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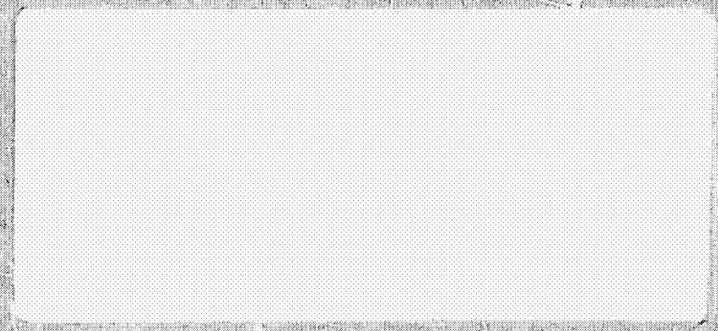

FINAL REPORT
RADIATIVE ASPECTS OF LUNAR MATERIALS
CONTRACT NO. NAS8-21044

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RADIATIVE ASPECTS OF LUNAR MATERIALS

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PREFACE

In this report we present the results of four interrelated investigations on the nature of the intensity and angular distribution of visible and infrared radiation leaving the lunar surface. The chapters are paginated independently and may be read independently.

Chapter I of the report describes a crater model for the lunar thermal radiation. This theory applies when the lunar surface normal is in the same plane as the incident solar ray and the ray being observed. Agreement between theory and experiment is very good.

In Chapter II is given a compact mathematical expression, which has been developed on a phenomenological basis, and which fits the data on lunar infrared radiation under all conditions except within a few degrees of the terminator and on the dark side. This expression may be used for accurate machine calculation of view factors and thermal fluxes near the lunar surface.

Chapter III consists of a detailed analysis of the photometric function originally derived by Hapke. Hapke's model is modified in order to conform more closely to the physical conditions in which

the model is applicable and also to provide a rationale for separating the derivation of Hapke's retrodirective function from an independent treatment of the effect of random local variations in the slopes of the lunar surface material. The latter phase of this work remains to be completed but will follow the methods of Chapter I.

Chapter IV describes the beginning of a longer-range program whose purpose is to treat the lunar surface ab initio as a random scattering, absorbing, and reflecting composite medium which interacts with the incident electromagnetic radiation. In order to properly pursue this line of investigation it has been necessary to develop new techniques for treating stochastic processes and stochastic differential equations. These techniques are discussed in this chapter.

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CHAPTER I

A CRATER MODEL FOR LUNAR THERMAL RADIATION

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1. Introduction:

The directional characteristics of lunar infrared radiation have been measured with great precision by Saari and Shorthill.[1] Ashby and Burkhard[2] have derived a phenomenological expression which describes these measurements rather accurately. Smith[3] has attempted to explain the strong anisotropy of the radiation with the help of a statistical model of lunar roughness. Excellent agreement is obtained for the full moon data for a 20° r.m.s. slope. However, no agreement has been obtained when comparison was made with the variation of the subsolar radiance with phase. A very serious drawback of Smith's method is, furthermore, that no general expression has been obtained for the lunar infrared radiation and separate calculations are necessary for each individual situation (phase). Buhl[4] has used a different approach. He considered the effects of craters in altering the emission characteristics of the lunar surface. Buhl studied several crater distributions but was unable to obtain satisfactory agreement with the data.

Both theoretical papers have in common that a Lambertian cosine-law was used to describe the directional characteristics of a uniformly heated flat surface consisting of lunar material. There is no reason for this assumption to hold. Electromagnetic theory predicts that a uniformly heated dielectric material radiates predominantly in the direction of the surface normal, although the exact amount of this enhancement is difficult to calculate.

Here we shall modify Buhl's crater model by generalizing the basic ansatz for the radiation law. Our model assumes, furthermore, a lunar surface covered in part with distribution of craters of various depth to diameter ratios, as determined by Surveyor measurement. This is also an improvement over Buhl's work, since he was able to deal with craters of a uniform depth to diameter ratio only.

A further very important improvement of the approach presented here over previous work consists in the extensive use of group theory and invariant tensor methods. These methods enable us to determine which features of the directional characteristics of the lunar infrared radiation are model dependent, i.e., depend on the details of the crater distribution, shape, et. It turns out that the thermal behavior of the lunar rim are the only features of the radiation which depend strongly on these details.

The outline of this paper is the following: In Section 2 we shall firstly summarize the relevant aspects of previous work.

Sections 3 and 4 deal with the radiation emitted by unshadowed craters. In these two sections the group theoretical method is developed and explained in detail. In these sections all quantities are calculated up to and including effects of second order in the average surface slope. In the subsequent sections only first order effects are taken into account. The purpose of this method is to estimate the effect of neglecting these higher order terms, as will be done in our final calculations. It turns out that this approximation tends to underestimate all temperatures by 2-3°K. This is therefore the limit of accuracy of our theoretical expressions.

