MRI REPORT

DIFFERENTIAL-DIFFERENCE PROPERTIES OF GENERALIZED JACOBI POLYNOMIALS

INTERIM REPORT
15 January 1969

Contract No. NAS9-7641

MRI Project No. 3162-P

For

NASA Manned Spacecraft Center
General Research Procurement Branch
Houston, Texas 77058

Attn: J.W. Carlson/BG731(48)

NASA CR 92470

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by

Jet Wimp

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PREFACE

This report, written by Jet Wimp, Analysis and Applied Mathematics Section, Midwest Research Institute, covers work performed from 1 July 1968 to 17 January 1969 on Contract No. NAS9-7641.

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17 January 1969

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I. INTRODUCTION

In this paper we derive a differential-difference equation satisfied by the hypergeometric polynomials

$$P_n(x) = {}_{p+2}F_q\left(\left.\begin{matrix} -n, n+\lambda, a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}\right| x\right), \quad n = 0,1,2,\ldots \qquad (1)$$

Throughout, we employ the shorthand notation

$$(a_p+n) = \prod_{j=1}^{p} (a_j+n), \text{ etc.,} \qquad (2)$$

see [1]. In general, where any variable is subscripted by a p or q, it is to be understood that the shorthand notation has been invoked.

II. RESULTS

Theorem

Let i) $\lambda \neq 1,2,\ldots$;

ii) none of the quantities b_j, λ, $\lambda+1-b_j$ be negative integers or zero, $j = 1,2,\ldots,q$;

iii) no b_j = any a_h, $h = 1,2,\ldots,p$; $j = 1,2,\ldots,q$.

Then the polynomials $P_n(x)$ satisfy the differential-difference equation

$$(\epsilon x-\delta x^2)\frac{dP_n(x)}{dx} = \sum_{\nu=0}^{\sigma} (A_\nu+xB_\nu)P_{n-\nu}(x) \quad , \qquad (3)$$

where

$$\delta = \begin{cases} 1, p+1 \geq q , \\ 0, p+1 < q , \end{cases} \qquad \epsilon = \begin{cases} 0, p+1 > q , \\ 1, p+1 \leq q , \end{cases} \qquad \sigma = \max\left\{p+1, q\right\} , \tag{4}$$

and no such equation of lower order $\sigma' < \gamma$ exists. The A_ν's and B_ν's are unique and

$$A_\nu = \begin{cases} (-n)_\nu \left[(1-n-\lambda)_\nu\right]^{-1} (2\nu-2n-\lambda) \left\{(-)^{\nu+1}\epsilon + \dfrac{(-)^{\sigma+1}}{\nu!} \right. \\ \qquad\qquad \left. \times \displaystyle\sum_{s=0}^{\nu} \frac{(-\nu)_s (n-s)(b_q+n-s-1)}{(\nu+s-2n-\lambda)_{\sigma+1-\nu}} \right\} , \ \nu > 0 ; \\ n \left\{\epsilon - \dfrac{(b_q+n-1)}{(2n+\lambda-\sigma)_\sigma} \right\} , \ \nu = 0 \ ; \end{cases} \tag{5}$$

$$B_\nu = \begin{cases} (-n)_\nu \left[(1-n-\lambda)_\nu\right]^{-1} (2\nu-2n-\lambda) \left\{(-)^{\nu}\delta + \dfrac{(-)^{\sigma+1}}{\Gamma(\nu)} \right. \\ \qquad\qquad \left. \times \displaystyle\sum_{s=0}^{\nu-1} \frac{(1-\nu)_s (a_p+n-s-1)}{(\nu+s+1-\lambda-2n)_{\sigma-\nu}} \right\} , \ \nu > 0 , \\ -\delta n , \ \nu = 0 \ . \end{cases} \tag{6}$$

