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ON THE FORMATION OF THE MOON BY ACCRETION OF SOLAR OR PLANETARY MATERIAL

by

J. Anthony Burke

INTRODUCTION

A mechanism for the genesis of planetary systems proposed by Burke (1968) also suggests the formation of satellites associated with the protoplanets so that we are led to consider this mechanism as a possible description of the development of our own "proto-moon". In Burke's picture material lost from a collapsing proto-star flows more or less radially outward until it impinges on the distant remnants of the rotating disc shaped cloud of material from which the star condensed. Accretion of this ejected material by the inner edge of the disc decreases the angular momentum per unit mass of the latter, causing it to move inward. Continued accretion and inward motion by this ring of material increases its density until, exceeding a critical value, it divides into an inner and an outer portion. The inner portion continues to accrete material and moves inwards, until it divides again. Successive divisions in the same fashion finally leave a series of concentric rings in the plane of the original disc (and perpendicular to the proto-star's axis of rotation) which may ultimately develop into planets.

According to this picture, satellite formation could proceed in an analogous fashion beginning with a disc of material left behind by a collapsing proto-planet. Here the outward flow of material could either be provided by centrifugal ejection from the proto-planet, or by focusing of the stellar stream by the proto-planet.

If such a mechanism is actually viable, then it provided a theory of the formation of our moon. The moon would then be comprised of material derived from the original disc around the proto-star and from material flowing out from the star itself. The initial compositions of these two supplies would be identical since they both ultimately derive from the same original collapsing cloud. However, the final composition of the moon would be subject to at least three separating processes; certain elements may be preferentially ejected from the proto-star; certain elements may be preferentially ejected from the proto-earth; of the elements present in a ring-shaped pre-moon, only certain ones may ultimately end up in the moon (for example, lighter elements are lost in conventional evaporative processes, etc.).

In the development of such an evolutionary scheme it is necessary to evaluate the conditions in which material would be accreted by a gravitating ring located in a fluid stream. In order to obtain a more readily solvable problem, we shall approximately represent the flow configuration by a stream, uniform at large distances, impinging on a cylindrical gravitating body.

II. ACCRETION FROM A FLUID STREAM BY A GRAVITATING CYLINDER

In dealing with the problem of accretion of a fluid by an infinite gravitating cylinder one is met with a mathematically intractable and physically unrealistic situation. The former difficulty derives from the logarithmic nature of the gravitational potential, while the latter arises because, of course, in all physical situations the cylinder must necessarily be finite. We are therefore led to modify the problem and to consider the gravitational potential to be truncated at some distance where other forces may become dominant and where the effects of the finite length become significant. Therefore, let us consider a potential of the form,

$$\Phi = \begin{cases} 2G\lambda \log\frac{r}{r_o}, & r \leq r_o , \\ 0, & r > r_o , \end{cases} \tag{1}$$

where r_o is some cut-off distance. This simplification permits us to treat the problem as two dimensional.

We shall further delimit the problem by presuming the absence of any magnetic field strong enough to affect the motion, and by taking the flow at distances greater than r_o

to be approximately uniform in direction, speed (V_o), and density (ρ_o).

1. Strong Field

Perhaps the simplest approach resembles the three dimensional cases considered by Hoyle and Lyttleton (1939) and Bondi and Hoyle (1944) where we neglect the internal energy of the fluid in comparison with the kinetic and gravitational energy and obtain a characteristic accretion radius r_c by equating the latter two. Thus, we write,

$$\tfrac{1}{2}V_o^2 = -2G\lambda\log\frac{r_c}{r_o} \tag{2}$$

where λ is the mass per unit length of the cylinder and G is the gravitational constant. Then the accretion rate of a cylinder of radius R is given by,

$$A \sim 2r_c\rho_o V_o.$$

If we take the effective cross section $2r_c$ to be given in terms of the geometric cross section by

$$r_c = f_s R, \tag{3}$$

then equation (2) gives

$$\frac{r_c}{r_o} = \frac{f_s}{\beta} = \exp\{-(M^2/4)/(G\lambda/c^2)\}, \tag{4}$$

where $\beta = r_o/R$, c is the velocity of sound, $c^2 = \gamma P_o/\rho_o$, and $M = V_o/c$ is the Mach number of the flow.