In Section 5 we calculate irradiation corrections and show that they must not be neglected. Sections 6-10 deal with the shadowing of lunar craters. Our final results are presented in Section 11, and excellent agreement with the experimental results is obtained.

2. The Energy Balance:

In this section we first summarize the work of Ashby[2] on the influence of thermal conductivity and the lunar albedo on the infrared radiation. In table 1 we have listed the total energy received per unit surface, the energy reflected optically and the energy lost by thermal conductivity as functions of the thermal latitude γ+).. The table shows that the main term that influences

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+) γ is the angle between the surface normal $\vec{N}$ and the direction of the incident radiation i, $\cos\gamma = (\vec{i},\vec{N})$

the energy balance is optical reflection; less than 1% of the incident energy is lost by thermal conductivity. That is rather fortuitous, since the conductivity loss depends not only on i, but also on the thermal longi .de. The values quoted in the table are averages over the longitude. Since the magnitude of the conduction term is sufficiently small we can neglect the longitude dependence within the required accuracy.

Subtracting these two energy losses from the total energy received we see that the energy available for thermal radiation is given to a high degree of accuracy by $S_0 \cos i$ (instead of $S \cdot \cos i$), where $S_0 = 1.88$ cal/cm^2 is the solar constant corrected for conduction and reflection. This result is rather important, since it allows us to take into account the two other energy transfer processes in a very simple manner.

Next we check the energy balance by integrating the energy radiated from the subsolar point into all directions. The temperature of the subsolar point is related to the energy by

$$T^4(\alpha) = \frac{E(\alpha)}{\cos\alpha} \qquad (2.1)$$

where α is the phase angle. The total energy E emitted by a unit surface at the subsolar point is

$$E = \int E(\alpha)\sin\alpha\, d\alpha\, d\Phi = \pi \int_0^{\frac{\pi}{2}} T^4(\alpha)\, d\cos^2\alpha \qquad (2.2)$$

where Φ is the thermal longitude. The evaluation of this integral leads to 1.89 cal/cm^2 min, agreeing rather precisely with the value 1.88 cal/cm^2 min following from the considerations discussed before. Furthermore, we find that the radiation in the direction orthogonal to the surface is increased by a factor 1.06±0.01 compared to a Lambertian radiator. The apparent temperature in the center of the lunar disk at full moon follows from this to be

$$T = 397 \pm 1^\circ K,$$

agreeing rather precisely with the data of Saari and Shorthill.[1]

The older value T = 407°K due to Pettit and Nichelson is incompatible with this.

3. Thermal Radiation from a Single Crater-Case of no Shadowing

Next we have to investigate the characteristics of the radiation emitted by a single crater. To calculate this we have to make an assumption about the behavior of a flat surface element. We shall assume that it radiates with an angular distribution proportional to

$$E(\vec{e},\vec{n}) \propto \rho(\vec{e},\vec{n}) = (1-r-t)(\vec{e},\vec{n}) + \frac{3}{2}r(\vec{e},\vec{n})^2 + 2t(\vec{e},\vec{n})^2 \qquad (3.1)$$

r and t are two constants, describing the deviation of the radiation from a purely Lambertian cos-law. $\vec{e}$ and $\vec{n}$ are unit vectors

in the direction of the emitted radiation and the local surface normal resp., $(\vec{e},\vec{n}) = \cos\epsilon$ denotes their scalar product.

The incident energy flux on a surface area f is $f(\vec{i},\vec{N})\cdot S$, where S is the solar constant, $S = 1.99$ csl/cm^2 sec. Part of this energy is reflected in the form of visible light and part is conducted away by thermal conductivity, as discussed in the previous section.

Therefore only a fraction $S_o df(\vec{i},\vec{N})$ is actually reradiated in the form of infrared radiation, $S_o = 1.88$ cal/cm^2 sec. The energy balance is fulfilled if we write the infrared energy emitted per unit solid angle unit time as

$$E(\vec{e},\vec{i},\vec{n}) = \frac{1}{\pi}S_o f(\vec{i},\vec{N})[(1-r-t)(\vec{e},\vec{n})+\frac{3}{2}r(\vec{e},\vec{n})^2+2t(\vec{e},\vec{n})^3] \qquad (3.2)$$

One easily checks that the total radiation emitted per unit time is

$$\int E(\vec{e},\vec{i},\vec{n})d\Omega_e = S_o f(\vec{i},\vec{N}) \qquad (3.3)$$

where $d\Omega_e$ is the solid angle element of the emitted radiation.

Next we shall calculate the radiation emitted from a surface of arbitrary slope distribution using the following approximations:

1) There are no preferred directions of the surface
2) We neglect reradiation
3) We neglect shadowing.

Approximation 1) is true for a surface covered by circular craters or hills, but also for much more general surfaces, provided that the asperities are not aligned in one dominant direction. This approximation is well fulfilled for the moon. The other assumptions concerning shadowing and reradiation will be eliminated in the second part of this paper by using the special surface model discussed above.

