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by T. G. Proctor and H. H. Suber

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ABSTRACT

Construction theorems are presented for the solutions of certain nonautonomous differential equations. Exponential series solutions are given for a family of periodic solutions of a Ricatti equation with odd periodic coefficients and finite Fourier series expansion. This result is generalized to the existence of a first order quasi-periodic vector differential equation which is odd in the independent variable. Applications are included.

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1. INTRODUCTION

We consider in this paper the construction of solutions for certain nonautonomous differential equations. The first result makes use of a technique developed by Golomb [5] and Wasow [10] for constructing solutions of some non-linear differential equations by means of series of exponential functions. The technique as employed here gives explicit formulae for a family of periodic solutions of a Ricatti equation with odd periodic coefficient and finite Fourier series expansion.

Following this is a theorem concerning the existence of a family of almost periodic solutions of the vector differential equation

$$\dot{y} = g(t, y).$$

Here y is an m-vector; g(t, y) is quasi-periodic and odd in t and satisfies certain other conditions. (A quasi-periodic function is a function almost periodic in t with a finite base of frequencies $\omega_1, \omega_2, \cdot \cdot \cdot, \omega_n$.) The theorem is a generalization of a result concerning periodic solutions when g(t, y) is periodic in t [2], in particular, the Ricatti case mentioned above. The proof of the theorem utilizes a method devised by Kolmogorov [7] to overcome the problem of arbitrarily small divisors and gives a method of constructing approximations to the almost periodic solutions. Since we assume that g has a finite base of frequencies we can present the system of equations in an autonomous form by considering a higher dimensional version of the differential equation. The theorem is as follows:

Let x be an n vector, let y be an m vector and consider the differential equations

$$\dot{x} = \omega \quad , \qquad \dot{y} = f(x, y) \ , \tag{1.1}$$

where $f(x, y) = -f(-x, y)$, and where the components of $f(x, y)$ are analytic for $|y| \leq R_1$ and $|\mathrm{Im} x| \leq R_2$, and where f has period 2π in each of the components of x. Suppose that the vector $\Omega = (\omega_1, \omega_2, \cdot \cdot \cdot, \omega_n)$ satisfies an inequality

$$|k \cdot \Omega| \geq \frac{K}{|k|^{\nu}} \tag{1.2}$$

for some positive constants K and ν and all vectors $k = (k_1, k_2, \cdot \cdot \cdot, k_n)$ with integer components where $k \cdot \Omega \equiv \sum_{i=1}^{n} k_i \omega_i$ and $|k| \equiv \sum_{i=1}^{n} |k_i| \neq 0$. Then if $f(x, y)$ is sufficiently small for $|y| \leq R_1$ and $|\mathrm{Im} x| \leq R_2$, there is a neighborhood of $y = 0$ such that all solutions of (1.1) starting in this neighborhood are almost periodic with base frequencies Ω.

We note the requirement on Ω (inequality (1.2)) is not stringent. If $\nu > n$, such a constant K exists for almost all Ω (in the sense of Lebesgue measure) [2]. The proof of this theorem is given by constructing an infinite sequence of coordinate transformations so that in the limiting set of coordinates the differential equations can be integrated in a neighborhood of $y = 0$.

The last section gives an application of the results mentioned above.

2. THE PERIODIC CASE

Let I_k denote the set of all k-tuples of non-negative integers. The elements of I_k can be counted according to the following technique. Let $(n_1, n_2, \cdots, n_k) \in I_k$, let $m = 2^{n_1}3^{n_2} \cdots p_i^{n_i} \cdots p_k^{n_k}$ where p_i is the ith prime number $i = 1, 2, \cdots, k$, then denote $(n_1, n_2, \cdots, n_k)$ by N_m. Note that $N_1 = (0, 0, \cdots, 0)$, $N_2 = (1, 0, \cdots, 0)$, etc. The natural ordering of the non-negative integers then orders all the elements of I_k, $N_1, N_2, \cdots$. For $N_n, N_m \in I_k$ we make the following remarks:

i) m is prime iff N_m is of the form $(0,0, \cdots,1, \cdots 0)$.

ii) Define addition in I_k component-wise. Then $N_n + N_m = N_\ell$ iff $nm = \ell$.

iii) We say that $N_n \leq N_m$ iff $n \leq m$ and $N_n < N_m$ iff $n < m$. From ii) it is clear that $N_n + N_m = N_\ell$ implies that $N_n \leq N_\ell$ and $N_m \leq N_\ell$.