For this case of the strong field ($G\lambda/c^2 > 1$) and with a small Mach number, equation (4) reduces to $f_s = \beta$, so that the accretion radius is determined entirely by the choice of potential cut-off distance, r_o. If the Mach number is somewhat larger than one, the gravitational and kinetic energies will dominate the internal energy, so that equation (4) should give a reasonable estimate of accretion cross section. Since we are neglecting the internal energy of the fluid, we are consistent in not being concerned with the effects on the flow caused by the formation of shock fronts.

2. Weak Field - Subsonic Flow

When the gravitational field is weak ($G\lambda/c^2 < 1$), we must account for the effects of pressure and presence of shock waves, so that a more elaborate analysis is needed. Fortunately, the less pronounced role of gravitation permits us to adopt a perturbation technique. The unperturbed flow will be taken to be entirely determined by pressure and inertial forces, subsonic, and without accretion. The gravitational field will be introduced as a first order effect, and the resultant flow modifications will permit us to evaluate an accretion rate and effective accretion cross section.

3. First Order Solution

We are thus treating the accretion problem as two dimensional and subsonic. Let the cylinder have radius unity and be centered on the origin of the z-plane having orthogonal axes ξ and η. Since the flow is symmetric about the ξ-axis, we need only concern ourselves with determining it in the upper half plane. The solution for the unperturbed flow of an inviscid, incompressible fluid is easily obtained by the usual complex transformation

$$W = z + \frac{1}{z} \tag{5}$$

(Milne-Thomson, 1960, for example) which maps the upper half of the z-plane exterior to the unit circle into the whole upper half of the w-plane (with coordinate axes x and y). The complex flow potential in the w-plane is V_oW, so that the vector velocity is simply $\vec{V} = V_o\hat{x}$. The perturbing gravitational force will bend the streamlines somewhat, so that a portion of the fluid will cross the x-axis in a downward direction. The part of this fluid crossing the line corresponding to the cylinder boundary we shall presume to be accreted.

In the w-plane the conventional equations of conservation of mass and momentum are, respectively,

$$\nabla\cdot(\rho\vec{V}) = 0, \tag{6}$$

$$(\vec{V}\cdot\nabla)\vec{V} = -\frac{1}{\rho}\nabla P - \nabla\Phi. \tag{7}$$

Let us put

$$\rho = \rho_o + \rho_1$$

and

$$\vec{V} = \vec{V}_o + \vec{V}_1 ,$$

where the subscript zero refers to unperturbed quantities, and the subscript one refers to small increments caused by the introduction of the gravitational potential Φ. We also presume the density and pressure to be related by a polytropic law,

$$\frac{P}{P_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma} .$$

With these substitutions, and remembering that the unperturbed quantities also obey equations (6) and (7), we obtain two equations for the first order perturbations:

$$\nabla \cdot (\rho_o \vec{V}_1 + \rho_1 \vec{V}_o) = 0,$$

$$(\vec{V}_o \cdot \nabla)\vec{V}_1 + (\vec{V}_1 \cdot \nabla)\vec{V}_o = -c^2 \frac{\nabla \rho_1}{\rho_o} + (\gamma - 2)c^2 \frac{\rho_1}{\rho_o} \frac{\nabla \rho_o}{\rho_o} - \nabla \Phi .$$

