NASA CONTRACTOR REPORT

NASA CR-61254

June 1968

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THE THEORY OF EXCEEDANCES

Prepared under Contract No. NAS 8-20082 by
N. E. Rich

LOCKHEED MISSILES AND SPACE COMPANY

For

NASA-GEORGE C. MARSHALL SPACE FLIGHT CENTER
Marshall Space Flight Center, Alabama

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Huntsville, Alabama

For

Aero-Astrodynamics Laboratory

NASA-GEORGE C. MARSHALL SPACE FLIGHT CENTER

FOREWORD

This report presents the results of work performed by Lockheed's Huntsville Research & Engineering Center while under subcontract to Northrop Nortronics (NSL PO 5-09287) for Marshall Space Flight Center (MSFC), Contract NAS8-20082. This task was conducted in response to the requirement of Appendix A-1, Schedule Order No. 23.

The NASA technical coordinator for this study is O.E. Smith of the Aerospace Environment Division of the Aero-Astrodynamics Laboratory.

SUMMARY

In this document the probability that the m^{th} largest of the past L observations will be exceeded k times in N future trials is derived. The expected number of exceedances and the variance are also found. An interpolative scheme is presented for defining probabilities associated with values not actually observed.

The EDNE computer program, which calculates the probabilities of exceedances and plots the interpolative solution, is discussed. A user's manual and listing of this program is also included.

Several asymptotic distributions for large future sample sizes are developed.

CONTENTS

Section 1
INTRODUCTION

In many situations, past observations or results are used to predict future events. If a variable, such as wind speed, has been observed L times in the past, it is often desirable to make probability statements about the values it will assume in N future observations. At times it is sufficient to study the number of times a given value is surpassed in the future trials. For instance, if the wind is great enough to blow over a building or a vehicle erect on the launch pad, the exact strength of the wind is not important. For such cases the theory of exceedances has been developed.

This document is intended to be a basic treatment of the theory of exceedances. The theoretical development is quite complete, and a practical example is included. For a more authoritative treatise of the subject, the reader is referred to Chapter 2 of E. J. Gumbel's Statistics of Extremes, Columbia University Press, New York, 1958.

Section 2
THEORETICAL DERIVATION OF THE DISTRIBUTION OF THE NUMBER OF EXCEEDANCES

A set of L observations — $x_1, x_2, \ldots, x_L$ — has been obtained in such a manner that the observations comprise a random sample; that is, the observations are independent random variables drawn from the same probability space with cumulative distribution function F [$F(x) = \Pr$ (an observation $\leq x$)]. The random variables are assumed to be continuous. The distribution function F, however, need not be known. The observations are arranged in descending order and written $x_{(1)}, x_{(2)}, \ldots, x_{(L)}$, where $x_{(1)} \geq x_{(2)} \geq \ldots \geq x_{(L)}$. Thus, $x_{(1)}$ is the largest observation, $x_{(2)}$ is the second largest, and $x_{(m)}$ is the m^{th} largest.

In the future a new set of N observations — $y_1, \ldots, y_N$ — will be drawn from the same population. Of these N observations, k of them will be greater than $x_{(m)}$, the m^{th} largest of the past L observations ($k = 0, 1, \ldots, N$; $m = 1, 2, \ldots, L$). The probability density function of k [$p(L, m, N, k)$ = probability that k observations out of N will exceed the m^{th} largest of L past observations] is derived below.

2.1 THE PROBABILITY DENSITY OF THE NUMBER OF EXCEEDANCES AS A FUNCTION OF THE INITIAL DISTRIBUTION

The m^{th} largest of the L values, $x_{(m)}$, partitions the real line into two sections. For any number y, either $y \leq x_{(m)}$ or $y > x_{(m)}$. If y is drawn from the underlying probability distribution of the observations, then

$$\Pr(y \leq x_{(m)}) = F(x_{(m)}) \equiv F_m$$

$$\Pr(y > x_{(m)}) = 1 - F(x_{(m)}) = 1 - F_m$$

where F_m is the distribution function for $x_{(m)}$. Thus, if one value is drawn in the new sample (N = 1), the probability that it does not or does exceed $x_{(m)}$ is

$$\Pr(x_{(m)} \text{ exceeded 0 times} \mid N = 1) = F_m$$

$$\Pr(x_{(m)} \text{ exceeded 1 time} \mid N = 1) = 1 - F_m$$

where $\Pr(A \mid B)$ denotes the conditional probability of A given B.

If two independent observations y_1 and y_2 are made (N = 2), then the following hold:

$$\begin{aligned}\Pr(x_{(m)} \text{ exceeded 0 times} \mid N = 2) &= \Pr(y_1 \le x_{(m)} \text{ and } y_2 \le x_{(m)}) \\ &= \Pr(y_1 \le x_{(m)}) \Pr(y_2 \le x_{(m)}) = F_m^2\end{aligned}$$

$$\begin{aligned}\Pr(x_{(m)} \text{ exceeded 1 time} \mid N = 2) &= \Pr(y_1 \le x_{(m)} \text{ and } y_2 > x_{(m)}) \\ &+ \Pr(y_1 > x_{(m)} \text{ and } y_2 \le x_{(m)}) \\ &= F_m(1 - F_m) + (1 - F_m) F_m \\ &= 2(1 - F_m) F_m\end{aligned}$$

$$\begin{aligned}\Pr(x_{(m)} \text{ exceeded 2 times} \mid N = 2) &= \Pr(y_1 > x_{(m)} \text{ and } y_2 > x_{(m)}) \\ &= (1 - F_m)^2\end{aligned}$$

Extending this procedure to N future observations,

$$\Pr(x_{(m)} \text{ exceeded 0 times} \mid N) = \Pr(y_1 \le x_{(m)}, \ldots, y_N \le x_{(m)})$$
$$= F_m^N$$

$$\Pr(x_{(m)} \text{ exceeded 1 time} \mid N) = \sum_{i=1}^{N} \Pr(y_i > x_{(m)} \text{ and all others} \le x_{(m)})$$
$$= N(1 - F_m)\, F_m^{N-1}$$

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•

$$\Pr(x_{(m)} \text{ exceeded N-1 times} \mid N) = \sum_{i=1}^{N} \Pr(y_i \le x_{(m)} \text{ and all others} > x_{(m)})$$
$$= N(1 - F_m)^{N-1}\, F_m$$

$$\Pr(x_{(m)} \text{ exceeded N times} \mid N) = \Pr(y_1 > x_{(m)}, \ldots, y_N > x_{(m)})$$
$$= (1 - F_m)^N$$

This can be written

$$\Pr(x_{(m)} \text{ exceeded k times} \mid N) = \binom{N}{k} (1 - F_m)^k\, F_m^{N-k}, \quad k = 0, 1, \ldots, N \quad m = 1, 2, \ldots, L \qquad (2.1\text{-}1)$$

where

$$\binom{N}{k} = \frac{N!}{k!\,(N-k)!}$$

A random variable which follows the above density function is said to have a binomial distribution.*

As the form given in Equation (2.1-1) depends upon the value F_m, the probability should be written as a conditional probability. The probability that the m^{th} largest of L past observations will be exceeded k times in N future observations, given the distribution F_m, is

$$p(L, m, N, k \mid F_m) = \binom{N}{k} (1 - F_m)^k \, F_m^{N-k} \tag{2.1-2}$$

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*The expression $\binom{A}{B} (1 - F_m)^B \, F_m^{A-B}$ is the (k+1) term in the expansion of $([1 - F_m] + F_m)^A$. Thus,

$$\binom{A}{B} = \frac{A!}{B!\,(A-B)!}$$

is called a binomial coefficient. The following identities are used repeatedly in the theoretical development:

$$\binom{A}{B} = \binom{A}{A-B}$$

$$\binom{A+1}{B} = \frac{A+1}{A-B+1}\binom{A}{B}$$

$$\binom{A}{B+1} = \frac{A-B}{B+1}\binom{A}{B}$$

$$\binom{A+1}{B+1} = \frac{A+1}{B+1}\binom{A}{B}$$

2.2 THE PROBABILITY DENSITY OF THE NUMBER OF EXCEEDANCES FREE OF THE INITIAL DISTRIBUTION

In many cases the value F_m is unknown and a form must be found that is independent of the distribution F_m. To accomplish this, the density function of F_m, $g(F_m)$, is derived. The marginal density function is then

$$p(L, m, N, k) = \int_0^1 p(L, m, N, k \mid F_m)\, g(F_m)\, dF_m \qquad (2.2\text{-}1)$$

The previous sample has been drawn and arranged in descending order – $x_{(1)}, \ldots, x_{(L)}$; these values are known and considered constant. Therefore, the sequence $F_1, \ldots, F_L = F(x_{(1)}), \ldots, F(x_{(L)})$ is also fixed. Consider taking a new sample and arranging it in descending order – $y_{(1)}, \ldots, y_{(L)}$. The probability distribution of F_m is computed as the probability that the m^{th} largest value in the new sample will not be greater than the m^{th} largest in the past sample.

Since $F(x_{(m)})$ is a distribution function, it is a nondecreasing function of $x_{(m)}$ and is bounded by 0 and 1. Thus, the distribution function of F_m is

$$G(F_m) = \Pr\left[F(y_{(m)}) \le F(x_{(m)})\right] = \Pr(y_{(m)} \le x_{(m)})$$

$$G(F_m) = \Pr(\text{at least } L - m+1 \text{ values of } L \text{ are } \le x_{(m)})$$

$$G(F_m) = \Pr(\text{at most } m-1 \text{ of } L \text{ exceed } x_{(m)})$$

$$G(F_m) = \sum_{k=0}^{m-1} \Pr(x_{(m)} \text{ exceeded } k \text{ times} \mid N = L)$$

By Equation (2.1-1), this is

$$G(F_m) = \sum_{k=0}^{m-1} \binom{L}{k} (1 - F_m)^k \, F_m^{L-k}$$

The density function of F_m is thus

$$g(F_m) = \frac{d}{dF_m} G(F_m) = \frac{d}{dF_m} \left[\sum_{k=0}^{m-1} \binom{L}{k} (1 - F_m)^k \, F_m^{L-k} \right]$$

$$g(F_m) = \frac{d}{dF_m} \left[F_m^L + \sum_{k=1}^{m-1} \binom{L}{k} (1 - F_m)^k \, F_m^{L-k} \right]$$

$$g(F_m) = L \, F_m^{L-1} + \sum_{k=1}^{m-1} \binom{L}{k} (-k) \, (1 - F_m)^{k-1} \, F_m^{L-k}$$

$$+ \sum_{k=1}^{m-1} \binom{L}{k} (L - k) \, (1 - F_m)^k \, F_m^{L-k}$$

$$g(F_m) = - \sum_{k=1}^{m-1} \binom{L}{k-1} (L - [k - 1]) \, (1 - F_m)^{k-1} \, F_m^{L-k}$$

$$+ \binom{L}{0} L \, (1 - F_m)^0 \, F_m^{L-0-1}$$

$$+ \sum_{k=1}^{m-1} \binom{L}{k} (L - k) \, (1 - F_m)^k \, F_m^{L-k-1}$$

$$g(F_m) = -\sum_{\substack{j=0 \\ j=k-1}}^{m-2} \binom{L}{j} (L - j)(1 - F_m)^j F_m^{L-j-1}$$

$$+ \sum_{\substack{j=0 \\ j=k}}^{m-1} \binom{L}{j} (L - j)(1 - F_m)^j F_m^{L-j-1}$$

$$g(F_m) = \binom{L}{m-1} (L - m + 1)(1 - F_m)^{m-1} F_m^{L-m}$$

$$g(F_m) = \binom{L}{m} m (1 - F_m)^{m-1} F_m^{L-m} , \quad 0 \le F_m \le 1 \qquad (2.2\text{-}2)$$

Substituting Equation (2.1-2) and Equation (2.2-2) into Equation (2.2-1), the marginal density function becomes

$$p(L, m, N, k) = \int_0^1 p(L, m, N, k \mid F_m)\, g(F_m)\, dF_m$$

$$p(L, m, N, k) = \int_0^1 \binom{N}{k} (1 - F_m)^k F_m^{N-k} \binom{L}{m} m (1 - F_m)^{m-1} F_m^{L-m}\, dF_m$$

$$p(L, m, N, k) = \binom{N}{k}\binom{L}{m} m \int_0^1 (1 - F_m)^{m+k-1} F_m^{N+L-(m+k)}\, dF_m$$

$$p(L, m, N, k) = \frac{\binom{N}{k}\binom{L}{m} m}{(N + L)\binom{N + L - 1}{k + m - 1}} \qquad (2.2\text{-}3)$$

The above equation gives the probability that the m^{th} largest among L past observations will be exceeded k times in N future trials. This formula is distribution free; i.e., it does not depend upon the underlying distribution function F.

Equation (2.2-3) can be written as

$$p(L, m, N, k) = \frac{\binom{N+L-m-k}{L-m}\binom{k+m-1}{m-1}}{\binom{N+L}{L}} \tag{2.2-4}$$

2.3 THE MEAN AND VARIANCE OF THE NUMBER OF EXCEEDANCES

Given m, L, and N, the expected or mean number of exceedances is computed as

$$E(k) = \sum_{k=0}^{N} k\, p(L, m, N, k) = \sum_{k=1}^{N} \frac{k\binom{N+L-m-k}{L-m}\binom{k+m-1}{m-1}}{\binom{N+L}{L}}$$

$$\binom{N+L}{L} E(k) = \sum_{\substack{j=0 \\ j=k-1}}^{N-1} (j+1)\binom{[N-1]+[L+1]-[m+1]-j}{[L+1]-[m+1]}\binom{j+[m+1]-1}{[m+1]-2}$$

$$\binom{N+L}{L} E(k) = \sum_{j=0}^{N-1} (j+1)\binom{[N-1]+[L+1]-[m+1]-j}{[L+1]-[m+1]}\binom{j+[m+1]-1}{[m+1]-1}\left(\frac{[m+1]-1}{j+1}\right)$$

$$\binom{N+L}{L} E(k) = m\binom{[N-1]+[L+1]}{[L+1]}\sum_{j=0}^{N-1} \frac{\binom{[N+1]+[L+1]-[m+1]-j}{[L+1]-[m+1]}\binom{j+[m+1]-1}{[m+1]-1}}{\binom{[N-1]+[L+1]}{[L+1]}}$$

By Equation (2.2-4), this is

$$\binom{N+L}{L} E(k) = m \binom{N+L}{L+1} \sum_{j=0}^{N-1} p(L+1, m+1, N-1, j)$$

$$\binom{N+L}{L} E(k) = m \binom{N+L}{L+1} = m \frac{N}{L+1} \binom{N+L}{L}$$

Therefore, the mean number of exceedances is

$$E(k) = \frac{mN}{L+1} \tag{2.3-1}$$

As would be expected, the mean number of exceedances increases with m.