For ω real let $\Omega = (\omega, -\omega, 2\omega, -2\omega, \cdot \cdot \cdot, k\omega, -k\omega)$. Now for $N_m = (n_1, n_2, \cdot \cdot \cdot, n_{2k-1}, n_{2k}) \varepsilon I_{2k}$ let m' be the integer so that $N_{m'} = (n_2, n_1, \cdot \cdot \cdot, n_{2k}, n_{2k-1})$. We observe that $N_m \cdot \Omega = -N_{m'} \cdot \Omega$. Let O_k denote the class of odd periodic functions of the real variable t with period $2\pi\omega$ which have finite Fourier series containing only terms of the type $f_j^* e^{ij\omega t}$ for $j = \pm 1, \pm 2, \cdot \cdot \cdot, \pm k$. Using the above notation we may represent functions in O_k in the form

$$f(t) = \sum_{\substack{N_n \varepsilon I_{2k} \\ n \text{ prime}}} f_n e^{iN_n \cdot \Omega t} \tag{2.1}$$

where $f_n = -f_n$ since f is odd.

<u>Theorem</u> <u>2.1</u>. For $\eta > 0$ sufficiently small the differential equation

$$y' = \eta(a(t) + b(t)y + c(t)y^2), \tag{2.2}$$

where a, b and c ε O_k, has an even periodic solution y(t) of the form

$$y(t) = \sum_{N_n \varepsilon I_{2k}} y_n e^{iN_n \cdot \Omega t} \quad . \tag{2.3}$$

<u>Proof</u>. Assume that (2.2) has a solution of the form indicated. Then formally we have

$$\sum_{N_n \varepsilon I_{2k}} iN_n \cdot \Omega y_n e^{iN_n \cdot \Omega t} =$$

$$\eta \sum_{N_n \varepsilon I_{2k}} \left[a_n + \sum_{\substack{mp=n \\ p \text{ prime}}} y_m b_p + \sum_{\substack{\ell mp=n \\ p \text{ prime}}} y_\ell y_m c_p \right] e^{iN_n \cdot \Omega t}, \tag{2.4}$$

where a_j, b_j and c_j, j prime are the coefficients of the series representations (2.1) of a, b, and c respectively.

Now in case $N_n \cdot \Omega \neq 0$ we write

$$y_n = \begin{cases} \dfrac{\eta}{iN_n \cdot \Omega} a_n , & n \text{ prime} \\ \dfrac{\eta}{iN_n \cdot \Omega} \left[\sum\limits_{\substack{mp=n \\ p \text{ prime}}} y_m b_p + \sum\limits_{\substack{\ell mp=n \\ p \text{ prime}}} y_\ell y_m c_p \right]. & n \text{ not prime} \end{cases} \tag{2.5}$$

and for $N_n \cdot \Omega = 0$,

$$y_n = 0 . \tag{2.5a}$$

Suppose that the terms containing $e^{iN_n \cdot \Omega t}$ on the right side of (2.4) vanish whenever $N_n \cdot \Omega = 0$. Then by remark iii) above we see that (2.5) defines y_n recursively so that (2.3) will be a formal solution. To show that this is indeed the case we present the following.

Lemma 2.2. Let $\{y_n\}$ be the sequence of numbers defined by 2.5. Then $y_n = y_{n^*}$.

Proof. The proof is by induction. If $N_n \cdot \Omega = 0$ then clearly $y_n = y_{n'}$ and so in particular for $n = 1$. Now suppose that $y_m = y_{m'}$ for all $m < n$. Then we may write

$$y_n = \frac{1}{iN_n \cdot \Omega}\left[\sum_{mp=n} y_m b_p + \sum_{\ell mp=n} y_\ell y_m c_p\right]$$

$$= \frac{1}{-iN_{n'} \cdot \Omega}\left[\sum_{mp=n} y_{m'}(-b_{p'}) + \sum_{\ell mp=n} y_{\ell'} y_{m'}(-c_{p'})\right]$$

where p is prime. But $mp = n$ iff $m'p' = n'$. So we see that $y_n = y_{n'}$ which proves the lemma.