Finally we put $\vec{V}_o = V_o \hat{x}$, ρ_o = constant, and $\rho_1/\rho_o = \tilde{\rho}$ to obtain

$$\frac{1}{V_o}\left(\frac{\partial V_{1x}}{\partial x} + \frac{\partial V_{1y}}{\partial y}\right) + \frac{\partial \tilde{\rho}}{\partial x} = 0, \qquad (8)$$

$$V_o \frac{\partial V_{1x}}{\partial x} = -c^2 \frac{\partial \tilde{\rho}}{\partial x} - \frac{\partial \Phi}{\partial x} , \qquad (9)$$

$$V_o \frac{\partial V_{1y}}{\partial x} = -c^2 \frac{\partial \tilde{\rho}}{\partial y} - \frac{\partial \Phi}{\partial y} , \qquad (10)$$

where V_{1x} means the x component of V, etc. Equation (9) may be integrated with respect to x to obtain

$$V_o V_{1x} = -c^2 \tilde{\rho} - \Phi + k(y) . \qquad (11)$$

The arbitrary function k(y) may be seen to be identically zero since at large distances from the origin we require the perturbations in the flow to vanish. If we differentiate equation (11) by y and subtract it from equation (10) we recover the curl-free flow condition:

$$\frac{\partial V_{1x}}{\partial y} = \frac{\partial V_{1y}}{\partial x} \qquad (12)$$

If we use equation (11) to eliminate $\tilde{\rho}$ from equation (8), we have

$$\frac{\partial V_{1x}}{\partial x}(1-M^2) + \frac{\partial V_{1y}}{\partial y} - \frac{V_o}{c^2} \frac{\partial \Phi}{\partial x} = 0 \quad .$$

Differentiation of this equation by y and use of equation (12) yields,

$$\frac{\partial^2 V_{1y}}{\partial x^2}(1-M^2) + \frac{\partial^2 V_{1y}}{\partial y^2} - \frac{V_o}{c^2} \frac{\partial^2 \Phi}{\partial x \partial y} = 0 \quad . \qquad (13)$$

The substitution $\chi = (1-M^2)^{\frac{1}{2}} x$ changes the equation to the form

$$\frac{\partial^2 V_{1y}}{\partial \chi^2} + \frac{\partial^2 V_{1y}}{\partial y^2} - \frac{V_o}{c^2}(1-M^2)^{-\frac{1}{2}} \frac{\partial^2 \Phi}{\partial \chi \partial y} = 0 \quad ,$$

which may be solved by a Green's function. We obtain

$$V_{1y}(\vec{S}') = \frac{1}{2\pi c} \cdot \frac{M}{(1-M^2)^{\frac{1}{2}}} \iint \log|S-S'| \frac{\partial^2 \Phi}{\partial \chi \partial y} d\chi dy \quad ,$$

where the integrals are taken to cover the entire χ-y plane, and

$$|\vec{S}-\vec{S}'|^2 = (\chi-\chi')^2 + (y-y')^2 \quad .$$

Two integrations by parts, remembering that Φ vanishes at large χ or y, gives

$$V_{1y}(\vec{S}') = \frac{1}{2\pi c} \frac{M}{(1-M^2)^{\frac{1}{2}}} \iint \frac{\partial^2}{\partial \chi \partial y}(\log|\vec{S}-\vec{S}'|) \Phi(\chi,y) d\chi dy \qquad (14)$$

Ultimately we seek the fluid flux across a line corresponding to the cylinder boundary in the z-plane. This flux defines the accretion rate A, so that

$$A = 2\rho_o \int_{\substack{\text{one} \\ \text{quadrant}}} V_{\perp}(z) dl(z)$$

where $V_{\perp}(z)$ is the velocity component perpendicular to the cylinder boundary in the z-plane, and $dl(z)$ is arc length along the boundary. From the transformation equation (5) we find that $V_{\perp} = 2\eta V_{1y}$ and $2\eta dl = dx$ so that

$$A = 2\rho_o \int_0^2 V_{1y}(\hat{S}')dx' = 2\rho_o(1-M^2)^{\frac{1}{2}} \int_0^{\chi_2} V_{1y}(\hat{S}')d\chi' \quad , \qquad (15)$$

where $\chi_2 = 2(1-M^2)^{-\frac{1}{2}}$. We are presuming at this point that in an actual situation where flow crossing the x-axis for $x>2$ would be prohibited, the flow across the boundary $0 \leq x \leq 2$ would not be enormously changed. Substitution of equation (14) into equation (15), and integration with respect to χ', since $\frac{\partial}{\partial \chi} = -\frac{\partial}{\partial \chi'}$, yields

$$A = -\frac{\rho_o}{\pi c}M \iint \left[\frac{\partial}{\partial y}(\log|\hat{S}-\hat{S}'|)\Phi(\chi,y)d\chi dy\right]_0^{\chi_2} .$$

