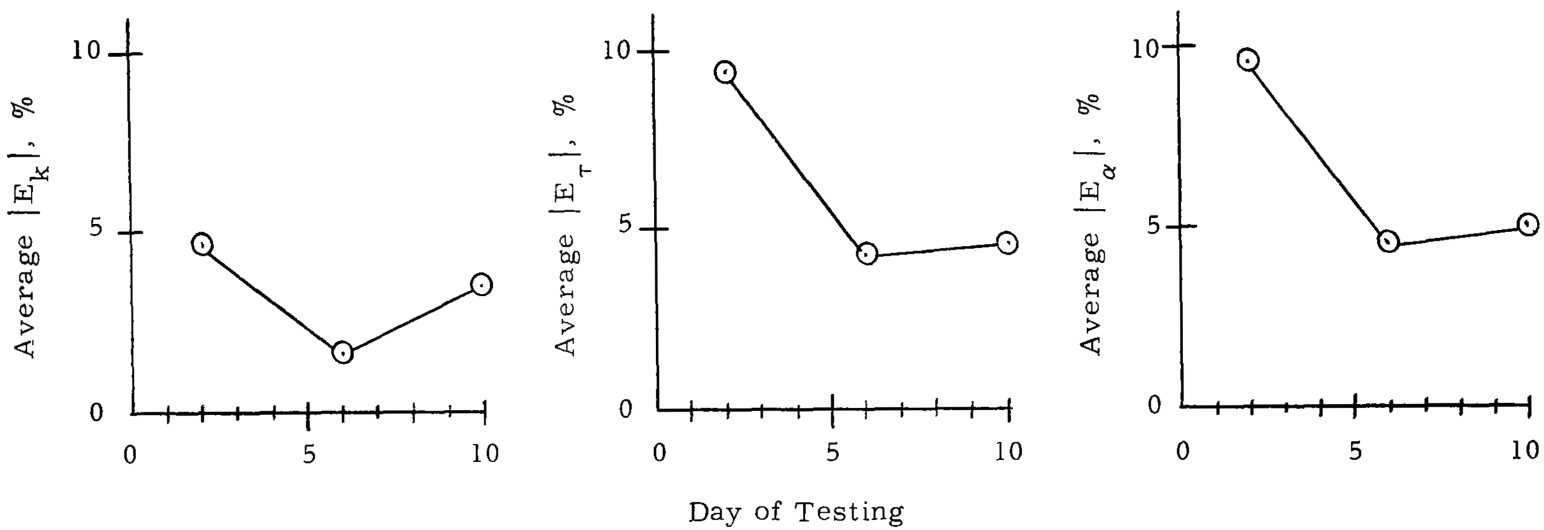


Figure 5.5-2 Difference Between Iterative Parameter Estimates and Noniterative On-Line Parameter Estimates, $Y_C(p) = 5/p$

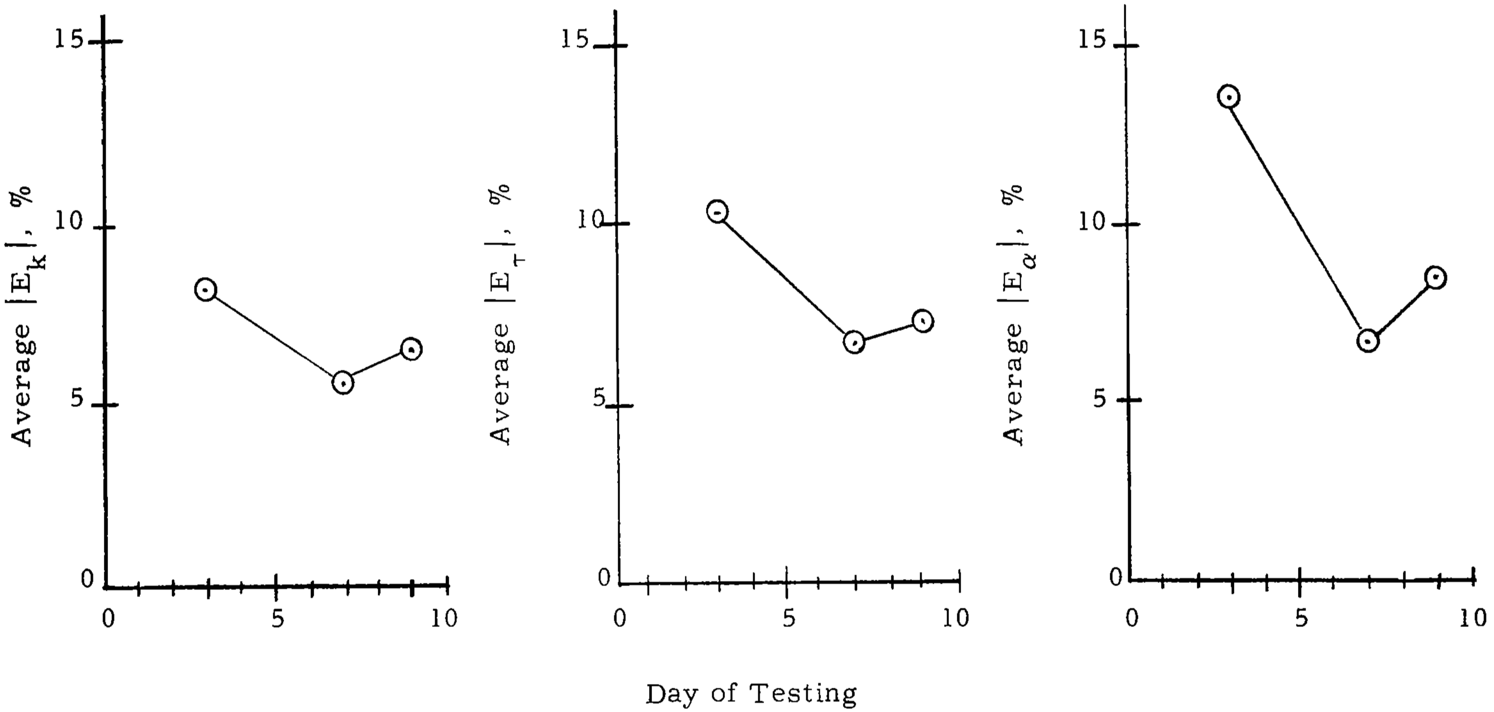


Figure 5.5-3 Difference Between Iterative Parameter Estimates and Noniterative On-Line Parameter Estimates, $Y_C(p) = 5/p^2$

CHAPTER 6

SUMMARY AND RECOMMENDATIONS

This concluding chapter has two purposes. One is to review and summarize the research discussed in the preceding chapters. The second purpose is to suggest directions for additional research. The research results are divided into two areas. One area consists of the development and analysis of the regression analysis parameter identification technique. The second research area is devoted to the analysis of human operator compensatory tracking experiments.

6.1 Summary of Regression Analysis Parameter Identification

In Chapter 2 a review is presented of the statistical background of regression analysis. This statistical principle is then developed into a parameter identification technique for dynamical systems. The technique consists of obtaining estimates for the difference between unknown parameters of a system under study and known parameters of a model. The estimate of the system parameters, $\tilde{c}$, is

$$\underline{\tilde{c}} = \underline{\hat{c}} + \underline{b}_e$$

where $\underline{\hat{c}}$ is the known model parameter vector and $\underline{b}_e$ is the estimate obtained from

$$\underline{b}_e = R^{-1}\underline{v}$$

It is shown that this parameter identification technique yields satisfactory results in a single computation if the difference between the system and model parameter values is sufficiently small. In general, since the system parameters are known only approximately, a single computation does not give a satisfactory result and the computation must be iterated. One feature that is introduced in Chapter 2 is the use of implicit inversion of the matrix R. By the use of implicit matrix inversion, the regression analysis technique is amenable to implementation on an analog computer.

The method developed in Chapter 2 for a general system is applied to the specific case of the crossover model of the human operator in Chapter 3. The application is quite straightforward. However, there are subtleties in the implementation which are significant. One problem that became apparent during the development is that of scaling the elements of the matrix R. It was found that over a wide range of model parameter values, typical magnitudes of the elements of R vary considerably. Thus system gain settings which are acceptable for one set of model parameter values may be unacceptable for another set of model parameter values. The most satisfactory method of overcoming this problem in an analog implementation is to provide automatic scaling of the elements of R based on the values of the model parameters. This was done in the present implementation and satisfactory results were obtained. A second feature of the implementation was the use of the fact that the covariance of the parameter influence coefficients for the crossover model is zero. Thus if sufficiently long data intervals are used, the off-diagonal elements of R can be neglected. This results in a much simpler implementation. Both the necessity of automatic scaling of R and the fact that the parameter influence coefficients have zero covariance are results which may have application to the identification of systems other than the human operator.

In Chapter 4 several sources of error in the application of regression analysis to the crossover model are discussed. As mentioned previously, a satisfactory result is obtained from a single computation if the parameter values of the system being identified are known sufficiently well. To determine how well the system parameter values must be known for the single computation to give acceptable results, a study based on an infinite interval of data was made. It was found that if the difference between the assumed system parameter values, i.e., the model parameter values, and the true values was not larger than 8% for both K and ω, the single computation gives results which are in error by not more than 1%. For the case of larger initial differences between the model and system parameter values it was shown experimentally that the iterative computational technique converged to the proper value in less than ten iterations.

Another source of error is the use of a finite length of data. This error is in the form of a statistical difference between the values for the elements of R and $\underline{v}$ based on an infinite interval of data and values which are based on a finite length of data. It is found that the variance of the parameter estimate due to this effect is bounded by a term which is inversely proportional to the length of the data interval. The results of numerical calculations indicate that the standard deviation of the error due to data intervals as short as five seconds is less than 10% for the estimates of both K and α.

The model of the human operator used throughout this research characterizes the system output as the response of a linear time-invariant system plus an uncorrelated additive noise term. It is important to know the effect of the equivalent noise on the estimates of the linear system parameter values. It is shown that the expected value of the parameter estimates obtained by regression analysis are unaffected by the additive noise, i.e., the regression analysis parameter values are unbiased estimates of the true system parameters in the presence of additive noise. It is further shown that if the additive noise is white, then the variance of the parameter estimates is directly proportional to the amount of noise present and inversely proportional to the length of the data interval.

A final source of error that is discussed in Chapter 4 is that due to erroneous model initial conditions at the start of a data interval. It is pointed out that the elements of R and $\underline{v}$ should be based on a steady-state response of the model. However, in the process of obtaining the parameter estimates, it is necessary to change the value of the model parameters at the start of each data interval. This change in parameter values causes a transient in the model response which in turn gives erroneous values for R and $\underline{v}$. The suggested solution to this problem is to delay the start of the calculation of the elements of R and $\underline{v}$ for approximately three time-constants of the model after the parameter values have been changed.

6.2 Human Operator Experimental Results

The impetus for developing the regression analysis parameter estimation technique was the desire to analyze human operator performance data. The analysis of the human operator data was on the basis of twenty second data intervals. Using this data interval it was possible to obtain parameter estimates for twenty-five intervals for each subject during each day of testing. The results based on a study of the mean and variance of the random variables, K and τ are:

(1) Average human operator gain, K, increases with learning.

(2) Average human operator time-delay, τ, decreases with learning.

These trends are interpreted as being the direct result of the subject learning to do a better job of tracking.

The real power of the regression analysis technique becomes apparent in studying the variance of the parameters. To study time-variation of the parameters it is necessary to have estimates for short intervals of data. The regression analysis technique can be used to obtain such estimates while such methods as continuous parameter adjustment techniques can not.

By making use of the fact that the total variance of both K and τ can be separated into a within-subject and a between-subject component for each day of testing, the following results were obtained.

(1) The human operator adopts a more consistent perceptual-motor signal processing path as he learns the tracking task.

(2) For the single integrator controlled element, the average value of K is a better indication of individuality in the trained human operator than is the average time-delay.

(3) For the more difficult to control double integrator controlled element, the subjects adopt more uniform average values of gain and time-delay than for easier control tasks.

(4) The increased remnant for a double integrator controlled element over that for a single integrator controlled element appears to be mainly due to sources other than time variation of the human operator.

(5) The variance of τ appears to be a more sensitive indicator of learning than the average value of either K or τ.

(6) There appears to be an inherent variability in the human operator gain on which learning has little effect.

6.3 Recommendations

There are three recommended areas for extension of the research reported here.

(1) Verification of the results on the variance of K and τ: The results of Chapter 5 are based on six subjects. The concept of analyzing the variance of the gain and time-delay is novel and hence these results should be verified by similar analysis on data from more subjects. Also the variance of parameters based on data intervals shorter than twenty seconds should be analyzed.

(2) Use of regression analysis in study of nonlinearities as a source of remnant: To be meaningful this analysis requires a model which includes nonlinearities in the forward loop. Such models are most easily analyzed by the equation error technique. Thus a combination of the regression analysis and a technique such as that described by Wingrove and Edwards [43] for eliminating the bias in the equation error method would seem to be a likely approach.

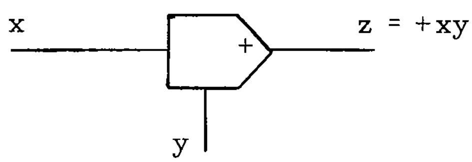
(3) Use of regression analysis in more difficult tasks: A possible application of regression analysis would be in a study

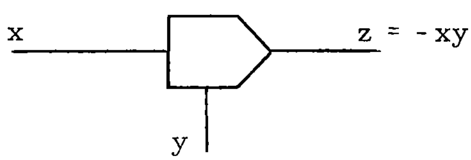
of emphasis in a two-axis control situation. The gain and time-delay or equivalent parameters for each axis could be determined for short time intervals. The parameter values could then be studied for changes of attention or emphasis from one axis to the other. A similar application would be parameter identification in natural settings, e.g., automobile driving.

APPENDIX A
ANALOG COMPUTER CIRCUITS

The analog computer circuits used in the data analysis are presented in this appendix. The notation used is basically that recommended by Simulation Councils, Inc. as reported in the December 1967 issue of Simulation. It was necessary to make several modifications and extensions to this notation which are summarized in Fig. A-1. All variables are expressed in terms of "machine units". Thus the computer reference voltage, 100 volts for the computer used here, corresponds to one machine unit, i.e., +100 v = +1.0 m.u.

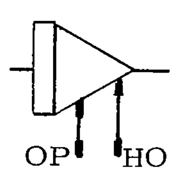
(1) Multiplier (Two types)





(2) Mode Control

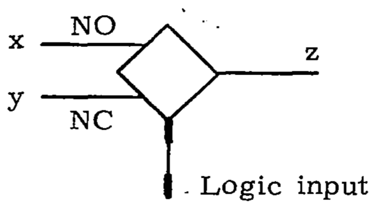
Integrator mode of operation is defined by the following diagram and truth-table.



Logic level OP	Logic level HO	Mode
0	0	Reset
1	0	Operate
1	1	Hold
0	1	Undefined

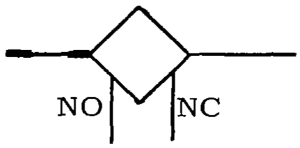
Figure A-1 Circuit Diagram Notation

(3) SPDT Switch



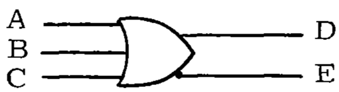
Logic level	z
0	-y
1	-x

or



(4) Complemented output

All logic elements have both true and complemented binary output. An example of the notation is given here for an "or" gate.

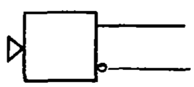


$$D = A + B + C$$

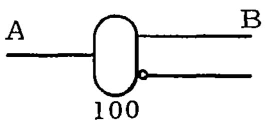
$$E = \overline{A + B + C}$$

(5) Latching Push-button

The logic output of this device changes state each time the button is depressed.



(6) Pulser or "One-shot"



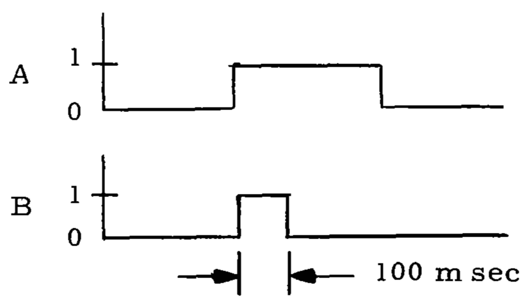
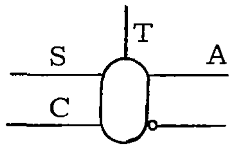
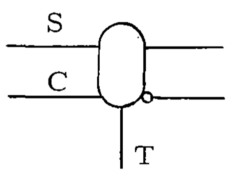


Figure A-1 Circuit Diagram Notation (continued)

(7) Flip-flop



S	C	A
1	0	1
0	1	0
1	1	0
0	0	*



*With S = 0 and C = 0 , a 0 → 1 transition on the trigger line, T, will cause the output to change state.

Figure A-1 Circuit Diagram Notation (continued)

The circuits shown in Figs. A-2, A-4 and A-5 are the circuit diagrams corresponding to the computational block diagrams of Figs. 3.3-2, 3.4-1 and 3.4-2 respectively. The circuits given in Figs. A-6, A-7 and A-8 correspond to the modified block diagrams of Figs. 3.5-3, 3.5-4 and 3.5-5 respectively.

As a check on the computer setup of Fig. A-2 the steady-state sinusoidal response of $u_0(t)$, $u_1(t)$ and $u_2(t)$ was compared with the theoretical functions. For the case of $\hat{K} = 3$, $\hat{\alpha} = 6$ and $\psi(t) = 0.3 \sin 2t$, the magnitude and phase of $u_0(t)$, $u_1(t)$ and $u_2(t)$ are:

$$|u_0(t)| = 0.373$$

$$\angle u_0(t) = -41.6°$$

$$|u_1(t)| = 0.103$$

$$\angle u_1(t) = +43.6°$$

$$|u_2(t)| = 0.031$$

$$\angle u_2(t) = +136.3°$$

The corresponding computer time histories are given in Fig. A-3.