The calculation proceeds as follows. We have to integrate the radiation emitted over a surface with varying surface normal $\vec{n}(f)$ where f stands for the surface element considered. The total radiation becomes

$$E(\vec{N},\vec{i},\vec{e}) = \frac{S_o}{\pi}\int df(\vec{i},\vec{n})[(1-r-t)(\vec{e},\vec{n})+\frac{3}{2}r(\vec{e},\vec{n})^2+2t(\vec{e},\vec{n})^3] \quad (3.4)$$

where $\vec{N}$ is the <u>average</u> surface normal. This integral can be evaluated rigorously. Let us consider e.g. the first term (we introduce tensor notation: n_i, i = 1,2,3 are the components of the unit vector $\vec{n}$)

$$(1-r-t)\frac{S_o}{\pi}i_k e_m \int df n_k n_m \quad (3.5)$$

The integral can only be of the form

$$\int df n_k n_m = a_1 N_k N_m + a_2 \delta_{km} \quad (3.6)$$

This can be shown formally with the help of elementary group theory[5] (the result follows from the rotation invariance assumed, i.e. that there are no preferred directions on the moon). It is, however, intuitively clear since N is the only quantity that can enter our result besides the invariant unit tensor δ_{ik}.

By contracting (3.6) over k and m (multiplying by δ_{km}) we obtain

$$\int df = a_1 + 3a_2 \tag{3.7}$$

and by multiplying (3.6) with $N_k N_m$ we have

$$\int df (\vec{N},\vec{n})^2 = a_1 + a_2$$

$(\vec{N},\vec{n}) = \cos\alpha$ is the cos of the slope of the surface, i.e.,

$$\int df (\vec{N},\vec{n}) = \hat{F}\,\overline{\cos^2\alpha} \equiv \hat{F} c_2 \tag{3.9}$$

where the bar denotes averaging over the surface and $\hat{F}$ is the total surface considered:

$$\int df = \hat{F} \tag{3.10}$$

We will however, re-express our results in terms of the apparent surface area (we introduce the notation $\overline{\cos^n\alpha} = c_n$).

$$F = \int df' = \int df\cos\alpha = \hat{F}c \tag{3.11}$$

i.e.

$$a_1+3a_2 = \hat{F} = \frac{F}{c_1} \tag{3.12}$$

$$a_1+a_2 = \frac{Fc_2}{c_1} \tag{3.13}$$

i.e.

$$\int df n_k n_m = \frac{F}{c_1}\left[\frac{3c_2-1}{2}N_kN_m + \frac{1-c_2}{2}\delta_{km}\right] \tag{3.14}$$

This completes the evaluation of the integral (3.6). Inserting the result into (3.5) we have for this term

$$(1-r-t)\frac{S_o}{\pi}\frac{F}{2c_1}[(3c_2-1)(\vec{i},\vec{N})(\vec{e},\vec{N})+(1-c_2)(\vec{i},\vec{e})] \tag{3.15}$$

Similarly the other terms can be evaluated in terms of the slope distribution of the surface. The relevant tensors are of the form

$$\int df n_i n_k n_m = b_1N_iN_kN_m+b_2(\delta_{ik}N_m+\delta_{im}N_k+\delta_{km}N_i) \tag{3.16}$$

and

$$\int df n_i n_k n_m n_r = d_1N_iN_kN_mN_r+d_2(\delta_{ik}N_mN_r+\delta_{im}N_kN_r+\delta_{ir}N_mN_k+\delta_{mr}N_iN_k$$
$$+\delta_{kr}N_iN_m+\delta_{mk}N_iN_r)+d_3(\delta_{ik}\delta_{mr}+\delta_{im}\delta_{kr}\delta_{ir}\delta_{mk}) \tag{3.17}$$

This follows from symmetry and rotation invariance. The constants can be determined as before to be

$$b_1 = \frac{F}{2c_1}(5c_3-3c_1) \qquad b_2 = \frac{F}{2c_1}(c_1-c_3)$$

$$d_1 = \frac{F}{8c_1}(3-30c_2+35c_4)$$

$$d_2 = \frac{F}{8c_1}(6c_2-1-5c_4)$$

$$d_3 = \frac{F}{8c_1}(1-2c_2+c_4) \qquad (3.18)$$

The values of the constants c_1 can, of course, only be given for a special surface model, i.e. the crater model. Here the final result is that, neglecting shadowing and reradiation, the radiation from a surface with slope distribution $\overline{\cos^n\alpha} = c_n$ is given by

