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## ON A PROBLEM OF THERMO-VISCOELASTICITY

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## 1. INTRODUCTION

A special problem area related to solid propellant rocket design is the analysis of viscoelastic materials under transient thermal conditions. Because these materials exhibit rheological properties which are sensitive to temperature, analysis is generally quite difficult and therefore numerical methods are indicated.<sup>1-3</sup> In evaluating the latter, it is important to have analytic check-cases. This note provides one such case.

Besides providing a check-case for numerical procedures, the problem to be considered can also be used as a convenient laboratory experiment to evaluate a common assumption in thermo-viscoelastic stress analysis, namely that of thermorheological simplicity. The latter is often taken to imply that there exists a differential temperature reduction of time given by<sup>4, 5</sup>

$$dt' = \frac{dt}{a_T [T(x, t)]} \quad (1)$$

This relation is devoid of the temperature rates  $\frac{d^n T}{dt^n}$  for  $n=1, 2, \dots$  and there is, as yet, no experimental justification for leaving out temperature rates.

In this note, analytical results are predicted on the basis of assumption (1). The problem considered may be quite easily and accurately modelled in the laboratory. Tests may then be run with variable thermal rates, leading to transient temperature distributions. By comparing laboratory results with analytical predictions, one may check the validity of assumption (1), i. e., whether temperature rates may be excluded from consideration.

## 2. PROBLEM DESCRIPTION

The problem considered is that of a hollow or solid circular cylinder of thermorheologically simple material, infinite in extent in the axial direction, with a temperature distribution which may be any prescribed function of radius and time (cf. Figure 1). Normal stresses which are prescribed functions of time only,  $\sigma_a(t)$  and  $\sigma_b(t)$ , act on the lateral surfaces and a uniform axial strain  $e_z(t)$  may be prescribed as any arbitrary function of time. It is also assumed that  $\nu_{rel} = \text{const.} = \nu = \frac{1}{2}$ . A complete and closed form analytic solution for the stresses and displacements appears to be very difficult for this problem. However, a relatively simple expression may be found for the total axial force  $F_z(t)$ . Although the force  $F_z(t)$  is an integral of the stress distribution, it is exact and may therefore provide a useful check for numerical calculations. Furthermore, if one is interested in experiments related to thermorheological material simplicity, then one is interested in this force because it is the primary quantity determined experimentally.

### 3. ANALYSIS

Due to the symmetry about the axis and the uniformity in the axial direction, the shear strains and shear stresses are zero. The only equilibrium equation not identically satisfied is, therefore,

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \quad (2)$$

The stress-strain law for a thermorheologically simple material is given by

$$\sigma'_{ij}(r, t) = 2 \int_0^t \mu_{rel} [t'(r, t) - \xi'(r, \xi)] \frac{\partial e'_{ij}(r, \xi)}{\partial \xi} d\xi \quad (3a)$$

$$\sigma_{\alpha\alpha}(r, t) = 3 \int_0^t K_{rel} [t'(r, t) - \xi'(r, \xi)] \frac{\partial}{\partial \xi} [e_{\alpha\alpha}(r, \xi) - \Theta(r, \xi)] d\xi \quad (3b)$$

$$\text{where } \sigma_{\alpha\alpha} = \sigma_r + \sigma_\theta + \sigma_z \quad (4a)$$

$$e_{\alpha\alpha} = e_r + e_\theta + e_z \quad (4b)$$

$$\sigma'_{ij}(r, t) = \sigma_{ij}(r, t) - \frac{1}{3} \delta_{ij} \sigma_{\alpha\alpha}(r, t) \quad (4c)$$

$$e'_{ij}(r, t) = e_{ij}(r, t) - \left(\frac{1}{3}\right) \delta_{ij} e_{\alpha\alpha}(r, t) \quad (4d)$$

$$\Theta(r, t) = \int_0^t \alpha [T(r, \tau)] \frac{\partial T(r, \tau)}{\partial \tau} d\tau \quad (4e)$$

$$t'(r, t) - \xi'(r, t) = \int_\xi^t \frac{d\tau}{a_T [T(r, \tau)]} \quad (4f)$$

$$\alpha = \text{volume coefficient of thermal expansion} \quad (4g)$$

Also, for later reference, note that if  $\nu_{\text{rel}}$  is a constant, the following relation holds:

$$\mu_{\text{rel}} = \frac{3(1-2\nu)}{2(1+\nu)} K_{\text{rel}} \quad (\nu = \text{const}). \quad (5)$$

Let the axial strain be a prescribed function of time

$$e_z = e'_z(r, t) + \frac{1}{3} e_{\alpha\alpha}(r, t) = e_o(t). \quad (6)$$

Upon solving (7) for  $e'_z(r, t)$  and substituting into (3a) one obtains:

$$\sigma'_z(r, t) = f(r, t) - \frac{2}{3} \int_0^t \mu_{\text{rel}}(t' - \xi') \frac{\partial}{\partial \xi} [e_{\alpha\alpha}(r, \xi) - \Theta(r, \xi)] d\xi \quad (7)$$

where

$$f(r, t) = z \int_0^t \mu_{\text{rel}}(t' - \xi') \frac{\partial}{\partial \xi} [e_o(\xi) - \frac{1}{3} \Theta(r, \xi)] d\xi. \quad (8)$$