<u>Proof</u>: By equating coefficients of x^{k+1} in (3) we find

$$(k+1) \left\{\epsilon(k-n)(a_p+k)\beta_{-1}(k) - \delta k(b_q+k)\beta_0(k)\right\}$$

$$\equiv (a_p+k) \sum_{\nu=0}^{\sigma} C_\nu \alpha_{\nu+1}(k)\beta_{\nu-1}(k) + (k+1)(b_q+k) \sum_{\nu=0}^{\sigma} D_\nu \alpha_\nu(k)\beta_\nu(k) \quad , \tag{7}$$

where

$$\begin{bmatrix} C_\nu \\ D_\nu \end{bmatrix} = \frac{(-)^\nu (1-n-\lambda)_\nu}{(-n)_\nu} \begin{bmatrix} A_\nu \\ B_\nu \end{bmatrix} \ , \ \alpha_\nu(k) = (k-n)_\nu \ , \ \beta_\nu(k) = (n+\lambda+k-\sigma)_{\sigma-\nu} \quad . \tag{8}$$

The above can be considered an identity between polynomials in the (generally complex-valued) variable k . If $p+1 > q$, (7) requires that two polynomials of degree $p+\sigma+2$ be identical, and this condition furnishes $p+\sigma+3$ equations in $2\sigma+2$ unknowns so we must have $\sigma \leq p+1$. If $p+1 = q$, we similarly find $\sigma \leq p+1$, while if $p+1 < q$, we find that $\sigma \leq q$. Thus

$$\sigma \leq \max \left\{ p+1, \ q \right\} \tag{9}$$

Now if we assume equality above, the A_ν and B_ν (if they exist) are unique. Suppose there is another such recurrence relation with coefficients A_ν^* and B_ν^* . Subtracting these two, we have

$$0 = \sum_{\nu=0}^{\sigma} \left[(A_\nu - A_\nu^*) + x(B_\nu - B_\nu^*) \right] P_{n-\nu}(x) \tag{10}$$

but this is impossible, under the hypothesis (ii) and (iii), since Wimp has shown that in this case any linear difference equation satisfied by $P_n(x)$ must be of order $\sigma+1$ at least, see [1].

Now, if $q = p+1$, (7) holds if and only if

$$(k+1)\left[\beta_0(k+1)\epsilon - (b_q+k)\right] \equiv \sum_{\nu=0}^{\sigma} C_\nu \alpha_\nu(k+1)\beta_\nu(k+1) \quad , \tag{11}$$

$$(n+\lambda+k-\sigma)\left[-\delta k \ \beta_1(k+1) + (k-n)(a_p+k)\right] \equiv \sum_{\nu=0}^{\sigma} D_\nu \alpha_\nu(k)\beta_\nu(k) \quad . \tag{12}$$

(Note that a suitable linear combination of (11) and (12) gives (7), i.e., multiply (11) by $(k-n)(n+\lambda+k-\sigma)(a_p+k)$ and (12) by $(k+1)(b_q+k)$ and add.) To establish (11) for $p+1 = q$, we observe that it asserts an identity between two polynomials in k , each of degree $q+2$ and each having two identical factors. It only remains to show (11) holds for $q+1$ distinct values of k . Assume all the quantities -1, $-b_j$, $j = 1,2,\ldots,q$, are distinct and let k have these values in (7). The result is (11) evaluated at these values.

Likewise to show (12) for $p+1 = q$, we need only prove it holds for the $p+2$ values (assumed distinct) n, $\sigma-n-\lambda$, $-a_j$, $j = 1,2,\ldots,p$. This is true, since (7) and (12) for these values are the same.

(The requirement that the values of k chosen above be distinct may be relaxed by continuity.)

Now replace x by x/a_j, $j = p'+1, p'+2,\ldots,q-1$ in (3), where $p' < q-1$. This shows that

$$\epsilon x \frac{dP_n'(x)}{dx} = \sum_{\nu=0}^{\sigma} (C_\nu + xD_\nu')P_{n-\nu}(x)\ (-)^\nu (1-n-\lambda)_\nu/(-n)_\nu \quad , \qquad (13)$$

where $P_n'(x)$ is $P_n(x)$ with p replaced by p' and

$$D_\nu' = \lim_{a_u\to\infty}\ \lim_{a_{u+1}\to\infty} \cdots \lim_{a_v\to\infty} \left[D_\nu/a_u a_{u+1}\cdots a_v\right], \quad u = p'+1,\ v = q-1 . \qquad (14)$$

The same limit process applied to (12) yields the following equation for the determination of D_ν' :

$$(k-n)(n+\lambda+k-\sigma)(a_{p'}+k) = \sum_{\nu=0}^{\sigma} D_\nu'\alpha_\nu(k)\beta_\nu(k) \quad . \qquad (15)$$

The equation for C_ν in this case is (11) as it stands. But (11) and (15) together are (11) and (12), respectively, written for $p+1 < q$.