Lemma 2.3. If $N_n \varepsilon I_{2k}$ is such that $N_n \cdot \Omega = 0$ then

$$\sum_{mp=n} y_m b_p + \sum_{\ell mp=n} y_\ell y_m c_p + \sum_{mp=n'} y_m b_p + \sum_{\ell mp=n'} y_\ell y_m c_p = 0,$$

where p is prime.

Proof. By Lemma 2.2 we have $y_n = y_{n'}$ for all n and since b and $c \ \varepsilon \ O_k$, we may write

$$\sum_{mp=n} y_m b_p = \sum_{mp=n} y_{m'}(-b_{p'}) = -\sum_{mp=n'} y_m b_p$$

and

$$\sum_{\ell mp=n} y_\ell y_m c_p = \sum_{\ell mp=n} y_{\ell'} y_{m'}(-c_{p'}) = -\sum_{\ell mp=n'} y_\ell y_m c_p,$$

where p is prime.

Now in order to show that (2.2) is a solution of equation (2.1) we will prove that formal series (2.3) with y_n defined by (2.5) converges uniformly and absolutely for all t and η sufficiently small.

Let C represent the complex plane, for $z = (z_1, z_2, \cdot \cdot \cdot, z_{2k}) \varepsilon C^{2k}$ and $N_n \varepsilon I_{2k}$ define $z^{N_n} = z_1^{n_1} z_2^{n_2} \cdot \cdot \cdot z_{2k}^{n_{2k}}$ and $|z| = \max_j |z_j|$. A function f mapping C^{2k} into C is analytic in the polydisk $\{z \varepsilon C^{2k} : |z_i| \leq r\}$ of radius r about the origin iff f has the representation,

$$f(z) = \sum_{N_n \varepsilon I_{2k}} a_n z^{N_n}$$

where the sum is uniformly and absolutely convergent in the polydisk. In case f is analytic, Cauchy's inequality gives for $|z| \leq \delta$

$$|a_n| \leq M/\delta^{|N_n|}$$

where $|N_n| = n_1 + n_2 + \cdot \cdot \cdot + n_{2k}$ and $M = \sup_{|z|=\delta} |f(z)|$.

Now for $z \varepsilon C^{2k}$ let

$$a^*(z) = \sum_{j=1}^{2k} |a_j| z_j \ ,$$

$$b^*(z) = \sum_{j=1}^{2k} |b_j| z_j \ , \tag{2.6}$$

$$c^*(z) = \sum_{j=1}^{2k} |c_j| z_j \ ;$$

and let $u(z) = f(a^*(z), b^*(z), c^*(z))$ where

$$f(a,b,c) = \begin{cases} \frac{1-\eta b}{2\eta c} - \frac{1-\eta b}{2\eta c}\left[1 - \frac{4\eta^2 ac}{(1-\eta b)^2}\right]^{\frac{1}{2}}, & c \neq 0, \\ \frac{\eta a}{1-\eta b}, & c = 0. \end{cases}$$

Note that u is the solution of the equation

$$u(z) = \eta[c^* u^2(z) + b^* u(z) + a^*] \tag{2.7}$$

which vanishes when $a^* = b^* = c^* = 0$. We see that $u(z)$ is an analytic function of z in any region which does not include zeros of the function

$$g(z) = (1-\eta b^*(z))^2 - 4\eta^2 a^*(z) c^*(z).$$

Now for $\delta > 0$ choose $\eta_o > 0$ so that $|g(z)| > 0$ whenever $|z| \leq 1 + \delta$; e.g. for $L = \max_j\{|a_j|, |b_j|, |c_j|\}$, let $\eta_o < \frac{1}{8Lk(1+\delta)}$, then $\eta_o|a^*|$, $\eta_o|b^*|$, $\eta_o|c^*| < \frac{1}{4}$ and we see that in this case $|g(z)| > 0$. Now for all η, $0 < \eta < \eta_o$ we have u analytic in the polydisk $|z| \leq 1 + \delta$. Hence, in this polydisk u has the representation

$$u(z) = \sum_{N_n \varepsilon I_{2k}} u_n z^{N_n}, \tag{2.8}$$

where

$$|u_n| \leq M/(1+\delta)^{|N_n|}$$

with $M = \sup_{|z_j|=1+\delta} |u(z)|$.