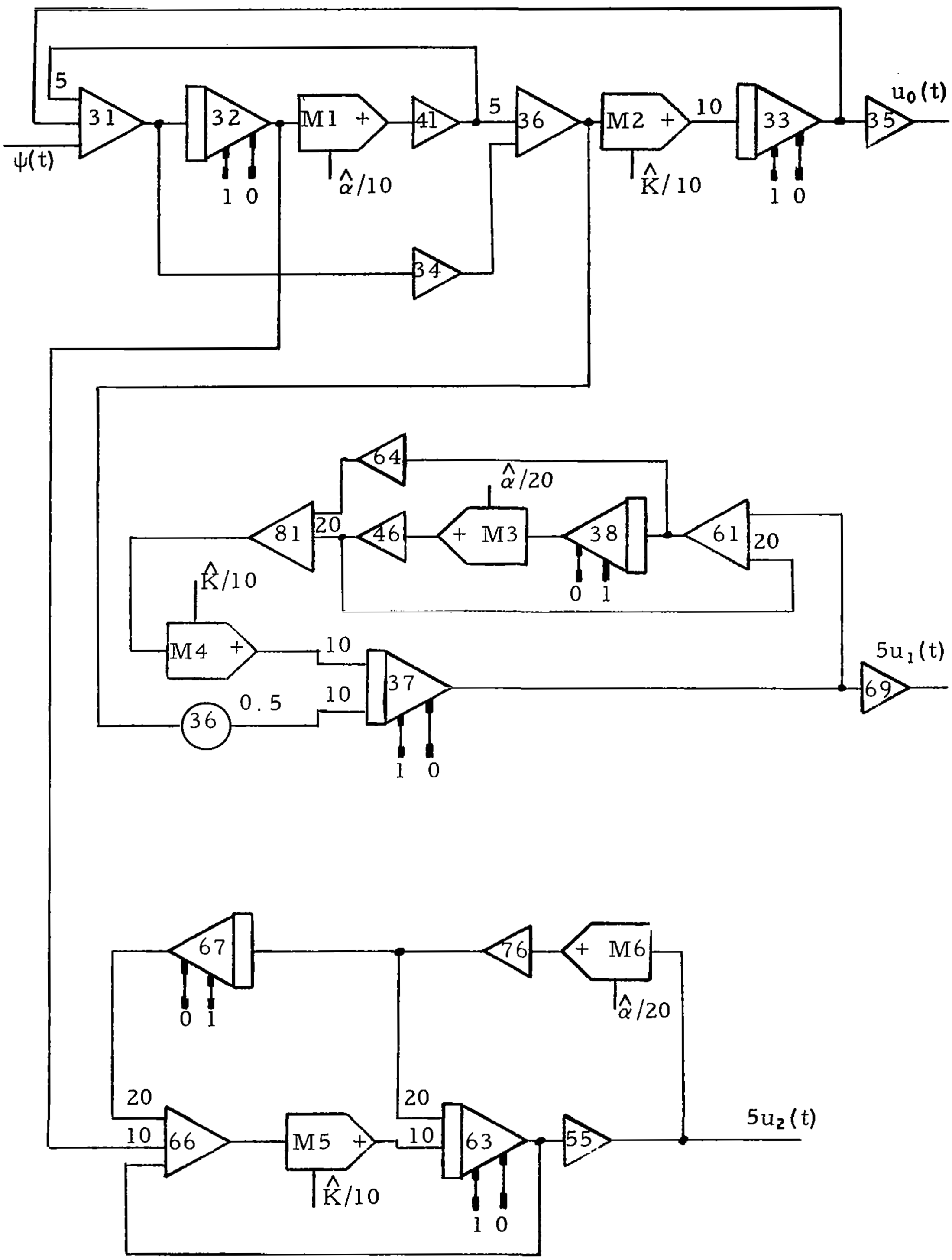


Figure A-2 Circuit Diagram for Crossover Model Implementation

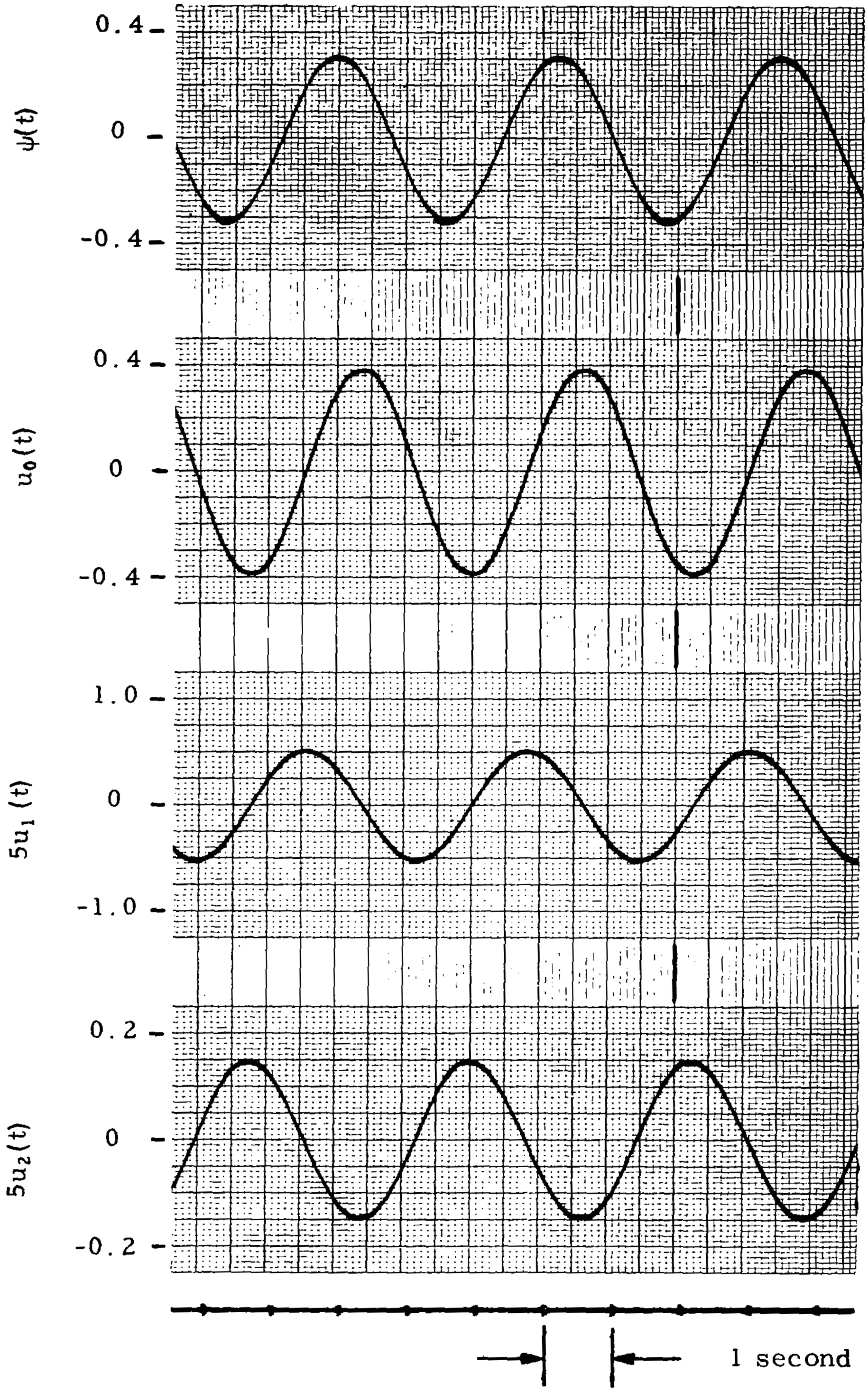


Figure A-3 Dynamic Check Time Histories

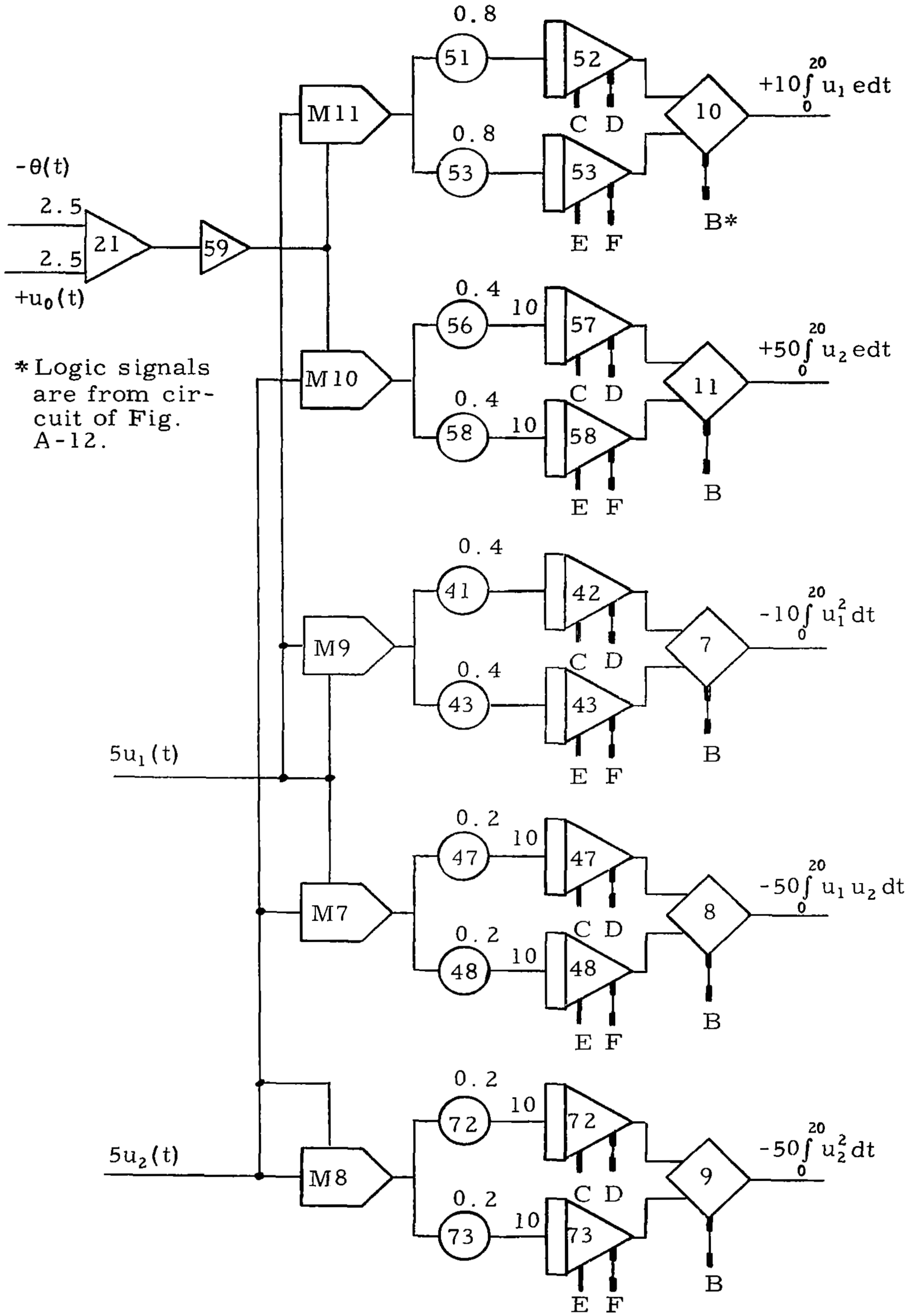


Figure A-4 Circuit for Evaluating R and v

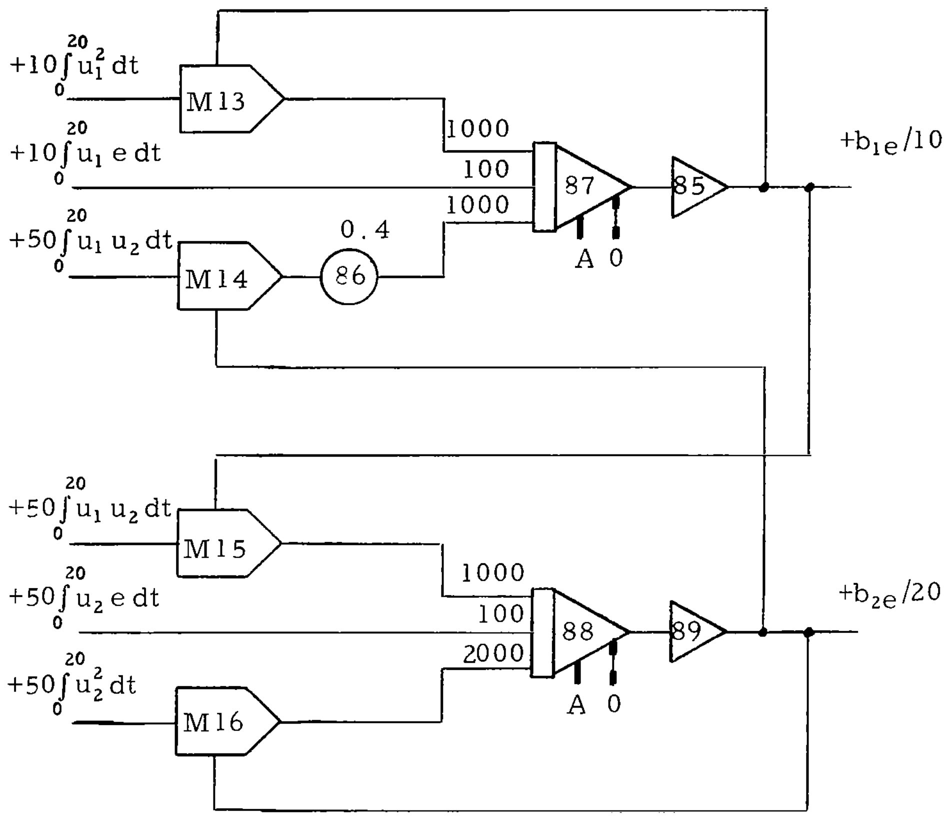


Figure A-5 Circuit for Solving $\underline{b}' + kR\underline{b} = k\underline{v}$

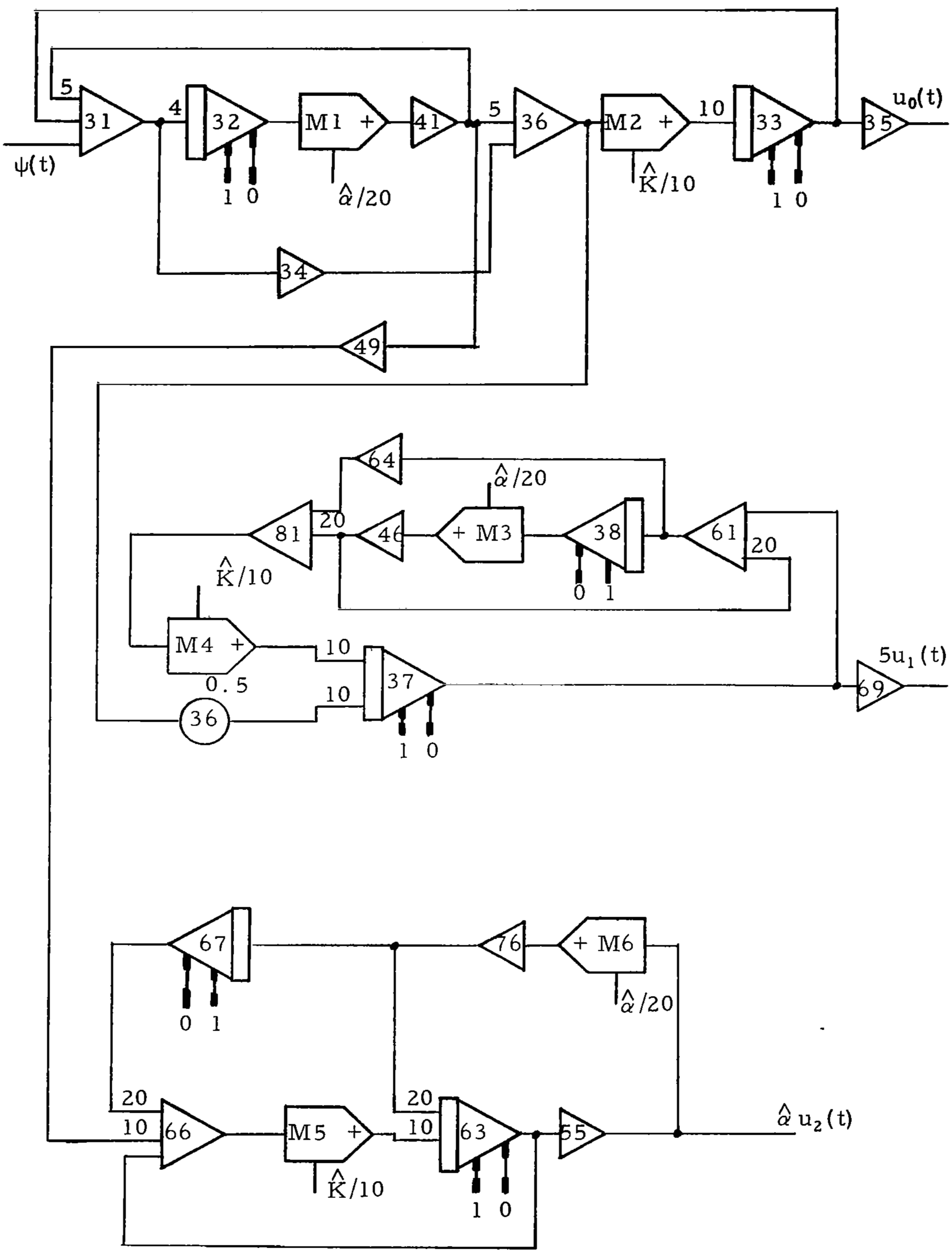


Figure A-6 Modified Circuit for Crossover Model Implementation

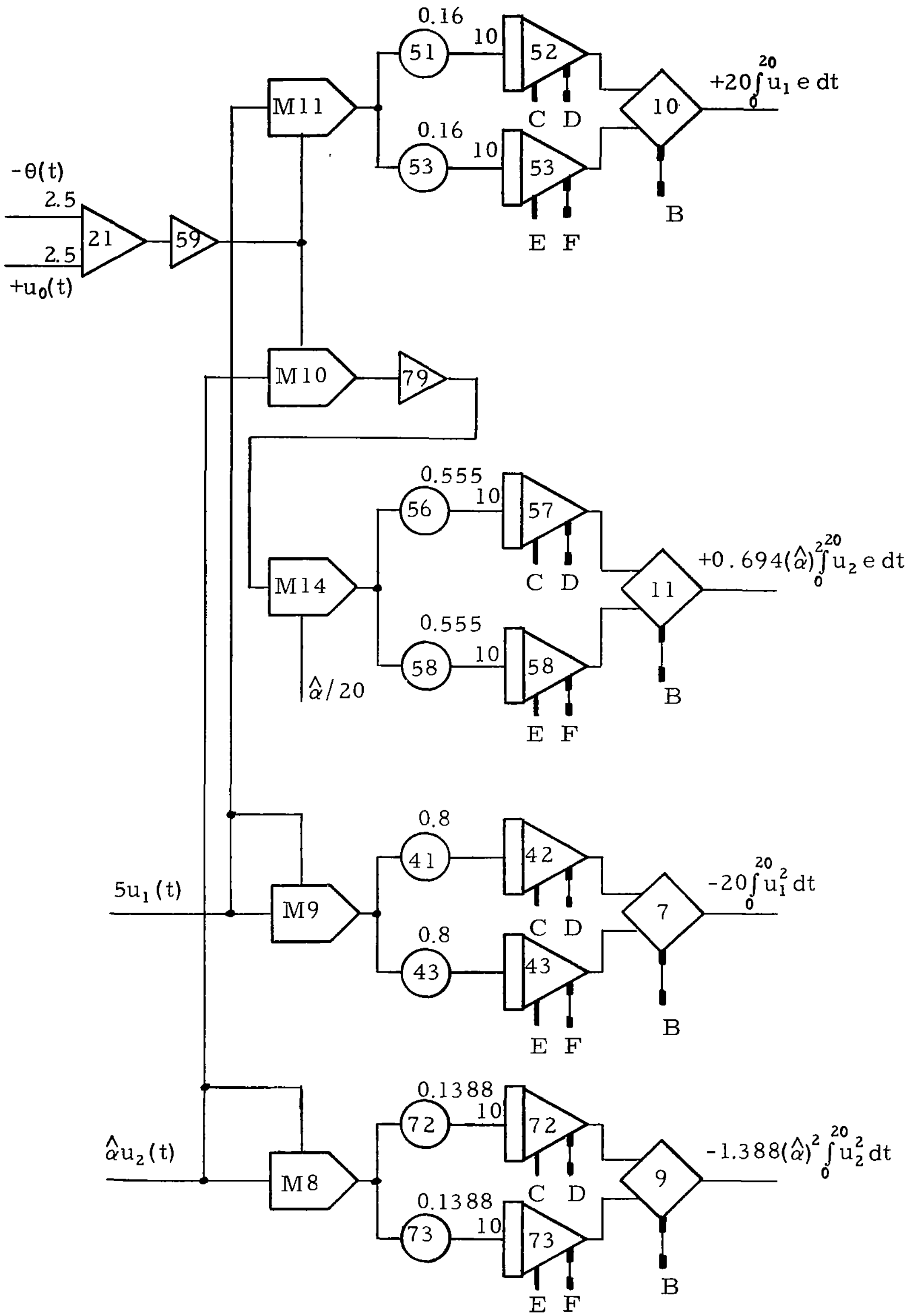


Figure A-7 Modified Circuit for Evaluating R and v

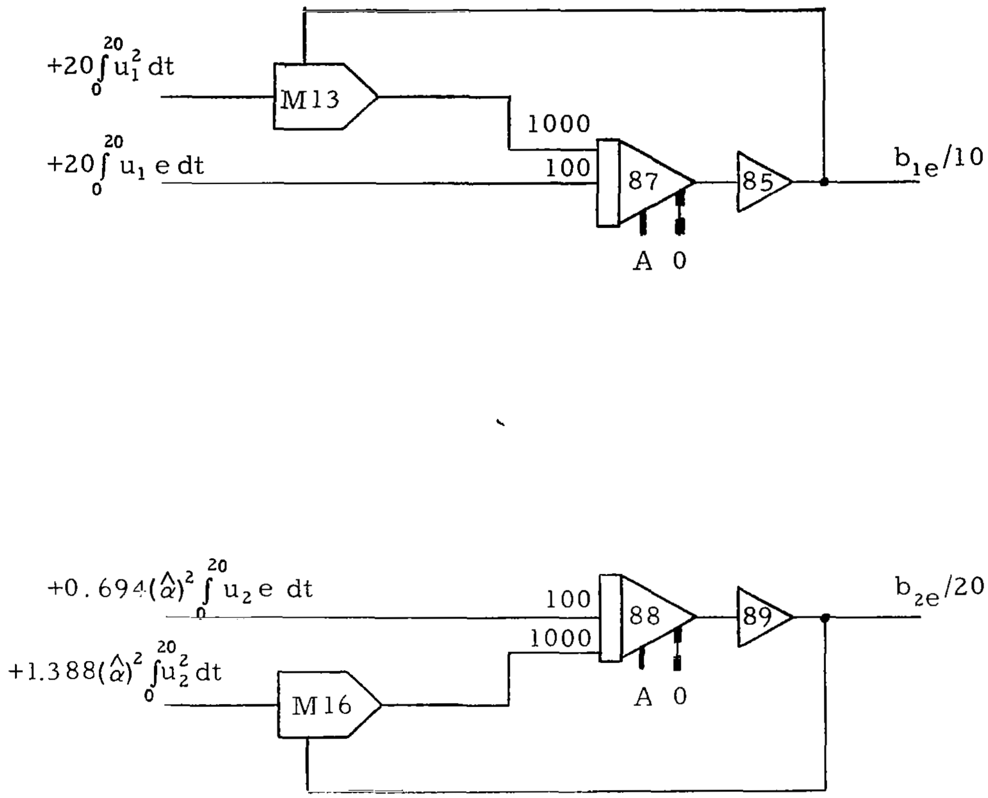


Figure A-8 Modified Circuit for Solving $\underline{b}' + k R \underline{b} = k \underline{v}$

In the experimental data analysis two approaches were used in setting the model parameter values: one was iterative, the other was a noniterative on-line method.