The second central moment is computed as

$$E(k^2) = \sum_{k=0}^{N} k^2 \frac{\binom{N+L-m-k}{L-m}\binom{k+m-1}{m-1}}{\binom{N+L}{L}}$$

For values of N greater than one, this can be expanded as

$$\binom{N+L}{L} E(k^2) = \sum_{k=2}^{N} k(k-1)\binom{N+L-m-k}{L-m}\binom{k+m-1}{m-1} + \sum_{k=0}^{N} k \binom{N+L-m-k}{L-m}\binom{k+m-1}{m-1}$$

$$\binom{N+L}{L} E(k^2) = \sum_{\substack{j=0 \\ j=k-2}}^{N-2} (j+2)(j+1) \binom{[N-2] + [L+2] - [m+2] - j}{[L+2] - [m+2]} \binom{j + [m+2] - 1}{[m+2] - 3}$$

$$+ \binom{N+L}{L} E(k)$$

$$\binom{N+L}{L} E(k^2) = \sum_{j=0}^{N-2} \left\{ (j+2)(j+1) \binom{[N-2] + [L+2] - [m+2] - j}{[L+2] - [m+2]} \binom{j + [m+2] - 1}{[m+2] - 1} \frac{([m+2] - 2)([m+2] - 1)}{(j+2)(j+1)} \right\} + \binom{N+L}{L} \frac{mN}{L+1}$$

$$\binom{N+L}{L} E(k^2) = (m+1)\, m \binom{[N-2] + [L+2]}{[L+2]} \sum_{j=0}^{N-2} p(L+2, m+2, N-2, j) + \binom{N+L}{L} \frac{mN}{L+1}$$

$$\binom{N+L}{L} E(k^2) = (m+1)\, m \binom{L+1}{L+2} + \binom{N+L}{L} \frac{mN}{L+1} .$$

$$E(k^2) = \frac{(m+1)\, mN(N-1)}{(L+1)(L+2)} + \frac{mN}{L+1}$$

The variance is, therefore,

$$\text{var}(k) = E(k^2) - [E(k)]^2$$

$$\text{var}(k) = \frac{mN(N+L+1)(L-m+1)}{(L+1)^2\,(L+2)} , \quad N > 1$$

For the case in which N = 1, the density function is

$$p(L, m, N, k) = \frac{\binom{L+1-m-k}{L-m}\binom{k+m-1}{m-1}}{L+1}, \quad k = 0, 1, \quad N = 1$$

The variance is

$$\mathrm{var}(k) = E(k^2) - [E(k)]^2 = 0^2 \cdot p(L, m, 1, 0) + 1^2 \cdot p(L, m, 1, 1) - \frac{1}{(L+1)^2}$$

$$\mathrm{var}(k) = \frac{1}{L+1} - \frac{1}{(L+1)^2} = \frac{L}{(L+1)^2}, \quad N = 1$$

Thus, the form given below holds for all positive integers N.

$$\mathrm{var}(k) = \frac{mN(N+L+1)(L-m+1)}{(L+1)^2 (L+2)} \tag{2.3-2}$$

Actually, Equations (2.3-1) and (2.3-2) should be written in the conditional form.

$$E(k \mid L, m, N) = \frac{mN}{L+1} \tag{2.3-3}$$

$$\mathrm{var}(k \mid L, m, N) = \frac{mN(N+L+1)(L-m+1)}{(L+1)^2 (L+2)} \tag{2.3-4}$$

It can be shown that $\mathrm{var}(k \mid L, m, N) = \mathrm{var}(k \mid L, L-m+1, N)$; i.e., the variance of the m^{th} largest is also the variance of the (m-1) smallest observation. Note that if L and N are held constant, the variance is smaller for small or large values of m than it is for values near N/2. In fact, the variance is at a minimum for m = 1 or m = L. This is contrary to what one would at first expect.

In summary, the probability of having k values in N future observations exceed the m^{th} largest among L past observations is given by:

$$p(L, m, N, k) = \frac{m\binom{L}{m}\binom{N}{k}}{(N+k)\binom{N+L-1}{k+m-1}} = \frac{\binom{N+L-m-k}{L-m}\binom{k+m-1}{m-1}}{\binom{N+L}{L}}$$

The expected number of exceedances and the variance are

$$E(k) = \frac{mN}{L+1}$$

$$\text{var}(k) = \frac{mN(N+L+1)(L-m+1)}{(L+1)^2 (L+2)}$$

Section 3
PRACTICAL TECHNIQUES FOR REAL DATA

Several questions that arise when the theory is applied to real data must be considered. For instance, a convenient method of displaying the probabilities must be found; the difficulty in interpretation caused by repeated values in the initial sample must be overcome; and a procedure should be developed for estimating the probability of exceeding a value that was not actually obtained in the past sample. Suggested solutions to these problems are presented in terms of the example below.

The annual peak winds, in knots, at 10 meters above the ground at Cape Kennedy, Florida, for the years 1950 through 1966, are the following:

Year	'50	'51	'52	'53	'54	'55	'56	'57	'58	'59	'60
Wind Speed	58	43	59	62	53	43	60	47	43	44	46

Year	'61	'62	'63	'64	'65	'66
Wind Speed	42	39	43	53	48	48

Arranged in descending order, the past sample is:

m	1	2	3	4	5	6	7	8	9	10	11
Wind Speed	62	60	59	58	53	53	48	48	47	46	44

m	12	13	14	15	16	17
Wind Speed	43	43	43	43	42	39

In this case the size of the past sample, L, is 17. If four future years are of interest, then N is equal to 4.

3.1 DISPLAYING THE PROBABILITIES

A convenient way to present the probability density function is in the form of a table, which for the above example is shown in Table 1. Thus, the probability that in four future years the peak wind for exactly one year will exceed 60 knots, found in the column m = 2 and the row k = 1, is 0.273. The probability of at most 2 exceedances over 59 knots is the sum of the probabilities of 0, 1 and 2 exceedances over the third largest observation (0.511 + 0.341 + 0.120 = 0.972). The expected number of exceedances over the smallest value of 39 knots is 3.78. Each column (exclusive of mean and standard deviation) adds to one, since it is a true probability density of k, the number of exceedances.

Note that the table depends only upon the sample sizes L and N and not upon the actual observations themselves. In this sense, the exceedances approach is "probabilistic" and not "statistical."

3.2 REPETITIONS IN THE INITIAL SAMPLE

A problem arises in determining the probability that 53 knots will be exceeded k times in the future. This is because both the fifth and sixth largest observations are equal to 53 knots.

In the example given, as well as in most practical situations, the assumption that each observation is drawn from a continuous distribution is not valid. If the underlying distribution is truly continuous, the probability of obtaining two equal observations is zero. However, because of the limits of the precision of the measuring device, the observation is actually a discrete variable. The two observations of 53 knots would, with probability one, not be equal if a sufficiently accurate measuring instrument were used.

Table 1

THE PROBABILITY THAT THE M^{TH} LARGEST AMONG 17 PAST OBSERVATIONS WILL BE EXCEEDED K TIMES IN 4 FUTURE TRIALS

M

K	1	2	3	4
0	0.80952381E 00	0.64761904E 00	0.51127818E 00	0.39766081E 00
1	0.16190476E 00	0.27268170E 00	0.34085212E 00	0.37426899E 00
2	0.25563909E-01	0.68170425E-01	0.12030075E 00	0.17543859E 00
3	0.28404343E-02	0.10693400E-01	0.25062655E-01	0.46783623E-01
4	0.16708437E-03	0.83542186E-03	0.25062655E-02	0.58479529E-02
MEAN	0.22222222E 00	0.44444444E 00	0.66666666E 00	0.88888889E 00
SDEV	0.49296546E 00	0.67634303E 00	0.80204417E 00	0.89471772E 00

K	5	6	7	8
0	0.30409355E 00	0.22807017E 00	0.16725145E 00	0.11946532E 00
1	0.38011694E 00	0.36491226E 00	0.33450291E 00	0.29406848E 00
2	0.22807016E 00	0.27368420E 00	0.30877191E 00	0.33082704E 00
3	0.76023387E-01	0.11228069E 00	0.15438595E 00	0.20050123E 00
4	0.11695905E-01	0.21052630E-01	0.35087716E-01	0.55137839E-01
MEAN	0.11111111E 01	0.13333333E 01	0.15555556E 01	0.17777778E 01
SDEV	0.96393713E 00	0.10145146E 01	0.10491495E 01	0.10693922E 01

K	9	10	11	12
0	0.82706761E-01	0.55137839E-01	0.35087716E-01	0.21052630E-01
1	0.24812028E 00	0.20050123E 00	0.15438595E 00	0.11228069E 00
2	0.33834584E 00	0.33082704E 00	0.30877191E 00	0.27368420E 00
3	0.24812028E 00	0.29406848E 00	0.33450291E 00	0.36491226E 00
4	0.82706759E-01	0.11946532E 00	0.16725145E 00	0.22807017E 00
MEAN	0.20000000E 01	0.22222222E 01	0.24444444E 01	0.26666667E 01
SDEV	0.10760552E 01	0.10693922E 01	0.10491495E 01	0.10145146E 01

K	13	14	15	16
0	0.11695905E-01	0.58479529E-02	0.25062655E-02	0.83542186E-03
1	0.76023387E-01	0.46783623E-01	0.25062655E-01	0.10693400E-01
2	0.22807016E 00	0.17543859E 00	0.12030075E 00	0.68170425E-01
3	0.38011694E 00	0.37426899E 00	0.34085212E 00	0.27268170E 00
4	0.30409355E 00	0.39766081E 00	0.51127818E 00	0.64761904E 00
MEAN	0.28888889E 01	0.31111111E 01	0.33333333E 01	0.35555556E 01
SDEV	0.96393713E 00	0.89471772E 00	0.80204417E 00	0.67634303E 00

K	17
0	0.16708437E-03
1	0.28404343E-02
2	0.25563909E-01
3	0.16190476E 00
4	0.80952381E 00
MEAN	0.37777778E 01
SDEV	0.49296546E 00

The safest way of quoting the probability of all four future values being less than 53 ($k = 0$) is "between 0.228 and 0.304."

3.3 INTERPOLATIVE PROCEDURE FOR VALUES BETWEEN THE GIVEN OBSERVATIONS

Suppose that the probability of not exceeding 61 knots ($k = 0$) is desired. Since 61 knots was not actually observed, the probability cannot be read from the table. If L, N and k are held constant, $p(L, m, N, k)$ can be considered as a function of m. This is not a probability density function on m, since it does not total one.

As 61 lies between the largest and the second largest observations, the probability of not exceeding 61 knots would logically lie between 0.648 and 0.810. The rank of $m = 1\frac{1}{2}$ could artificially be assigned to 61 knots. For purposes of interpolation, the author suggests

$$p(L, m, N, k) = \frac{1}{\binom{N+L}{L}} \frac{\Gamma(N+L-m-k+1)\Gamma(k+m)}{\Gamma(L-m+1)\,\Gamma(N-k+1)\,\Gamma(m)\,\Gamma(k+1)}, \qquad 1 \leq m \leq L$$

Here the gamma function is defined

$$\Gamma(u) = \int_0^\infty t^{u-1} e^{-t}\, dt = \int_0^1 \left(\ln \frac{1}{t}\right)^{u-1} dt$$

Also, $\Gamma(u+1) = u\Gamma(u)$ and, if u is an integer, $\Gamma(u+1) = u!$

The advantages of this approximation is that a smooth curve is obtained that matches the values for which m is exactly an integer. The five curves ($k = 0, 1, 2, 3, 4$) for this example ($L = 17$, $N = 4$) are shown in Figures 1 through 5.