On the other hand, substituting from (2.8) into (2.7) and using (2.6) we obtain

$$u_n = \begin{cases} \eta|a_n| & n \text{ prime} \\ \eta \sum\limits_{\substack{mp=n \\ p \text{ prime}}} u_m|b_p| + \eta \sum\limits_{\substack{\ell mp=n \\ p \text{ prime}}} u_\ell u_m c_p, & n \text{ not prime.} \end{cases} \tag{2.9}$$

Comparing this with the recursion formula (2.4), with $y_1 = 0$, we see immediately that

$$|y_n| \leq \frac{1}{\omega}|u_n|, \quad n = 1, 2, \cdot \cdot \cdot.$$

Since $|e^{iN_n \cdot \Omega t}| = 1$ for all t we have

$$\Big| \sum_{N_n \varepsilon I_{2k}} y_n e^{iN_n \cdot \Omega t} \Big| \leq \frac{1}{\omega} \sum_{N_n \varepsilon I_{2k}} |u_n| ,$$

$$\leq \frac{M}{\omega} \sum_{N_n \varepsilon I_{2k}} \frac{1}{(1+\delta)^{|N_n|}} ,$$

$$\leq \frac{M}{\omega} \left(\frac{1+\delta}{\delta}\right)^{2k} ,$$

which not only proves absolute and uniform convergence, but also gives a bound for the solution y(t).

Remarks:

i) In the proof of Lemma 2.2 we showed that $y_n = y_{n'}$ for all $N_n \varepsilon I_{2k}$ such that $N_n \cdot \Omega \neq 0$. From this we conclude that the solution found above is even in t. Note also that the solution has zero mean value.

ii) The particular order relation used here for I_k is not essential to the proof. See Golomb [5] and Wasow [10] for different schemes.

iii) It is possible to use the result in this section directly to obtain solutions with mean value other than zero. Let f(t, y) represent the right side of equation (2.1) and suppose that y(t) is the solution of

$$y' = f(t, y)$$

given above. For any fixed constant c, let $z = y + c$ in (2.1). The theorem gives a technique for obtaining a solution of the new equation

$$z' = f^*(t, z),$$

where $f^*(t, z) = f(t, z - c)$ with zero mean value. This in turn gives a solution to the original equation with mean value $-c$.

iv) Let y be an n-vector, let $p_\ell(t)$ be an n-vector with components $p_\ell^{(j)}(t) \varepsilon O_k$, $j = 1,2,\cdots,n$, $\ell = 1,2,\cdots$. Then the differential equation

$$y' = \eta \sum_{\substack{N_\ell \varepsilon I_{2k} \\ \ell<\infty}} p_\ell(t) y^{N_\ell} \qquad (2.10)$$

where the right side converges for $|y| \leq r$ may be solved using the techniques of this section. The only essential difference occurs when one attempts to find an analytic solution of the corresponding equation (2.8). Here one may use the implicit function theorem to show existence of such a solution for η sufficiently small.

v) The existence of periodic solutions of equations of the form (2.10), for η small is shown by Hale [6, p. 45].

3. THE QUASIPERIODIC CASE

For any positive integer n let J_n denote the set of all n-tuples of integers, for $\alpha = (\alpha_1, \alpha_2, \cdot \cdot \cdot, \alpha_n) \varepsilon J_n$ let $|\alpha| = \sum_{i=1}^{n} |\alpha_i|$ and let C^m be all m vectors $(y_1, \cdots, y_m)$ where each component is a complex number. For simplicity we will treat only the case where y is a m = 1 vector.