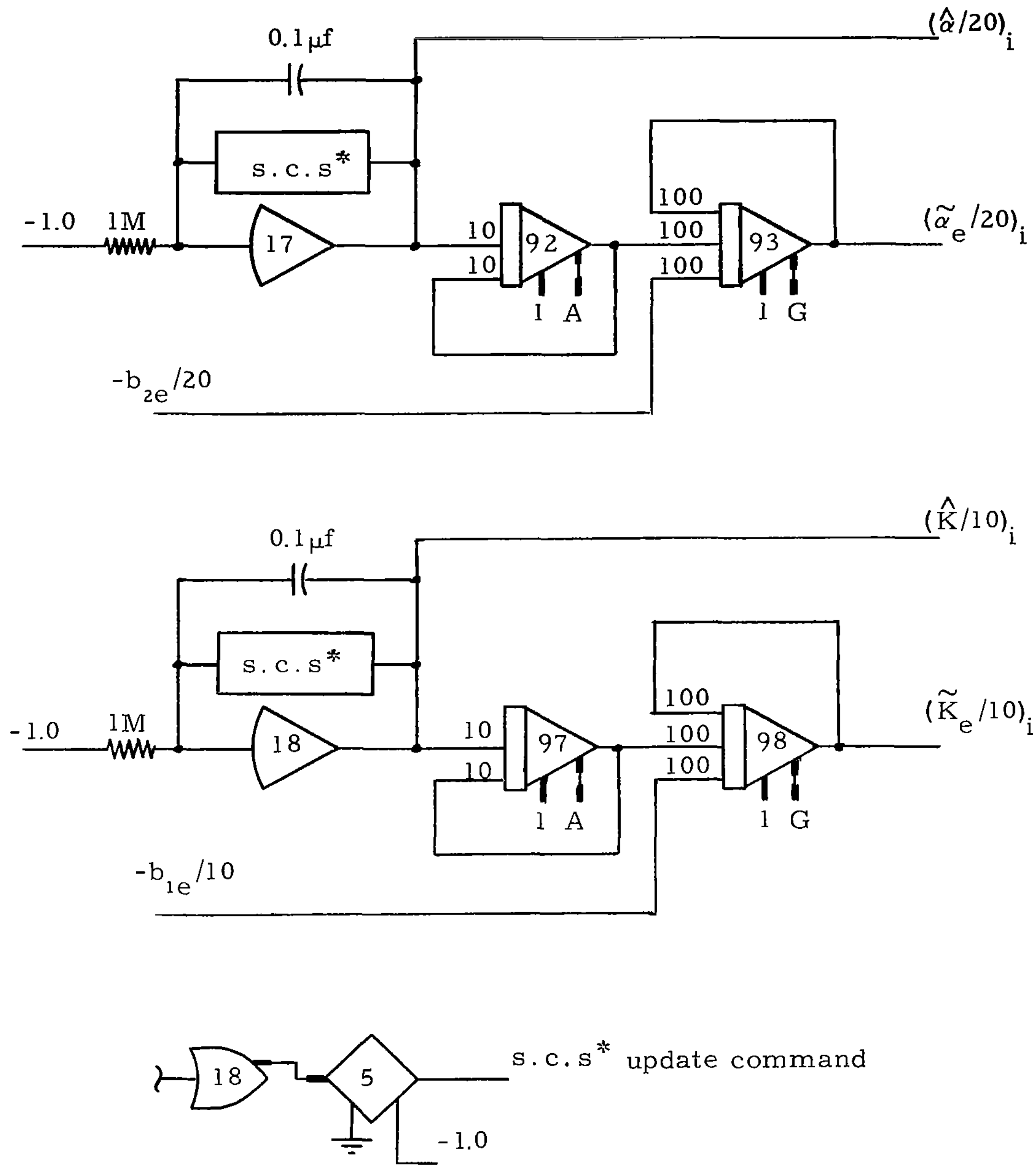
The circuitry for the iterative analysis is presented in Figs. A-9 and A-10. The model parameter values for each of the T-second intervals are preprogrammed using the "sequential coefficient selector" (s.c.s.) presented in Fig. A-10. As shown in Fig. A-9 the parameter values are obtained by using the s.c.s. as feedback resistors of summing amplifiers which have a constant input. The resistance corresponding to the parameter value for a given data interval is set prior to the analysis of data using the manual switches shown in Fig. A-10. Then during the calculation, the s.c.s. automatically steps to the proper resistance value at the beginning of each interval.

In the noniterative on-line analysis the model parameter values for each interval are obtained using the circuit shown in Fig. A-11. As indicated in this figure, the model parameter values for a given interval are set equal to the estimate from the preceding interval, i.e.,

$$\hat{K}_{i+1} = \tilde{K}_{ei}$$

In the noniterative analysis, two limits are imposed on the model parameter values. One limit, imposed due to the potentially large errors in the noniterative estimates of the human operator parameter values, assures that the model equations are stable. To guarantee that the model represented by Eq. 3.3-4 is stable, the parameter values are limited such that the damping ratio of the second order system is greater than 0.2. This limit requires that $\hat{\alpha} \geqq 1.5\hat{K}$. The second limitation is imposed to assure that $\hat{\alpha}$ does not exceed the valid range of the multipliers. It was found that a similar limit is not necessary for $\hat{K}$. The implementation of these two limits is seen in the comparator-switch combinations of Fig. A-11.

The logic control circuitry for the analog computational circuits is given in Fig. A-12 and typical logic time histories are presented in Fig. A-13.



*sequential coefficient selector

Figure A-9 Circuit for Obtaining Iterated Parameter Values

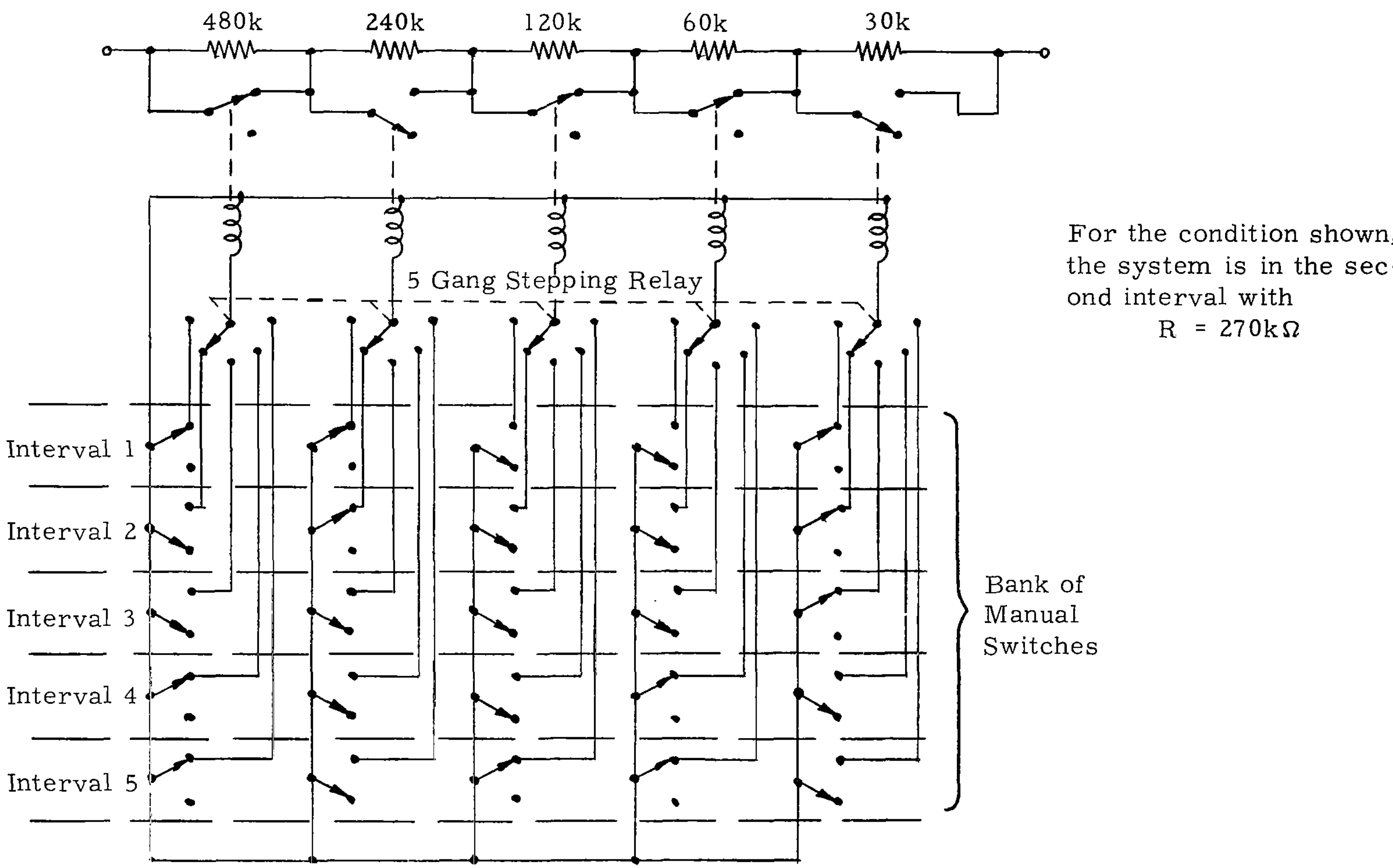


Figure A-10 Sequential Coefficient Selector

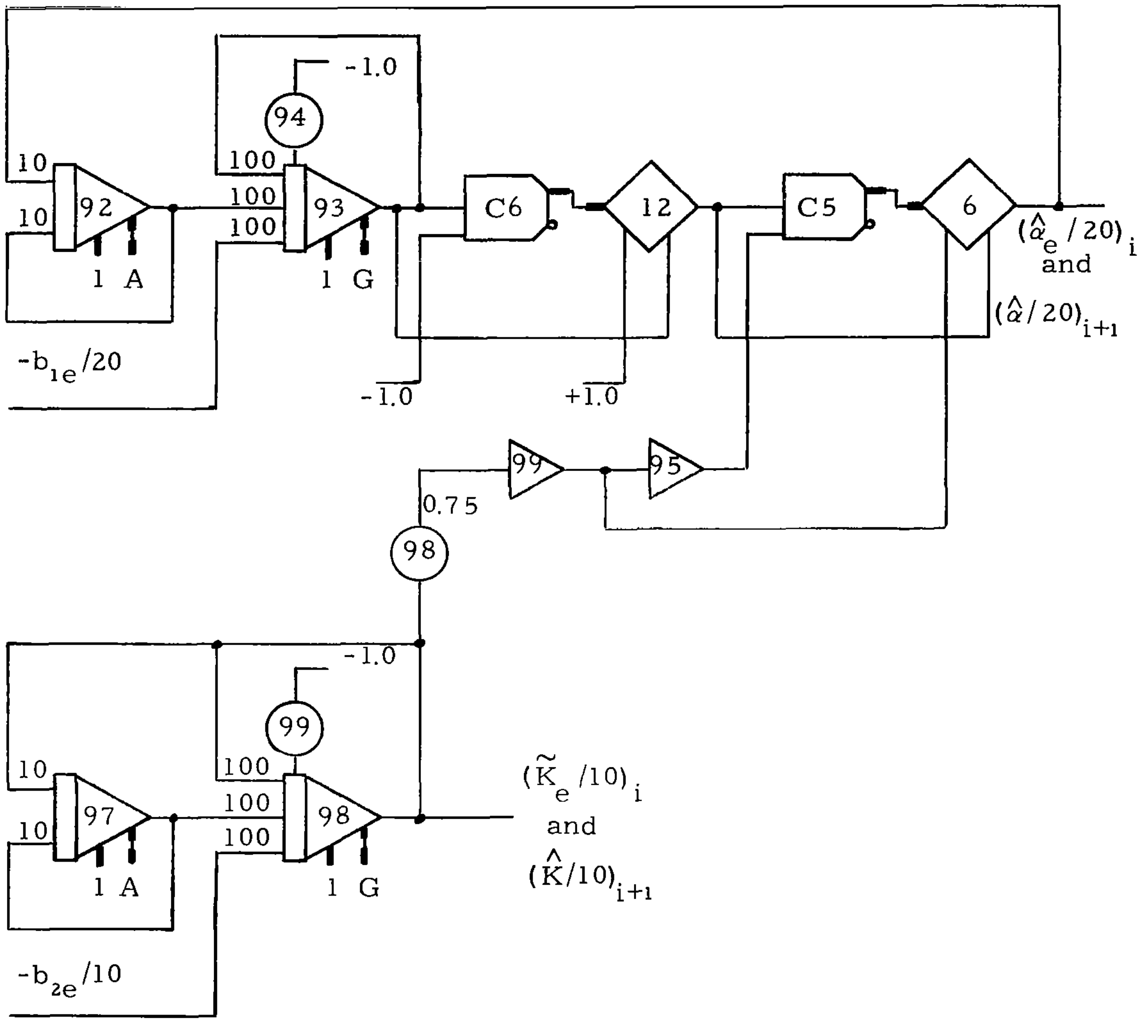


Figure A-11 Circuit for Obtaining Noniterated On-line Parameter Values

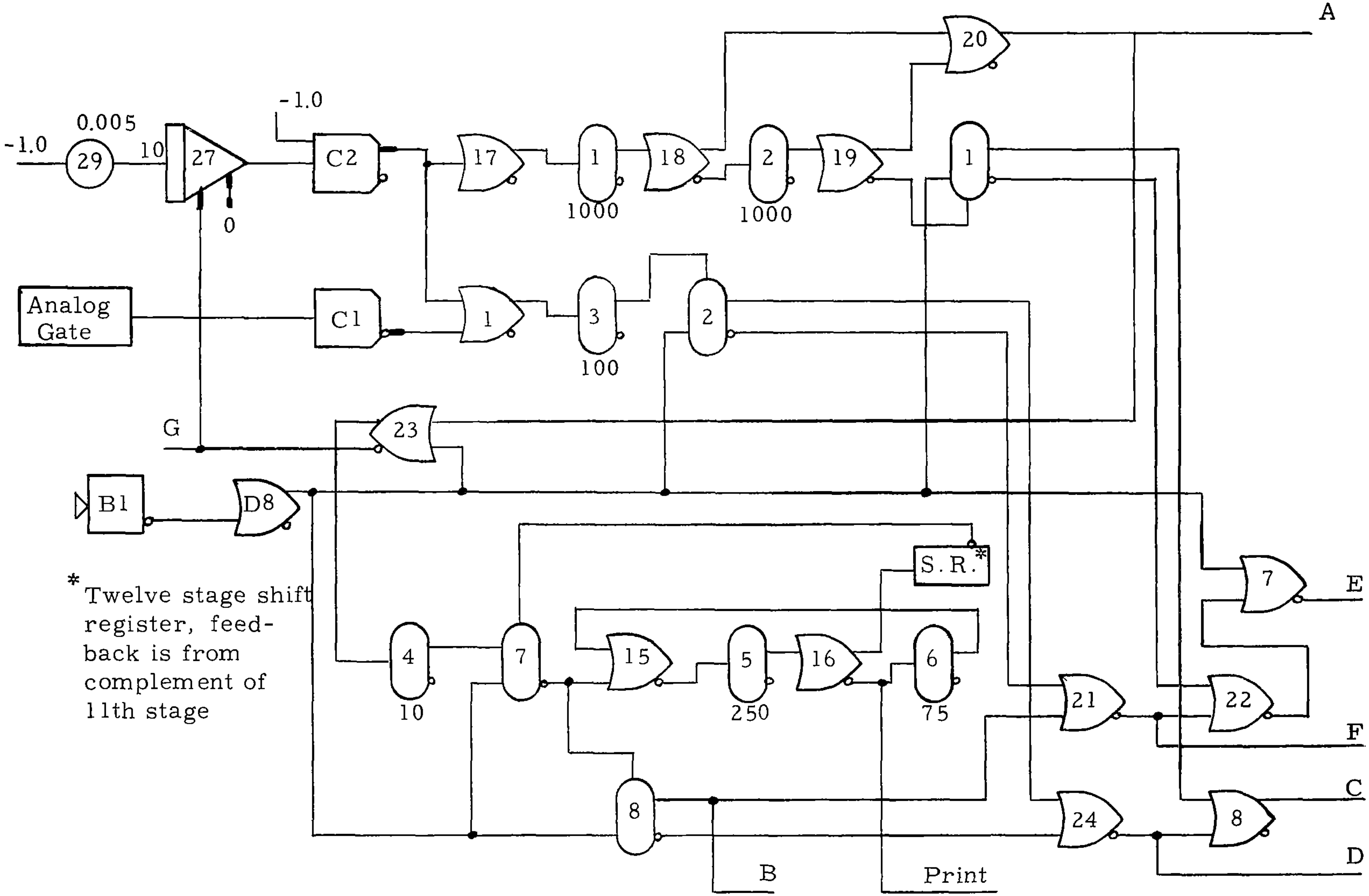


Figure A-12 Logic Circuitry

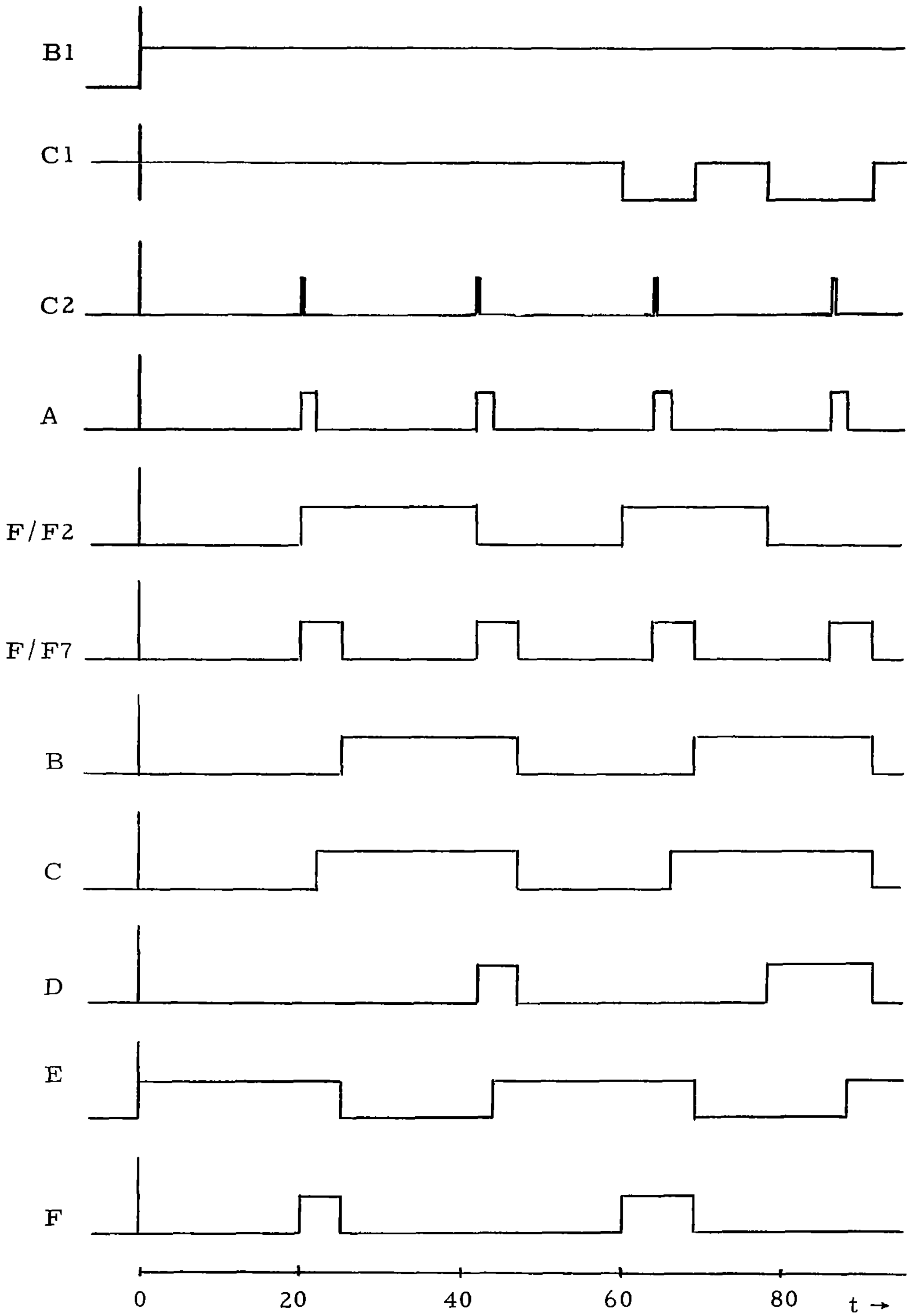


Figure A-13 Typical Logic Time Histories

The "analog gate" shown in Fig. A-14 is used to assure that none of the elements of R and $\underline{v}$ exceeds the valid range of the computer components. In operation the analog gate, when used with a comparator, produces a logic "1" output if the absolute value of any element of R or $\underline{v}$ exceeds 100 volts. As is seen in Fig. A-12, this logic signal is then used to place the integrators of all elements of R and $\underline{v}$ into Hold prior to the end of the given 20-second interval.