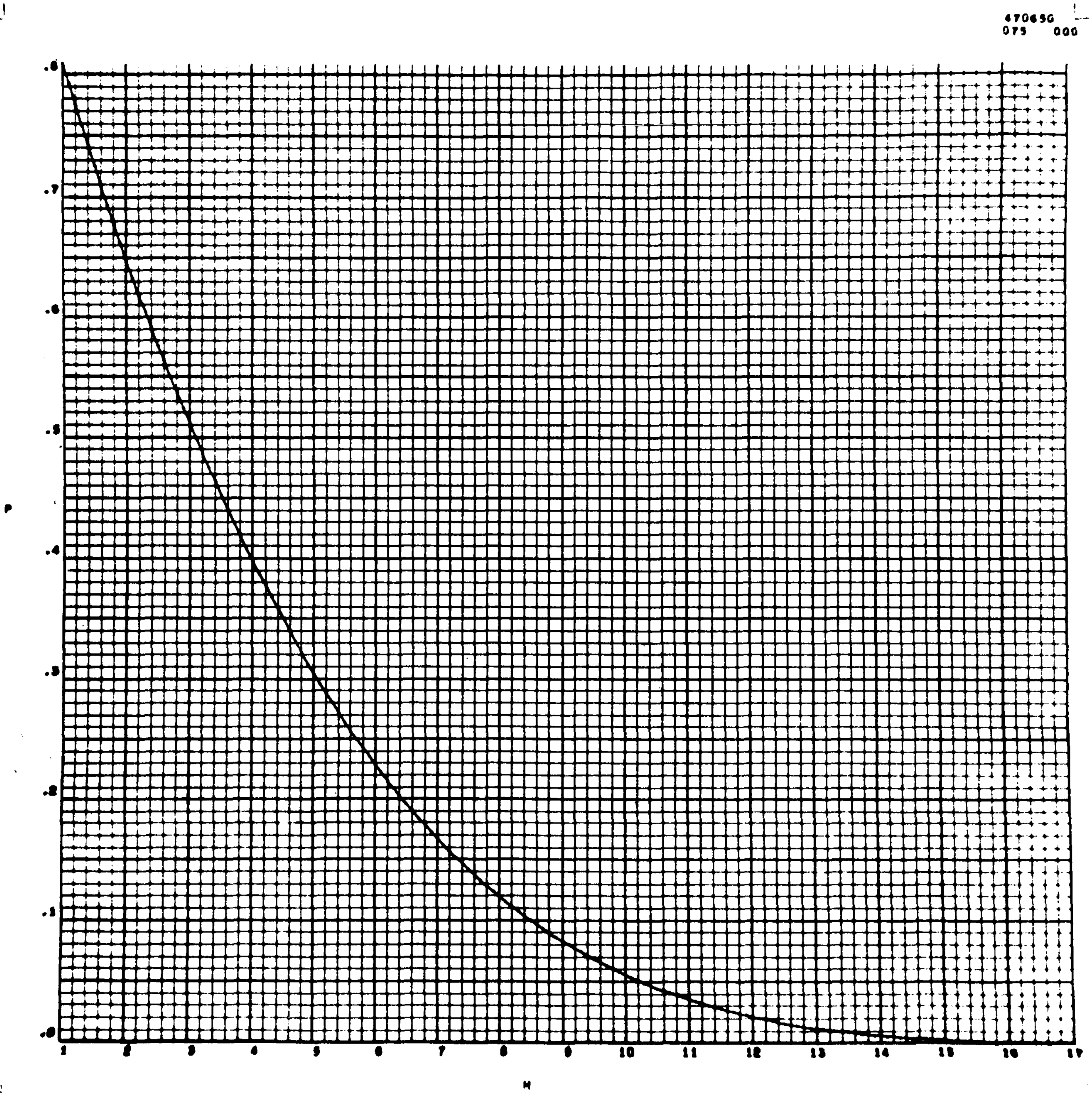


Figure 1 - Interpolative Approximation, L = 17, N = 4, K = 0

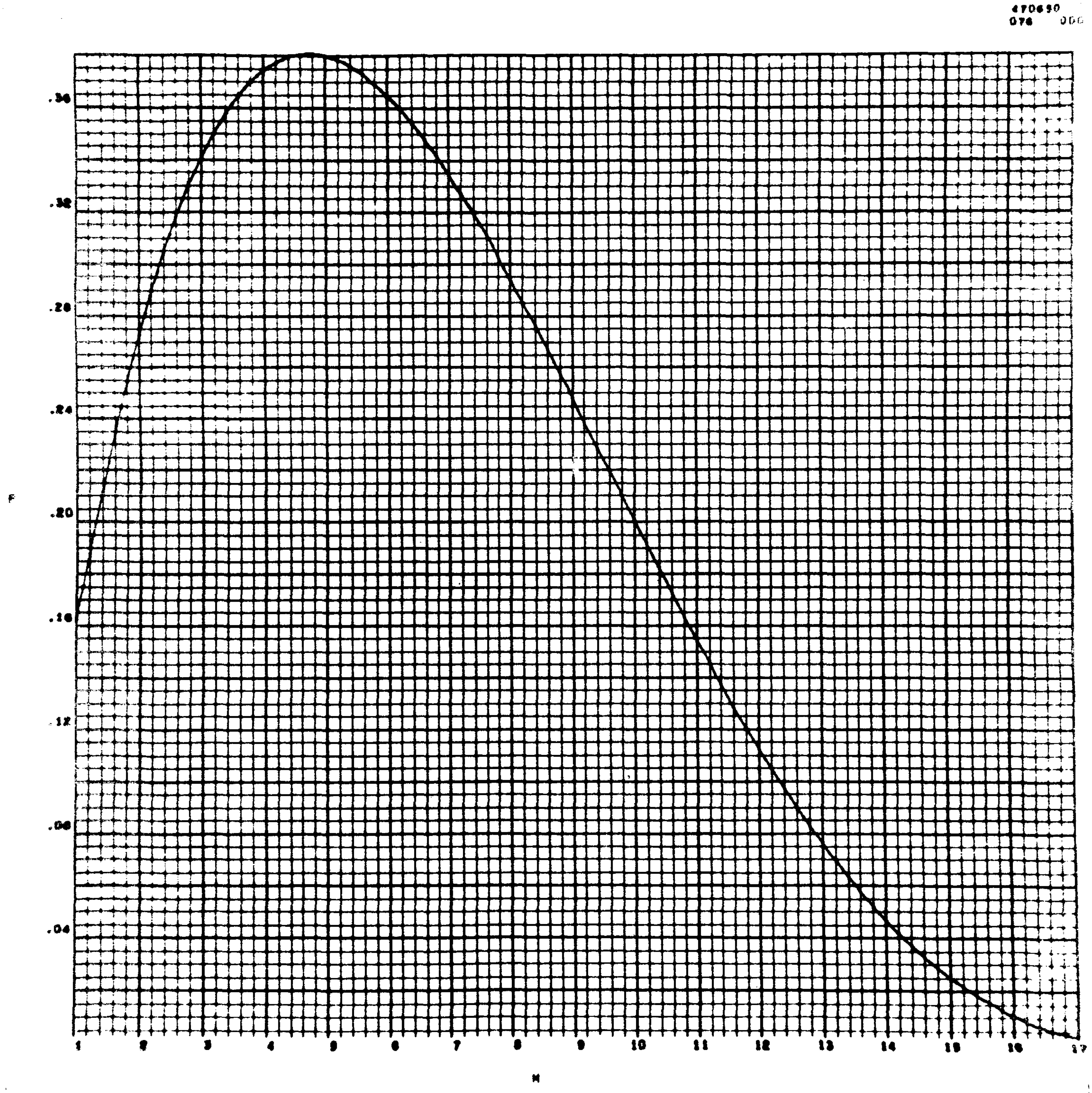


Figure 2 - Interpolative Approximation, L = 17, N = 4, k = 2

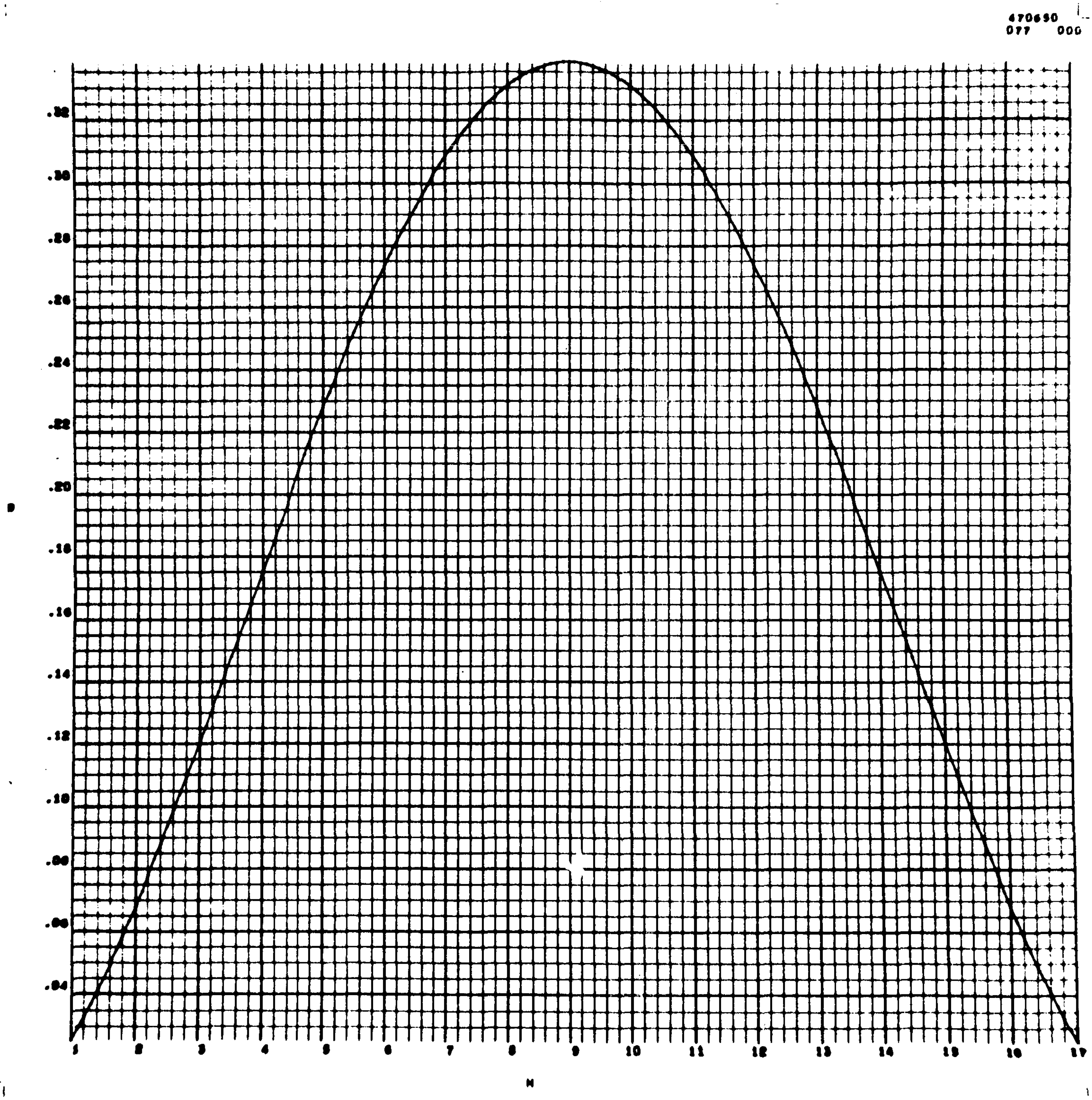


Figure 3 - Interpolative Approximation, L = 17, N = 4, k = 2

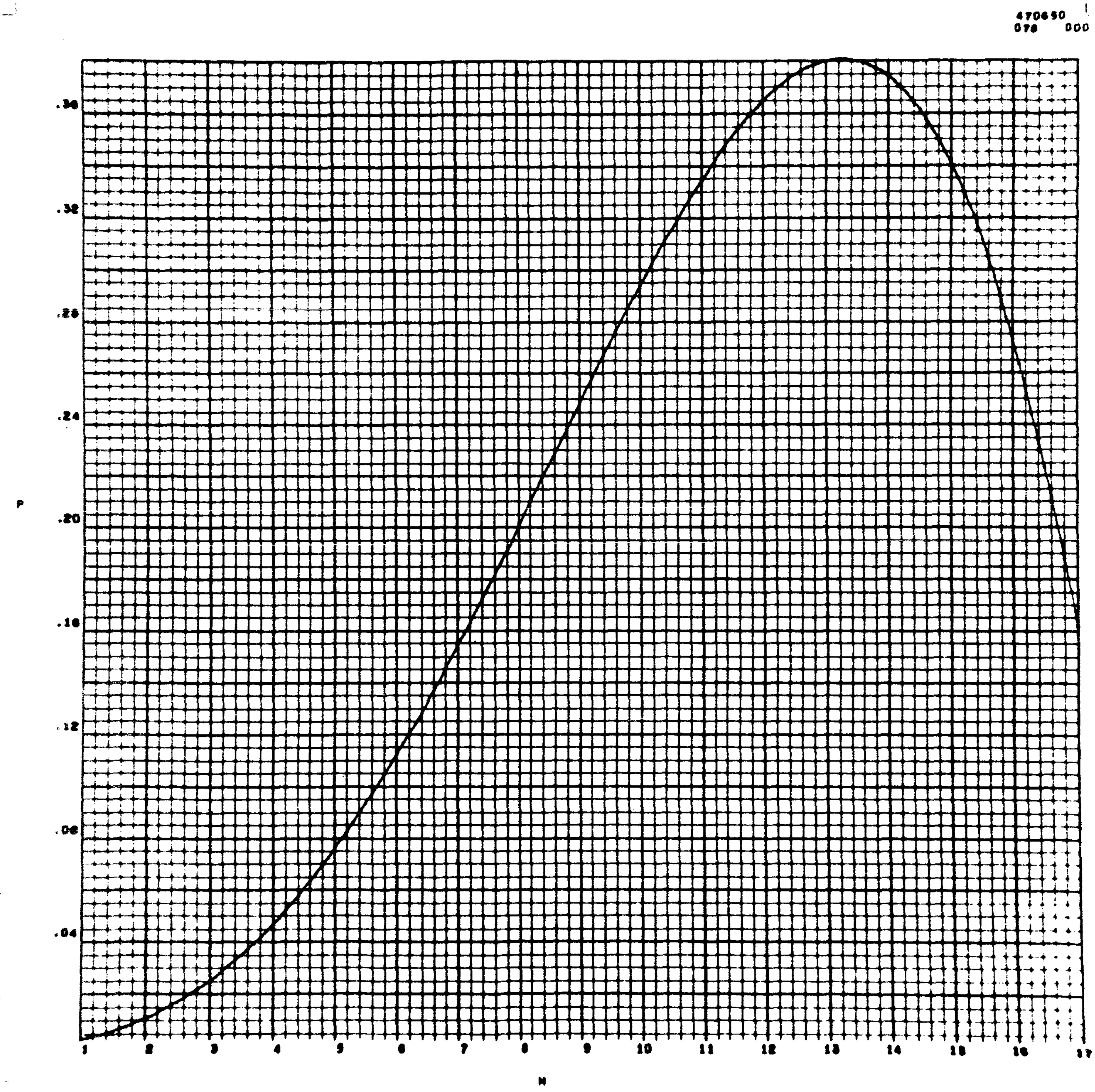


Figure 4 - Interpolative Approximation, L = 17, N = 4, k = 3

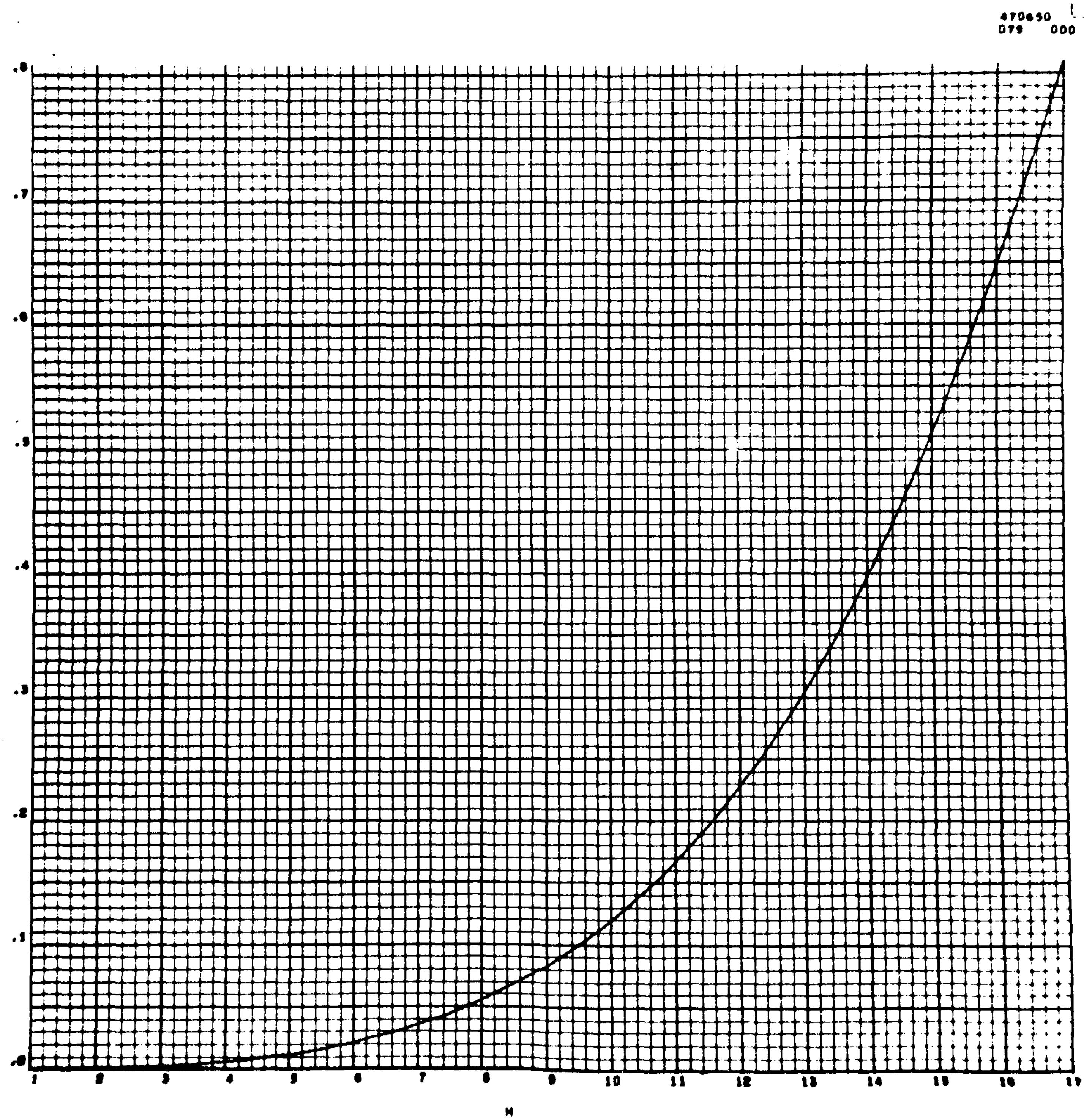


Figure 5 - Interpolative Approximation, L = 17, N = 4, k = 4

The interpolative scheme cannot be used for values greater than the largest observation or less than the smallest observation. However, it can be said, for example, that the probability that the annual peak wind will not exceed 63 knots in four years is at least 0.810.

Note that this approximation is the author's own invention and, as far as she knows, has not been developed by others.

Section 4
THE EDNE PROGRAM

The Exact Distribution of the Number of Exceedances (EDNE) computer program calculates the probability that the m^{th} largest among L past observations will be exceeded k times in N future trials. The mean number of exceedances and the standard deviation of the number of exceedances is determined for each value of m. The means and the standard deviations are computed by the formulas:

$$E(k) = \frac{mN}{L+1}$$

$$\text{s. dev. } (k) = [\text{var}(k)]^{1/2} = \left[\frac{mN\,(N+L+1)\,(L-m+1)}{(L+1)^2\,(L+2)}\right]^{1/2}$$

The probabilities are based upon the equation:

$$p(L, m, N, k) = \frac{m\binom{L}{m}\binom{N}{k}}{(N+L)\binom{N+L-1}{k+m-1}}$$

To avoid overflow in the computer, however, the following set of relationships is utilized:

$$p(L, 1, N, 0) = \frac{L}{N+L}$$

$$p(L, m+1, N, k) = \frac{(L-m)\,(k+m)}{m(N+L-m-k)}\,p(L, m, N, k)$$

$$p(L, m, N, k+1) = \frac{(k+m)(N-k)}{(N+L-m-k)(k+1)} p(L, m, N, k)$$

$$p(L, L-m+1, N, N-k) = p(L, m, N, k)$$

The program will also, upon request, supply plots of the approximating function discussion in Section 3.3. The probability is plotted as a function of m, where m varies from 1 to L in increments of Δm, an input quantity.

The program was written for the IBM 7094 digital computer and the IBM 4020 plotter.

4.1 INPUT

The first data card contains the four positive integers L1, L2, N1 and N2 in a 4I3 format. These are the bounds on L and N. For each pair L and N, where $L1 \leq L \leq L2$ and $N1 \leq N \leq N2$, the program will output the corresponding table and/or set of plots. The second card contains the real numbers OPT1, OPT2, DELM in a 3E12.8 format. The tables will be output only if OPT1 = 1, and the plots will be given only if OPT2 = 1. The value DELM determines the increment of m for the plots.

After a set of data, the program reads another card under an I3 format. If this card has an integer greater than zero, the program will read another case; if this field is blank, the program will exit.