We shall be concerned in this section with functions defined and analytic on (x, y) subsets of $C^n \times C^1$ into C^1 which are periodic of period 2π in each component of x. These subsets will be of the form

$$D(r, \rho) = \{(x,y) \varepsilon C^{1+n}: \ |\mathrm{Im} x| \leq \rho, \ |y| \leq r\}$$

where the norm $| \ |$ of the vector x denotes the maximum of the absolute value of its components. We denote the class of such functions by $P(r, \rho)$ and note that any $g \varepsilon P(r, \rho)$ has a Fourier-Taylor series representation

$$g(x, y) = \sum_{|\alpha|, |\beta|=0}^{\infty} g_{\alpha\beta} e^{i\alpha \cdot x} y^{\beta},$$

where the $g_{\alpha\beta}$ are complex numbers and where the sum is taken over all $\alpha \varepsilon J_n$ and $\beta \varepsilon I_1$.

Several lemmas are listed below without proof. The proofs are elementary and are similar to those given in [2].

Lemma 3.1. Let $h \varepsilon P(R_1, R_2)$ and let $|h(x,y)| \leq M$, $M > 0$ in $D(R_1, R_2)$. The Fourier-Taylor coefficients, given by

$$h_{\alpha\beta} = \frac{1}{(2\pi)^n} \iint \cdots \int h_\beta(x) e^{-i\alpha \cdot x} dx_1 dx_2 \cdots dx_n, \quad \alpha \varepsilon J_n, \quad \beta \varepsilon I_1$$

where

$$h_\beta(x) = \frac{1}{\beta!} \frac{\partial^\beta}{\partial y^\beta} h(x, y)\bigg|_{y=0} ,$$

and where the jth integral is taken from $x_j = 0$ to $x_j = 2\pi$, satisfy the inequality

$$|h_{\alpha\beta}| \leq \frac{Me^{-|\alpha|R_2}}{R_1^{|\beta|}} .$$

If $h(-x, y) = -h(x, y)$ in the above we have $h_{-\alpha\beta} = -h_{\alpha\beta}$ and conversely. If $h(-x, y) = h(x, y)$ we have $h_{-\alpha\beta} = h_{\alpha\beta}$ and conversely.

Lemma 3.2. If the elements of the sequence $\{h_{\alpha\beta}\}$ $\alpha \varepsilon J_n$, $\beta \varepsilon I_1$ satisfy

$$|h_{\alpha\beta}| \leq \frac{Me^{-|\alpha|R_2}}{R_1^{|\beta|}} ,$$

then

$$h(x,y) = \sum_{|\alpha|,|\beta|=0}^{\infty} h_{\alpha\beta} e^{i\alpha\cdot x} y^{\beta}$$

is analytic for $|y| \leq R_1 e^{-\delta}$, $|\text{Im} x| \leq R_2 - \delta$ for any positive $\delta < 1$ such that $R_2 - \delta > 0$; and in this domain we have

$$|h(x, y)| \leq \frac{2^{2n+1} M}{\delta^{n+1}} .$$

Lemma 3.3. For all positive numbers m, ν, δ we have

$$m^{\nu} \leq \left(\frac{\nu}{e}\right)^{\nu} \frac{e^{m\delta}}{\delta^{\nu}} .$$

Lemma 3.4. (Cauchy's Inequality). If the complex valued function h(z) is analytic and bounded by M for $|z| \leq R$, $M,R > 0$, then for $|z| \leq Re^{-\delta}$, $0 < \delta < 1$, we have

$$\left|\frac{dh}{dz}\right| \leq \frac{2M}{R\delta} .$$

The proof of the main result of this section depends almost entirely on the following considerations.

Let $f \in P(R_1, R_2)$ and satisfy $f(-x, y) = -f(x, y)$, let f satisfy

$$|f(x, y)| \leq M \equiv \delta^{2(n+\nu)+7} \tag{3.1}$$

in $D(R_1, R_2)$ where δ is specified below, let ω satisfy (1.2) for some positive constants K and ν and all $\alpha \in J_n$ with $|\alpha| \neq 0$ and consider the differential equations

$$\dot{x} = \omega, \qquad \dot{y} = f(x, y). \tag{3.2}$$

Lemma 3.5. For each x in $|\mathrm{Im}x| \leq R_2 - 2\delta$ there exists an invertible transformation U defined on a subset of C into C, given by $U(\eta) = y$ where

$$y = \eta + u(x, \eta), \quad |u(x, \eta)| \leq \frac{M}{\delta^{n+\nu+2}} \tag{3.3}$$