The circuitry for computing the power match, PM, discussed in Section 5.4 is presented in Fig. A-15.

$$PM = 1 - \frac{\int_0^{120} e^2 dt}{\int_0^{120} \theta^2 dt}$$

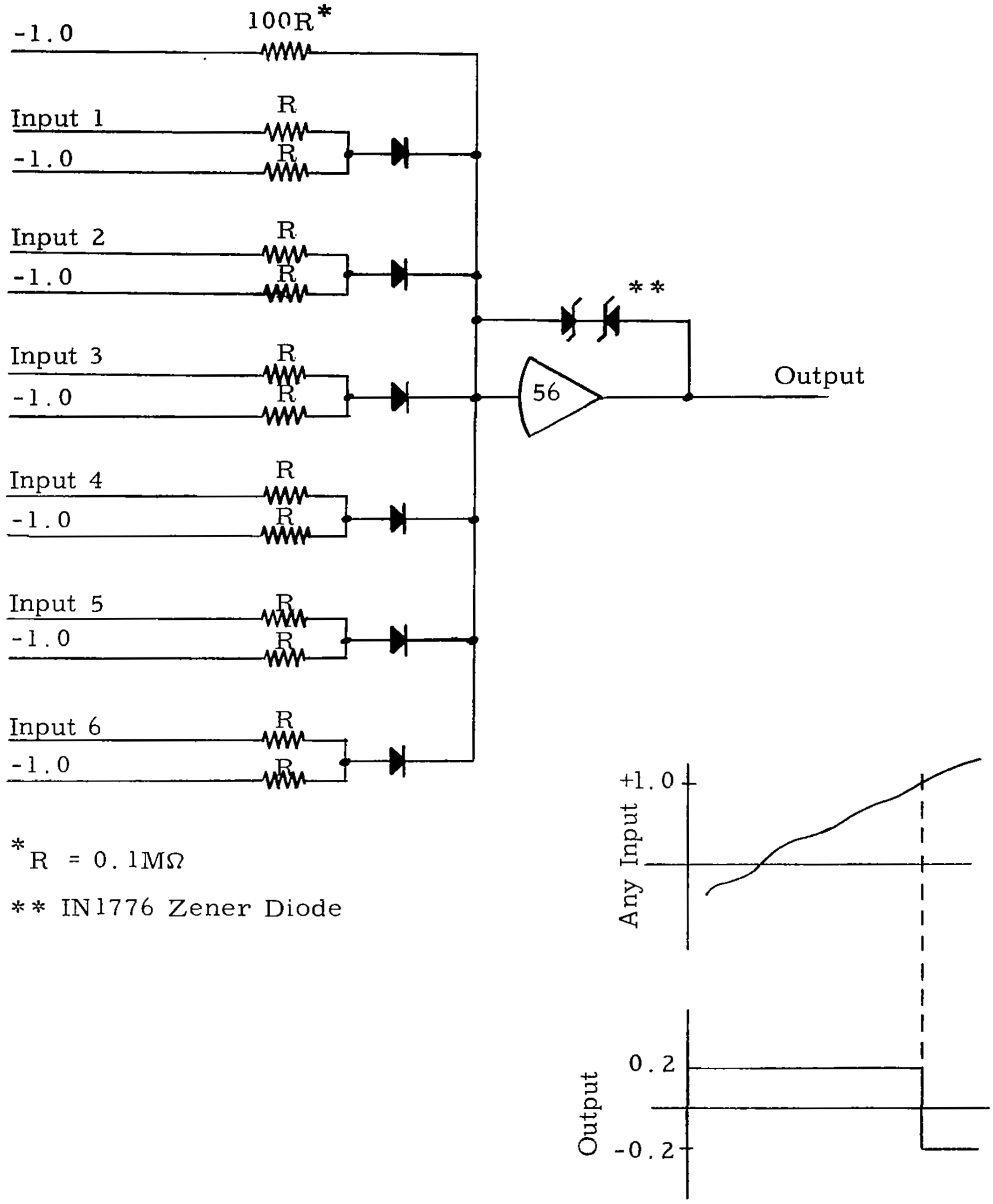


Figure A-14 Analog Gate

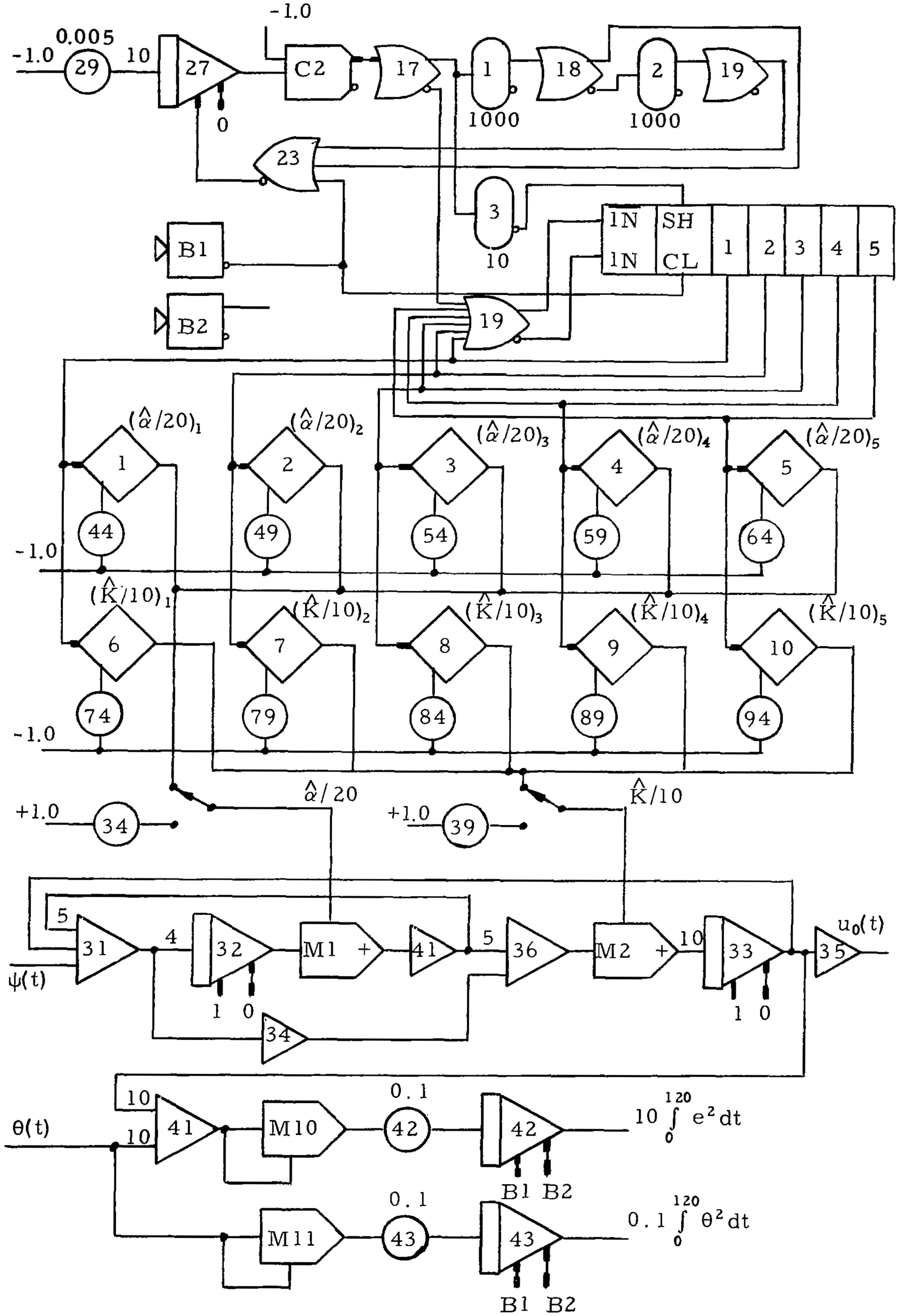


Figure A-15 Power Match Circuitry

APPENDIX B

ANALYSIS OF INFINITE DATA INTERVAL

Let us initially consider a general linear time-invariant system of the form shown in Fig. B-1. In this preliminary analysis, it is desired to obtain an expression for the covariance of the two signals, $y_1(t)$ and $y_2(t)$. If the two sections of the system in Fig. B-1 have weighting functions $g_1(t)$ and $g_2(t)$ respectively, then $y_1(t)$ and $y_2(t)$ can be expressed as convolutions.

$$y_1(t) = \int_{-\infty}^{\infty} g_1(\sigma)\psi(t-\sigma)\,d\sigma \qquad \text{B-1}$$

$$y_2(t) = \int_{-\infty}^{\infty} g_2(\sigma)\psi(t-\sigma)\,d\sigma \qquad \text{B-2}$$

The time-cross-correlation function, $\Gamma_{y_1y_2}(\nu)$, [6] for $y_1(t)$ and $y_2(t)$ is then given by:

$$\Gamma_{y_1y_2}(\nu) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} y_1(t)y_2(t-\nu)\,dt \qquad \text{B-3}$$

Similarly the statistical cross-correlation function for the two functions, $\phi_{y_1y_2}(\nu)$, is given by:

$$E[y_1(t)y_2(t-\nu)] = \phi_{y_1y_2}(\nu) \qquad \text{B-4}$$

If the system input is a stationary and ergodic random process, then

$$\Gamma y_1y_2(\nu) = \phi_{y_1y_2}(\nu) \qquad \text{B-5}$$

with probability one.

The expression for $\phi_{y_1y_2}(\nu)$ can be expanded as follows:

$$\phi_{y_1y_2}(\nu) = E\left\{\left[\int_{-\infty}^{\infty} g_1(\sigma)\psi(t-\sigma)\,d\sigma\right]\left[\int_{-\infty}^{\infty} g_2(\mu)\psi(t-\nu-\mu)\,d\mu\right]\right\} \qquad \text{B-6}$$

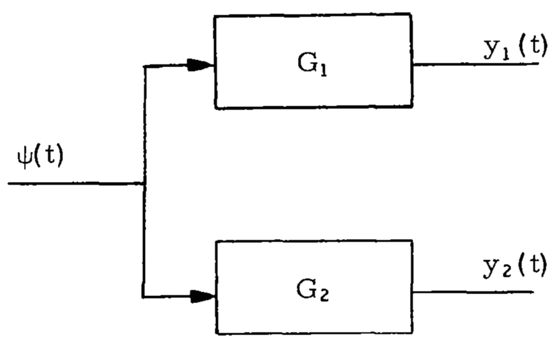


Figure B-1 Block Diagram for Linear Time-Invariant System

$$\phi_{y_1y_2}(\nu) = \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(\sigma) g_2(\mu)\ E[\psi(t-\sigma)\psi(t-\nu-\mu)] d\mu d\sigma \right\} \qquad \text{B-7}$$

$$\phi_{y_1y_2}(\nu) = \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(\sigma) g_2(\mu)\, \phi_{\psi}(\nu+\mu-\sigma)\, d\mu d\sigma \right\} \qquad \text{B-8}$$

The power spectral density $\Phi_{y_1y_2}(\omega)$ which is defined as the Fourier transform of the correlation function [6] is then:

$$\Phi_{y_1y_2}(\omega) = \int_{-\infty}^{\infty} \phi_{y_1y_2}(\nu)\, e^{-j\omega\nu}\, d\nu \qquad \text{B-9}$$

$$\Phi_{y_1y_2}(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(\sigma) g_2(\mu)\, \phi_{\psi}(\nu+\mu-\sigma)\, e^{-j\omega\nu} d\mu d\sigma d\nu \qquad \text{B-10}$$

Define a new variable, γ, as:

$$\gamma = \nu + \mu - \sigma \qquad \text{B-11}$$

then

$$e^{-j\omega\nu} = {}^{-j\omega(\gamma-\mu+\sigma)} = e^{-j\omega\gamma}\, e^{+j\omega\mu}\, e^{-j\omega\sigma} \qquad \text{B-12}$$

and

$$\Phi_{y_1y_2}(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_2(\sigma)\, e^{-j\omega\sigma}\, g_2(\mu)\, e^{+j\omega\mu} \phi_{\psi}(\gamma)\, e^{-j\omega\gamma} d\mu d\sigma d\gamma \qquad \text{B-13}$$

$$\Phi_{y_1y_2}(\omega) = G_1(j\omega)\, G_2^*(j\omega)\, \Phi_{\psi}(\omega) \qquad \text{B-14}$$

where $\phi_{\psi}(t)$ is the autocorrelation function for $\psi(t)$, and $\Phi_{\psi}(j\omega)$ is the corresponding autospectral density. $G_1(j\omega)$ is the transfer function corresponding to $y_1(t)$ and $G_2(j\omega)$ is the transfer function corresponding to $y_2(t)$.

The cross-correlation function, $\phi_{y_1y_2}(\nu)$, can now be expressed as the inverse Fourier transform of $\Phi_{y_1y_2}(j\omega)$.

$$\phi_{y_1y_2}(\nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{y_1y_2}(j\omega)\, e^{j\omega\nu} d\omega \qquad \text{B-15}$$

$$\phi_{y_1y_2}(\nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_1(j\omega)\, G_2^*(j\omega)\,\Phi_{\psi}(j\omega)\, e^{j\omega\nu}\, d\omega \qquad \text{B-16}$$

The covariance of $y_1(t)$ and $y_2(t)$ corresponds to $\phi_{y_1y_2}(0)$ which can be expressed as:

$$\phi_{y_1y_2}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{y_1y_2}(j\omega)\, d\omega \qquad \text{B-17}$$

Consider now rearranging Eq. B-17 to a form which is more amenable to numerical calculation. Note that,

$$\Phi_{y_1y_2}(j\omega) = \Phi^*_{y_2y_1}(j\omega) = \Phi^*_{y_1y_2}(-j\omega) \qquad \text{B-18}$$

then

$$\phi_{y_1y_2}(0) = \frac{1}{2\pi}\left\{\int_{-\infty}^{0}\Phi_{y_1y_2}(j\omega)\, d\omega + \int_{0}^{\infty}\Phi_{y_1y_2}(j\omega)\, d\omega\right\} \qquad \text{B-19}$$

$$\phi_{y_1y_2}(0) = \frac{1}{2\pi}\left\{\int_{0}^{\infty}\Phi^*_{y_1y_2}(j\omega)\, d\omega + \int_{0}^{\infty}\Phi_{y_1y_2}(j\omega)\, d\omega\right\} \qquad \text{B-20}$$

$$\phi_{y_1y_2}(0) = \frac{1}{2\pi}\int_{0}^{\infty}[\Phi_{y_1y_2}(j\omega) + \Phi^*_{y_1y_2}(j\omega)]\, d\omega \qquad \text{B-21}$$

Substituting Eq. B-14 into Eq. B-21 yields:

$$\phi_{y_1y_2}(0) = \frac{1}{2\pi}\int_{0}^{\infty} G_1(j\omega)\, G_2^*(j\omega)\,\Phi_{\psi}(j\omega) + G_1^*(j\omega)\, G_2(j\omega)\,\Phi^*_{\psi}(j\omega)]\, d\omega \qquad \text{B-22}$$

Since $\psi(t)$ is a real process.

$$\Phi_{\psi}(j\omega) = \Phi^*_{\psi}(j\omega) \qquad \text{B-23}$$

and

$$\phi_{y_1 y_2}(0) = \frac{1}{2\pi} \int_0^\infty [G_1(j\omega) G_2^*(j\omega) + G_1^*(j\omega) G_2(j\omega)] \Phi_\psi(j\omega)\, d\omega \qquad \text{B-24}$$

$$\phi_{y_1 y_2}(0) = \frac{1}{\pi} \int_0^\infty \mathrm{Re}[G_1^*(j\omega) G_2(j\omega)] \Phi_\psi(j\omega)\, d\omega \qquad \text{B-25}$$

From Eq. B-5 it is seen that the time covariance, i.e., $\Gamma_{y_1 y_2}(0)$, can be written as:

$$\Gamma_{y_1 y_2}(0) = \frac{1}{\pi} \int_0^\infty \mathrm{Re}[G_1^*(j\omega) G_2(j\omega)] \Phi_\psi(j\omega)\, d\omega \qquad \text{B-26}$$

Consider now the application of Eq. B-26 to the analysis of the crossover model. The system that is to be studied is given in Fig. B-2. In Chapter 3, it was shown that the best estimate of ΔK and $\Delta\alpha$ is given by:

$$\underline{b}_e = R^{-1} \underline{v} \qquad \text{B-27}$$

where,

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \qquad \text{B-28}$$

and

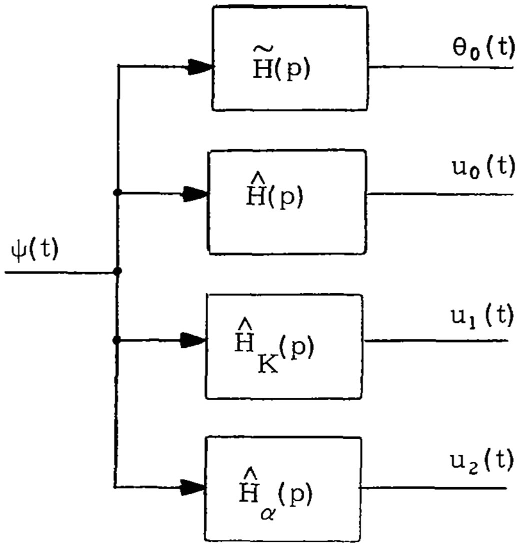
$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \text{B-29}$$

The elements of R and $\underline{v}$ have the form:

$$r_{ij} = \frac{1}{T} \int_{-T/2}^{T/2} [u_i(t) u_j(t)] dt \qquad i = 1, 2 \quad j = 1, 2 \qquad \text{B-30}$$

and

$$v_i = \frac{1}{T} \int_{-T/2}^{T/2} u_i(t)\, e(t)\, dt \qquad i = 1, 2 \qquad \text{B-31}$$



$$\tilde{H}(p) = - \frac{\tilde{K}(p - \tilde{\alpha})}{p^2 + (\tilde{\alpha} - \tilde{k})p + \tilde{\alpha}\tilde{k}}$$

$$\hat{H}(p) = - \frac{\hat{K}(p - \hat{\alpha})}{p^2 + (\hat{\alpha} - \hat{K})p + \hat{\alpha}\hat{K}}$$

$$\hat{H}_K(p) = \left.\frac{\partial H}{\partial \alpha}\right|_{\wedge} = \left\{ - \frac{p^3 + (\hat{\alpha})^2 p}{[p^2 + (\hat{\alpha} - \hat{K})p + \hat{\alpha}\hat{K}]^2} \right\}$$

$$\hat{H}_\alpha(p) = \left.\frac{\partial H}{\partial \alpha}\right|_{\wedge} = \left\{ \frac{2\hat{K}p^2}{[p^2 + (\hat{\alpha} - \hat{K})p + \hat{\alpha}\hat{K}]^2} \right\}$$

Figure B-2 Block Diagram for Crossover Model Analysis where

$$e(t) = \theta(t) - u_0(t) \qquad \text{B-32}$$

Expressions for the limiting case of $T \to \infty$ can be obtained by applying Ex. B-26 to Eqs. B-30 and B-31. The expressions for the limiting case are then:

$$\lim_{T\to\infty} r_{11} = \frac{1}{\pi}\int_0^\infty |\hat{H}_K(j\omega)|^2 \Phi_\psi(j\omega)\,d\omega \qquad \text{B-33}$$

$$\lim_{T\to\infty} r_{12} = \frac{1}{\pi}\int_0^\infty \mathrm{Re}[\hat{H}_K^*(j\omega)\hat{H}_\alpha(j\omega)]\Phi_\psi(j\omega)\,d\omega \qquad \text{B-34}$$

$$\lim_{T\to\infty} r_{22} = \frac{1}{\pi}\int_0^\infty |\hat{H}_\alpha(j\omega)|^2 \Phi_\psi(j\omega)\,d\omega \qquad \text{B-35}$$

$$\lim_{T\to\infty} v_1 = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\{[\hat{H}_K^*(j\omega)][\tilde{H}(j\omega) - \hat{H}(j\omega)]\}\Phi_\psi(j\omega)\,d\omega \qquad \text{B-36}$$

$$\lim_{T\to\infty} v_2 = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\{[\hat{H}_\alpha^*(j\omega)][\tilde{H}(j\omega) - \hat{H}(j\omega)]\}\Phi_\psi(j\omega)\,d\omega \qquad \text{B-37}$$

The value of $\underline{b}_e$ for the limiting case of $T \to \infty$ is denoted by $\underline{b}_\infty$. Thus,

$$\underline{b}_\infty = \lim_{T\to\infty} [R^{-1}\underline{v}] \qquad \text{B-38}$$

To obtain the value of $\underline{b}_\infty$ it is necessary to evaluate the integrals of Eqs. B-33 through B-37. To numerically perform these integrations it is necessary to obtain algebraic expressions for each of the integrands.