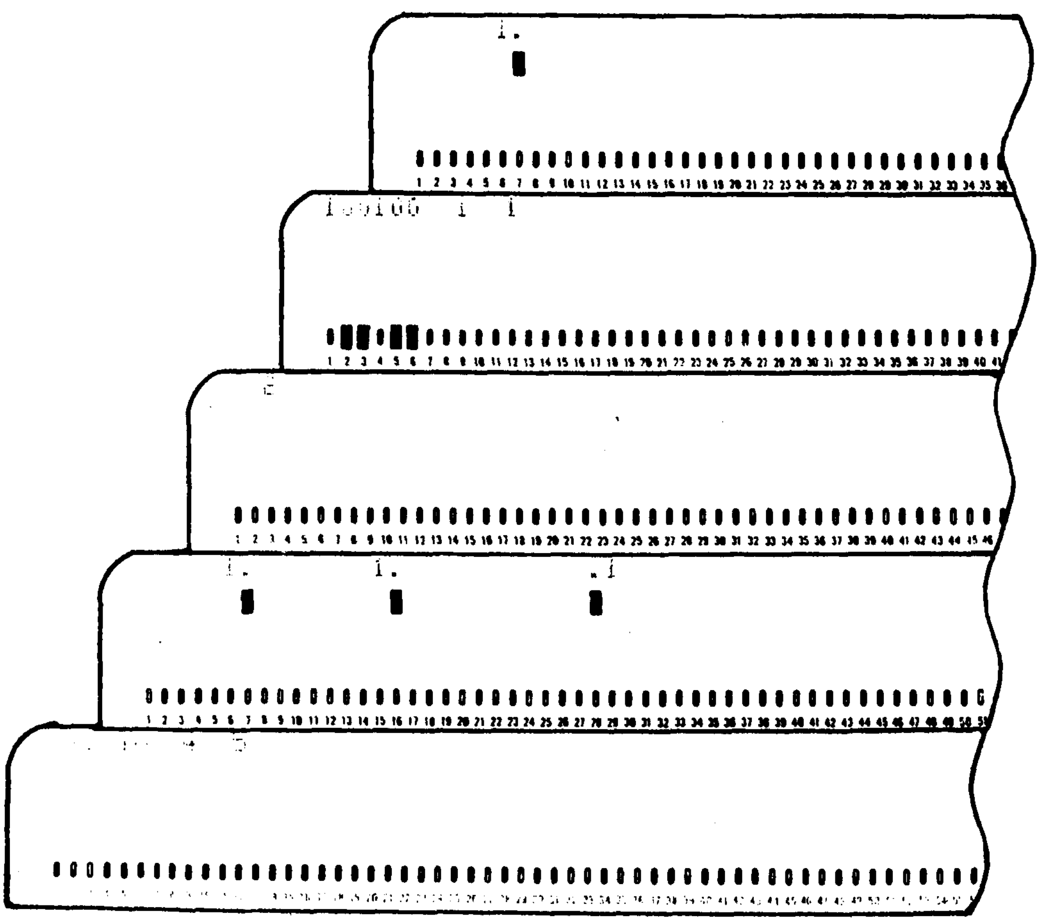
4.2 OUTPUT

For a given pair of positive integers L and N and for OPT1 = 1., the output is a (N + 3) by L table. The columns are identified by the rank m of the past L observations. The last two rows list the mean and standard deviation of the number of exceedances for the respective value of m. The first N + 1 rows are identified by the number of exceedances, k. Thus, the number in the $(k+1)^{st}$ row and m^{th} column is $p(L, m, N, k)$, the probability that k values of N will exceed the m^{th} largest of L past observations.

If OPT2 = 1., N+1 plots will be made for each pair L and N. Each plot corresponds to a value of k between 0 and N. The number of points on each plot is (L - 1)/DELM + 1; this should not exceed 1024.

4.3 SAMPLE CASE

Shown in Tables 2 through 5 and Figures 6 through 16 is a sample case from the EDNE program. For this case, the data consists of the five cards below, followed by a blank card:



Data for Sample Case

Table 2

THE PROBABILITY THAT THE M^{TH} LARGEST AMONG 16 PAST OBSERVATIONS WILL BE EXCEEDED K TIMES IN 4 FUTURE TRIALS

M

K	1	2	3	4
0	0.80000000E 00	0.63157894E 00	0.49122806E 00	0.37564499E 00
1	0.16842105E 00	0.28070175E 00	0.34674921E 00	0.37564499E 00
2	0.28070175E-01	0.74303403E-01	0.13003095E 00	0.18782249E 00
3	0.33023735E-02	0.12383900E-01	0.28895767E-01	0.53663569E-01
4	0.20639834E-03	0.10319917E-02	0.30959750E-02	0.72239419E-02
MEAN	0.23529412E 00	0.47058823E 00	0.70588235E 00	0.94117647E 00
SDEV	0.50829338E 00	0.69600938E 00	0.82352941E 00	0.91633894E 00

K	5	6	7	8
0	0.28173374E 00	0.20660474E 00	0.14757461E 00	0.10216718E 00
1	0.37564498E 00	0.35417955E 00	0.31785344E 00	0.27244580E 00
2	0.24148606E 00	0.28606809E 00	0.31785344E 00	0.33436529E 00
3	0.86687301E-01	0.12714137E 00	0.17337460E 00	0.22291019E 00
4	0.14447883E-01	0.26006190E-01	0.43343650E-01	0.68111447E-01
MEAN	0.11764706E 01	0.14117647E 01	0.16470588E 01	0.18823529E 01
SDEV	0.98430591E 00	0.10323488E 01	0.10631719E 01	0.10782531E 01

K	9	10	11	12
0	0.68111447E-01	0.43343650E-01	0.26006190E-01	0.14447883E-01
1	0.22291019E 00	0.17337460E 00	0.12714137E 00	0.86687301E-01
2	0.33436529E 00	0.31785344E 00	0.28606809E 00	0.24148606E 00
3	0.27244580E 00	0.31785344E 00	0.35417955E 00	0.37564498E 00
4	0.10216718E 00	0.14757461E 00	0.20660474E 00	0.28173374E 00
MEAN	0.21176470E 01	0.23529412E 01	0.25882353E 01	0.28235294E 01
SDEV	0.10782531E 01	0.10631719E 01	0.10323488E 01	0.98430591E 00

K	13	14	15	16
0	0.72239419E-02	0.30959750E-02	0.10319917E-02	0.20639834E-03
1	0.53663569E-01	0.28895767E-01	0.12383900E-01	0.33023735E-02
2	0.18782249E 00	0.13003095E 00	0.74303403E-01	0.28070175E-01
3	0.37564499E 00	0.34674921E 00	0.28070175E 00	0.16842105E 00
4	0.37564499E 00	0.49122806E 00	0.63157894E 00	0.80000000E 00
MEAN	0.30588235E 01	0.32941176E 01	0.35294117E 01	0.37647059E 01
SDEV	0.91633894E 00	0.82352941E 00	0.69600938E 00	0.50829338E 00

Table 3

THE PROBABILITY THAT THE M^{TH} LARGEST AMONG 16 PAST OBSERVATIONS WILL BE EXCEEDED K TIMES IN 5 FUTURE TRIALS

M

K	1	2	3	4
0	0.76190476E 00	0.57142857E 00	0.42105263E 00	0.30409356E 00
1	0.19047619E 00	0.30075187E 00	0.35087718E 00	0.35775712E 00
2	0.40100250E-01	0.10025062E 00	0.16511867E 00	0.22359820E 00
3	0.66833750E-02	0.23588382E-01	0.51599585E-01	0.89439279E-01
4	0.78627940E-03	0.36856846E-02	0.10319917E-01	0.22359820E-01
5	0.49142463E-04	0.29485477E-03	0.10319917E-02	0.27519778E-02
MEAN	0.29411764E 00	0.58823529E 00	0.88235293E 00	0.11764706E 01
SDEV	0.58166263E 00	0.79647435E 00	0.94240116E 00	0.10486072E 01

K	5	6	7	8
0	0.21465427E 00	0.14757481E 00	0.93383207E-01	0.63246347E-01
1	0.33539730E 00	0.29514962E 00	0.24595802E 00	0.19460414E 00
2	0.26831784E 00	0.29514962E 00	0.30271756E 00	0.29190622E 00
3	0.13415892E 00	0.18163054E 00	0.22703817E 00	0.26536928E 00
4	0.41279667E-01	0.68111450E-01	0.10319917E 00	0.14595311E 00
5	0.61919499E-02	0.12383900E-01	0.22703816E-01	0.38920828E-01
MEAN	0.14705882E 01	0.17647059E 01	0.20588235E 01	0.23529412E 01
SDEV	0.11263848E 01	0.11813624E 01	0.12166347E 01	0.12338928E 01

K	9	10	11	12
0	0.38920828E-01	0.22703816E-01	0.12383900E-01	0.61919499E-02
1	0.14595311E 00	0.10319917E 00	0.68111450E-01	0.41279667E-01
2	0.26536928E 00	0.22703817E 00	0.18163054E 00	0.13415892E 00
3	0.29190622E 00	0.30271756E 00	0.29514962E 00	0.26831784E 00
4	0.19460414E 00	0.24595802E 00	0.29514962E 00	0.33539730E 00
5	0.63246347E-01	0.93383207E-01	0.14757481E 00	0.21465427E 00
MEAN	0.26470588E 01	0.29411764E 01	0.32352941E 01	0.35294117E 01
SDEV	0.12338928E 01	0.12166347E 01	0.11813624E 01	0.11263848E 01

K	13	14	15	16
0	0.27519778E-02	0.10319917E-02	0.29485477E-03	0.49142463E-04
1	0.22359820E-01	0.10319917E-01	0.36856846E-02	0.78627940E-03
2	0.89439279E-01	0.51599585E-01	0.23588382E-01	0.66833750E-02
3	0.22359820E 00	0.16511867E 00	0.10025062E 00	0.40100250E-01
4	0.35775712E 00	0.35087718E 00	0.30075187E 00	0.19047619E 00
5	0.30409356E 00	0.42105263E 00	0.57142857E 00	0.76190476E 00
MEAN	0.38235294E 01	0.41176470E 01	0.44117646E 01	0.47058823E 01
SDEV	0.10486072E 01	0.94240116E 00	0.79647435E 00	0.58166263E 00

Table 4

THE PROBABILITY THAT THE M^{TH} LARGEST AMONG 100 PAST OBSERVATIONS WILL BE EXCEEDED K TIMES IN 1 FUTURE TRIAL

M

K	1	2	3	4
0	0.99009901E 00	0.98019800E 00	0.97029701E 00	0.96039601E 00
1	0.99009899E-02	0.19801980E-01	0.29702969E-01	0.39603959E-01
MEAN	0.99009901E-02	0.19801980E-01	0.29702970E-01	0.39603960E-01
SDEV	0.99009900E-01	0.13931928E 00	0.16976661E 00	0.19502689E 00

K	5	6	7	8
0	0.95049502E 00	0.94059402E 00	0.93069302E 00	0.92079202E 00
1	0.49504948E-01	0.59405938E-01	0.69306926E-01	0.79207915E-01
MEAN	0.49504950E-01	0.59405940E-01	0.69306930E-01	0.79207920E-01
SDEV	0.21691932E 00	0.23638290E 00	0.25397535E 00	0.27006300E 00

K	9	10	11	12
0	0.91089103E 00	0.90099003E 00	0.89108903E 00	0.88118803E 00
1	0.89108904E-01	0.99009893E-01	0.10891088E 00	0.11881187E 00
MEAN	0.89108910E-01	0.99009900E-01	0.10891089E 00	0.11881188E 00
SDEV	0.28490088E 00	0.29867531E 00	0.31152738E 00	0.32356702E 00

K	13	14	15	16
0	0.87128703E 00	0.86138604E 00	0.85148504E 00	0.84158405E 00
1	0.12871286E 00	0.13861384E 00	0.14851483E 00	0.15841582E 00
MEAN	0.12871287E 00	0.13861386E 00	0.14851485E 00	0.15841584E 00
SDEV	0.33488137E 00	0.34554313E 00	0.35560960E 00	0.36513047E 00

K	17	18	19	20
0	0.83168305E 00	0.82178206E 00	0.81188107E 00	0.80198007E 00
1	0.16831681E 00	0.17821779E 00	0.18811878E 00	0.19801977E 00
MEAN	0.16831683E 00	0.17821782E 00	0.18811881E 00	0.19801980E 00
SDEV	0.37414739E 00	0.38269600E 00	0.39080701E 00	0.39850716E 00

K	21	22	23	24
0	0.79207908E 00	0.78217608E 00	0.77227709E 00	0.76237609E 00
1	0.20792076E 00	0.21782174E 00	0.22772273E 00	0.23762372E 00
MEAN	0.20792079E 00	0.21782178E 00	0.22772277E 00	0.23762376E 00
SDEV	0.40581983E 00	0.41276561E 00	0.41936274E 00	0.42562743E 00

Table 4 (Continued)

K	25	26	27	28
0	0.75247510E 00	0.74257410E 00	0.73267311E 00	0.72277211E 00
1	0.24752470E 00	0.25742568E 00	0.26732667E 00	0.27722766E 00
MEAN	0.24752475E 00	0.25742574E 00	0.26732673E 00	0.27722772E 00
SDEV	0.43157415E 00	0.43721588E 00	0.44256429E 00	0.44762988E 00

K	29	30	31	32
0	0.71287112E 00	0.70297012E 00	0.69306913E 00	0.68316814E 00
1	0.28712864E 00	0.29702963E 00	0.30693061E 00	0.31683160E 00
MEAN	0.28712871E 00	0.29702970E 00	0.30693069E 00	0.31683168E 00
SDEV	0.45242216E 00	0.45694973E 00	0.46122038E 00	0.46524119E 00

K	33	34	35	36
0	0.67326714E 00	0.66336615E 00	0.65346515E 00	0.64356416E 00
1	0.32673258E 00	0.33663356E 00	0.34653454E 00	0.35643553E 00
MEAN	0.32673267E 00	0.33663366E 00	0.34653465E 00	0.35643564E 00
SDEV	0.46901858E 00	0.47255839E 00	0.47586593E 00	0.47894600E 00

K	37	38	39	40
0	0.63366316E 00	0.62376218E 00	0.61386118E 00	0.60396019E 00
1	0.36633651E 00	0.37623750E 00	0.38613848E 00	0.39603946E 00
MEAN	0.36633663E 00	0.37623762E 00	0.38613861E 00	0.39603960E 00
SDEV	0.48180297E 00	0.48444078E 00	0.48686300E 00	0.48907283E 00

K	41	42	43	44
0	0.59405919E 00	0.58415820E 00	0.57425720E 00	0.56435621E 00
1	0.40594044E 00	0.41584143E 00	0.42574240E 00	0.43564339E 00
MEAN	0.40594059E 00	0.41584158E 00	0.42574257E 00	0.43564356E 00
SDEV	0.49107314E 00	0.49286647E 00	0.49445508E 00	0.49584095E 00

K	45	46	47	48
0	0.55445522E 00	0.54455423E 00	0.53465323E 00	0.52475224E 00
1	0.44554437E 00	0.45544535E 00	0.46534633E 00	0.47524730E 00
MEAN	0.44554455E 00	0.45544554E 00	0.46534653E 00	0.47524752E 00
SDEV	0.49702575E 00	0.49801094E 00	0.49879769E 00	0.49938694E 00

K	49	50	51	52
0	0.51485125E 00	0.50495026E 00	0.49504926E 00	0.48514829E 00
1	0.48514829E 00	0.49504926E 00	0.50495026E 00	0.51485125E 00
MEAN	0.48514851E 00	0.49504950E 00	0.50495049E 00	0.51485148E 00
SDEV	0.49977938E 00	0.49997549E 00	0.49997549E 00	0.49977938E 00

Table 4 (Continued)

K	53	54	55	56
0	0.47524730E 00	0.46534633E 00	0.45544535E 00	0.44554437E 00
1	0.52475224E 00	0.53465323E 00	0.54455423E 00	0.55445522E 00
MEAN	0.52475247E 00	0.53465346E 00	0.54455445E 00	0.55445544E 00
SDEV	0.49936694E 00	0.49879769E 00	0.49801094E 00	0.49702575E 00

K	57	58	59	60
0	0.43564339E 00	0.42574240E 00	0.41584143E 00	0.40594044E 00
1	0.56435621E 00	0.57425720E 00	0.58415820E 00	0.59405919E 00
MEAN	0.56435643E 00	0.57425742E 00	0.58415841E 00	0.59405940E 00
SDEV	0.49584095E 00	0.49445508E 00	0.49286647E 00	0.49107314E 00

K	61	62	63	64
0	0.39603946E 00	0.38613848E 00	0.37623750E 00	0.36633651E 00
1	0.60396019E 00	0.61386118E 00	0.62376218E 00	0.63366316E 00
MEAN	0.60396039E 00	0.61386138E 00	0.62376237E 00	0.63366336E 00
SDEV	0.48907283E 00	0.48686300E 00	0.48444078E 00	0.48180297E 00

K	65	66	67	68
0	0.35643553E 00	0.34653454E 00	0.33663356E 00	0.32673258E 00
1	0.64356416E 00	0.65346515E 00	0.66336615E 00	0.67326714E 00
MEAN	0.64356435E 00	0.65346534E 00	0.66336633E 00	0.67326732E 00
SDEV	0.47894600E 00	0.47586593E 00	0.47255839E 00	0.46901858E 00