for $|\eta| \leq R_1 e^{-4\delta}$, $|\mathrm{Im}x| \leq R_2 - 2\delta$. Putting $\eta = U^{-1}(y)$ in (3.3) we obtain the differential equations (3.2) in the new coordinates

$$\dot{x} = \omega \,, \qquad \dot{\eta} = f^*(x, \eta); \tag{3.4}$$

and in $D(R_1 e^{-4\delta}, R_2 - 2\delta)$

$$|f^*(x, \eta)| \leq M^{3/2}. \tag{3.5}$$

Further we have $f^*(x, \eta) = -f^*(-x, \eta)$ and $f^* \in P(R_1 e^{-4\delta}, R_2 - 2\delta)$. Here δ is taken as a positive number satisfying

$$\delta \leq \min\left\{ \left(\frac{R_1}{16}\right)^{1/n+\nu+4}, \; R_2, \; \frac{R_1 K}{2^{2n+3}}\left(\frac{e}{\nu}\right)^{\nu}, \; \frac{K}{2^{2n+1}}\left(\frac{e}{\nu}\right)^{\nu}, \; \left(\frac{1}{2}\right)^{1/n+\nu+\delta} \right\} .$$

Proof. a) Definition of $u(x, \eta)$: Choose $u(x, \eta)$ as the solution with mean value zero of

$$\frac{\partial u}{\partial x}\,\omega = f(x, \eta) \tag{3.6}$$

where $\frac{\partial u}{\partial x}$ is the vector with elements $\frac{\partial u}{\partial x_j}$ in the ith row and jth column. This gives

$$u(x, \eta) = \sum_{|\alpha|, |\beta|=0}^{\infty} {}_{\alpha\beta} e^{i\alpha\cdot x} \eta^{\beta}$$

where

$$u_{\alpha\beta} = \frac{f_{\alpha\beta}}{i\alpha\cdot\omega} \qquad \alpha \neq 0, \qquad u_{o\beta} = 0.$$

Since $|f(x, \eta)| \leq M$ for $|y| \leq R_1$, $|Imx| \leq R_2$, Lemma 3.1, inequality (1.2) and the above imply

$$|u_{\alpha\beta}| \leq \frac{Me^{-|\alpha|R_2}}{R_1^{|\beta|}} \quad \frac{|\alpha|^{\nu}}{K} .$$

By Lemma 3.3

$$|u_{\alpha\beta}| \leq \frac{M}{K\delta^{\nu}} \left(\frac{\nu}{e}\right)^{\nu} \frac{e^{-|\alpha| (R_2-\delta)}}{R_1^{|\beta|}} .$$

Hence, using Lemma 3.2 we have $u(x, \eta)$ defined and analytic for $|\eta| \leq R_1 e^{-\delta}$, $|Imx| \leq R_2 - 2\delta$ and bounded in this domain by

$$| u(x, \eta)| \leq \frac{2^{2n+1}}{\delta^{n+1+\nu}} \frac{M}{K} \left(\frac{\nu}{e}\right)^{\nu} . \tag{3.7}$$

Thus, inequality (3.3) is valid and we note that $u(x, \eta) = u(-x, \eta)$ and $u \in P(R_1 e^{-\delta}, R_2 - 2\delta)$.

b) The transformation U : By equation (3.3) the set $D = \{\eta \in C \ : |\eta| \leq R_1 e^{-2\delta}\}$ is mapped into a set containing $A = \{y \in C \ : |y| \leq R_1 e^{-3\delta}\}$; i.e. $U(D) \supseteq A$. Since

$$\frac{\partial u}{\partial \eta} \geq \frac{1}{2} > 0$$

for $|\eta| \leq R_1 e^{-2\delta}$, $|\text{Im}x| \leq R_2 - 2\delta$, we see that U^{-1} is defined on A. Thus for $|\eta| \leq R_1 e^{-4\delta}$, $|\text{Im}x| \leq R_2 - 2\delta$, $u(x, \eta)$ is defined and (3.7) holds.

c) The function $f^*(x, \eta)$: Substituting from (3.3) into (3.2) gives

$$(1 + \frac{\partial u}{\partial \eta})\dot{\eta} = f(x, y) - f(x, \eta),$$

so that

$$f^*(x,\eta) = (1 + \frac{\partial u}{\partial \eta}(x,\eta))^{-1}(f(x,\eta + u(x,\eta)) - f(x,\eta)); \quad (3.8)$$

and we note here that $-f^*(-x,\eta) = f^*(x,\eta)$.