The expressions for the integrands are obtained by direct substitution of the expressions that are given as part of Fig. B-2 into Eqs. B-33 through B-37. To simplify the necessary algebraic manipulation, the following auxillary variables are defined.

$P1$ = Input Filter Cut-off Frequency (rad/sec)

$P2 = \tilde{K}$

$P3 = \tilde{\alpha}$

$P4 = \hat{K}$

$P5 = \hat{\alpha}$

$AN1 = [(P5)^2 + \omega^2]\omega$

$AN2 = -2(P4)\,\omega^2$

$AN3 = (P2)(P3)$

$AN4 = -(P2)\omega$

$AN5 = (P4)(P5)$

$AN6 = -(P4)\omega$

$AN7 = [(AN3) - \omega^2]$

$AN8 = [(P3) - (P2)]\omega$

$AN9 = [(AN5) - \omega^2]$

$AN10 = [(P5) - (P4)]\omega$

$D1 = [(AN7)^2 + (AN8)^2]$

$D2 = [(AN9)^2 + (AN10)^2]$

After considerable algebraic manipulation, the expressions for the integrands are found to be:

$$A1 = |\hat{H}_K(j\omega)|^2 = \frac{1}{(D2)^2}[\omega^3 + (P5)^2\omega]^2 \qquad \text{B-39}$$

$$\mathrm{Re}[\hat{H}_K^*(j\omega)\hat{H}_\alpha(j\omega)] \equiv 0 \qquad \text{B-40}$$

$$A2 = |\hat{H}_\alpha(j\omega)|^2 = \frac{1}{(D2)^2}[4(P4)^2\omega^4] \qquad \text{B-41}$$

$$A3 = \mathrm{Re}\{[\hat{H}_K^*(j\omega)][\tilde{H}(j\omega) - \hat{H}(j\omega)]\}$$

$$= \frac{1}{(D2)^2}\Big\{(AN1)\Big[\frac{1}{(D1)}\{[(AN7)(AN4) - (AN8)(AN3)]$$

$$\times [(AN9)^2 - (AN10)^2] + [(AN3)(AN7) + (AN4)(AN8)]$$

$$\times [2(AN9)(AN10)]\} - \{(AN9)(AN6) + (AN10)(AN5)\}\Big]\Big\} \qquad \text{B-42}$$

$$A4 = \mathrm{Re}\{[\hat{H}^*_\alpha(j\omega)][\tilde{H}(j\omega) - \hat{H}(j\omega)]\}$$

$$= \frac{1}{(D2)^2}\left\{(AN2)\left[\frac{1}{(D1)}\{[(AN3)(AN7) + (AN4)(AN8)][(AN9)^2 - (AN10)^2] - [(AN4)(AN7) - (AN3)(AN8)][2(AN9)(AN10)]\} - \{(AN5)(AN9) - (AN6)(AN10)\}\right]\right\} . \quad \text{B-43}$$

The input used in the numerical calculations was a white noise signal which was passed through a linear filter. The filter used was a critically damped second order filter with a cut-off frequency of 2 rad/sec. Thus $\Phi_\psi(j\omega)$ has the form,

$$A5 = \Phi_\psi(j\omega) = \frac{4}{\omega^4 + 8\omega^2 + 16} \quad \text{B-44}$$

or

$$A5 = \frac{1}{[1 + (\frac{\omega}{P1})^2]^2} \quad \text{B-45}$$

where the input cut-off frequency is fixed at:

$$P1 = 2 \text{ rad/sec}$$

A Fortran II program was written to perform the necessary numerical calculations discussed in this appendix. The computer used to implement the program was a SDS-940 digital computer. A copy of the program is included here.

```
-QED
VERSION 4-12-68
*APPEND /AUGALL/

*P
      DIMENSION P(5),AN(10),D(2),R(4),U(2),S(30),IS(5),A(2)
       DIF=.000001
       N5=0
       OPEN (2,INPUT,/DATA/)
10     READ 2,1,(P(I)-I=1,5)
       TYPE 4, (P(I),I=1,5)
       IR=0
12       TEMV=0.0
       IDONE = 0
       IR=IR+1
       EL=0.0
       ER=19.0
       NORD=7
       IS(5)=1
      PI=3.1415927
20     CALL ROMBERG(EL,ER,NORD,W,V,S,IS)
       IF (IS(5)) 25,40,25
25       AN(1)=(P(5)**2 + W**2)*W
      AN(2)=-2*P(4)*W**2
      AN(3)=P(2)*P(3)
      AN(4)=-P(2)*W
      AN(5)=P(4)*P(5)
      AN(6)=-P(4)*W
      AN(7)=AN(3)-W**2
      AN(8)=(P(3)-P(2))*W
      AN(9)=AN(5)-W**2
      AN(10)=(P(5)-P(4))*W
      D(1)=AN(7)**2 + AN(8)**2
      D(2)=AN(9)**2 + AN(10)**2
      DTWSQ=1/(D(2)**2)
      HKSQ=DTWSQ*((W**3 + W*P(5)**2)**2)
      HASQ=DTWSQ*(4*W**4*P(4)**2)
      Q=AN(4)*AN(7)-AN(3)*AN(8)
      E=AN(3)*AN(7) + AN(4)*AN(8)
      F=AN(9)**2-AN(10)**2
      G=2*AN(9)*AN(10)
      H=1/D(1)

      REHK=DTWSQ*AN(1)*(H*(Q*F+E*G)-(AN(9)*AN(6)+AN(10)*AN(5)))
       REHA=DTWSQ*AN(2)*(H*(E*F-Q*G)-(AN(5)*AN(9)-AN(6)*AN(10)))
      SX=1/((1+(W/P(1))**2)**2)
       IF (IR-1) 27,26,27
26     V=HKSQ*SX
       GO TO 20
27     IF (IR-2) 29,28,29
28     V=HASQ*SX
       GO TO 20
29     IF (IR-3) 31,30,31
30     V=REHK*SX
       GO TO 20
31     V=REHA*SX
       GO TO 20
40     TEMV=TEMV + V
       IF (IDONE-0) 44,42,44
42     EL=ER
       ER=ER+1
       IS(5)=1
       IDONE=1
       GO TO 20
44     IF (ABSF(V)-DIF) 35,35. 42
35     TYPE 5,TEMV
       R(IR)=TEMV
49       IF (IR-4) 12,50,12
50     A(1)=R(3)/R(1)
       A(2)=R(4)/R(2)
       TYPE 3, A(1),A(2)
       N5=N5 + 1
       IF (N5-5) 10,52,52
52       CLOSE (2)
1     FORMAT (5F6.2)
2     FORMAT (F13.8)
4      FORMAT (8H INPUT  ,5F7.2)
5      FORMAT (/F13.8//)
3      FORMAT (8H A(1) = ,F11.6,4X,8H A(2) = ,F11.6//)
       STOP
       END
*WRITE /AUGALL/
OLD FILE?
528 WORDS.
```

APPENDIX C

EFFECT OF FINITE DATA INTERVAL

Let us initially consider a general linear time-invariant system such as is discussed in Section 2.6. The description of the system contains a parameter vector, $\underline{c}$, of dimension L. The system with corresponding model can be represented as in Fig. C-1. In this figure, $\tilde{\underline{c}}$ is the value of the parameter vector for the system and $\hat{\underline{c}}$ is the value of the parameter vector for the model.

It is seen in Section 2.6 that under appropriate conditions on $\tilde{\underline{c}}$ and $\hat{\underline{c}}$ that the estimate of the difference between $\tilde{\underline{c}}$ and $\hat{\underline{c}}$ obtained by regression analysis, $\underline{b}_e$, is defined by:

$$\underline{b}_e = R^{-1}\underline{v} \tag{C-1}$$

The matrix R is defined by:

$$R = \frac{1}{T}\int_{-T/2}^{T/2} [\underline{u}(t)\,\underline{u}^{\#}(t)]\,dt \tag{C-2}$$

where,

$$\underline{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_L(t) \end{bmatrix} \tag{C-3}$$

and the vector $\underline{v}$ is defined by:

$$\underline{v} = \frac{1}{T}\int_{-T/2}^{T/2} [\underline{u}(t)\,e(t)]\,dt \tag{C-4}$$

Let us now define a new variable, $\underline{z}(t)$, by:

$$\underline{z}(t) = R^{-1}\underline{u}(t) \tag{C-5}$$

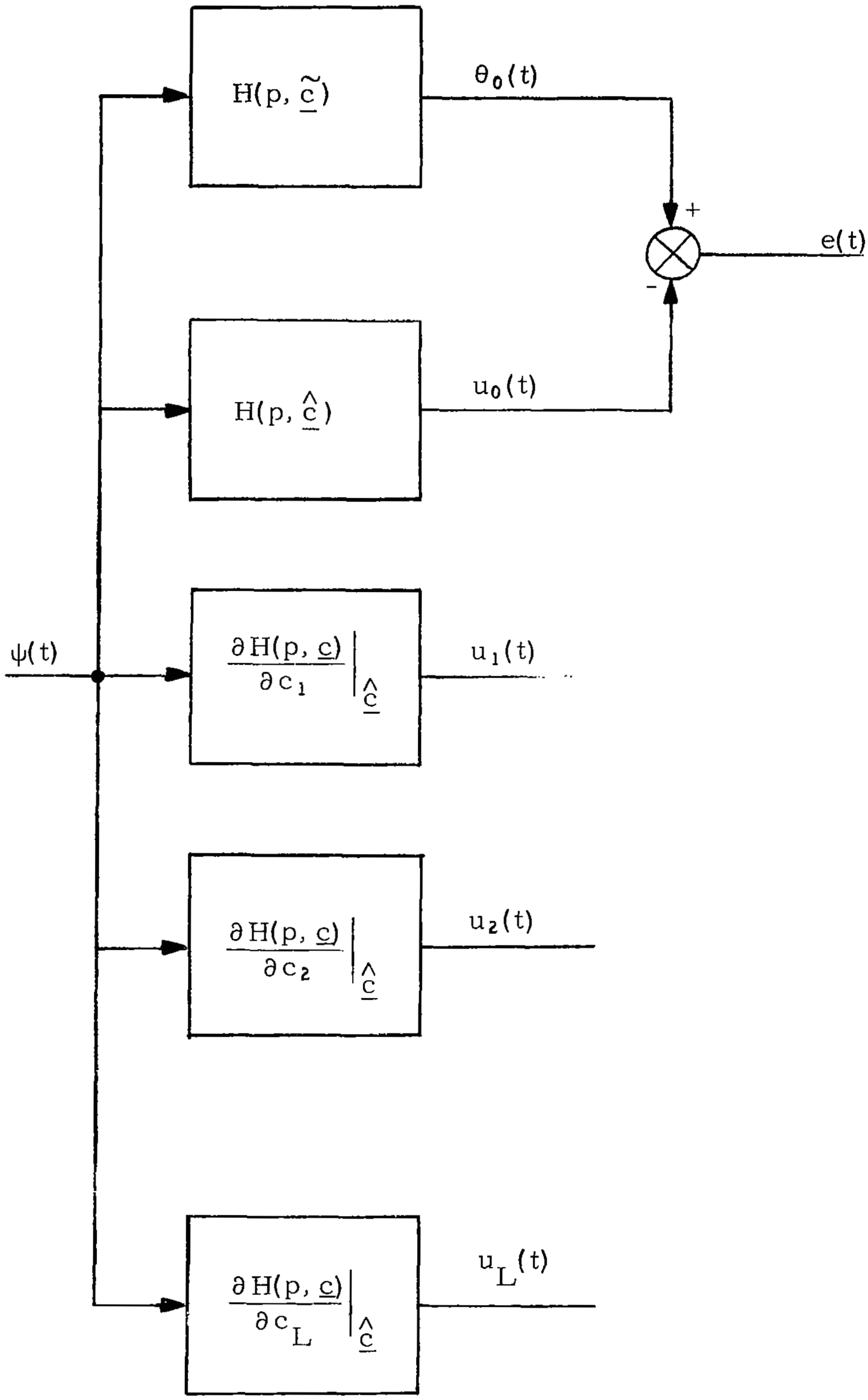


Figure C-1 Block Diagram for Linear System with Corresponding Model

Then,

$$\underline{b}_e = \frac{1}{T} \int_{-T/2}^{T/2} [\underline{z}(t)\, e(t)]\, dt \qquad \text{C-6}$$

Let us also at this point define the value of $\underline{b}_e$ for the case of an infinite interval of data. As in Appendix B this is defined by:

$$\underline{b}_\infty = \lim_{T \to \infty} \underline{b}_e \qquad \text{C-7}$$

Then for the case where $\psi(t)$ is a stationary ergodic random process,

$$\underline{b}_\infty = \phi_{\underline{z}e}(0) \qquad \text{C-8}$$

Consider now another expression for the value of $\underline{b}_e$. Eq. C-6 can be rewritten in the form of a convolution as follows:

$$\underline{b}_e = \int_{-\infty}^{\infty} [\underline{z}(t-\sigma)\, e(t-\sigma)\, w(\sigma)]\, d\sigma \qquad \text{C-9}$$

where,

$$w(\sigma) = \begin{cases} \frac{1}{T} & -\frac{T}{2} \leqq \sigma \leqq \frac{T}{2} \\ 0 & |\sigma| > \frac{T}{2} \end{cases} \qquad \text{C-10}$$

The expected value of $\underline{b}_e$ is then obtained as follows:

$$E[\underline{b}_e] = E\left\{ \int_{-\infty}^{\infty} [\underline{z}(t-\sigma)\, e(t-\sigma)\, w(\sigma)]\, d\sigma \right\}$$

$$= \int_{-\infty}^{\infty} E[\underline{z}(t-\sigma)\, e(t-\sigma)]\, w(\sigma)\, d\sigma$$

$$= \int_{-\infty}^{\infty} \phi_{\underline{z}e}(0)\, w(\sigma)\, d\sigma$$

$$= \phi_{\underline{z}e}(0) \int_{-\infty}^{\infty} w(\sigma)\, d\sigma$$

$$E[\underline{b}_e] = \phi_{\underline{z}e}(0) \qquad \text{C-11}$$

Consider now the effect on the value of $\underline{b}_e$ of a finite interval of data, i.e., $T < \infty$. Gilbert [14] has shown that this effect can be considered as a perturbation from the value of $\underline{b}_\infty$ given by Eq. C-7, then,

$$\underline{b}_e = \underline{b}_\infty + \underline{\delta} \qquad \text{C-12}$$

where $\underline{\delta}$ represents the effect of the finite data interval. Note that since

$$E[\underline{b}_e] = E[\underline{b}_\infty + \delta] = \underline{b}_\infty \qquad \text{C-13}$$

it follows that,

$$E[\underline{\delta}] = 0 \qquad \text{C-14}$$

Consider now the covariance matrix for $\underline{b}_e$, which from Eq. C-12 is given by:

$$E[\underline{b}_e \underline{b}_e^{\#}] = E[\underline{b}_\infty \underline{b}_\infty^{\#} + \underline{b}_\infty \underline{\delta}^{\#} + \underline{\delta}\,\underline{b}_\infty^{\#} + \underline{\delta}\,\underline{\delta}^{\#}] \qquad \text{C-15}$$

and since $\underline{b}_\infty$ is a constant

$$E[\underline{b}_e \underline{b}_e^{\#}] = \underline{b}_\infty \underline{b}_\infty^{\#} + E[\underline{\delta}\,\underline{\delta}^{\#}] \qquad \text{C-16}$$

Eq. C-16 can be rewritten in terms of the covariance matrix for $\underline{\delta}$ as:

$$E[\underline{\delta}\,\underline{\delta}^{\#}] = E[\underline{b}_e \underline{b}_e^{\#}] - \underline{b}_\infty \underline{b}_\infty^{\#} \qquad \text{C-17}$$

If the convolution expression for $\underline{b}_e$ is substituted in Eq. C-17, then

$$E[\underline{\delta}\,\underline{\delta}^{\#}] = E\left\{ \int_{-\infty}^{\infty} [\underline{z}(t-\sigma_1)\, e(t-\sigma_1)\, w(\sigma_1)\, d\sigma_1 \int_{-\infty}^{\infty} [\underline{z}^{\#}(t-\sigma_2)\, e(t-\sigma_2)\, w(\sigma_2)\, d\sigma_2 \right\} - \underline{b}_\infty \underline{b}_\infty^{\#}$$

C-18

Expanding Eq. C-18 and interchanging the integration and expectation operations yields:

$$E[\underline{\delta}\,\underline{\delta}^{\#}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[\underline{z}(t-\sigma_1)\,\underline{z}^{\#}(t-\sigma_2)\, e(t-\sigma_1)\, e(t-\sigma_2)]\, w(\sigma_1)\, w(\sigma_2)\, d\sigma_1 d\sigma_2$$

$$- \underline{b}_\infty \underline{b}_\infty^{\#} \qquad \text{C-19}$$

At this point it is necessary to make one more assumption, namely that e(t) and all of the $z_i(t)$ are jointly gaussian. Under this assumption, a typical element of the matrix $E[\underline{z}(t-\sigma_1)\underline{z}^{\#}(t-\sigma_2)e(t-\sigma_1)e(t-\sigma_2)]$ can be written as

$$\begin{aligned} E[z_i(t-\sigma_1)z_j(t-\sigma_2)e(t-\sigma_1)e(t-\sigma_2)] &= E[z_i(t-\sigma_1)z_j(t-\sigma_2)]E[e(t-\sigma_1)e(t-\sigma_2)] \\ &+ E[z_i(t-\sigma_1)e(t-\sigma_1)]E[z_j(t-\sigma_2)e(t-\sigma_2)] \\ &+ E[z_i(t-\sigma_1)e(t-\sigma_2)]E[z_j(t-\sigma_2)e(t-\sigma_1)] \\ & i = 1, 2, \ldots, L; \\ & j = 1, 2, \ldots, L \end{aligned} \qquad \text{C-20}$$

or, in terms of correlation functions,

$$\begin{aligned} E[z_i(t-\sigma_1)z_j(t-\sigma_2)e(t-\sigma_1)e(t-\sigma_2)] &= \phi_{z_i z_j}(\beta)\phi_e(\beta) + \phi_{z_i e}(0)\phi_{z_j e}(0) \\ &+ \phi_{z_i e}(\beta)\phi_{z_j e}(-\beta) \end{aligned} \qquad \text{C-21}$$

where,

$$\beta = \sigma_1 - \sigma_2$$

Now the matrix, $E[\underline{z}(t-\sigma_1)\underline{z}^{\#}(t-\sigma_2)e(t-\sigma_1)e(t-\sigma_2)]$, can be written as

$$\begin{aligned} E[\underline{z}(t-\sigma_1)\underline{z}^{\#}(t-\sigma_2)e(t-\sigma_1)e(t-\sigma_2)] &= E[\underline{z}(t-\sigma_1)\underline{z}^{\#}(t-\sigma_1)]E[e(t-\sigma_1)e(t-\sigma_2)] \\ &+ E[\underline{z}(t-\sigma_1)e(t-\sigma_1)]E[\underline{z}^{\#}(t-\sigma_2)e(t-\sigma_2)] \\ &+ E[\underline{z}(t-\sigma_1)e(t-\sigma_2)]E[\underline{z}^{\#}(t-\sigma_2)e(t-\sigma_1)] \end{aligned}$$