$$\begin{aligned} E(\vec{i},\vec{e},\vec{N}) = \frac{S_o}{\pi}\frac{F}{2c_1}\{&(1-r-t)[(3c_2-1)(\vec{i},\vec{N})(\vec{e},\vec{N})+(1-c_2)(\vec{i},\vec{e})] \\ &+\frac{3}{2}r(5c_3-3c_1(N,i)(N,e)^2+\frac{3}{2}r(c_1-c_3)[2(\vec{i},\vec{e})(\vec{e},\vec{N})+(\vec{i},\vec{N})] \\ &+\frac{2t}{4}(3-30c_2+35c_4)(N,i)(N,e)^3+\frac{2\cdot 3t}{4}(6c_2-5c_4-1) \\ &\cdot[(i,e)(N,e)^2+(i,N)(N,e)]+\frac{2\cdot 3t}{4}(1-2c_2+c_4)(\vec{i},\vec{e})\} \end{aligned} \qquad (3.19)$$

4. Geometry of Spherical Craters

Next we specialize to a surface covered in part by craters. The special model which we shall use is that the crater is part of a sphere (see Fig. 1). For these one can easily calculate the constants c_i as a function of the depth/diameter ratio of the craters. This can be easily done with the help of a stereographic projection on the plane through the center of the sphere of which the crater is part. (See Fig. 2). We can describe a point on the sphere either by the three Cartesian coordinates (x,y,z) or by its stereographic coordinates (ξ,η). From the figure one easily reads off the connection

$$\xi = \frac{x\,R}{R-x} \qquad \eta = \frac{R\,y}{R-z} \tag{4.1}$$

and

$$x = 2\xi\,\frac{R^2}{R^2+\xi^2+\eta^2} \qquad y = 2\eta\,\frac{R^2}{R^2+\xi^2+\eta^2}$$

$$z = -R\,\frac{R^2-\xi^2-\eta^2}{R^2+\xi^2+\eta^2} \tag{4.2}$$

The surface element df of the sphere becomes in terms of (ξ,η)

$$df = \frac{4R^4 d\xi d\eta}{(R^2+\xi^2+\eta^2)^2} \tag{4.3}$$

Denoting the radius of the crater by a and the projected radius by $\bar{a}$ (see Fig. 4) we have from (4.2)

$$a = 2\bar{a}\,\frac{R^2}{R^2+\bar{a}^2} \qquad \bar{a} = \frac{Ra}{R+\sqrt{R^2-a^2}} \tag{4.4}$$

This enables us to find the surface $\hat{F}$ of the crater

$$\hat{F} = 4R^4 \int_0^{\xi^2+\eta^2-a^2} \frac{d\xi\,d\eta}{(R^2+\xi^2+\eta^2)} = \frac{4\pi R^2\bar{a}^2}{R^2+\bar{a}^2} \tag{4.5}$$

Introducing the parameter s by

$$s = h/2R \tag{4.6}$$

we find

$$\bar{a} = R\sqrt{\frac{s}{1-s}} \qquad a = 2R\sqrt{s(1-s)}$$

and thus for the true and apparent areas of the crater

$$\hat{F} = 4\pi R^2 s \tag{4.7}$$

$$F = \pi a^2 = 4\pi R^2 s(1-s) \tag{4.8}$$

After these preliminaries we are able to calculate the constants c_i necessary for the explicit evaluation of (3.19). The unit normal vector has the components

$$\vec{u} = -(x/R,\ y/R,\ z/R) \tag{4.9}$$

Therefore the constants c_n become

$$c_n = 4R^2 \int_0^{\xi^2+\eta^2-\bar{a}^2} \frac{d\xi\, d\eta}{(R^2+\xi^2+\eta^2)^2} \left(\frac{2R^2}{R^2+\xi^2+\eta^2} - 1\right)^n \frac{1}{F} . \tag{4.10}$$

Introducing polar coordinates this becomes

$$c_n = \frac{1}{s} \int_0^{\bar{a}^2} \frac{dz}{(R^2+z)^2} \left(\frac{2R^2}{R^2+z} - 1\right)^n \tag{4.11}$$

These integrals can be evaluated with the help of the substitution $(R+z)^{-1} = z'$ to be

$$c_n = \frac{1}{2(n+1)s}\left[1-(1-2s)^{n+1}\right] \tag{4.12}$$

i.e.

$$c_0=1 \qquad c_1=1-s \qquad c_2=1-2s+\frac{4}{3}s^2 \tag{4.13}$$

$$c_3=1-3s+4s^2-2s^2 \qquad c_4=1-4s+8s^2-8s^3+3.2s^4$$

From $c_2 = \overline{\cos^2\alpha} = 1-\sin^2\bar{\alpha} = 1-2s+\frac{4}{3}s^2$ we obtain (neglecting the s^2 term) the simple interpretation

$$s \simeq \tfrac{1}{2}\,\overline{\sin^2\alpha}\ . \qquad (4.14)$$

(4.14) relates s and the square of the average surface slope. We expect that the average s of the lunar surface will be in the range $0.05 \leq s \leq 0.20$, i.e. s is a small parameter. Therefore it is justified to neglect terms of third and higher order in s, since they will contribute less than 1% to the total radiation energy, corresponding to a contribution of 1°K to the apparent surface temperature. Inserting (4.13) into (3.19) we obtain for the total energy emitted:

$$\begin{aligned} E(\vec{i},\vec{e},\vec{N}) = \frac{S_o F}{\pi}\{&(i,e)[(1-r-t)s(1+s/3)]+(i,e)(N,e)3rs(1-s) \\ &+(i,N)\tfrac{3}{2}rs(1-s)+(i,N)(e,N)[(1-r)(1-2s)] \\ &+(i,N)(e,N)^2\ \tfrac{3}{2}r(1-5s+5s^2)\} \end{aligned} \qquad (4.15)$$

In this way we are able to express the infrared energy in terms of the coefficients r,t of the radiation law for a flat surface and in terms of the square of the average surface slope, 2s.

5. Reradiation Corrections:

In this and the subsequent sections we shall neglect all s^2 and t terms, since the results derived so far will be simplified to estimate the order of magnitude of these contributions. It turns out that they are actually negligible.

Then we shall calculate the contribution of the radiation emitted by one part of the crater surface and subsequently absorbed and re-emitted by another part of the crater.

The geometrical situation is shown in Fig. 3, $\vec{e}$, $\vec{i}$, $\vec{e}_{10} = -\vec{e}_{01}, \vec{n}_1, \vec{n}_o$ are unit vectors in the directions indicated. The radiation emitted into the direction e from point 1 becomes (ρ is defined by (3.1))

$$S_o df_o \left(\frac{in_o}{\pi}\right) \cdot \rho \left(\vec{e}_{01} \cdot \vec{n}_o\right) \left(\frac{\vec{e}_{01} \vec{n}_1}{\pi r_{01}^2}\right) \xi \left(\vec{e} \vec{n}_1\right) df_1 \qquad (5.1)$$

We have to select those terms from (5.1) which lead to terms of first order in s(we shall also include terms proportional to $s^{3/2}$). To evaluate (5.1) we first note that

$$\left(\vec{e}_{ik} \vec{n}_i\right) = \left(\vec{e}_{ki} \vec{n}_k\right) = \frac{r_{ik}}{2k} \qquad (5.2)$$

as can be seen from elementary geometrical considerations. Therefore ρ becomes

$$\rho \left(e_{01} n_o\right) = (1-r) \frac{r_{01}}{2R} + \frac{3}{2} r \frac{r_{01}^2}{4R^2} \qquad (5.3)$$

The reradiation correction of first order becomes

$$S_o df_o \frac{(in_o)}{4\pi R^2}[(1-r)+\frac{3}{2}r\ \frac{r_{01}}{2R}]\frac{1}{\pi}\rho(\vec{en}_1)df_1 \qquad (5.4)$$

The term proportional to 1-r is easily evaluated since

$$\int df_o(in_o) = F(\vec{iN}) \qquad (5.5)$$

It contributes

$$S_o \frac{F}{4\pi R^2}(iN)\int df_1\rho(\vec{en}_1)(1-r)$$

$$= \frac{FS_o s}{\pi}(1-r)(\vec{iN})[(1-r)(\vec{eN})+\frac{3}{2}rs+\frac{3}{2}r(1-2s)(Ne)^2] \qquad (5.6)$$

to the total radiation. The next term, proportional to r, is slightly more complicated:

$$\frac{3S_o r}{16\pi^2 R^3}\int df_o\int df_1(in_o)r_{01}[(1-r)(\vec{en}_1)+\frac{3}{2}r(\vec{en}_1)^2] \qquad (5.7)$$

The integral contained in (5.7) can be evaluated to be

$$\int df_o\int df_1 r_{01} = a^5\pi^2\cdot\frac{128}{45\pi} = 0.905(\pi^2a^5) \qquad (5.8)$$

This completes our discussion of reradiation. The total infrared energy becomes, therefore, adding (4.15), (5.6) and (5.7) (neglecting s^2 and t terms)

$$E(i,e,N) = \frac{S_o F_o}{\pi} \cdot \{(1-r)s(\vec{i}\vec{e}) + (\vec{i}\vec{e})(\vec{N}\vec{e})3rs + (\vec{i}\vec{N}) \cdot \frac{3}{2}rs + (\vec{i}\vec{N})(\vec{e}\vec{N})$$

$$\times[(1-r)(1-2s) + s(1-r)^2 + \frac{3}{2}r(1-r)s^{3/2}]$$

$$+ (\vec{i}\vec{N})(\vec{e}\vec{N})^2[\frac{3}{2}r(1-5s+5s^2) + \frac{3}{2}rs]\} \qquad (5.9)$$

6. Shadowing: Geometrical Considerations

We now turn to the shadowing of the craters. Two different types of shadowing have to be distinguished, i.e., actual and apparent shadowing.