K	69	70	71	72
0	0.31683160E 00	0.30693061E 00	0.29702963E 00	0.28712864E 00
1	0.68316814E 00	0.69306913E 00	0.70297012E 00	0.71287112E 00
MEAN	0.68316831E 00	0.69306930E 00	0.70297029E 00	0.71287128E 00
SDEV	0.46524119E 00	0.46122038E 00	0.45694973E 00	0.45242216E 00

K	73	74	75	76
0	0.27722766E 00	0.26732667E 00	0.25742568E 00	0.24752470E 00
1	0.72277211E 00	0.73267311E 00	0.74257410E 00	0.75247510E 00
MEAN	0.72277227E 00	0.73267326E 00	0.74257425E 00	0.75247524E 00
SDEV	0.44762988E 00	0.44256429E 00	0.43721588E 00	0.43157415E 00

K	77	78	79	80
0	0.23762372E 00	0.22772273E 00	0.21782174E 00	0.20792076E 00
1	0.76237609E 00	0.77227709E 00	0.78217808E 00	0.79207906E 00
MEAN	0.76237623E 00	0.77227722E 00	0.78217821E 00	0.79207920E 00
SDEV	0.42562743E 00	0.41936274E 00	0.41276561E 00	0.40581983E 00

Table 4 (Concluded)

K	81	82	83	84
0	0.19801977E 00	0.18811878E 00	0.17821779E 00	0.16831681E 00
1	0.80198007E 00	0.81188107E 00	0.82178206E 00	0.83168305E 00
MEAN	0.80198019E 00	0.81188118E 00	0.82178217E 00	0.83168316E 00
SDEV	0.39850716E 00	0.39080701E 00	0.38269600E 00	0.37414739E 00
K	85	86	87	88
0	0.15841582E 00	0.14851483E 00	0.13861384E 00	0.12871286E 00
1	0.84158405E 00	0.85148504E 00	0.86138604E 00	0.87128703E 00
MEAN	0.84158415E 00	0.85148514E 00	0.86138613E 00	0.87128712E 00
SDEV	0.36513047E 00	0.35560960E 00	0.34554313E 00	0.33488187E 00
K	89	90	91	92
0	0.11881187E 00	0.10891088E 00	0.99009893E-01	0.89108904E-01
1	0.88118803E 00	0.89108903E 00	0.90099003E 00	0.91089103E 00
MEAN	0.88118811E 00	0.89108910E 00	0.90099009E 00	0.91089108E 00
SDEV	0.32356702E 00	0.31152738E 00	0.29867531E 00	0.28490088E 00
K	93	94	95	96
0	0.79207915E-01	0.69306926E-01	0.59405938E-01	0.49504948E-01
1	0.92079202E 00	0.93069302E 00	0.94059402E 00	0.95049502E 00
MEAN	0.92079207E 00	0.93069306E 00	0.94059405E 00	0.95049504E 00
SDEV	0.27006300E 00	0.25397535E 00	0.23638290E 00	0.21691982E 00
K	97	98	99	100
0	0.39603959E-01	0.29702969E-01	0.19801980E-01	0.99009899E-02
1	0.96039601E 00	0.97029701E 00	0.98019800E 00	0.99009901E 00
MEAN	0.96039603E 00	0.97029702E 00	0.98019801E 00	0.99009901E 00
SDEV	0.19502689E 00	0.16970661E 00	0.13931928E 00	0.99009900E-01

Table 5

THE PROBABILITY THAT THE M^TH LARGEST AMONG 17 PAST OBSERVATIONS WILL BE EXCEEDED K TIMES IN 5 FUTURE TRIALS

K	M = 1	2	3	4
0	0.77272727E 00	0.58874458E 00	0.44155844E 00	0.32535885E 00
1	0.16398268E 00	0.29437229E 00	0.34859876E 00	0.36150982E 00
2	0.36796535E-01	0.92959669E-01	0.15493278E 00	0.21265283E 00
3	0.58099792E-02	0.20657704E-01	0.45568465E-01	0.79744812E-01
4	0.64555324E-03	0.30378976E-02	0.85440371E-02	0.18607123E-01
5	0.37973720E-04	0.22784232E-03	0.79744812E-03	0.21265283E-02
MEAN	0.27777778E 00	0.55555555E 00	0.83333333E 00	0.11111111E 01
SDEV	0.56353913E 00	0.77316930E 00	0.91636600E 00	0.10228069E 01

K	5	6	7	8
0	0.23498139E 00	0.16586921E 00	0.11403508E 00	0.76023398E-01
1	0.34556086E 00	0.31100477E 00	0.26608155E 00	0.21720968E 00
2	0.25917064E 00	0.29027111E 00	0.30409354E 00	0.30075186E 00
3	0.12094630E 00	0.16586921E 00	0.21052630E 00	0.25062655E 00
4	0.34556084E-01	0.57416263E-01	0.87719290E-01	0.12531327E 00
5	0.47846885E-02	0.95693771E-02	0.17543858E-01	0.30075185E-01
MEAN	0.13888889E 01	0.16666667E 01	0.19444444E 01	0.22222222E 01
SDEV	0.11019358E 01	0.11597539E 01	0.11993473E 01	0.12224880E 01

K	9	10	11	12
0	0.48872177E-01	0.30075185E-01	0.17543858E-01	0.95693771E-02
1	0.16917292E 00	0.12531327E 00	0.87719290E-01	0.57416263E-01
2	0.28195486E 00	0.25062655E 00	0.21052630E 00	0.16586921E 00
3	0.28195486E 00	0.30075186E 00	0.30409354E 00	0.29027111E 00
4	0.16917291E 00	0.21720968E 00	0.26608155E 00	0.31100477E 00
5	0.48872175E-01	0.76023388E-01	0.11403508E 00	0.16586921E 00
MEAN	0.25000000E 01	0.27777778E 01	0.30555555E 01	0.33333333E 01
SDEV	0.12301048E 01	0.12224880E 01	0.11993473E 01	0.11597539E 01

K	13	14	15	16
0	0.47846885E-02	0.21265283E-02	0.79744812E-03	0.22784232E-03
1	0.34556084E-01	0.18607123E-01	0.85440371E-02	0.30378976E-02
2	0.12094630E 00	0.79744812E-01	0.45568465E-01	0.20657704E-01
3	0.25917064E 00	0.21265283E 00	0.15493278E 00	0.92959669E-01
4	0.34556086E 00	0.36150982E 00	0.34859876E 00	0.29437229E 00
5	0.23498139E 00	0.32535885E 00	0.44155844E 00	0.58874458E 00
MEAN	0.36111111E 01	0.38888889E 01	0.41666666E 01	0.44444444E 01
SDEV	0.11019358E 01	0.10228069E 01	0.91636600E 00	0.77316930E 00

Table 5 (Continued)

M

K	17
0	0.37973720E-04
1	0.64555324E-03
2	0.58099792E-02
3	0.36796535E-01
4	0.18398268E 00
5	0.77272727E 00
MEAN	0.47222221E 01
SDEV	0.56353913E 00