C-22

If it is noted that,

$$\phi_{z_j e}(-\beta) = \phi_{e z_j}(\beta)$$

then Eq. C-22 can be written in terms of correlation function as,

$$\begin{aligned} E[\underline{z}(t-\sigma_1)\underline{z}^{\#}(t-\sigma_2)e(t-\sigma_1)e(t-\sigma_2)] &= \phi_{\underline{z}\,\underline{z}^{\#}}(\beta)\phi_e(\beta) + \phi_{\underline{z}e}(0)\phi_{\underline{z}^{\#}e}(0) \\ &+ \phi_{\underline{z}e}(\beta)\phi_{e\underline{z}^{\#}}(\beta) \end{aligned} \qquad \text{C-23}$$

Note now that,

$$\underline{b}_{-\infty}\underline{b}^{\#}_{-\infty} = \phi_{\underline{z}e}(0)\,\phi_{\underline{z}^{\#}e}(0) \qquad \text{C-24}$$

The covariance matrix for $\underline{\delta}$ can now be written as:

$$E[\underline{\delta}\,\underline{\delta}^{\#}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\phi_{\underline{z}\,\underline{z}^{\#}}(\beta)\,\phi_{e}(\beta) + \phi_{\underline{z}\,e}(\beta)\,\phi_{e\underline{z}^{\#}}(\beta)]\,w(\sigma_1)\,w(\sigma_2)\,d\sigma_1 d\sigma_2 \qquad \text{C-25}$$

where it has been noted that

$$\beta = \sigma_1 - \sigma_2$$

or

$$\sigma_2 = \sigma_1 - \beta$$

For convenience, let,

$$\Omega(\beta) = [\phi_{\underline{z}\,\underline{z}^{\#}}(\beta)\,\phi_{e}(\beta) + \phi_{\underline{z}\,e}(\beta)\phi_{e\underline{z}^{\#}}(\beta)] \qquad \text{C-26}$$

Then,

$$E[\underline{\delta}\,\underline{\delta}^{\#}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Omega(\beta)\,w(\sigma_1)\,w(\sigma_1 - \beta)\,d\sigma_1\,d\beta \qquad \text{C-27}$$

or,

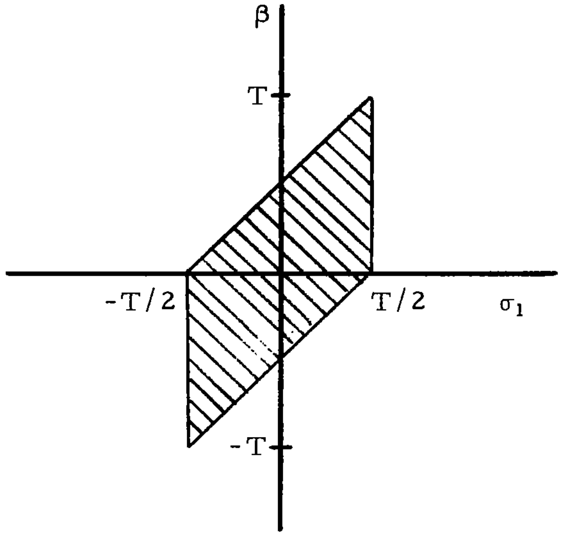
$$E[\underline{\delta}\,\underline{\delta}^{\#}] = \int_{-\infty}^{\infty}\Omega(\beta)\int_{-\infty}^{\infty}w(\sigma_1)\,w(\sigma_1 - \beta)\,d\sigma_1\,d\beta \qquad \text{C-28}$$

Consider now the evaluation of $\int_{-\infty}^{\infty} w(\sigma_1)\,w(\sigma_1 - \beta)\,d\sigma_1$. This evaluation is aided by the sketch presented in Fig. C-2. The value of $w(\sigma)$ for this figure is from the definition of $w(\sigma)$ given in Eq. C-10. From Fig. C-2 it is seen for the case of $\beta = 0$, that,

$$\int_{-\infty}^{\infty} w(\sigma_1)\,w(\sigma_1)\,d\sigma_1 = \int_{-T/2}^{T/2}\left(\frac{1}{T^2}\right)d\sigma_1 = \frac{1}{T} \qquad \text{C-29}$$

Also from Fig. C-2, it is seen that for any given value of β,

$$\int_{-\infty}^{\infty} w(\sigma_1)\,w(\sigma_1 - \beta)\,d\sigma_1 \leqq \int_{-\infty}^{\infty} w(\sigma_1)\,w(\sigma_1)\,d\sigma_1$$



$$w(\sigma_1)\,w(\sigma_1 - \beta) = \begin{cases} \frac{1}{T^2} & \text{Shaded area} \\ 0 & \text{Elsewhere} \end{cases}$$

Figure C-2 Graphical Description of $w(\sigma_1)\,w(\sigma_1 - \beta)$

Therefore the following inequality is satisfied,

$$\int_{-\infty}^{\infty} w(\sigma_1)\, w(\sigma_1 - \beta)\, d\sigma_1 \leqq \frac{1}{T} \qquad \text{C-30}$$

From this point on let us consider only the diagonal terms of $E[\underline{\delta}\,\underline{\delta}^{\#}]$, i.e., the variance of the individual elements of $\underline{\delta}$. Define the variance of δ_i by:

$$E[\delta_i^2] = \sigma_{\delta_i}^2 \qquad \text{C-31}$$

Also denote the diagonal terms of the matrix $\Omega(\beta)$ by:

$$\Omega_i(\beta) = [\phi_{z_i z_i}(\beta)\phi_e(\beta) + \phi_{z_i e}(\beta)\phi_{e z_i}(\beta)] \qquad \text{C-32}$$

Then,

$$\sigma^2_{\delta_i} = \int_{-\infty}^{\infty} \Omega_i(\beta) \int_{-\infty}^{\infty} w(\sigma_1)\, w(\sigma_1 - \beta)\, d\sigma_1\, d\beta \qquad \text{C-33}$$

From Eq. C-30,

$$\sigma^2_{\delta_i} \leqq \frac{1}{T} \int_{-\infty}^{\infty} \Omega_i(\beta)\, d\beta \qquad \text{C-34}$$

For the remainder of the discussion of this appendix, let us consider the crossover model for the human operator. The general system given in Fig. C-1 is then replaced by the specific system which is presented in Fig. B-2. From Fig. B-2 and other discussion it is seen that L = 2 for the crossover model.

Consider now obtaining expressions for $\Omega_1(\beta)$ and $\Omega_2(\beta)$. From Eq. C-32 the expression for $\Omega_1(\beta)$ is:

$$\Omega_1(\beta) = \phi_{z_1 z_1}(\beta)\, \phi_e(\beta) + \phi_{z_1 e}(\beta)\, \phi_{e z_1}(\beta) \qquad \text{C-35}$$

Since the processes under discussion here are assumed to be stationary and ergodic, the expressions of Eq. C-35 can be written as,

$$\phi_{z_1 e}(\beta) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} z_1(t - \beta e(t)\, dt \qquad \text{C-36}$$

The necessary expressions for $z_1(t)$ and $z_2(t)$ are obtained from the definition given in Eq. C-5. From this definition,

$$\underline{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \left\{ \begin{matrix} \dfrac{\left[\frac{1}{T}\int_{-T/2}^{T/2} (u_2)^2 dt\right] u_1(t) - \left[\frac{1}{T}\int_{-T/2}^{T/2} u_1 u_2 dt\right] u_2(t)}{\left[\frac{1}{T}\int_{-T/2}^{T/2} (u_1)^2 dt\right]\left[\frac{1}{T}\int_{-T/2}^{T/2} (u_2)^2 dt\right] - \left[\frac{1}{T}\int_{-T/2}^{T/2} u_1 u_2 dt\right]^2} \\ \dfrac{\left[\frac{1}{T}\int_{-T/2}^{T/2} (u_1)^2\right] u_2(t) - \left[\frac{1}{T}\int_{-T/2}^{T/2} u_1 u_2 dt\right] u_1(t)}{\left[\frac{1}{T}\int_{-T/2}^{T/2} (u_1)^2 dt\right]\left[\frac{1}{T}\int_{-T/2}^{T/2} (u_2)^2 dt\right] - \left[\frac{1}{T}\int_{-T/2}^{T/2} u_1 u_2 dt\right]^2} \end{matrix} \right\} \qquad \text{C-37}$$

Substituting the expressions of Eq. C-37 into Eqs C-35 and C-36 and rearranging terms yields:

$$\Omega_1(\beta) = \left[\frac{1}{\phi^2_{u_1}(0)\phi^2_{u_2}(0) - 2\phi_{u_1}(0)\phi_{u_2}(0)\phi^2_{u_1u_2}(0) + \phi^4_{u_1u_2}(0)} \right]$$

$$\times \Big\{ \phi^2_{u_2}(0)\phi_{u_1}(\beta)\phi_e(\beta) - \phi_{u_2}(0)\phi_{u_1u_2}(0)[\phi_{u_1u_2}(\beta) + \phi_{u_2u_1}(\beta)]\phi_e(\beta)$$

$$+ \phi^2_{u_1u_2}(0)\phi_{u_2}(\beta)\phi_e(\beta) + \phi^2_{u_2}(0)\phi_{u_1e}(\beta)\phi_{eu_1}(\beta)$$

$$- \phi_{u_2}(0)\phi_{u_1u_2}(0)[\phi_{u_1e}(\beta)\phi_{eu_2}(\beta) + \phi_{u_2e}(\beta)\phi_{eu_1}(\beta)]$$

$$+ \phi^2_{u_1u_2}(0)\phi_{u_2e}(\beta)\phi_{eu_2}(\beta)\Big\} \qquad \text{C-38}$$

An expression for $\Omega_2(\beta)$ can be obtained by similar manipulations.

$$\Omega_2(\beta) = \left[\frac{1}{\phi^2_{u_1}(0)\phi^2_{u_2}(0) - 2\phi_{u_1}(0)\phi_{u_2}(0)\phi^2_{u_1u_2}(0) + \phi^4_{u_1u_2}(0)} \right]$$

$$\times \Big\{ \phi^2_{u_1u_2}(0)\phi_{u_1}(\beta)\phi_e(\beta) - \phi_{u_1}(0)\phi_{u_1u_2}(0)[\phi_{u_1u_2}(\beta) + \phi_{u_2u_1}(\beta)]\phi_e(\beta)$$

$$+ \phi^2_{u_1}(0)\phi_{u_2}(\beta)\phi_e(\beta) + \phi^2_{u_1u_2}(0)\phi_{u_1e}(\beta)\phi_{eu_1}(\beta)$$

$$- \phi_{u_1}(0)\phi_{u_1u_2}(0)[\phi_{u_1e}(\beta)\phi_{eu_2}(\beta) + \phi_{u_2e}(\beta)\phi_{eu_1}(\beta)]$$

$$+ \phi^2_{u_1}(0)\phi_{u_2e}(\beta)\phi_{eu_2}(\beta)\Big\} \qquad \text{C-39}$$

In Appendix B it is shown that,

$$\phi_{u_1 u_2}(0) = \lim_{T \to \infty} r_{12} \equiv 0 \tag{C-40}$$

This fact can be used very beneficially in simplifying Eqs. C-38 and C-39. When Eq. C-40 is substituted into these equations, the expressions for $\Omega_1(\beta)$ and $\Omega_2(\beta)$ become:

$$\Omega_1(\beta) = \frac{1}{\phi^2_{u_1}(0)} [\phi_{u_1}(\beta)\phi_e(\beta) + \phi_{u_1 e}(\beta)\phi_{eu_1}(\beta)] \tag{C-41}$$

$$\Omega_2(\beta) = \frac{1}{\phi^2_{u_2}(0)} [\phi_{u_2}(\beta)\phi_e(\beta) + \phi_{u_2 e}(\beta)\phi_{eu_2}(\beta)] \tag{C-42}$$

Combining Eqs. C-34, C-41 and C-42 yields:

$$\sigma^2_{\delta_1} \leqq \frac{1}{T} \int_{-\infty}^{\infty} \left\{ \frac{1}{\phi^2_{u_1}(0)} [\phi_{u_1}(\beta)\phi_e(\beta) + \phi_{u_1 e}(\beta)\phi_{eu_1}(\beta)] \right\} d\beta \tag{C-43}$$

$$\sigma^2_{\delta_2} \leqq \frac{1}{T} \int_{-\infty}^{\infty} \left\{ \frac{1}{\phi^2_{u_2}(0)} [\phi_{u_2}(\beta)\phi_e(\beta) + \phi_{u_2 e}(\beta)\phi_{eu_2}(\beta)] \right\} d\beta \tag{C-44}$$

To obtain a value for the upper bounds given by Eqs. C-43 and C-44, the integration is more easily carried out in the frequency domain than in the time domain. Thus if Plancherel's theorem is applied to the infinite integrals of Eqs. C-43 and C-44, these equations become:

$$\sigma^2_{\delta_1} \leqq \frac{1}{T} \left\{ \frac{\frac{1}{2\pi} \int_{-\infty}^{\infty} [\Phi^*_{u_1}(j\omega)\Phi_e(j\omega) + \Phi^2_{eu_1}(j\omega)] d\omega}{[\frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{u_1}(j\omega)\, d\omega]^2} \right\} \tag{C-45}$$

$$\sigma^2_{\delta_2} \leqq \frac{1}{T} \left\{ \frac{\frac{1}{2\pi} \int_{-\infty}^{\infty} [\Phi^*_{u_2}(j\omega)\Phi_e(j\omega) + \Phi^2_{eu_2}(j\omega)] d\omega}{[\frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{u_2}(j\omega)\, d\omega]^2} \right\} \tag{C-46}$$

After sufficient algebraic manipulation the integrals of Eqs. C-45 can be put in the following forms.

$$\int_{-\infty}^{\infty} \Phi^*_{u_1}(j\omega)\, \Phi_e(j\omega)\, d\omega = 2\int_0^{\infty} \Phi_{u_1}(j\omega)\, \Phi_e(j\omega)\, d\omega \qquad \text{C-47}$$

$$\int_{-\infty}^{\infty} \Phi^2_{eu_1}(j\omega)\, d\omega = 2\int_0^{\infty} \mathrm{Re}[\Phi^2_{eu_1}(j\omega)]\, d\omega \qquad \text{C-48}$$

$$\int_{-\infty}^{\infty} \Phi_{u_1}(j\omega\, d\omega = 2\int_0^{\infty} [\Phi_{u_1}(j\omega)]\, d\omega \qquad \text{C-49}$$

Completely analogous expressions can be obtained for the integrals of Eq. C-46. Substituting Eqs. C-47 through C-49 and the corresponding integrals for Eq. C-46 yields:

$$\sigma^2_{\delta_1} \leqq \frac{\pi}{T}\left\{\frac{\int_0^{\infty}\{\Phi_{u_1}(j\omega)\,\Phi_e(j\omega) + \mathrm{Re}[\Phi^2_{eu_1}(j\omega)]\}\,d\omega}{[\int_0^{\infty}\Phi_{u_1}(j\omega)\,d\omega]^2}\right\} \qquad \text{C-50}$$

$$\sigma^2_{\delta_2} \leqq \frac{\pi}{T}\left\{\frac{\int_0^{\infty}\{\Phi_{u_2}(j\omega)\,\Phi_e(j\omega) + \mathrm{Re}[\Phi^2_{eu_2}(j\omega)]\}\,d\omega}{[\int_0^{\infty}\Phi_{u_2}(j\omega)\,d\omega]^2}\right\} \qquad \text{C-51}$$

For the case of the crossover model, the integrands of Eq. C-50 can be expanded using the transfer operators given in Fig. B-2.

$$\Phi u_1(j\omega) = |\hat{H}_K(j\omega)|^2 \Phi_\psi(j\omega) \qquad \text{C-52}$$

$$\Phi_e(j\omega) = [\tilde{H}(j\omega) - \hat{H}(j\omega)][\tilde{H}(j\omega) - \hat{H}(j\omega)]^* \Phi_\psi(j\omega)$$

$$= \{|\tilde{H}(j\omega)|^2 + |\hat{H}(j\omega)|^2 - 2\mathrm{Re}[\tilde{H}(j\omega)\hat{H}(j\omega)]\}\Phi_\psi(j\omega) \qquad \text{C-53}$$

$$\mathrm{Re}[\Phi^2_{eu_1}(j\omega)] = \mathrm{Re}[\Phi_{eu_1}(j\omega)]^2 - \mathrm{Im}[\Phi_{eu_1}(j\omega)]^2 \qquad \text{C-54}$$

$$\Phi_{eu_1}(j\omega) = \{\hat{H}^*_K(j\omega)[\tilde{H}(j\omega) - \hat{H}(j\omega)]\}\Phi_\psi(j\omega) \qquad \text{C-55}$$

Again, completely analogous expressions can be obtained for the integrands of Eq. C-51. To complete the integrations of Eqs. C-50 and C-51 it is convenient to define several auxiliary variables. In addition to the variables P1 through P5, AN1 through AN10, D1, D2 and A1 through A5 defined in Appendix B, it is necessary to define the following variables.

$$(A6) = \mathrm{Im}\{\hat{H}_k^*(\tilde{H} - \hat{H})\}$$

$$= \frac{(AN1)}{(D2)^2}\left\{\frac{1}{(D1)}\left[((AN7)(AN4) - (AN8)(AN3))(2(AN9)(AN10)) - ((AN3)(AN7) + (AN4)(AN8))((AN9)^2 - (AN10)^2)\right] + \left[(AN5)(AN9) - (AN6)(AN10)\right]\right\} \qquad \text{C-56}$$

$$(A7) = \mathrm{Im}\{\hat{H}_\alpha^*(\tilde{H} - \hat{H})\}$$

$$= \frac{(N2)}{(D2)^2}\left\{\frac{1}{(D1)}\left[((AN7)(AN4) - (AN8)(AN3))((AN9)^2 - (AN10)^2) + ((AN3)(AN7) + (AN4)(AN8))(2(AN9)(AN10))\right] - \left[(AN9)(AN6) + (AN10)(AN5)\right]\right\} \qquad \text{C-57}$$

$$(A8) = \{(\tilde{H} - \hat{H})(\tilde{H} - \hat{H})^*\} = \{|\tilde{H}|^2 + |\hat{H}|^2 - 2\mathrm{Re}[\tilde{H}\hat{H}]\}$$

$$(A8) = \left\{\frac{1}{(D1)}\left[(AN3)^2 + (AN4)^2\right] + \frac{1}{(D2)}\left[(AN5)^2 + (AN6)^2\right] - \frac{2}{(D1)(D2)}\left[((AN3)(AN5) + (AN4)(AN6))((AN7)(AN9) + (AN8)(AN10)) - ((AN4)(AN5) - (AN6)(AN3))((AN7)(AN10) - (AN9)(AN8))\right]\right\} \qquad \text{C-58}$$

$$(I1) = \int_0^\infty \{(A1)(A8) + (A3)^2 - (A6)^2\}(A5)^2\, d\omega \qquad \text{C-59}$$

$$(I2) = \int_0^\infty \{(A2)(A8) + (A4)^2 - (A7)^2\}(A5)^2\, d\omega\ . \qquad \text{C-60}$$

The integrals of Eqs. C-59 and C-60 represent the numerator of Eqs. C-50 and C-51 respectively. The denominator integrals are obtained from Eqs. B-39, B-41 and B-45.