Actual shadowing is due to the fact that the angle of incidence exceeds a limiting value, determined by the depth to diameter ratio of the crater. In this case part of the crater will be shadowed.

Apparent shadowing occurs when the angle of observation exceeds a critical value. In this case part of the crater surface will be invisible from the direction of observation. In general both effects, apparent as well as actual shadowing, are present simultaneously. (See Fig. 4), leading to rather complicated geometrical configurations.

We have restricted ourselves therefore to situations which can be dealt with easily, i.e. the following (see Fig. 5).

a) Only apparent or actual shadowing present

b) Apparent and actual shadows do not overlap

c) Apparent and actual shadows overlap completely.

The simplest situation is obviously a), we shall deal with it first.

We specialize again to spherical craters. For these it is well known that the shadowed area is bordered by a circle with radius equal to the one of the crater rim. This can also be read off from Fig. 6, by observing the symmetry of the figure with respect to reflections at the $(\vec{J},\vec{N})$ plane. Therefore the shadowed area has the form shown in Fig. 7 (seen from the direction $\vec{J}$). The size of the shadowed area can be determined from Fig. 6.

$$dg\eta = \frac{x}{R-h} = dgi \tag{6.1}$$

$$\cos(90-i) = \frac{b}{a-x} \tag{6.2}$$

$$b = (a-x)\cos(90-i) = [a-(R-h)ctgi]\sin i = a[\sin i - \frac{R-h}{a}\cos i] \tag{6.3}$$

The ratio (R-h)a can be expressed in terms of the parameter $s = h/2R$ introduced in (4.6), using $a = 2R\sqrt{s(1-s)}$

$$\frac{R-h}{a} = \frac{1-2s}{2\sqrt{s(1-s)}} = \sqrt{\frac{1}{4s-4s^2} - 1} \equiv t. \tag{6.4}$$

The length b, characterizing the size of the shadowed area is therefore given by

$$b = a[\sin i - t\cos i] \tag{6.5}$$

b has to be positive, otherwise the shadow vanishes. Shadowing sets in at an angle of incidence for which b = 0, i.e.

$$\tan i = t \tag{6.6}$$

This completes the discussion of the geometrical configuration of the shadowed area.

7. Shadowing of Lambertian Radiation

Here we shall calculate the influence of shadowing on the Lambertian part of the radiation. We have to evaluate the integral (3.6) for this purpose, taking into account that the region of integration has to be extended over the unshadowed surface area only. This is done best by writing

$$\int_{\text{unshadowded}} df n_k n_m = \int_c df n_k n_m - \int_s df n_k n_m \tag{7.1}$$

(c--crater, s--shadowed part of carter)

The first term on the right hand side has been evaluated in (3.14). The symmetry of the shadowed area can be used to simplify the integration of the second term. For this purpose we rotate the coordinate systemused in such a way that the x-axis points in the direction of the incident solar light (Fig. 8). Then the vector $\vec{i}$ has the components (1,0,0), i.e. only an x-component, so that only terms

$$\int_s df n_x n_m \qquad (7.2)$$

are needed for the evaluation of (3.5). From the symmetry of the shadowed area we see that

$$\int_s df n_x n_y = \int_s df n_x n_z = 0 \qquad (7.3)$$

Therefore

$$i_x e_x \int_s df n_x^2 = (\vec{i}\vec{e}) \int df n_x^2 \qquad (7.4)$$

is the only remaining term which has to be subtracted from (3.5) in order to take shadowing into account.

If we define

$$v_i = \frac{1}{sF} \int_s df n_x^2 \qquad (7.5)$$

We obtain the following modification of the Lambertian radiation law (neglecting reradiation at the moment)

$$E_i(\vec{i},\vec{e},\vec{N}) = \frac{S_o F}{\pi}\{(\vec{i}\vec{e})s(1-v_i)+(\vec{i}\vec{N})(\vec{e}\vec{N})(1-2s)(1-r)\} \qquad (7.6)$$

For apparent shadowing all calculations remain unchanged, except that the shadowed area has to be replaced by the invisible one, i.e. we have to define

$$sFv_e = \int_{\text{inv.}} dfn_x^2 \qquad (7.7)$$

(inv.--invisible area). v_i and v_e are of course the same function, differing only by their argument.

If both apparent and actual shadowing are present the situation is more complicated. If the two shadows are situated as in Fig. 5b they can be treated independently, and we have to replace v_i by v_i+v_e in this case. If the two shadows overlap completely, as in Fig. 5c only the larger one has to be taken into account, since it is irrelevant that e.g. part of the invisible area is shadowed.

The correct expression for the radiation law for all situations shown in Fig. 5 is therefore

$$E_L = \frac{S_o F}{\pi}\{(\vec{i}\vec{e})s(1-v_i U v_e)+(\vec{i}\vec{N})(\vec{e}\vec{N})(1-r)(1-2s)\} \qquad (7.8)$$

where $v_i U v_e$ is the set theoretic union of the areas v_i and v_e.