Similarly, replacing x by xb_j, $j = q'+1, q'+2,\ldots,p+1$, $q' \le p$ and letting $b_j \to \infty$ in (3) gives

$$-x^2 \frac{dP_n''(x)}{dx} = \sum_{\nu=0}^{\sigma} (C_\nu' + xD_\nu)P_{n-\nu}''(x)(-)^\nu(1-n-\lambda)_\nu/(-n)_\nu \ . \qquad (16)$$

where

$$C'_\nu = \lim_{b_u \to \infty} \lim_{b_{u+1} \to \infty} \cdots \lim_{b_v \to \infty} (C_\nu / b_u b_{u+1} \cdots b_v) \;, \; u = q'+1, \; v = p+1, \quad (17)$$

and $P''_n(x)$ is $P_n(x)$ with q replaced by q' . This limit process applied to (11) gives

$$-(k+1)(b_q+k) = \sum_{\nu=0}^{\sigma} C'_\nu \alpha_\nu (k+1) \beta_\nu (k+1) \quad , \qquad (18)$$

and (12) is used unchanged for D_ν . These two equations, though, are just (11) and (12) for $p+1 > q$.

Thus (11) and (12) are established for all p,q and we have succeeded in "uncoupling" Eq. (7) to give Eqs. (11) and (12) which involve C_ν and D_ν alone, respectively.

Next, we solve these two equations.

In Eq. (11), let $k+1-n = -s$, $s = 0,1,2,\ldots,\sigma$. The result can be written

$$\sum_{\nu=0}^{s} \frac{(-)^\nu C_\nu (-s)_\nu}{(s+1-2n-\lambda)_\nu} = \epsilon(n-s) + \frac{(-)^{\sigma+1}(n-s)(n+b_q-s-1)}{(s+1-2n-\lambda)_{\sigma-s}} \;,$$

$$s = 0,1,2,\ldots,\sigma \quad . \qquad (19)$$

But if $1-2n-\lambda \neq 0,-1,-2,\ldots,$ the above equation can be solved for C_ν by applying a Lemma of Wimp [1]. After some algebra and evaluation of ${}_2F_1$'s of unit argument, one arrives at (5). To find the D_ν's, let $k-n = -s$ in (12) and proceed in a similar fashion.

III. CONCLUDING REMARKS

If $p+1 = q$ and $x = 1$ in (3), we get a recursion relation for $P_n(1)$ of order $\max(p+1,q)$. Note that this is of order one less than that obtained by putting $x = 1$ in the homogeneous linear difference equation satisfied by $P_n(x)$ given in [1]. If $p = 1$, the resulting recursion relation for

$$ {}_3F_2\left(\begin{matrix}-n,n+\lambda,a_1\\ b_1,b_2\end{matrix}\middle|\,1\right) \tag{20}$$

is that given by Bailey [3], which in turn is Watson's result [2] slightly rewritten. For $p+1 = q$ and x general, (3) of course provides a generalization of the classical differential-difference formula for the Jacobi polynomials, see [4, p. 170 (15)].

A differential-difference relation for the polynomials

$$Q_n(x) = {}_{p+1}F_q\left(\begin{matrix}-n,a_1,a_2,\ldots,a_p\\ b_1,b_2,\ldots,b_q\end{matrix}\middle|\,x\right) \tag{21}$$

can be easily obtained from (3) by replacing x by x/λ and letting $\lambda \rightarrow \infty$.

We point out that the conditions of the theorem can be relaxed considerably. If λ is a positive integer m, we can write

$$(-n)_\nu/(1-n-\lambda)_\nu = n!\,(n+1-\nu)_{m-1}/\Gamma(n+m) \tag{22}$$

which is well-defined for all n, so condition (i) is not essential to the analysis.

Also, if any of the quantities (ii) are negative integers or zero, limits may be taken after the equation has been multiplied by a suitable factor, see [1]. The quantity n can even be nonintegral when $q > p+1$ or when $q = p+1$ and $|\arg(1-x)| < \pi$, by the permanence principle for functional equations. (It may be necessary in this case to multiply the equation by a factor $(r-n-\lambda)$ to make the coefficients well-defined.)

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