A modification to the computer program given in Appendix B was written to evaluate Eqs. C-50 and C-51 using the auxiliary variables presented here and in Appendix B. A copy of this modified program is included here.

```
      DIMENSION P(5),AN(10),D(2),U(2),S(30),IS(5)
       DIF=.0000001
       NORD=7
       N5=0
       OPEN (2,INPUT,/DATA/)
10     READ 2,1,(P(I),I=1,5)
       TYPE 4,(P(I),I=1,5)
       IR=0
12       TEMV=0.0
       IR=IR+1
       EL=0.0
       ER=19.0
       IDONE=0
       IS(5)=1
20     CALL ROMBERG(EL,ER,NORD,W,V,S,IS)
       IF (IS(5)) 25,30,25
25      AN(1)=(P(5)**2 + W**2)*W
      AN(2)=-2*P(4)*W**2
      AN(3)=P(2)*P(3)
      AN(4)=-P(2)*W
      AN(5)=P(4)*P(5)
      AN(6)=-P(4)*W
      AN(7)=AN(3)-W**2
      AN(8)=(P(3)-P(2))*W
      AN(9)=AN(5)-W**2
      AN(10)=(P(5)-P(4))*W
      D(1)=AN(7)**2 + AN(8)**2
      D(2)=AN(9)**2 + AN(10)**2
      DTWSQ=1/(D(2)**2)
      HKSQ=DTWSQ*((W**3 + W*P(5)**2)**2)
      HASQ=DTWSQ*(4*W**4*P(4)**2)
      Q=AN(4)*AN(7)-AN(3)*AN(8)
      E=AN(3)*AN(7) + AN(4)*AN(8)
      F=AN(9)**2-AN(10)**2
      G=2*AN(9)*AN(10)
      H=1/D(1)
       Y1=AN(5)*AN(9)-AN(6)*AN(10)
       Y2=AN(9)*AN(6)+AN(10)*AN(5)
       Y3=AN(3)**2+AN(4)**2
       Y4=AN(5)**2+AN(6)**2
       Y5=AN(5)*AN(3)+AN(4)*AN(6)
       Y6=AN(7)*AN(9)+AN(8)*AN(10)
       Y7=AN(4)*AN(5)-AN(6)*AN(3)
       Y8=AN(7)*AN(10)-AN(9)*AN(8)

      REHK=DTWSQ*AN(1)*(H*(Q*F+E*G)-Y2)
       REHA=DTWSQ*AN(2)*(H*(E*F-Q*G)-Y1)
      SX=100/((1+(W/P(1))**2)**2)
       AMHK=(AN(1)/(D(2)**2))*(H*(Q*G-E*F)+Y1)
       AMHA=(AN(2)/(D(2)**2))*(H*(Q*F+E*G)-Y2)
       HDIF=H*Y3+(1/D(2))*Y4-(2/(D(1)*D(2)))*(Y5*Y6-Y7*Y8)

       IF (IR-1) 27,26,27
26     V=(HKSQ*HDIF+REHK**2-AMHK**2)*SX**2
       GO TO 20
27     V=(HASQ*HDIF+REHA**2-AMHA**2)*SX**2
       GO TO 20
30     TEMV=TEMV + V
       IF(IDONE-0) 44,42,44
42     EL=ER
       ER=ER+1
       IS(5)=1
       IDONE=1
       GO TO 20
44     IF (ABSF(V)-DIF) 35,35,42
35       TYPE 5, TEMV
45     IF (IR-2) 12,50,12
50     N5=N5 + 1
       IF (N5-6) 10,52,52
52     CLOSE (2)
5      FORMAT (/F14.8//)
6      FORMAT (F14.8/)
1     FORMAT (5F6.2)
2     FORMAT (F13.10)
4      FORMAT (8H INPUT  ,5F7.2)
       STOP
       END
*
```

APPENDIX D

COMPONENTS OF VARIANCE

Let us consider a population A which is defined by the density function p(x) of the random variable X. In addition, let the population A be composed of N mutually exclusive subpopulations $A_1, A_2, \ldots, A_N$ which have individual density functions $p_1(x), p_2(x), \ldots, p_N(x)$. It can be shown [48] that:

$$p(x) = \sum_{i=1}^{N} \eta_i p_i(x) \qquad \text{D-1}$$

where,

$$\eta_i = \text{Prob}[X \subset A_i] \qquad \text{D-2}$$

Now let μ and σ^2 represent the mean and variance respectively of X for $X \subset A$. Also let μ_i and σ_i^2 represent the mean and variance for $X \subset A_i$. It is then desired to obtain expressions which relate these parameters of p(x) and the $p_i(x)$.

Consider first the expression for the mean, μ.

$$\mu = \int_{-\infty}^{\infty} x p(x)\, dx \qquad \text{D-3}$$

From Eq. D-1,

$$\mu = \int_{-\infty}^{\infty} x \sum_{i=1}^{N} \eta_i p_i(x)\, dx \qquad \text{D-4}$$

Interchanging the integration and summation operators yields:

$$\mu = \sum_{i=1}^{N} \eta_i \int_{-\infty}^{\infty} x\, p_i(x)\, dx \qquad \text{D-5}$$

Then,

$$\mu = \sum_{i=1}^{N} \eta_i \mu_i \qquad \text{D-6}$$

Now consider the expression for the variance, σ^2.

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 p(x)\, dx \qquad \text{D-7}$$

Again substituting the expression of Eq. D-1 and interchanging the integration and summation operators yields:

$$\sigma^2 = \sum_{i=1}^{N} \eta_i \int_{-\infty}^{\infty} (x - \mu)^2 p_i(x)\, dx \qquad \text{D-8}$$

$$\sigma^2 = \sum_{i=1}^{N} \eta_i \int_{-\infty}^{\infty} [(x - \mu_i) + (\mu_i - \mu)]^2 p_i(x)\, dx \qquad \text{D-9}$$

Noting that the cross-product terms in the squared expression vanish yields:

$$\sigma^2 = \sum_{i=1}^{N} \eta_i [\sigma_i^2 + (\mu_i - \mu)^2] \,. \qquad \text{D-10}$$

If it is assumed that:

$$\eta_1 = \eta_2 = \ldots = \eta_N = \frac{1}{N} \qquad \text{D-11}$$

then,

$$\mu = \frac{1}{N} \sum_{i=1}^{N} \mu_i \qquad \text{D-12}$$

and,

$$\sigma^2 = \frac{1}{N} \sum_{i=1}^{N} \sigma^2 + \frac{1}{N} \sum_{i=1}^{N} (\mu_i - \mu)^2 \qquad \text{D-13}$$

From Eq. D-13 it is seen that σ^2 can be thought of as having two components. One component, σ_W^2, is due to the variance with the individual subpopulations and the second component, σ_B^2, is due to the difference in the means of the individual subpopulations. The two components of σ^2 are defined by:

$$\sigma_W^2 = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 \qquad \text{D-14}$$

$$\sigma_B^2 = \frac{1}{N} \sum_{i=1}^{N} (\mu - \mu_i)^2 \qquad \text{D-15}$$

Then,

$$\sigma^2 = \sigma_w^2 + \sigma_B^2 \tag{D-16}$$

Let us now consider the problem of obtaining unbiased estimates of the elements of Eq. D-16. It is assumed that M independent samples are taken from each of the subpopulations $A_1, A_2, \ldots, A_N$. Thus a total of $N \times M$ samples are taken from the population A. Denote the samples from subpopulation A_i by $x_{i1}, x_{i2}, \ldots, x_{iM}$.

As a preliminary to obtaining estimates of the elements of Eq. D-16, let us consider some properties of the general sample, x_{ij}. Rather than carry through the density function notation introduced earlier, let us express x_{ij} as follows:

$$x_{ij} = \alpha_1 x_{1j} + \alpha_2 x_{2j} + \cdots + \alpha_N x_{Nj} \tag{D-17}$$

where,

$$\alpha_i = \begin{cases} 1 \text{ with probability } \frac{1}{N} \\ 0 \text{ with probability } (1 - \frac{1}{N}) \end{cases}$$

The α_i form a set of mutually exclusive and exhaustive random variables which are independent of the x_{ij}. Thus.

$$\text{Prob}(\alpha_1 = 1 \text{ or } \alpha_2 = 1 \text{ or } \ldots \text{or } \alpha_N = 1) = \text{Prob}(\alpha_1 = 1) + \text{Prob}(\alpha_2 = 1) + \ldots + \text{Prob}(\alpha_N = 1) \tag{D-18}$$

and

$$\text{Prob}(\alpha_1 = 1 \text{ or } \alpha_2 = 1 \text{ or } \ldots \text{or } \alpha_N = 1) = 1 \tag{D-19}$$

Then

$$\begin{aligned} E[x_{ij}] &= E[x_{ij}/\alpha_1 = 1]\text{Prob}(\alpha_1 = 1) + E[x_{ij}/\alpha_2 = 1]\text{Prob}(\alpha_2 = 1) \\ &\quad + \cdots + E[x_{ij}/\alpha_N = 1]\text{Prob}(\alpha_N = 1) \\ &= \frac{1}{N}\mu_1 + \frac{1}{N}\mu_2 + \cdots + \frac{1}{N}\mu_N \end{aligned} \tag{D-20}$$

$$E[x_{ij}] = \frac{1}{N}\sum_{i=1}^{N}\mu_i \tag{D-21}$$

From Eq. D-12, it is seen that,

$$E[x_{ij}] = \mu \qquad \text{D-22}$$

Consider now $E[x_{ij}x_{\ell m}]$. Note that the subpopulations are assumed to be mutually exclusive and the samples within a subpopulation are assumed to be independent. Consider the following four conditions.

1) $i = \ell,\ j = m$

$$E[x_{ij}x_{\ell m}] = E[(x_{ij})^2] = \sigma_i^2 + \mu_i^2 \qquad \text{D-23}$$

2) $i = \ell,\ j \neq m$

$$E[x_{ij}x_{\ell m}] = \mu_i^2 \qquad \text{D-24}$$

3) $i \neq \ell,\ j = m$

$$E[x_{ij}x_{\ell m}] = \mu_i\,\mu_\ell \qquad \text{D-25}$$

4) $i \neq \ell,\ j \neq m$

$$E[x_{ij}x_{\ell m}] = \mu_i\,\mu_\ell \qquad \text{D-26}$$

Let us now consider obtaining an unbiased estimate of the total variance, σ^2. As a likely candidate for such an estimate consider MS_1, which is defined by

$$MS_1 = \frac{1}{MN - 1} \sum_{i=1}^{N} \sum_{j=1}^{M} (x_{ij} - \overline{G})^2 \qquad \text{D-27}$$

where

$$\overline{G} = \frac{1}{MN} \sum_{i=1}^{N} \sum_{j=1}^{M} x_{ij} \qquad \text{D-28}$$

or

$$\overline{G} = \frac{1}{N} \sum_{i=1}^{N} \overline{G}_i \qquad \text{D-29}$$

where

$$\overline{G}_i = \frac{1}{M} \sum_{j=1}^{M} x_{ij} \qquad i = 1, 2, \ldots, N \qquad \text{D-30}$$

Note that

$$E[\overline{G}] = \frac{1}{MN} \sum_{i=1}^{N} \sum_{j=1}^{M} E[x_{ij}] = \frac{1}{MN} \sum_{i=1}^{N} \sum_{j=1}^{M} \mu_i = \frac{M}{MN} \sum_{i=1}^{N} \mu_i \qquad \text{D-31}$$

$$E[\overline{G}] = \mu \qquad \text{D-32}$$

and

$$E[\overline{G}_i] = \frac{1}{M} \sum_{j=1}^{M} E[x_{ij}] \qquad i = 1, 2, \ldots, N \qquad \text{D-33}$$

$$= \frac{1}{M} \sum_{j=1}^{M} \mu_i$$

$$E[\overline{G}_i] = \mu_i \qquad \text{D-34}$$

Thus $\overline{G}$ is an unbiased estimate of μ and the $\overline{G}_i$ are unbiased estimates of the μ_i.

Let us now consider if MS_1 is an unbiased estimate of σ^2. First rewrite MS_1 as:

$$MS_1 = \frac{1}{MN - 1} \sum_{i=1}^{N} \sum_{j=1}^{M} [(x_{ij} - \mu) + (\mu - \overline{G})]^2 \qquad \text{D-35}$$

$$MS_1 = \frac{1}{MN - 1} \sum_{i=1}^{N} \sum_{j=1}^{M} (x_{ij} - \mu)^2 + \frac{2}{MN - 1} \sum_{i=1}^{N} \sum_{j=1}^{M} (x_{ij} - \mu)(\mu - \overline{G})$$

$$+ \frac{1}{MN - 1} \sum_{i=1}^{N} \sum_{j=1}^{M} (\mu - \overline{G})^2 \qquad \text{D-36}$$

Consider now the expected value of the individual terms of Eq. D-36. Denote these terms by A, B and C. Then,

$$E[A] = E\left\{ \frac{1}{MN - 1} \sum_{i=1}^{N} \sum_{j=1}^{M} (x_{ij} - \mu)^2 \right\} \qquad \text{D-37}$$

$$= \frac{1}{MN - 1} \sum_{i=1}^{N} \sum_{j=1}^{M} E[(x_{ij} - \mu)^2]$$

$$E[A] = \frac{1}{MN - 1} \sum_{i=1}^{N} \sum_{j=1}^{M} \left\{ E[x_{ij}^2] - 2\mu E[x_{ij}] + \mu^2 \right\} \qquad \text{D-38}$$

From Eqs. D-22 and D-23 it is seen that E[A] is:

$$E[A] = \frac{1}{MN-1} \sum_{i=1}^{N} \sum_{j=1}^{M} \left\{ (\sigma_i^2 + \mu_i^2) - 2\mu\mu_i + \mu^2 \right\} \qquad \text{D-39}$$

$$= \frac{M}{MN-1} \left\{ \sum_{i=1}^{N} \sigma_i^2 + \sum_{i=1}^{N} [\mu_i^2 - 2\mu\mu_i + \mu^2] \right\}$$

$$E[A] = \frac{M}{MN-1} \left\{ \sum_{i=1}^{N} \sigma_i^2 + \sum_{i=1}^{N} (\mu - \mu_i)^2 \right\} \qquad \text{D-40}$$

Consider now the E[B] which is defined by:

$$E[B] = \frac{2}{MN-1} \sum_{i=1}^{N} \sum_{j=1}^{N} E[x_{ij} - \mu)(\mu - \overline{G})] \qquad \text{D-41}$$

$$E[B] = \frac{2}{MN-1} \sum_{i=1}^{N} \sum_{j=1}^{M} \left\{ E[x_{ij}\mu] - E[\mu^2] + E[\mu\overline{G}] - E[x_{ij}\overline{G}] \right\} \qquad \text{D-42}$$

The first three terms inside the double summation are, respectively, $\mu_i\mu$, $-\mu^2$ and μ^2.

Now consider the fourth term within the summations.

$$E[x_{ij}\overline{G}] = E\left[x_{ij}\left(\frac{1}{N}\sum_{\ell=1}^{N} \overline{G}_\ell\right)\right] = \frac{1}{N}\sum_{\ell=1}^{N} E[x_{ij}\overline{G}_\ell] \qquad \text{D-43}$$

When $\ell = i$,

$$E[x_{ij}\overline{G}_\ell] = E[x_{ij}\overline{G}_i] = E\left[x_{ij}\left(\frac{1}{M}\sum_{m=1}^{M} x_{im}\right)\right]$$

$$= \frac{1}{M}\sum_{m=1}^{M} E[x_{ij}\, x_{im}]$$

From Eqs. D-23 and D-24 this becomes,

$$E[x_{ij}\overline{G}_i] = \frac{1}{M}\left\{\sigma_i^2 + \mu_i^2 + \sum_{m \neq j} \mu_i^2\right\} = \frac{1}{M}\sigma_i^2 + \mu_i^2 \qquad \text{D-44}$$

when $\ell \neq i$,

$$E[x_{ij}\overline{G}_\ell] = \mu_i\,\mu_\ell \qquad \text{D-45}$$

Substituting Eqs. D-44 and D-45 into Eq. D-43 yields:

$$E[x_{ij}\overline{G}_\ell] = \frac{1}{N}\left\{\frac{1}{M}\sigma_i^2 + \mu_i^2 + \sum_{\ell \neq i} \mu_i \mu_\ell\right\} \quad \text{D-46}$$

$$= \frac{1}{N}\left\{\frac{1}{M}\sigma_i^2 + \mu_i\left[\sum_{\ell=1}^{N} \mu_\ell\right]\right\}$$

$$= \frac{1}{NM}\sigma_i^2 + \mu_i\left[\frac{1}{N}\sum_{\ell=1}^{N} \mu_\ell\right]$$

$$E[x_{ij}\overline{G}] = \frac{1}{NM}\sigma_i^2 + \mu_i \mu \quad \text{D-47}$$

Eq. D-42 can now be expressed as:

$$E[B] = \frac{2}{MN-1}\sum_{i=1}^{N}\sum_{j=1}^{M}\left[\mu_i \mu - \mu^2 + \mu^2 - \frac{1}{NM}\sigma_i^2 - \mu_i \mu\right] \quad \text{D-48}$$

$$= \left(\frac{2}{MN-1}\right)\left(\frac{M}{MN}\right)\sum_{i=1}^{N}[-\sigma_i^2]$$

and,

$$E[B] = \frac{-2}{MN-1}\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^2\right) \quad \text{D-49}$$

Finally, E[C] is given by:

$$E[C] = \frac{1}{MN-1}\sum_{i=1}^{N}\sum_{j=1}^{M} E[(\mu - \overline{G})^2] \quad \text{D-50}$$

$$E[C] = \frac{1}{MN-1}\sum_{i=1}^{N}\sum_{j=1}^{M} E\left\{\left[\frac{1}{N}\sum_{\ell=1}^{N}(\mu_\ell - \overline{G}_\ell)\right]^2\right\} \quad \text{D-51}$$

The cross terms of the multiplication of Eq. D-51 have the form,

$$E[(\mu_\ell - \overline{G}_\ell)(\mu_{k\neq\ell} - \overline{G}_{k\neq\ell})]$$

$$= E[\mu_\ell \mu_k - \mu_\ell \overline{G}_k - \overline{G}_\ell \mu_k + \overline{G}_\ell \overline{G}_k]$$

$$= 0 \quad \text{D-52}$$

Thus Eq. D-51 can be rewritten as:

$$E[C] = \frac{1}{MN-1} \sum_{i=1}^{N} \sum_{j=1}^{M} \left\{ \frac{1}{N^2} \sum_{\ell=1}^{N} E[(\mu_\ell - \overline{G}_\ell)^2] \right\} \qquad \text{D-53}$$

The variance of the sample mean is known [11] to be related to the variance of the corresponding random variable by

$$E[(\overline{G}_\ell - \mu_\ell)^2] = \frac{1}{M}\sigma_\ell^2 \qquad \text{D-54}$$

Then Eq. D-53 can be rewritten as,

$$E[C] = \left(\frac{1}{MN-1}\right) \sum_{i=1}^{N} \sum_{j=1}^{M} \left[\frac{1}{N^2} \sum_{\ell=1}^{N} \left(\frac{1}{M} \sigma_\ell^2 \right) \right] \qquad \text{D-55}$$

$$E[C] = \left(\frac{1}{MN-1}\right) \left(\frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 \right) \qquad \text{D-56}$$

Combining Eqs. D-40, D-49 and D-56 yields:

$$E[MS_1] = \left\{ \frac{M}{MN-1} \sum_{i=1}^{N} \sigma_i^2 + \frac{M}{MN-1} \sum_{i=1}^{N} (\mu - \mu_i)^2 \right.$$

$$\left. - \frac{2}{N(MN-1)} \sum_{i=1}^{N} \sigma_i^2 + \frac{1}{N(MN-1)} \sum_{i=1}^{N} \sigma_i^2 \right\} \qquad \text{D-57}$$

$$E[MS_1] = \frac{MN-2+1}{MN-1} \left(\frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 \right) + \frac{M}{MN-1} \sum_{i=1}^{N} (\mu - \mu_i)^2 \qquad \text{D-58}$$

$$E[MS_1] = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 + \frac{1}{N - (\frac{1}{M})} \sum_{i=1}^{N} (\mu - \mu_i)^2 \qquad \text{D-59}$$

Inspection of Eqs. D-13 and D-59 shows that, in general, MS_1 is not an unbiased estimate of σ^2. Further inspection of these two equations shows that for two special cases, MS_1 is an unbiased estimate of σ^2. These cases are:

(1) $\mu_1 = \mu_2 = \ldots = \mu_N$

(2) limit as $M \to \infty$

Now consider modifying MS_1 so that an unbiased estimate of σ^2 is obtained. The most apparent modification is to subtract $\frac{1}{MN-1}$ times an unbiased estimate of $\frac{1}{N}\sum_{i=1}^{N}(\mu - \mu_i)^2$ from MS_1.