8. Shadowing: Lambertian Reradiation

The reradiation corrections to the Lambertian term is given by (5.5) to be

$$\frac{S_o F s}{\pi}(1-r)^2(\vec{i}\vec{N})(\vec{e}\vec{N}) \qquad (8.1)$$

First we consider the influence of apparent shadowing on (8.1). If part of the surface of the crater is invisible the

integral $\int df_1$ in (5.5) has to be extended over the observable part of the surface only. Denoting the fraction of the crater area which is invisible at a particular angle of observation by

$$u_e = \frac{1}{F} \int_{\text{inv.}} df \tag{8.2}$$

we see that (8.1) has to be replaced by

$$\frac{FS_o s}{\pi}(1-r)^2(\vec{iN})(\vec{eN})(1-u_e). \tag{8.3}$$

The situation is slightly different for actual shadowing. There the integral over df_o as well as the one over df_1 have to be extended over the unshadowed fraction of the surface only, giving rise to a factor $(1-u_i)^2$, where u_i is defined by

$$u_i = \frac{1}{F} \int_s df \tag{8.4}$$

The generalizatim of our result to all situations shown in Fig. 5 is

$$\frac{FS_o s}{\pi}(1-r)^2(\vec{iN})(\vec{eN})(1-u_e \cup u_i)(1-u_i) \tag{8.5}$$

This completes the discussion of Lambertian radiation.

9. Shadowing: Non-Lambertian Terms

In order to find the appropriate modifications of the non-Lambertian terms we have to evaluate expressions of the form

$$i_k e_m e_n \int_{\text{unshadowed}} dfn_k n_m n_n \,. \tag{9.1}$$

We decompose the integral as in (7.1). The symmetries can again be used to simplify the integration. In the case of actual shadowing we use the coordinate system shown in Fig. 8. Integrals of the form

$$\int_s dfn_x n_m n_n \tag{9.2}$$

have to be evaluated. The only non-vanishing integral is (due to symmetry)

$$\int_s dfn_x^2 n_z \tag{9.3}$$

Therefore we have to subtract

$$-2i_x e_x e_z \int dfn_x^2 n_z \tag{9.4}$$

from the total radiation. The factor 2 is due to the fact that the terms n=x m=z and n=z m=x contribute the same amounts to (9.4). The integral $\int_s dfn_x^2 n_z$ is proportional to s because of the factor

n_x^2 appearing in it. Since n_z deviates from 1 only by terms of order s

$$n_z = 1-0(s) \tag{9.5}$$

we can replace (9.4) in the desired approximation (neglecting terms of order s^2) by

$$-2i_x e_x e_z \int df n_x^2 = -2i_x e_x e_z v_i sF \tag{9.6}$$

Next we have to transform from the special coordinate system used to general coordinates, which will be done by writing (9.6) in an invariant, vectorial form. $i_x e_x$ becomes in this way $(\vec{i}\vec{e})$ as before. e_z is given by the scalar product $(\vec{e}\vec{j})$ where $\vec{j}$ (shown in Fig. 6) is the unit vector normal to $\vec{i}$:

$$\vec{j} = \frac{\vec{i}x(\vec{N}x\vec{i})}{|\vec{i}x(\vec{N}x\vec{i})|} = \frac{\vec{i}x(\vec{N}x\vec{i})}{\sin i} \ . \tag{9.7}$$

The shadowing term (9.6) becomes therefore

$$-2(\vec{i}\vec{e})[(\vec{N}\vec{e})-(\vec{i}\vec{e})(\vec{N}\vec{i})] \frac{v_i}{\sin i} sF. \tag{9.8}$$

We define a third shadowing function w_i by

$$w_i = \frac{v_i}{\sin i} \ . \tag{9.9}$$

Multiplying (9.8) by $\frac{3}{2}S_o r/\pi$ we obtain the contribution of this shadowing term to the total radiation

$$\frac{3rS_o}{\pi}[-(\vec{i}\vec{e})(\vec{N}\vec{e})+(\vec{i}\vec{e})^2(\vec{i}\vec{N})]w_i sF. \qquad (9.10)$$

Next we consider apparent shadowing. Since the non-Lambertian radiation is not symmetric in i and e we cannot simply replace i by e in all equations. We choose the x-axis of the coordinate system in the direction of observation $\vec{e}$. The shadowing contribution becomes

$$-i_k \frac{e_x e_x}{i}\int dfn_k n_x^2 = -i_z\int dfn_x^2 n_z \cong$$

$$= -i_z\int dfn_x^2 = -i_z sFv_i \qquad (9.11)$$

To rewrite this in invariant vectorial form we proceed as before and obtain

$$-\frac{3}{2}rs\frac{FS_o}{\pi}[(\vec{i}\vec{N})-(\vec{i}\vec{e})(\vec{N}\vec{e})]w_e \qquad (9.12)$$

w_e is defined by (9.9), the argument i being replaced by e.