Substituting (5) into (7), and using (3b), one finds

$$\sigma'_z(r, t) = f(r, t) - \frac{(1-2\nu)}{3(1+\nu)} \sigma_{\alpha\alpha} \quad (9)$$

while use of (4a), (4c) and (9) renders

$$\sigma_z = (1+\nu) f(r, t) + \nu (\sigma_r + \sigma_\theta). \quad (10)$$

If one combines this result with the equilibrium equation (2) to eliminate  $\sigma_\theta$ , one obtains

$$\sigma_z = (1+\nu) f(r, t) + \nu \frac{1}{r} \frac{\partial}{\partial r} (r^2 \sigma_r) \quad (11)$$

and upon substitution of this expression into the relation

$$F_z(t) = \int_a^b \sigma_z(r, t) 2\pi r dr \quad (12)$$

it is found that the  $\sigma_r$  term appears as a perfect differential, which is easily integrated to give

$$F_z(t) = 2\pi(1+\nu) \int_a^b f(r, t) r dr + 2\pi\nu [b^2 \sigma_b(t) - a^2 \sigma_a(t)] \quad (13)$$

Written out fully, equation (13) reads

$$F_z(t) = -\frac{4\pi}{3} (1+\nu) \int_a^b \int_0^t \mu_{rel}(t'-\xi') \left\{ \alpha [T(r, \xi) \frac{\partial T(r, \xi)}{\partial \xi} - 3 \frac{\partial e_o(\xi)}{\partial \xi}] \right\} r d\xi dr + \\ + 2\pi\nu [b^2 \sigma_b(t) - a^2 \sigma_a(t)]. \quad (14)$$

#### 4. DISCUSSION

The solution (15) is in an integral form. For most practical cases, analytical integration will not be possible. Thus, some numerical calculations will be required. However, these computations are of a standard, straightforward type, and can be carried out to any desired degree of accuracy.

Furthermore, the solution (15) suggests a simple operational analog. Suppose

$$\sigma_a(t) = \sigma_b(t) = 0, \quad t \geq 0. \quad (15)$$

Then (15) can be written as

$$F_z(t) = \int_a^b \left\{ \int_0^t E_{rel} (t' - \xi') \frac{\partial}{\partial \xi} \left[ e_o(\xi) - \bar{\alpha} [T(r, \xi)] \right] d\xi \right\} 2\pi r dr \quad (16)$$

where

$$E_{rel} = 2(1+\nu) \mu_{rel} \quad (17)$$

$$\bar{\alpha} = \text{linear coeff. of thermal expansion} = \frac{\alpha}{3}$$

But the term in brackets

$$\int_0^t E_{rel} (t' - \xi') \frac{\partial}{\partial \xi} \left[ e_o(\xi) - \bar{\alpha} [T(r, \xi)] \right] d\xi \quad (18)$$

is just the force exerted on a uniaxial test specimen of unit cross-sectional area under the thermal history at radius  $r$ . Thus, the total force is formally equivalent to that exerted by a large number of concentric, cylindrical uniaxial test specimens. Each hypothetical uniaxial specimen, subjected to the thermal history at its radius  $r$ , produces a force per unit cross-sectional area.

In contrast the exact axial stress-distribution contains additional self-equilibrating stresses; therefore, the analog uniaxial stress distribution is not the time distribution, but fictitious.

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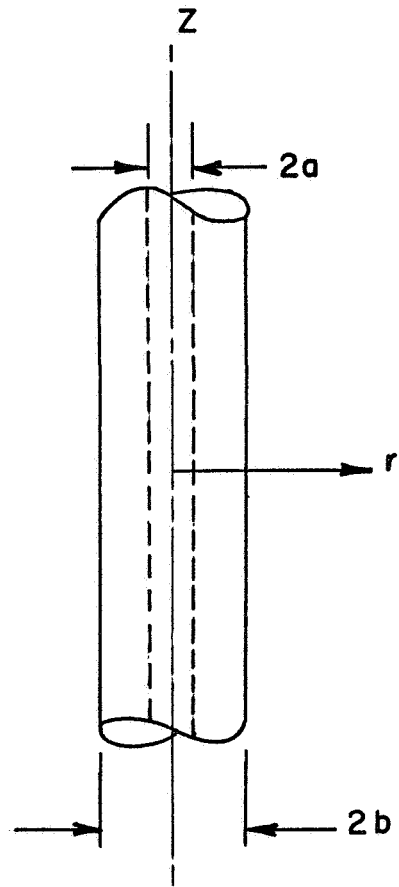


FIG. I DEFINITION OF NEW GEOMETRY