Finally, let us define a normalized potential ϕ by

$$\Phi = 2G\lambda \log\frac{r}{r_o} = 2G\lambda\phi \quad ,$$

(so that $\phi(x,y)$ is simply the w-plane representation of $\log(r/\beta)$, and let us express the accretion rate for a cylinder of radius R (hitherto we have let $R = 1$). The accretion rate is then

$$A = 2\rho_o G\lambda c^{-1} RMI(M,\beta) \quad , \qquad (16)$$

where

$$I(M,\beta) \equiv -\frac{1}{\pi}\iint \left[\frac{\partial}{\partial y}(\log|\hat{S}-\hat{S}'|)\phi(\chi,y)d\chi dy\right]_0^{\chi_2} . \qquad (17)$$

As in equation (3), take the effective accretion cross section to be related to the geometric cross section by a parameter f_w. Then,

$$A = 2f_w V_o \rho_o R \quad , \qquad (18)$$

so that

$$f_w = (\frac{G\lambda}{c^2}) I(M,\beta) \quad . \qquad (19)$$

Values of the function $I(M,\beta)$ are shown in Figure 1. We note that I varies only slowly with M and β, so that the accretion cross section is primarily dependent on the ratio of the gravitational to internal energy, manifest as $G\lambda/c^2$. Since the cut-off distance of the potential has only a relatively gentle effect on the accretion cross section, we need not be unduly concerned with its precise determination in any actualy physical situation.

4. Weak Field - Supersonic Flow

In many cases the uniform flow at large distances will be supersonic. Since the gravitational field is weak, and if the Mach number is not excessive, a detached bow shock front will form upstream of the cylinder. Its distance from the cylinder will be of the order of the radius of the cylinder, R (Liepmann and Roshko, 1957). The flow downstream of the front must be subsonic. There will be a transonic region in the flow near the η-axis, but this is getting away from the part of the cylinder where accretion will occur and so should not affect the problem too severely. Considering the other simplifications we have introduced, it is probably not unreasonable to regard the accretion in such a case to be roughly the same as for the subsonic case where M, V_o, ρ_o, and c correspond to the values just behind the front. The fact that the flow was initially supersonic will be reflected in the changed value of c.

III. CONCLUSIONS

It is immediately clear that if such an accretion mechanism as described in section I is to be responsible for the formation of the moon (or any satellite) then we may have significant accretion only if $G\lambda/c^2$ is not too small, which is to say that the temperature must not be too high. If a ring-shaped proto-moon has mass per unit length $\lambda \sim 10^{16}$ g/cm (weak field corresponding to total ring mass around three times the mass of the moon) then for significant accretion to occur the temperature must be only in the hundreds of degrees Kelvin. This is not likely to be the case for a stellar wind mass flow focused by the earth. Passage through a shock front is unlikely to help and will only raise the temperature if the shock is at all adiabatic. Therefore, we are led to believe that the flow must originate at low temperature. It appears that only at a very early era in the formation of the solar system would material be available which both flows outward and is cold. And a centrifugal loss from the proto-sun or proto-earth would satisfy these requirements better than any coronal discharge such as the solar wind. Thus, such a process would occur at an epoch when the sun and earth were forming. Even though subsequent condensation of the lunar ring might be slow, at least its birth would be approximately coeval with that of the earth. This conclusion is further strengthened if centrifugal ejection represents a relatively short phase in the life of a proto-planet or proto-star.

If the impinging gas stream is cool it is probably unionized and will be unaffected by magnetic fields, so that this potential obstacle in the way of effective accretion would not arise.