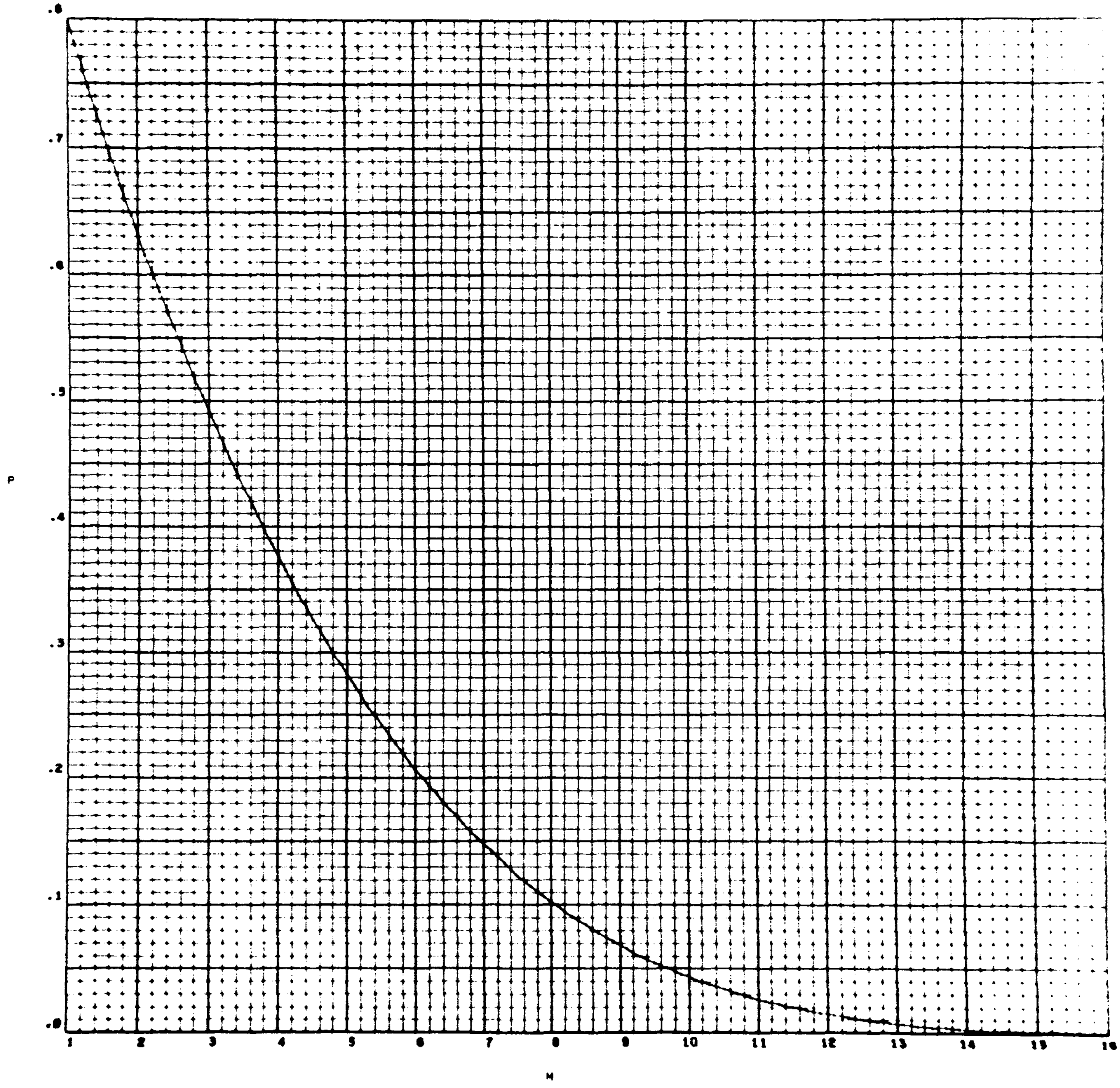


Figure 6 - Interpolative Approximation, L = 16, N = 4, k = 0

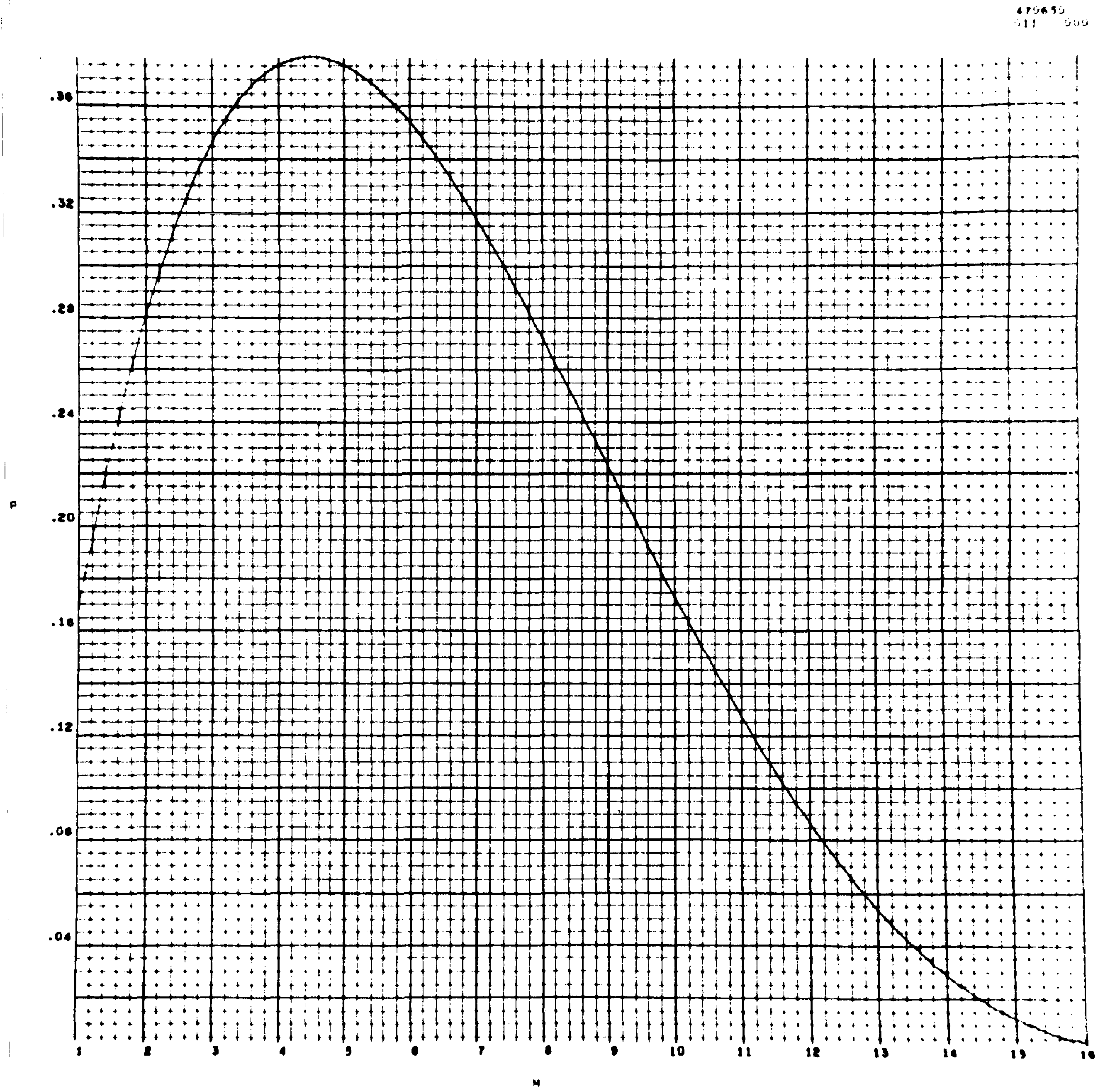


Figure 7 - Interpolative Approximation, L = 16, N = 4, k = 1

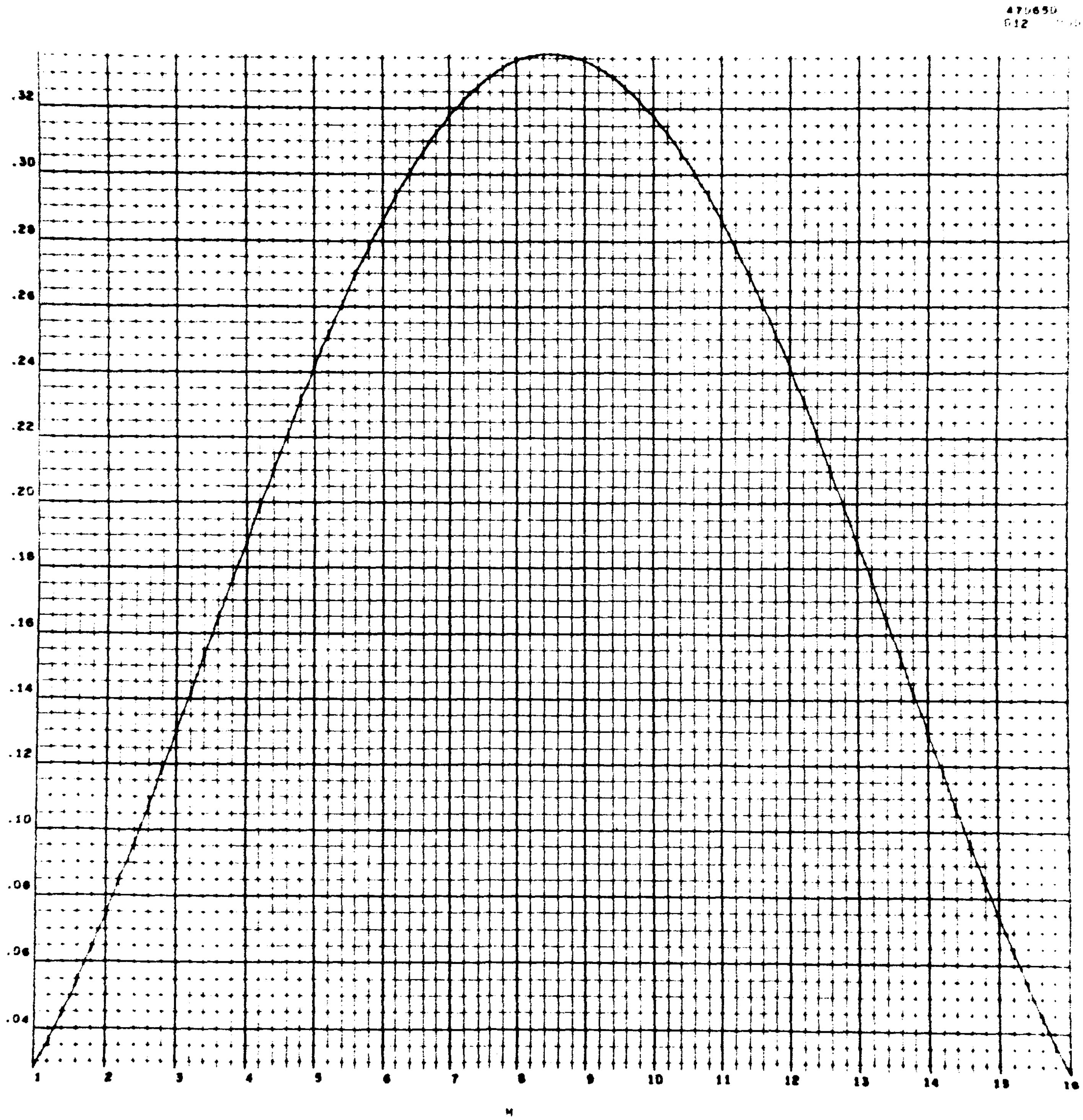


Figure 8 - Interpolative Approximation, L = 16, N = 4, k = 2

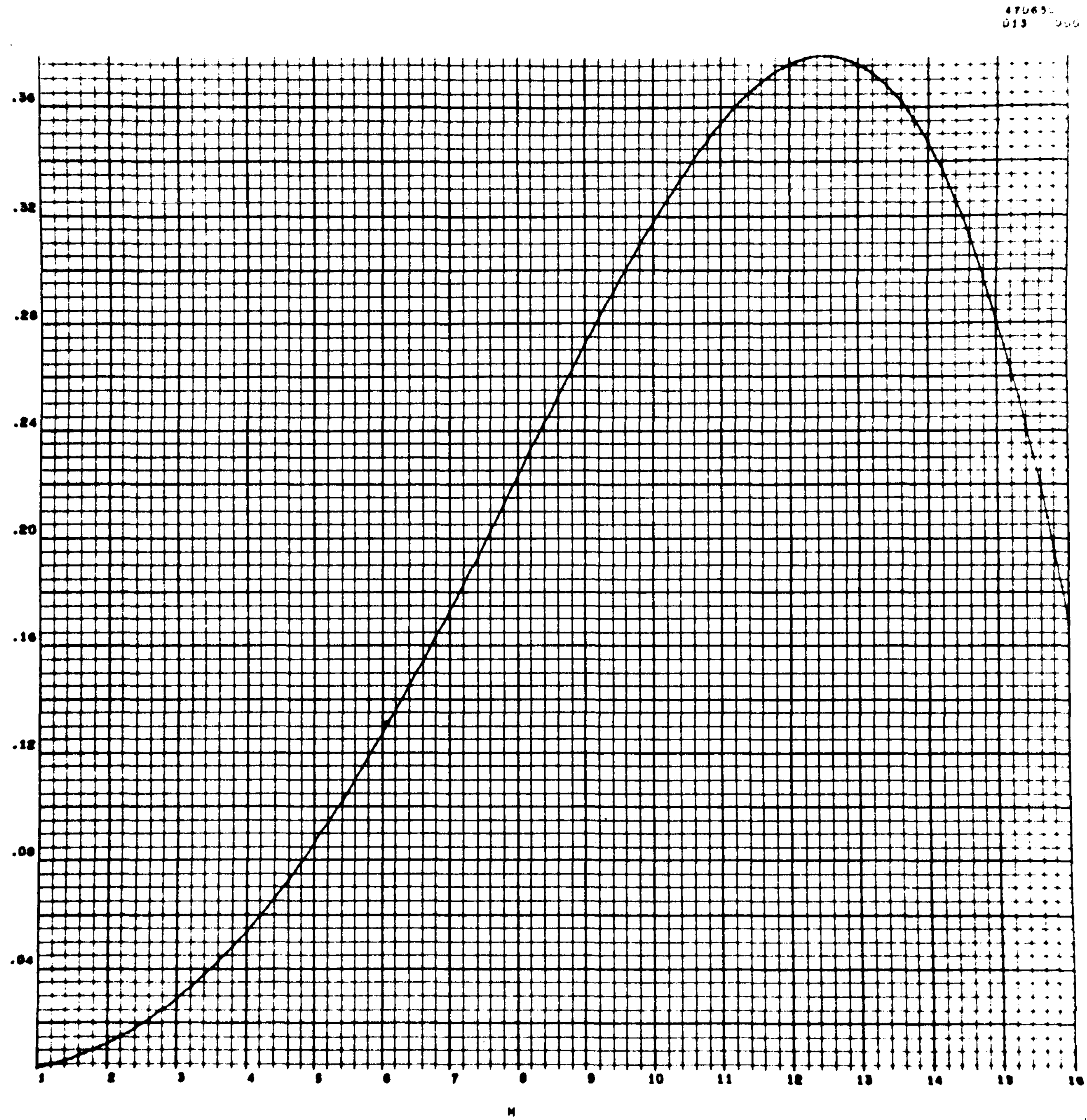


Figure 9 - Interpolative Approximation, L = 16, N = 4, k = 3

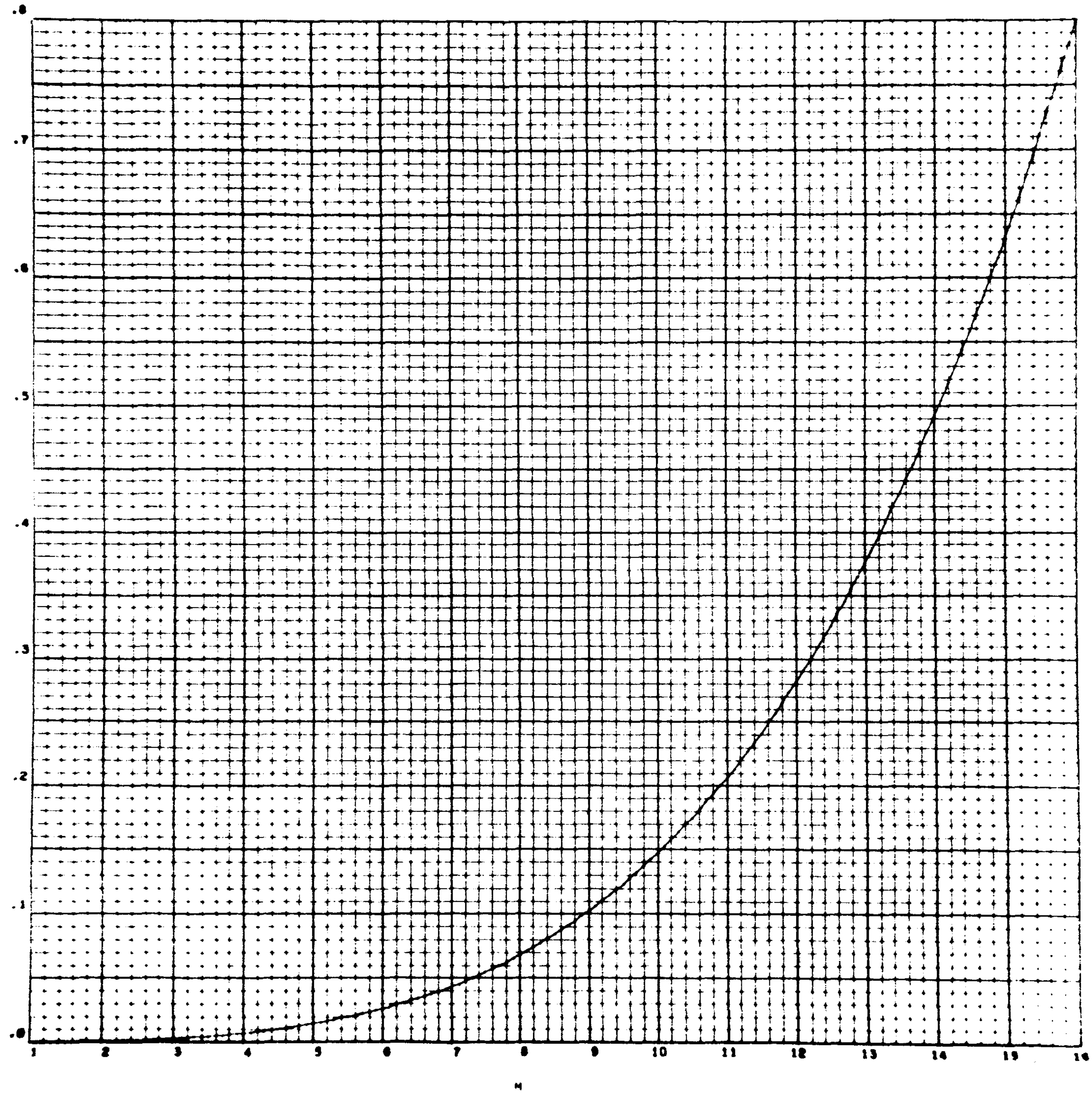


Figure 10 - Interpolative Approximation, L = 16, N = 4, k = 4

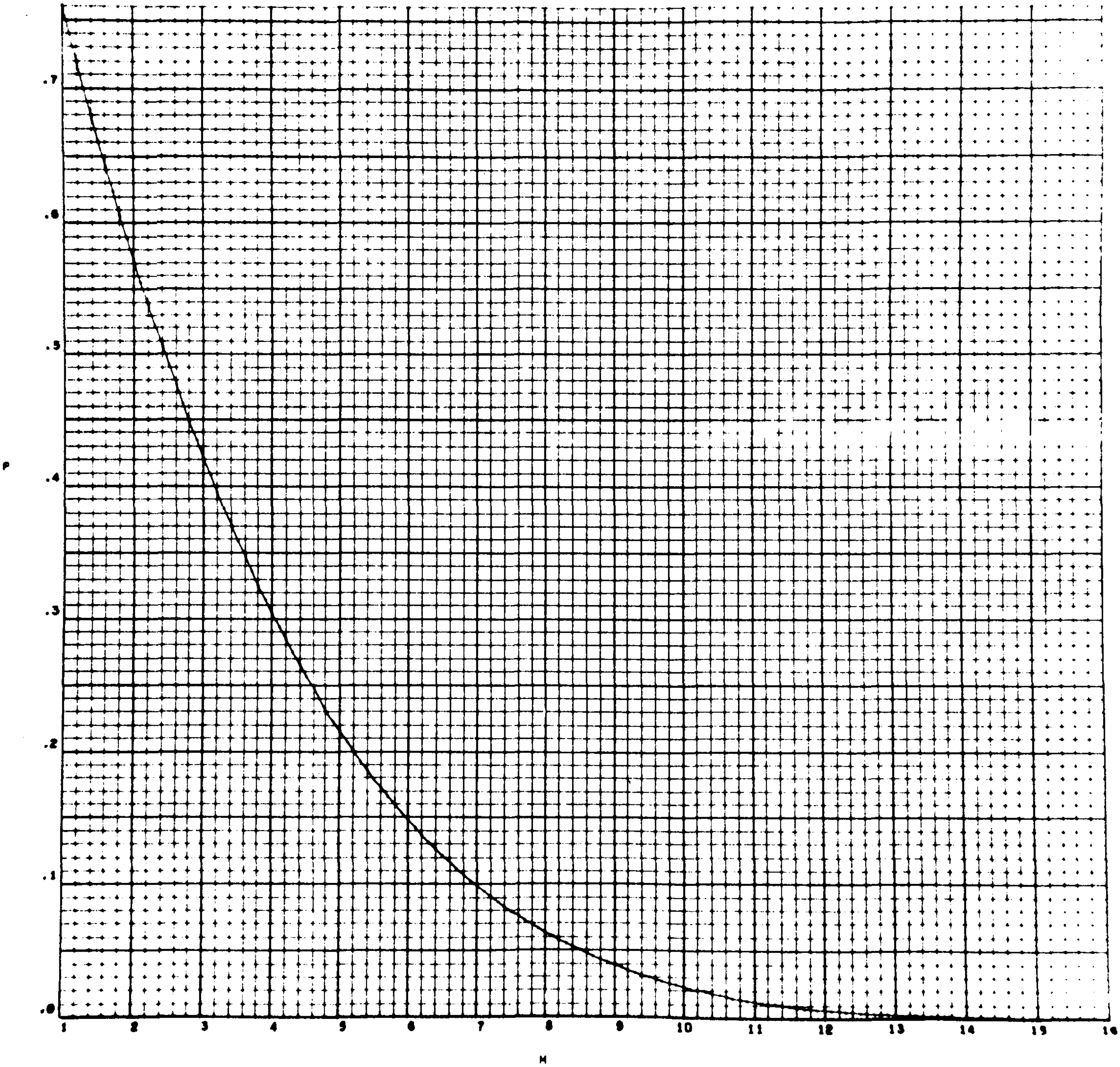


Figure 11 - Interpolative Approximation, L = 16, N = 5, k = 0

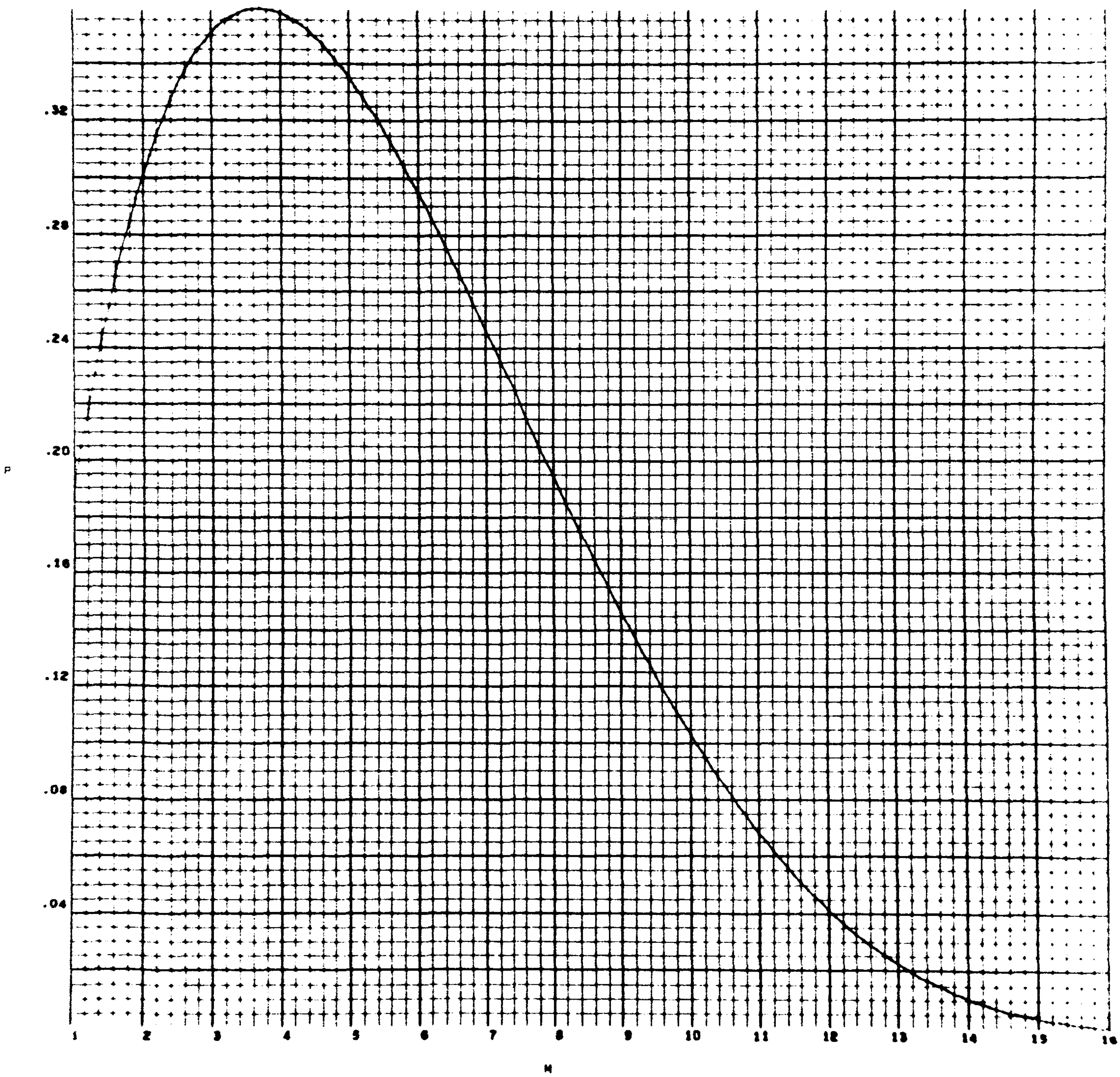


Figure 12 - Interpolative Approximation, L = 16, N = 5, k = 1

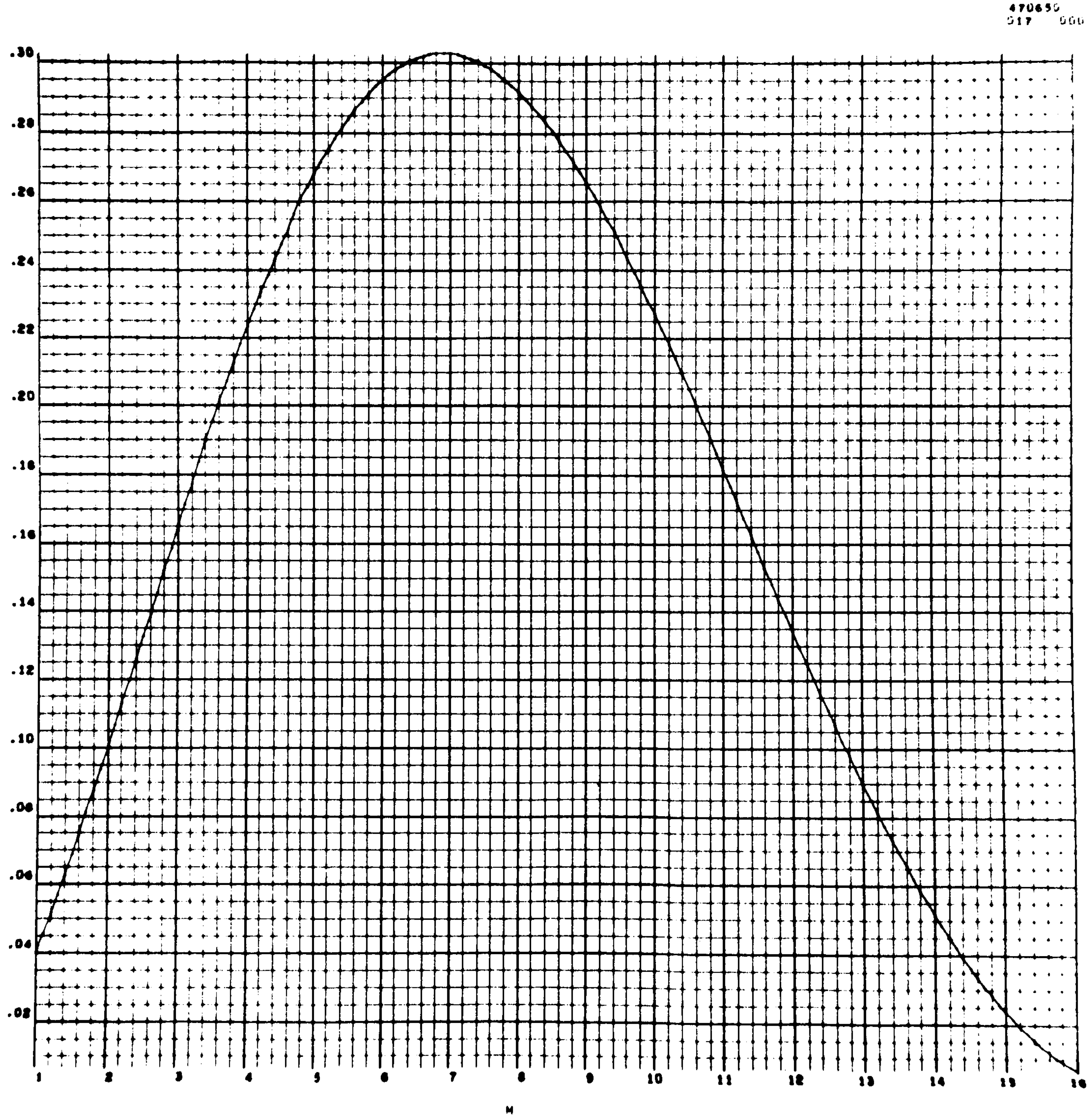


Figure 13 - Interpolative Approximation, L = 16, N = 5, k = 2

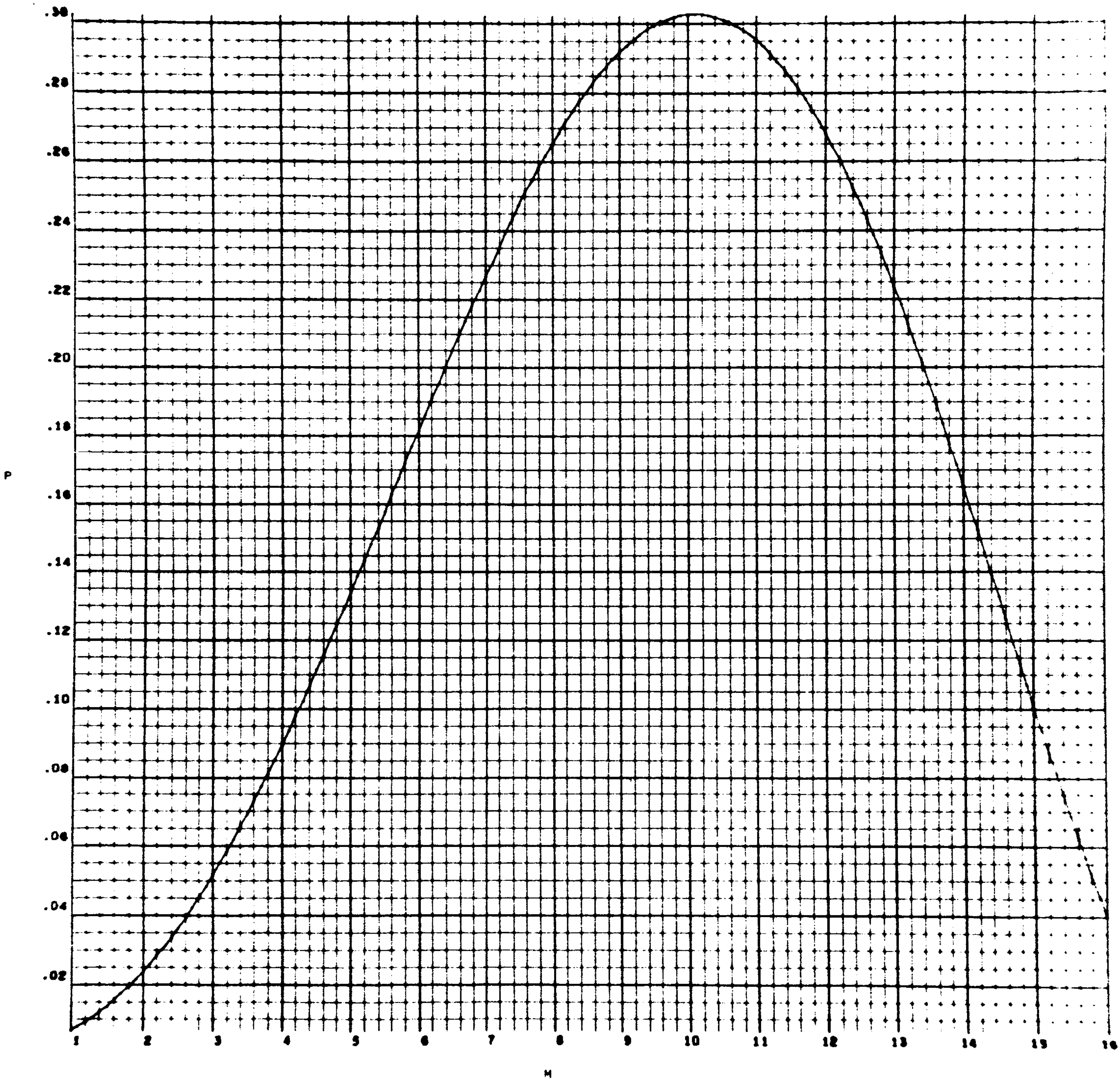


Figure 14 - Interpolative Approximation, L = 16, N = 5, k = 3

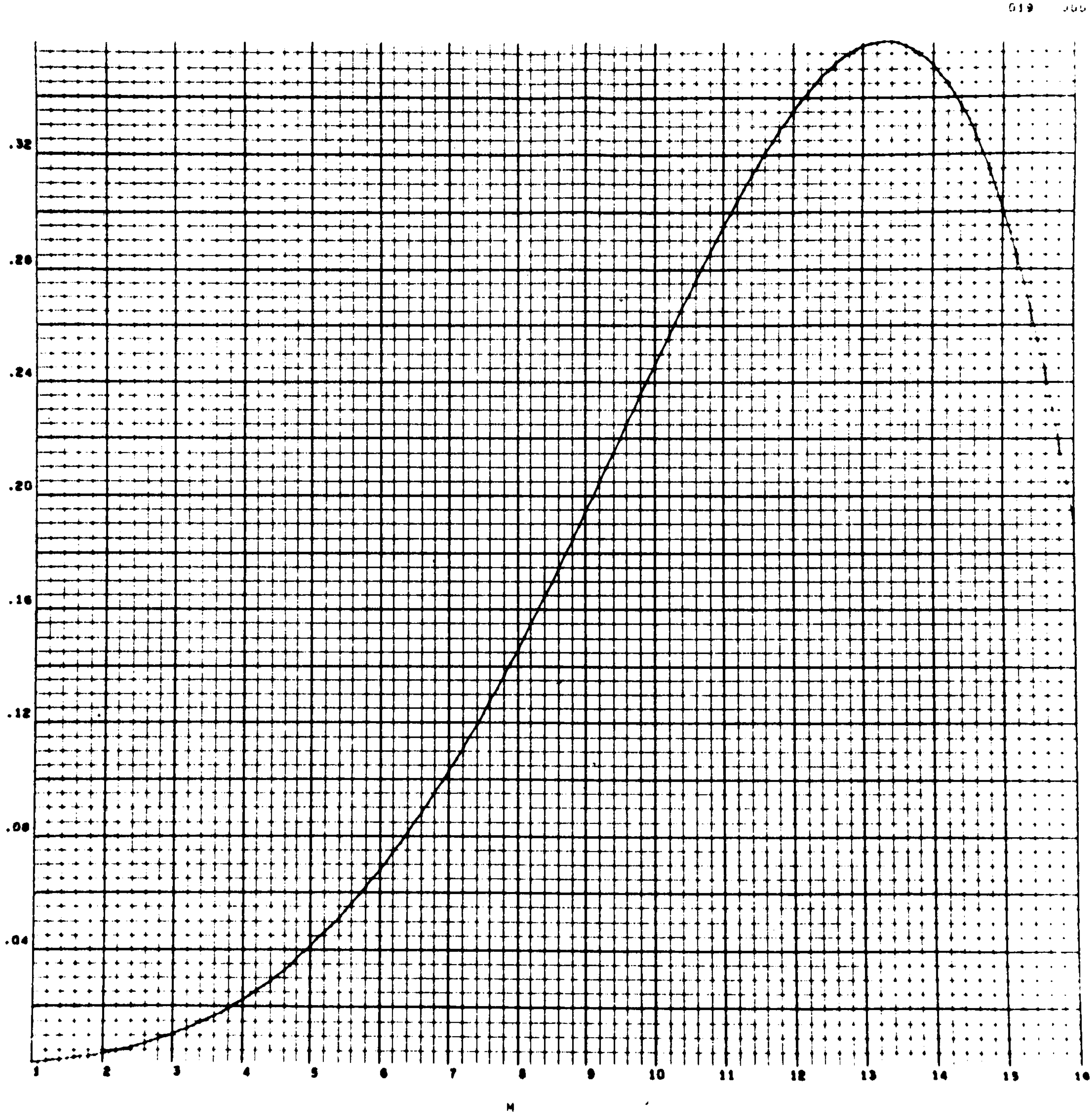


Figure 15 - Interpolative Approximation, L = 16, N = 5, k = 4

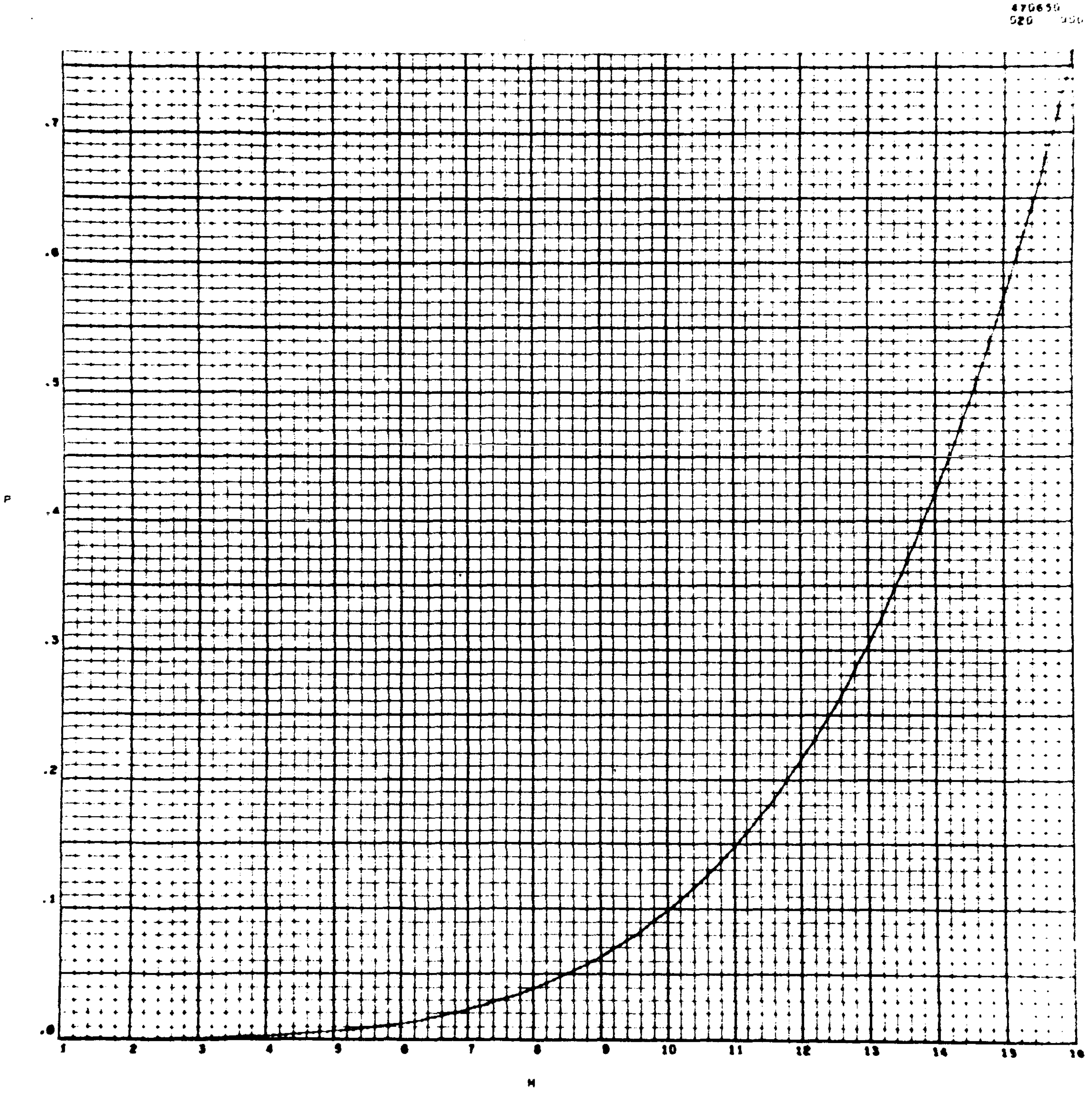


Figure 16 - Interpolative Approximation, L = 16, N = 5, k = 5

4.4 LISTING OF THE EDNE PROGRAM

```
$JOB              LMSC CARTER   BIN202,470650,00,12,140CE
$EXECUTE          IBJOB
$IBJOB
$IBFTC MAIN
C
C      THE EDNE PROGRAM
C
C        THE EXACT DISTRIBUTION OF THE NUMBER OF EXCEEDANCES (EDNE)
C      COMPUTER PROGRAM CALCULATES THE PROBABILITY THAT THE M'TH LARGEST
C      AMONG L PAST OBSERVATIONS WILL BE EXCEEDED K TIMES IN N FUTURE
C      TRIALS.   THE MEAN NUMBER OF EXCEEDANCES AND THE STANDARD DEVIATION
C      OF THE NUMBER OF EXCEEDANCES ARE DETERMINED FOR EACH VALUE OF M.
C
C      PROGRAMMER    - M. M. HANSING
C      STATISTICIAN - N. E. RICH
C
      DIMENSION P(200,51),  XMEAN(200), STDEV(200), VAR(200)
      DIMENSION X(1000),Y(1000)
      DIMENSION BTIL(12),STIL(12)
      DATA BTIL/6HM      ,11*6H      /
      DATA STIL/6HP      ,11*6H      /
C
   10 READ(5,110) L1,L2,N1,N2
      READ(5,111) OPT1,OPT2,DELM
C
      IF(OPT1.NE.1.) GO TO 801
C
      DO 800 L=L1,L2
      DO 800 N=N1,N2
      RL=L
      RN=N
      RLPN=L+N
      RNDLP1=RN/(RL+1.)
      RNDLP2=RN/((RL+1.)**2*(RL+2.))
      L12=(L+1)/2
      NP1=N+1
      L12P1=L12+1
C
```

```
      P(1,1)=RL/RLPN
C
      DO 30 M=1,L12
      RM=M
      XMEAN(M)=RM*RNDLP1
      VAR(M)=RM*(RL-RM+1.)*(RLPN+1.)*RNDLP2
      STDEV(M)=SQRT(VAR(M))
C
      DO 20 K=1,N
      RK=K-1
   20 P(M,K+1)=(RK+RM)*(RN-RK)/((RLPN-RM-RK)*(RK+1.))*P(M,K)
C
   30 P(M+1,1)=(RL-RM)/(RLPN-RM)*P(M,1)
C
      DO 40 I=L12P1,L
      K1=L-I+1
      RI=I
      XMEAN(I)=RI*RNDLP1
      STDEV(I)=STDEV(K1)
C
      DO 40 J=1,NP1
      K2=NP1-J+1
   40 P(I,J)=P(K1,K2)
C
C     WRITE STATEMENTS FOR CONTROLLED OUTPUT
C
C
      WRITE(6,700)
      WRITE(6,720) L,N
      IF(L.GT.4) GO TO 400
      WRITE(6,730) (ICOL,ICOL=1,L)
C
      DO 300 I=1,NP1
      IROW=I-1
  300 WRITE(6,740) IROW,(P(J,I),J=1,L)
      WRITE(6,750) (XMEAN(J),J=1,L)
      WRITE(6,760) (STDEV(J),J=1,L)
      GO TO 800
```

```
C
  400 ICOUNT=1
      DO 510 KFIR=1,L,4
      KLAST=KFIR+3
      IF(KLAST.GT.L) KLAST=L
      WRITE(6,730) (ICOL,ICOL=KFIR,KLAST)
C
      DO 500 I=1,NP1
      IROW=I-1
  500 WRITE(6,740) IROW,(P(J,I),J=KFIR,KLAST)
C
      WRITE(6,750) (XMEAN(J),J=KFIR,KLAST)
      WRITE(6,760) (STDEV(J),J=KFIR,KLAST)
      IF(KLAST.EQ.L) GO TO 800
      WRITE(6,710)
      ICOUNT=ICOUNT+1
      ICUT=(NP1+6)*ICOUNT
      IF(ICUT.LT.57) GO TO 510
      WRITE(6,700)
      WRITE(6,720) L,N
      ICOUNT=1
  510 CONTINUE
C
  800 CONTINUE
C
  801 IF(OPT2.NE.1.) GO TO 9999
      CALL CAMRAV(9)
      DO 900 L=L1,L2
      DO 900 N=N1,N2
      RL=L
      RN=N
      RLPN=L+N
      NP1=N+1
      NUMPTS = (RL-1.)/DELM  +1.
      DO 1000 KP1 = 1,NP1
      K = KP1 - 1
      RK = K
      DO 1010 I =1,NUMPTS
       |
```

```
      RI = I
      X(I)=1.+(RI-1.)*DELM
      IF(K.EQ.N) GO TO 1020
      IF(K.EQ.0) GO TO 1030
      FAC1 = 1.
      DO 1040 J = 1,K
      RJ = J
 1040 FAC1 = ((RN-RJ+1.)*(X(I)+RJ-1.))/(RJ*(RLPN-RJ+1.))*FAC1
      NMK = N-K
      FAC2 = 1.
      A = RLPN-X(I)-RK+1.
      B = RLPN-RK+1.
      DO 1050 J = 1,NMK