In an effort to obtain an unbiased estimate of $\frac{1}{N}\sum_i(\mu - \mu_i)^2$, which incidentally is σ_B^2 (Eq. D-16), let us consider MS_2 which is defined by

$$MS_2 = \frac{1}{N}\sum_{i=1}^{N}(\overline{G} - \overline{G}_i)^2 \qquad \text{D-60}$$

Eq. D-60 can be rewritten as:

$$MS_2 = \frac{1}{N}\sum_{i=1}^{N}[(\mu_i - \mu) + (\overline{G}_i - \mu_i) + (\mu - \overline{G})]^2$$

$$\begin{aligned} MS_2 = \frac{1}{N}\sum_{i=1}^{N} \{ & (\mu_i - \mu)^2 + (\overline{G}_i - \mu_i)^2 + (\mu - \overline{G})^2 \\ & + 2(\mu_i - \mu)(\overline{G}_i - \mu_i) + 2(\mu_i - \mu)(\mu - \overline{G}) \\ & + 2(\overline{G}_i - \mu_i)(\mu - \overline{G})\} \end{aligned} \qquad \text{D-61}$$

The $E[MS_2]$ can be determined from the expected value of the individual terms of Eq. D-61. Then,

$$E\left[\frac{1}{N}\sum_{i=1}^{N}(\mu_i - \mu)^2\right] = \frac{1}{N}\sum_{i=1}^{N}(\mu_i - \mu)^2 \qquad \text{D-62}$$

From Eq. D-54 it is seen that

$$E\left[\frac{1}{N}\sum_{i=1}^{N}(\overline{G}_i - \mu_i)^2\right] = \frac{1}{MN}\sum_{i=1}^{N}\sigma_i^2 \qquad \text{D-63}$$

From Eqs. D-51 through D-55, it is seen that

$$E\left[\frac{1}{N}\sum_{i=1}^{N}(\mu - \overline{G})^2\right] = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{1}{N^2M}\sum_{\ell=1}^{N}\sigma_\ell^2\right]$$

$$E\left[\frac{1}{N}\sum_{i=1}^{N}(\mu - \overline{G})^2\right] = \frac{1}{N^2M}\sum_{i=1}^{N}\sigma_i^2 \qquad \text{D-64}$$

The expression for the first cross-product term is

$$E\left[\frac{2}{N}\sum_{i=1}^{N}(\mu_i - \mu)(\overline{G}_i - \mu_i)\right] = \frac{2}{N}\sum_{i=1}^{N} E\left[\mu_i\overline{G}_i - \mu_i^2 - \mu_i\overline{G}_i + \mu_i\,\mu\right]$$

$$= \frac{2}{N}\sum_{i=1}^{N}\left[\mu_i^2 - \mu_i^2 - \mu\,\mu_i + \mu\,\mu_i\right] = 0 \qquad \text{D-65}$$

The expression for the second cross-product is:

$$E\left[\frac{2}{N}\sum_{i=1}^{N}(\mu_i - \mu)(\mu - \overline{G})\right] = \frac{2}{N}\sum_{i=1}^{N} E\left[\mu_i\mu - \mu_i\overline{G} - \mu^2 + \mu\overline{G}\right]$$

$$= \frac{2}{N}\sum_{i=1}^{N}\left[\mu_i\mu - \mu_i\mu - \mu^2 + \mu^2\right] = 0 \qquad \text{D-66}$$

The expression for the third cross-product term is:

$$E\left[\frac{2}{N}\sum_{i=1}^{N}(\overline{G}_i - \mu_i)(\mu - \overline{G})\right] = \frac{2}{N}\sum_{i=1}^{N} E\left[(\overline{G}_i - \mu_i)\left(\frac{1}{N}\sum_{\ell=1}^{N}(\mu_\ell - \overline{G}\ell)\right)\right]$$

$$= \frac{2}{N^2}\sum_{i=1}^{N}\sum_{\ell=1}^{N} E\left[(\overline{G}_i - \mu_i)(\mu_\ell - \overline{G}_\ell)\right] \qquad \text{D-67}$$

From Eq. D-54, for the case $\ell = i$,

$$E[(\overline{G}_i - \mu_i)(\mu_\ell - \overline{G}_\ell)] = -E[(G_i - \mu_i)^2]$$

$$= -\frac{1}{M}\sigma_i^2$$

For the case $\ell \neq i$,

$$E[(\overline{G}_i - \mu_i)(\mu_{\ell\neq i} - \overline{G}_{\ell\neq i})] = 0$$

Therefore, Eq. D-67 becomes

$$E\left[\frac{2}{N}\sum_{i=1}^{N}(\overline{G}_i - \mu_i)(\mu - \overline{G})\right] = -\frac{2}{MN^2}\sum_{i=1}^{N}\sigma_i^2 \qquad \text{D-68}$$

Combining Eqs. D-62 through D-68 yields

$$E[MS_2] = \frac{1}{N}\sum_{i=1}^{N}(\mu_i - \mu)^2 + \frac{1}{MN}\sum_{i=1}^{N}\sigma_i^2 + \frac{1}{MN^2}\sum_{i=1}^{N}\sigma_i^2 - \frac{2}{MN^2}\sum_{i=1}^{N}\sigma_i^2 \qquad \text{D-69}$$

$$E[MS_2] = \frac{1}{N}\sum_{i=1}^{N}(\mu - \mu_i)^2 + \frac{N-1}{MN}\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^2\right) \qquad \text{D-70}$$

From Eq. D-70 it is seen that MS_2 is not an unbiased estimate of σ_B^2. Again, the reasoning is that an unbiased estimate of σ_B^2 can be obtained by modifying MS_2.

To modify MS_2 consider obtaining an unbiased estimate of $\frac{1}{N}\sum_{i=1}^{N}\sigma_i^2$, which is also σ_W^2. As a candidate, consider MS_3 defined by

$$MS_3 = \frac{1}{N(M-1)} \sum_{i=1}^{N} \sum_{j=1}^{M} (x_{ij} - \overline{G}_i)^2 \qquad \text{D-71}$$

Then,

$$E[MS_3] = \frac{1}{N(M-1)} \sum_{i=1}^{N} \sum_{j=1}^{M} E[(x_{ij} - \mu_i) + (\mu_i - \overline{G}_i)]^2 \qquad \text{D-72}$$

$$= \frac{1}{N(M-1)} \sum_{i=1}^{N} \sum_{j=1}^{M} E[(x_{ij} - \mu_i)^2 + (\mu_i - \overline{G}_i)^2 + 2(x_{ij} - \mu_i)(\mu_i - \overline{G}_i)]$$

$$E[MS_3] = \frac{1}{N(M-1)} \sum_{i=1}^{N} \sum_{j=1}^{M} \{\sigma_i^2 + \frac{1}{M}\sigma_i^2 + 2E[(x_{ij} - \mu_i)(\mu_i - \overline{G}_i)]\}$$

The expected value inside the brackets is

$$E[(x_{ij} - \mu_i)(\mu_i - \overline{G}_i)] = E[x_{ij}\mu_i - \mu_i^2 + \mu_i\overline{G}_i - x_{ij}\overline{G}_i]$$

$$= \mu_i^2 - \mu_i^2 + \mu_i^2 - \frac{1}{M}\sum_{m=1}^{M} E(x_{ij}x_{im})$$

When $m = j$,

$$E[x_{ij}x_{im}] = \sigma_i^2 + \mu_i^2 \qquad \text{D-73}$$

When $m \neq j$,

$$E[x_{ij}x_{im}] = \mu_i^2 \qquad \text{D-74}$$

Then,

$$E[(\mu_i - \overline{G}_i)(x_{ij} - \mu_i)] = -\frac{1}{M}\sigma_i^2 \qquad \text{D-75}$$

and Eq. D-72 can be rewritten as

$$E[MS_3] = \frac{1}{N(M-1)} \sum_{i=1}^{N} \sum_{j=1}^{M} [\sigma_i^2 + \frac{1}{M}\sigma_i^2 - \frac{2}{M}\sigma_i^2]$$

$$= \frac{M}{N(M-1)} \sum_{i=1}^{N} \left(\frac{M-1}{M}\right)\sigma_i^2$$

$$E[MS_3] = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 \qquad \text{D-76}$$

From Eq. D-76 it is seen that MS_3 is an unbiased estimate of σ_W^2. Since MS_3 is an unbiased estimate of σ_W^2, let us denote this by MS_W. Then,

$$MS_W = MS_3 = \frac{1}{N(M-1)} \sum_{i=1}^{N} \sum_{j=1}^{M} (x_{ij} - \overline{G}_i) \qquad \text{D-77}$$

Now MS_W can be used in conjunction with MS_2 to obtain an unbiased estimate of σ_B^2. Define MS_B by

$$MS_B = MS_2 - \left(\frac{N-1}{MN}\right) MS_W \qquad \text{D-78}$$

Then from Eqs. D-70, D-76 and D-77 it is seen that

$$E[MS_B] = \sigma_B^2 \qquad \text{D-79}$$

Similarly define MS_{Total} by

$$MS_{Total} = MS_1 - \left(\frac{1}{MN-1}\right) MS_B \qquad \text{D-80}$$

Then from Eqs. D-16, D-59 and D-80 it is seen that

$$E[MS_{Total}] = \sigma^2 \qquad \text{D-81}$$

APPENDIX E

EXPERIMENTAL DATA

In the course of the experimental analysis, best estimates of crossover model gain, K, and time-delay, τ, were obtained. These values were obtained by the iterative regression analysis technique. The conditions analysed are:

Two controlled elements; $Y_C(p) = 5/p$, $Y_C(p) = 5/p^2$.

For each controlled element; three different subjects, five two minute trials per subject, five 20-second intervals within each trial.

The parameter estimates are presented in Table E.1.

Table E.1 EXPERIMENTAL DATA

$Y_C(p) = 5/p$ DAY 2   PARAMETER K

	SUBJECT 1	SUBJECT 2	SUBJECT 3
TRIAL 1	5.77	4.76	3.24
	4.35	5.14	2.85
	4.30	4.50	3.46
	3.83	4.92	2.88
	3.37	5.16	3.52
TRIAL 2	3.67	4.75	3.46
	3.76	4.46	2.91
	2.58	3.84	2.58
	2.82	3.25	2.69
	3.17	3.94	2.52
TRIAL 3	4.01	2.43	4.27
	3.08	2.84	3.47
	3.67	3.11	2.86
	3.81	4.09	2.92
	3.03	3.84	2.97
TRIAL 4	4.83	4.23	3.93
	3.67	3.68	3.51
	2.63	4.38	3.28
	3.25	4.53	2.43
	3.55	3.73	2.87
TRIAL 5	4.66	3.47	3.41
	4.04	3.74	2.98
	3.78	3.85	2.81
	3.89	4.42	2.79
	4.04	3.73	2.45

$Y_C(p) = 5/p$ DAY 6 PARAMETER K

	SUBJECT 1	SUBJECT 2	SUBJECT 3
TRIAL 1	5.95	6.55	4.41
	5.78	7.81	3.98
	5.91	7.89	3.30
	5.53	6.33	2.68
	4.43	6.15	3.47
TRIAL 2	4.72	6.61	4.13
	5.22	6.04	4.12
	4.61	5.59	3.97
	4.73	6.55	4.47
	5.22	6.51	3.49
TRIAL 3	5.11	7.21	3.94
	5.37	7.97	3.63
	5.28	7.81	3.48
	4.31	6.83	3.91
	6.22	7.55	3.40
TRIAL 4	4.38	6.06	4.13
	4.98	6.48	3.33
	4.71	6.51	3.35
	5.86	6.66	2.92
	4.27	6.91	3.34
TRIAL 5	4.35	7.09	3.67
	4.48	6.31	3.96
	4.94	6.51	3.49
	5.60	6.69	3.51
	5.18	5.53	3.58

$Y_C(p) = 5/p$ DAY 10 PARAMETER K

	SUBJECT 1	SUBJECT 2	SUBJECT 3
TRIAL 1	6.06	8.10	3.84
	6.05	10.42	2.71
	6.71	8.03	3.32
	6.61	8.81	3.44
	7.11	7.56	3.47
TRIAL 2	8.41	7.26	3.29
	7.43	7.69	2.82
	6.78	6.07	3.95
	6.28	8.07	3.73
	6.69	6.81	3.29
TRIAL 3	6.82	8.28	3.94
	6.88	7.69	3.63
	5.75	6.83	4.40
	5.97	6.89	4.62
	6.06	7.47	3.62
TRIAL 4	6.60	7.57	3.94
	6.27	7.52	3.66
	6.36	7.84	3.70
	5.75	6.52	3.43
	5.58	7.19	3.99
TRIAL 5	5.89	7.80	4.34
	6.15	7.82	3.97
	6.32	6.23	3.91
	6.93	7.04	3.24
	5.64	7.15	3.29

$Y_C(p) = 5/p^2$ DAY 3 PARAMETER K

	SUBJECT 1	SUBJECT 2	SUBJECT 3
TRIAL 1	1.72	3.03	2.93
	2.28	3.11	2.66
	1.60	3.15	3.08
	1.13	2.86	3.17
	1.81	3.08	3.11
TRIAL 2	2.28	2.87	3.81
	2.17	3.55	3.67
	2.31	3.33	3.73
	2.08	2.79	3.66
	2.25	3.12	3.23
TRIAL 3	2.62	2.76	2.58
	2.25	3.79	2.91
	2.78	2.95	3.80
	2.54	2.32	2.55
	1.85	3.09	2.45
TRIAL 4	1.09	3.15	2.67
	1.57	3.17	3.18
	1.85	3.03	2.47
	2.61	3.82	3.04
	1.95	3.65	3.41
TRIAL 5	1.42	3.04	3.69
	1.34	3.81	3.95
	1.77	3.21	3.89
	2.35	2.43	3.50
	2.30	3.55	3.59

$Y_C(p) = 5/p^2$ DAY 7 PARAMETER K

	SUBJECT 1	SUBJECT 2	SUBJECT 3
TRIAL 1	2.37	4.07	3.71
	2.41	3.03	3.57
	2.49	3.55	3.21
	2.52	4.30	3.27
	2.57	3.87	3.93
TRIAL 2	2.21	3.84	3.97
	2.48	3.48	3.41
	2.78	3.24	3.69
	2.68	3.15	3.67
	2.76	3.19	3.47
TRIAL 3	3.01	4.22	3.48
	2.81	3.85	3.19
	2.52	3.18	2.97
	2.65	3.07	2.65
	2.78	2.94	3.15
TRIAL 4	2.09	3.04	3.34
	2.58	2.92	2.68
	2.69	2.43	3.23
	2.37	2.45	3.66
	2.54	3.35	3.67
TRIAL 5	2.81	3.38	3.55
	3.22	3.31	2.94
	2.64	3.71	2.90
	2.87	3.42	3.57
	2.76	3.05	2.82

$Y_C(p) = 5/p^2$ DAY 9 PARAMETER K

	SUBJECT 1	SUBJECT 2	SUBJECT 3
TRIAL 1	3.33	3.28	3.26
	2.64	2.59	2.91
	2.57	3.43	3.65
	3.14	3.05	3.72
	3.07	2.77	3.72
TRIAL 2	3.94	3.13	3.68
	3.32	2.84	3.64
	3.35	3.01	3.83
	2.21	3.30	3.01
	2.68	3.03	3.20
TRIAL 3	2.65	2.82	2.44
	2.84	3.28	3.23
	2.45	2.97	3.12
	2.89	2.73	3.42
	2.90	2.95	3.77
TRIAL 4	3.57	3.42	3.79
	2.57	2.88	4.16
	3.52	2.41	3.19
	3.36	2.69	3.61
	3.41	3.22	3.75
TRIAL 5	3.09	3.43	2.44
	2.98	3.32	3.02
	3.15	3.32	3.21
	3.19	3.18	3.59
	3.32	3.46	3.85

$Y_C(p) = 5/p$ DAY 2 PARAMETER τ

	SUBJECT 1	SUBJECT 2	SUBJECT 3
TRIAL 1	0.231	0.276	0.268
	0.234	0.258	0.300
	0.258	0.261	0.299
	0.268	0.274	0.348
	0.281	0.221	0.281
TRIAL 2	0.286	0.267	0.277
	0.236	0.255	0.280
	0.216	0.218	0.246
	0.308	0.230	0.304
	0.212	0.210	0.266
TRIAL 3	0.261	0.330	0.288
	0.264	0.266	0.214
	0.205	0.268	0.268
	0.231	0.275	0.251
	0.264	0.252	0.171
TRIAL 4	0.238	0.272	0.251
	0.250	0.246	0.255
	0.208	0.206	0.244
	0.202	0.233	0.196
	0.244	0.240	0.192
TRIAL 5	0.242	0.254	0.208
	0.226	0.274	0.242
	0.218	0.236	0.253
	0.203	0.236	0.163
	0.215	0.224	0.266

$Y_C(p) = 5/p$ DAY 6 PARAMETER τ

	SUBJECT 1	SUBJECT 2	SUBJECT 3
TRIAL 1	0.205	0.191	0.280
	0.185	0.171	0.252
	0.202	0.169	0.277
	0.212	0.210	0.243
	0.184	0.201	0.195
TRIAL 2	0.218	0.196	0.216
	0.191	0.214	0.210
	0.203	0.206	0.250
	0.199	0.197	0.248
	0.210	0.181	0.242
TRIAL 3	0.255	0.185	0.262
	0.210	0.167	0.242
	0.209	0.171	0.243
	0.218	0.195	0.254
	0.193	0.176	0.260
TRIAL 4	0.214	0.216	0.218
	0.216	0.202	0.197
	0.205	0.204	0.237
	0.210	0.200	0.212
	0.233	0.193	0.214
TRIAL 5	0.210	0.188	0.240
	0.220	0.200	0.191
	0.190	0.192	0.210
	0.202	0.199	0.266
	0.218	0.198	0.242

$Y_C(p) = 5/p$ DAY 10 PARAMETER τ

	SUBJECT 1	SUBJECT 2	SUBJECT 3
TRIAL 1	0.216	0.165	0.220
	0.208	0.128	0.276
	0.199	0.166	0.214
	0.202	0.151	0.219
	0.188	0.175	0.252
TRIAL 2	0.158	0.183	0.241
	0.179	0.173	0.215
	0.197	0.193	0.199
	0.207	0.165	0.214
	0.186	0.194	0.210
TRIAL 3	0.195	0.161	0.211
	0.194	0.173	0.205
	0.200	0.179	0.231
	0.188	0.175	0.224
	0.188	0.178	0.240
TRIAL 4	0.192	0.167	0.241
	0.199	9,174	0.217
	0.203	0.170	0.228
	0.204	0.172	0.260
	0.200	0.170	0.258
TRIAL 5	0.214	0.167	0.218
	0.188	0.170	0.224
	0.207	0.181	0.202
	0.192	0.184	0.224
	0.205	0.174	0.207

$Y_C(p) = 5/p^2$ DAY 3 PARAMETER τ

	SUBJECT 1	SUBJECT 2	SUBJECT 3
TRIAL 1	0.589	0.411	0.455
	0.446	0.405	0.500
	0.544	0.372	0.433
	0.625	0.392	0.420
	0.453	0.410	0.427
TRIAL 2	0.470	0.362	0.351
	0.412	0.330	0.364
	0.484	0.339	0.357
	0.456	0.389	0.342
	0.420	0.334	0.395
TRIAL 3	0.395	0.422	0.461
	0.446	0.352	0.459
	0.380	0.414	0.325
	0.348	0.483	0.467
	0.476	0.398	0.490
TRIAL 4	0.274	0.342	0.439
	0.589	0.344	0.419
	0.578	0.406	0.541
	0.471	0.348	0.439
	0.633	0.352	0.392
TRIAL 5	0.585	0.396	0.361
	0.705	0.350	0.337
	0.666	0.368	0.305
	0.568	0.415	0.380
	0.581	0.365	0.372

$Y_C(p) = 5/p^2$ DAY 7 PARAMETER τ

	SUBJECT 1	SUBJECT 2	SUBJECT 3
TRIAL 1	0.550	0.327	0.360
	0.524	0.382	0.373
	0.461	0.375	0.365
	0.419	0 310	0.331
	0.518	0.344	0.332
TRIAL 2	0.602	0.342	0.336
	0.461	0.383	0.390
	0.401	0.411	0.361
	0.442	0.420	0.364
	0.391	0.394	0.364
TRIAL 3	0.375	0.316	0.384
	0.364	0.331	0.416
	0.422	0.345	0.449
	0.416	0.379	0.410
	0.318	0.383	0.421
TRIAL 4	0.427	0.385	0.358
	0.404	0.430	0.459
	0.471	0.465	0.411
	0.498	0.368	0.364
	0.450	0.362	0.364
TRIAL 5	0.460	0.364	0.365
	0.415	0.336	0.411
	0.425	0.330	0.388
	0.424	0.361	0.374
	0.470	0.384	0.470

$Y_C(p) = 5/p^2$ DAY 9 PARAMETER τ

	SUBJECT 1	SUBJECT 2	SUBJECT 3
TRIAL 1	0.400	0.405	0.365
	0.456	0.405	0.365
	0.400	0.389	0.365
	0.425	0.408	0.358
	0.388	0.421	0.358
TRIAL 2	0.339	0.361	0.358
	0.401	0.382	0.366
	0.383	0.374	0.348
	0.395	0.383	0.433
	0.383	0.374	0.410
TRIAL 3	0.410	0.470	0.529
	0.470	0.358	0.351
	0.440	0.361	0.369
	0.410	0.391	0.391
	0.433	0.416	0.342
TRIAL 4	0.307	0.340	0.352
	0.356	0.410	0.320
	0.355	0.351	0.411
	0.361	0.378	0.369
	0.314	0.404	0.351
TRIAL 5	0.360	0.352	0.497
	0.354	0.352	0.397
	0.395	0.354	0.398
	0.400	0.369	0.353
	0.401	0.386	0.334

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