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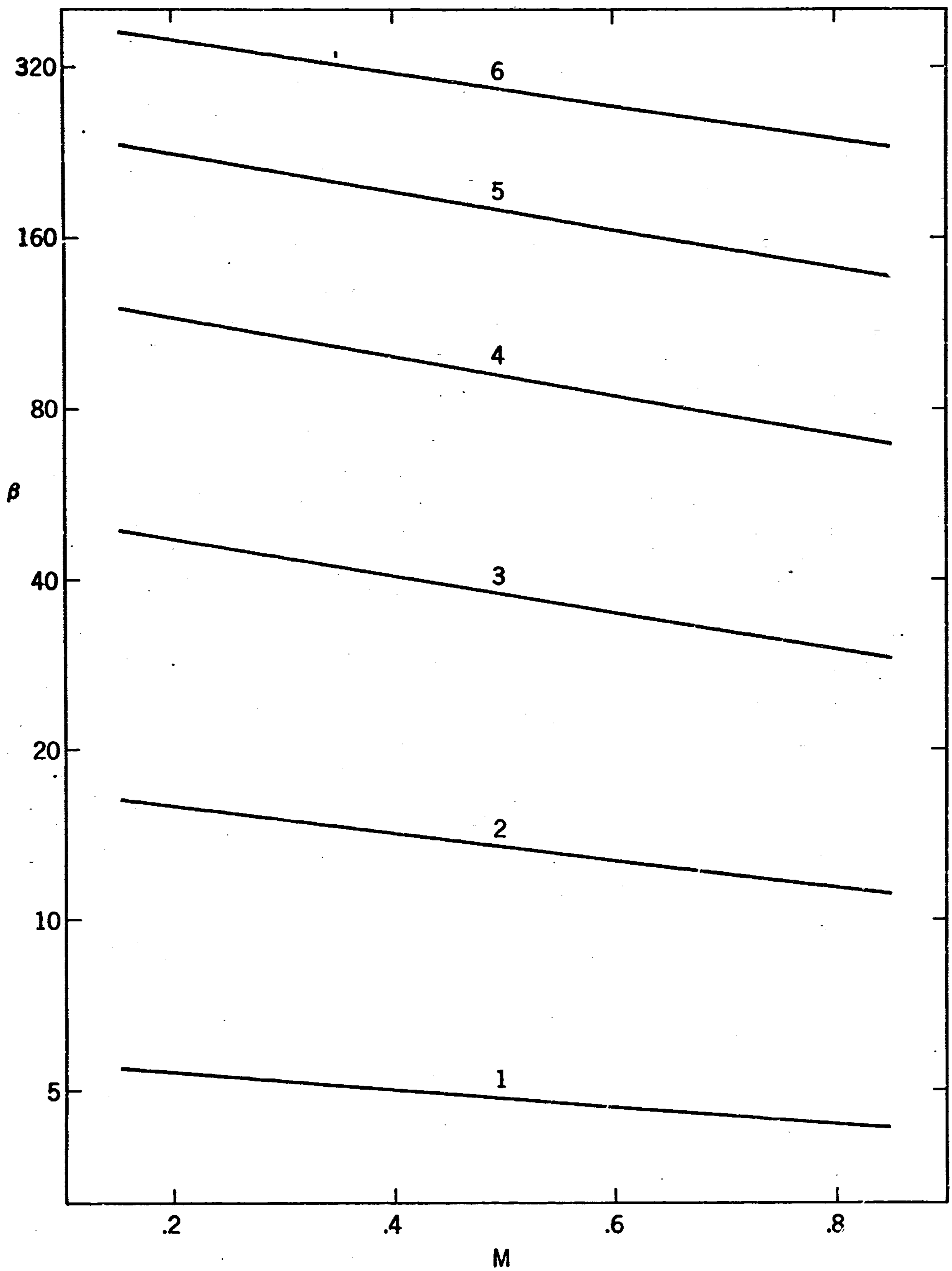

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