      RJ = J
 1050 FAC2 = ((A-RJ)/(B-RJ))*FAC2
      Y(I) = FAC1 * FAC2
      GO TO 1010
 1020 IF (N.NE.1) GO TO 1021
      Y(I) = X(I)/(RL+1.)
      GO TO 1010
 1021 Y(I) = 1.
      DO 1022 J = 1,N
      RJ = J
 1022 Y(I) = Y(I)*(RN+X(I)-RJ)/(RLPN-RJ+1.)
      GO TO 1010
 1030 Y(I) = 1.
      DO 1031 J = 1,N
      RJ = J - 1
 1031 Y(I) = Y(I)*(RLPN-X(I)-RJ)/(RLPN-RJ)
 1010 CONTINUE
      WRITE(6,770) (X(I),Y(I),I=1,NUMPTS)
  770 FORMAT(6E17.8)
      CALL QUIK3V(-1,42,BTIL,STIL,-NUMPTS,X,Y)
 1000 CONTINUE
  900 CONTINUE
  700 FORMAT(1H1)
  710 FORMAT(/)
  720 FORMAT(1H ,11X,43HTHE PROBABILITY THAT THE M'TH LARGEST AMONG,I3,
```

```
     118H PAST OBSERVATIONS//22X,27HWILL BE EXCEEDED K TIMES IN,I3,
     214H FUTURE TRIALS///46X,1HM)
  730 FORMAT(1H0,11X,1HK,I10,7X,I10,7X,I10,7X,I10/)
  740 FORMAT(1H ,9X,I3,4E17.8)
  750 FORMAT(1H0,8X,4HMEAN,4E17.8)
  760 FORMAT(1H ,8X,4HSDEV,4E17.8)
C
 9999 READ(5,110) ICASE
      IF(ICASE.GT.0) GO TO 10
      STOP
  110 FORMAT(4I3)
  111 FORMAT(3E12.8)
      END
$DATA
 16 16  4  5
      1.          1.                 .1
  2
100100  1  1
      1.
```

Section 5
SOME ASYMPTOTIC DISTRIBUTIONS

There are occasions when quick estimates are needed of the probabilities of the number of exceedances. The exact distribution is always superior; however, if a computer is not convenient, the necessary calculations may be forbidding. To this end, several asymptotic distributions have been derived. In this section are discussed the distributions of the number of exceedances over the larger past values (small m) for two cases. In the first case, the initial sample is small in comparison to the future sample ($L \ll N$). In the second case, both samples are large.

5.1 INITIAL SAMPLE SMALL AS COMPARED WITH FUTURE SAMPLE

Frequently, the past sample is quite small as compared to the future sample, $L \ll N$. Since N is large, the number of exceedances k and the proportion of exceedances $q = k/N$ $(0 \le q \le 1)$ can be considered as continuous variables. The density function of k, from Equation (2.2-3), is

$$p(L, m, N, k) = \frac{\binom{N}{k}\binom{L}{m} m}{(N+L)\binom{N+L-1}{k+m-1}}$$

The density function of q then becomes

$$h(q) = \left[\frac{dk}{dq}\right] p(L, m, N, Nq) = \frac{N\binom{N}{k}\binom{L}{m} m}{(N+L)\binom{N+L-1}{k+m-1}}$$

$$h(q) = \frac{Nm\binom{L}{m}(N+1)!\ (Nq+m-1)!\ (N-Nq+L-m)!}{(N+1)\ (N+L)!\ (Nq)!\ (N-Nq)!}$$

Substituting Stirling's formula,

$$J! \approx e^{-J} J^J \sqrt{2\pi J}$$

and combining terms leads to

$$h(q) \approx \frac{m\binom{L}{m} N(N+1)^{N+1} (Nq+m-1)^{Nq+m-1} (N-Nq+L-m)^{N-Nq+L-m}}{(N+L)^{N+L} (Nq)^{Nq} (N-Nq)^{N-Nq}}$$

$$\cdot \left[\frac{(N+1)(Nq+m-1)(N-Nq+L-m)}{(N+L)(Nq)(N-Nq)}\right]^{1/2}$$

Dividing numerator and denominator by $N^{2N+L} (N^3)^{1/2}$ leads to

$$h(q) \approx \frac{m\binom{L}{m}\left(1+\frac{1}{N}\right)^N \left(q+\frac{m-1}{N}\right)^{Nq+m-1} \left(1-q+\frac{L-m}{N}\right)^{N-Nq+L-m}}{\left(1+\frac{L}{N}\right)^{N+L} q^{Nq} (1-q)^{N-Nq}}$$

$$\cdot \left[\frac{\left(1+\frac{1}{N}\right)\left(q-\frac{m-1}{N}\right)\left(1-q+\frac{L-m}{N}\right)}{\left(1+\frac{L}{N}\right) q(1-q)}\right]^{1/2}$$

As N becomes large, this tends to a form independent of N;

$$h(q) = m\binom{L}{m} q^{m-1} (1-q)^{L-m}, \quad 0 \le q \le 1, \quad N \gg L \qquad (5.1\text{-}1)$$

Note that this is the same distribution as that of F_m, as discussed in Section 2.2.

The expected value and the variance of q are

$$E(q) = \int_0^1 q\, m \binom{L}{m} g^{m-1} (1 - q)^{L-m}\, dq = \frac{m \binom{L}{m} m!\, (L - m)!}{(L + 1)!}$$

$$E(q) = \frac{m}{L + 1} \qquad (5.1\text{-}2)$$

$$\operatorname{var}(q) = E(q^2) - [E(q)]^2 = \int_0^1 q^2\, m \binom{L}{m} q^{m-1} (1 - q)^{L-m}\, dq - \left(\frac{m}{L + 1}\right)^2$$

$$= \frac{m \binom{L}{m} (m + 1)!\, (L - m)!}{(L + 2)!} - \frac{m^2}{(L + 1)^2}$$

$$\operatorname{var}(q) = \frac{m(L - m + 1)}{(L + 1)^2 (L + 2)} \qquad (5.1\text{-}3)$$

Thus, the asymptotic density of the smallest values $(m = L)$ is

$$h(q) = L\, q^{L-1}, \quad m = L \qquad (5.1\text{-}4)$$

The distribution function

$$H(q) = q^L, \quad m = L \qquad (5.1\text{-}5)$$