Now we have

$$|(1 + \frac{\partial u}{\partial \eta})|^{-1} \leq 2 \; ; \qquad (3.9)$$

hence

$$|f^*(x, \eta)| \leq 2\,|f(x,\eta + u(x, \eta)) - f(x, \eta)|.$$

But

$$|\, f(x,\eta + u(x,\eta)) - f(x,\eta)| \leq \sup \{|\frac{\partial f}{\partial \eta}|\} \frac{2^{2n+1}}{\delta n+1+\nu} \frac{M}{K} (\frac{\nu}{e})^{\nu},$$

where the supremum is taken over $|y| \leq R_1 e^{-\delta}$, $|\mathrm{Im} x| \leq R_2$. By Cauchy's inequality

$$\left|\frac{\partial f}{\partial y}\right| \leq \frac{2M}{R_1 \delta} ,$$

thus

$$f^*(x, \eta) \leq \frac{2^{2n+3}}{KR_1 \delta^{n+\nu+2}} \left(\frac{\nu}{e}\right)^{\nu} M^2 \leq M^{3/2},$$

and the proof of the lemma is complete.

<u>Theorem</u> <u>3.1</u>. Let f be as in Lemma 3.5. Then if M (and thus δ) is sufficiently small for each x in $|\mathrm{Im} x| \leq R_2/2$ there exists an invertible transformation, V, defined and analytic on $\{\eta \in C : |\eta| \leq R_1 e^{-R_2}\}$ into C^m, given by $V(\eta) = y$ where

$$y = \eta + v(x, \eta). \tag{3.10}$$

Denoting the inverse transformation $\eta = V^{-1}(y)$ we obtain the differential equations (3.3) in new coordinates

$$\dot{x} = \omega, \quad \dot{\eta} = 0.$$

Furthermore we have $v(-x, \eta) = v(x, \eta)$.

Proof. Choose $\delta_1 > 0$ so that for $\delta_j = \delta_{j-1}^{3/2}$ we have

$$\sum_{j=1}^{\infty} \delta_j \le \frac{R_2}{4} .$$

$$\delta_1 \le \left\{ \left(\frac{R_1}{16}\right)^{1/n+\nu+4}, \ \frac{R_1 K}{2^{2n+3}}\left(\frac{e}{\nu}\right)^{\nu}, \ \frac{K}{2^{2n+1}}\left(\frac{e}{\nu}\right)^{\nu}, \ \left(\frac{1}{2}\right)^{1/n+\nu+5} \right\} .$$

Apply Lemma 3.5 iteratively j times. Let $u_i(x, y)$ denote the function in the transformation of coordinates at the ith step and let $f_i(x, \eta)$ denote the corresponding right side of the differential equation. We obtain the composite map $F_j(x, \eta) = y$ where

$$F_j(x,\eta) = \eta + u_j(x,\eta) + u_{j-1}(x,u_j(x,\eta))+\cdots+ u_1(x,\eta + u_j(x,\eta)+\cdots+ u_2(x,\eta + u_j(x, \eta) + \cdots + u_3(x, \eta)),$$

defined for $|\eta| \le R_1 \exp[-4 \sum_{i=1}^{j} \delta_i]$, $|\mathrm{Im}x| \le R_2 - 2 \sum_{i=1}^{j} \delta_i$,

where in the associated differential equations

$$\dot{x} = \omega \quad , \qquad \dot{\eta} = f_j(x, \eta) ,$$

the functions $f_j(x, \eta)$ satisfy

$$|f_j(x, \eta)| \le M_j = \delta_j^{\,2(n+\nu)+7} .$$

We observe that the composite transformations are defined for all j in $|\eta| \le R_1 e^{-R_2}$, $|\mathrm{Im}x| \le R_2/2$, and that

$$\sum_{i=1}^{\infty} |u_i| < \infty;$$

thus the limiting composite transformation

$$F(x, \eta) = \lim_{j\to\infty} F_j(x, \eta) = \eta + v(x, \eta), \qquad (3.11)$$