gives the probability that up to $q \cdot 100\%$ of a future large sample will exceed the smallest value of the L past observations.

For the largest value, $m = 1$, the asymptotic density and distribution function are

$$h(q) = L(1 - q)^{L-1} , \quad m = 1 \tag{5.1-6}$$

$$H(q) = 1 - (1 - q)^{L} , \quad m = 1 \tag{5.1-7}$$

Thus, the above formula gives the probability that at most a fraction q of a future large sample will be greater than the largest or less than the smallest of a past sample of size L.

5.2 INITIAL SAMPLE AND FUTURE SAMPLE BOTH LARGE

At times both the past and the future samples are large. A different asymptotic formula must be derived for this case.

The probability density from Equation (2.2-4)

$$p(L, m, N, k) = \frac{\binom{N+L-m-k}{L-m}\binom{k+m-1}{m-1}}{\binom{N+L}{L}}$$

can be written as

$$p(L, m, N, k) = \binom{k+m-1}{m-1} \frac{(N + L - m - k)!\ L!\ N!}{(L - m)!\ (N - k)!\ (N + L)!}$$

By Stirling's formula,

$$J! \approx J^{J} e^{-J} \sqrt{2\pi J} ,$$

the density is approximated by

$$p(L, m, N, k) \approx \binom{k+m-1}{m-1} \frac{(N+L-m-k)^{N+L-m-k}\, L^L\, N^N}{(L-m)^{L-m}\,(N-k)^{N-k}\,(N+L)^{N+L}} \cdot \left[\frac{(N+L-m-k)\, LN}{(L-m)(N-k)(N+L)}\right]^{1/2}$$

As N and L become large, the last factor goes to one. Thus, after a bit of rearranging,

$$p(L, m, N, k) \approx \left[\binom{k+m-1}{m-1} \frac{L^m\, N^k}{(L+N)^{m+k}}\right]\left[\frac{L^{L-m}\, N^{N-k}\,(N+L-m-k)^{N+L-m-k}}{(L-m)^{L-m}\,(N-k)^{N-k}\,(N+L)^{N+L-m-k}}\right].$$

If m and k are kept small L and N increase, the density approaches

$$p(L, m, N, k) \approx \binom{k+m-1}{m-1} \frac{L^m\, N^k}{(L+N)^{m+k}} \qquad (5.2\text{-}1)$$

Note that this formula should be used only for small values of m.

The probability that the m^{th} largest value is never exceeded $(k = 0)$ in N future observations is

$$p(L, m, N, 0) \approx \left(\frac{N}{N+L}\right)^m \qquad (5.2\text{-}2)$$

The probability that the largest value $(m = 1)$ will be exceeded k times is

$$p(L, 1, N, k) \approx \frac{L}{N+L}\left(\frac{N}{N+L}\right)^k \qquad (5.2\text{-}3)$$

In the case of equal sample sizes, N = L, Equation (4.2-1) becomes

$$p(L, m, L, k) \approx \binom{k+m-1}{m-1}\left(\frac{1}{2}\right)^{m+k} \qquad (5.2\text{-}4)$$

Since m is small compared to L, this is called the law of rare exceedances. For $L = N$, the probability that the m^{th} largest value will never be exceeded $(k = 0)$ is

$$p(L, m, L, 0) \approx \left(\frac{1}{2}\right)^{m} \qquad (5.2\text{-}5)$$

The probability that the largest value $(m = 1)$ will be exceeded k times is

$$p(L, 1, L, k) \approx \left(\frac{1}{2}\right)^{k+1} \qquad (5.2\text{-}6)$$

By procedures similar to those used in the derivation of Equation (2.3-1) and Equation (2.3-2), it can be shown that for $N = L$ the mean and variance are

$$E(k) = m \qquad (5.2\text{-}7)$$

$$\mathrm{var}(k) = 2m \qquad (5.2\text{-}8)$$

Section 6
CONCLUSIONS

The theory of exceedances should be applied in cases in which the probability of surpassing a previously attained value is of importance. If, in the example discussed in Section 2, the value of 60 knots was considered as critical, exceedances theory could be used to give the probability of exceeding the critical value in future events.

As another example, in a study done by the Systems Optimization Section of LMSC/HREC for NASA, R-AERO-DAP, the amount of flight propellant reserve (FPR) to be carried by a future Saturn missile was to be decided. One hundred flights with different values of perturbations in flight parameters were simulated on a high-speed computer and the necessary FPR for each flight was calculated. The largest of these was taken as a conservative estimate of the amount of reserve fuel needed for any one future flight. According to Table 4 in Section 4.3, the probability of exceeding the largest of the 100 FPR values in a future run is .0099.

The theory of exceedances can thus be used for both real and simulated data. It should not, however, be applied to cases in which the population distribution is of importance, e.g., for estimating the population mean or variance.

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