will exist in the above domain. In the coordinates defined by (3.11) the differential equation (3.3) becomes

$$\dot{x} = \omega \quad , \qquad \dot{\eta} = 0 \ .$$

4. APPLICATION

Adrianov [1] and Gelmand [4] outlined a procedure for finding a transformation $x = Z(t)y$ so a given differential equation

$$\frac{dx}{dt} = Q(t)x,$$

where $Z(t)$ $Q(t)$ are almost periodic $n \times n$ matrices and where P satisfies certain conditions and x in an n-vector, becomes

$$\dot{y} = Ay, \qquad A \text{ constant}$$

in the new coordinates. We shall follow this procedure and use Theorem 3.6 to effect the same transformation in circumstances where the earlier work fails to apply.

Let H be the class of all functions

$$f(t) = g(\omega_1 t, \omega_2 t, \cdot \cdot \cdot, \omega_n t)$$

where $g(u_1, u_2, \cdot \cdot \cdot, u_n)$ is real analytic and has period 2π in each u_i, $i = 1, 2, \cdot \cdot \cdot, n$. Consider the differential equations

$$\frac{dx_1}{dt} = [q(t) + \eta q_{11}(t)]x_1 + \eta q_{12}(t)x_2,$$

$$\frac{dx_2}{dt} = \eta q_{21}(t)x_1 + [q(t) + \eta q_{22}(t)]x_2, \tag{4.1}$$

where $\eta > 0$ and q, $q_{ij} \in H$ and $q_{ij}(t) = -q_{ij}(-t)$, $i,j = 1, 2$, and where $\omega = (\omega_1, \omega_2, \cdot \cdot \cdot, \omega_n)$ satisfies inequality (1.2). We make the change of coordinates

$$x_1 = y_1 + \tau y_2, \qquad x_2 = y_2 \tag{4.2}$$

where τ is any almost periodic solution of the differential equation

$$\dot{\tau} = \eta[q_{12} + (q_{11} - q_{22})\tau - q_{21}\tau^2]. \tag{4.3}$$

In the new coordinates (4.1) becomes

$$\frac{dy_1}{dt} = [q(t) + \eta(q_{11} - \tau q_{21})]y_1$$

$$\frac{dy_2}{dt} = \eta q_{21}y_1 + \eta(q_{21}\tau + q_{22} + q)y_2 \tag{4.4}$$

Theorem 3.6 guarantees that for η small enough equation (4.3) has almost periodic solutions, which belong to H. Equation (4.4) may now be integrated to obtain

$$y_1 = a^*(t)e^{a_o t}c_1$$

$$y_2 = b^*(t)e^{a_o t}c_1 + b^{**}(t)e^{b_o t}c_2$$

where a_o and b_o are the mean values of $q + \eta q_{11} - \eta\tau q_{21}$ and $q + \eta q_{22} + \eta q_{21}\tau$ respectively and a^*, b^* and $b^{**} \varepsilon H$.

Reversing the change of coordinates (4.2) we obtain a fundamental matrix solution of (4.1) of the form

$$\phi(t) = P(t)e^{At}, \qquad A = \begin{bmatrix} a_o & 0 \\ 0 & b_o \end{bmatrix},$$

where the elements of P(t) are almost periodic and belong to H.

The change of coordinates

$$x = P(t)z$$

in (4.1) yields

$$\dot{Z} = AZ .$$

The case treated by Gelmand [4] required that the linear term in the resulting differential equation for τ have mean value which dominates the other elements in order that there exists an almost periodic solution of this equation. Thus Theorem 3.6 permits us to consider a new situation.

If in 4.1 we require that $q, q_{ij} \varepsilon O_k$ (defined in section 2) i, j = 1, 2, then we may use Theorem 2.1 to obtain an explicit representation for periodic solutions of equation (4.3) for η sufficiently small. Then we may integrate equations (4.4) and obtain explicit solutions of (4.1). Note that if

$$\tau(t) = \sum_{n=1}^{\infty} \tau_n e^{iN_n \cdot \Omega t}$$

is the solution of (4.3) given by Theorem 2.1 and $\tau_1 = 0$ we have periodic solution of (4.1). The existence of these solutions was shown by Epstein [3].

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