If both actual and apparent shadowing are present and do not overlap both w_e and w_i have to be taken into account. For situations like the one shown in Fig. 5c only the larger one of the two shadowing functions is relevant, the other one has to be set equal to zero.

The only remaining effect which we have to consider is the shadowing of the non-Lambertian reradiation terms. Arguments similar to those presented before show that the effect of these terms adds

$$-\frac{3}{2}rs^{\frac{3}{2}}F\,\frac{S_o}{\pi}u_i U u_e \qquad (9.13)$$

to the total radiation.

Collecting all results we obtain the final expression for the radiation law ($(\vec{i}\vec{N}) = \cos i$, $(\vec{e}\vec{N}) = \cos e$, $(\vec{i}\vec{e}) = \cos\alpha$, α is the phase angle)

$$\begin{aligned}\frac{1}{\pi}\sigma T^4 = I(i,e,\alpha) = \frac{S_o}{\cos e\pi}\{&\cos\alpha\underline{(1-r)s(1-v_i-v_e}+\cos\alpha\cos e[\underline{3rs(1-v_i}+\tfrac{3}{2}rsv_e]\\ &+\cos^2\alpha\cos i 3rsv_i+\underline{\tfrac{3}{2}rs(1-v_e)}\cos i+\underline{\cos i(1-r)(1-2s)}\cdot\cos e\\ &+\cos i s(1-r)^2(1-2u_i+u_i^2-u_e)\cdot\cos e\\ &+\cos i\tfrac{3}{2}rs^{3/2}(i-u_i-u_e)\cdot\cos e+\cos i\cos^2 e\underline{\tfrac{3}{2}r(1-4s+5s^2)}\}\end{aligned} \qquad (9.14)$$

The terms underlined in (9.14) were obtained with the help of group theoretical arguments only. These terms are therefore independent of the detailed shape of the crater.

10. Evaluation of the Shadowing Functions

The functions u,v,w have to be calculated for various crater shapes. We note that all shadowing functions are multiplied by a factor s in the radiation law. It is therefore sufficient to evaluate u,v and w to lowest order in s only.

Let us start with u, defined by (8.2); au is the shadowed fraction of the circle, as shown in Fig. 7, i.e.

$$u = \frac{4}{\pi a^2} \int_0^{\eta^2+(\xi+c)^2=a} d\xi d\eta \qquad (10.1)$$

The factor 4 is due to the fact that the integral covers only one quadrant of the (ξ,η) plane. $-c=b-a$ is the ξ-coordinate of the center of the circle. The integral can be shown to be

$$u = \frac{2}{\pi}\{\text{arc cos}z - z\sqrt{1-z^2}\} \qquad (10.2)$$

where $z = c/a$. z has been computed as a function of i and e respectively in (6.4).

$$z = 1-\sin i+t\cos i \qquad (10.3)$$

The second shadowing function, v, can be derived with the help of (4.2) and (4.9):

$$v = 16R^2 \int_0^{\eta^2+(\varepsilon+c)^2=a^2} \frac{\xi^2}{(R^2+\xi^2+\eta^2)^2} \cdot \frac{1}{\partial F} \qquad (10.4)$$

Neglecting terms of order s this becomes after elementary integrations

$$v = \frac{2}{\pi}\{(\text{arc cos}z - z\sqrt{1-z^2})(1+4z^2) - \frac{10}{3}z^2(1-z^2)^{3/2}\} \qquad (10.5)$$

w is simply given by

$$w_i = \frac{v_i}{\sin i} \qquad (10.6)$$

The functions u and v are given in Tables 2 and 3. For practical purposes the approximation

$$v \approx w \qquad (10.7)$$

is sufficient and will be used for all numerical work.

The model of the lunar surface used is the following:

20% of surface area covered with craters with depth: diameter ratio 1 : 3

20% " " " " " " 1 : 4

20% " " " " " " 1 : 6

40% with large random asperities with 10° rms slope, as indicated by radar data.

This is in agreement with Surveyor observations, indicating that roughly 60% of the lunar surface is covered by craters. The various depth to diameter ratios correspond to a mixture of primary and secondary craters.

This distribution of craters and random surface features on the moon corresponds to a $\Phi = 30°$ rms surface slope, the averages being

$$s = 0.13 \tag{10.8}$$

The values of the shadowing functions u and v are calculated by adding the shadowing functions of the various craters with the appropriate weights indicated by (10.8). The resulting average shadowing functions are shown in Fig. 9.

11. Numerical Results

In order to evaluate (9.14) numerically, specific values for r,s,u and v have to be inserted. In the previous section we determined all but r from lunar surface data. The parameter r, characterizing the radiation law of a flat surface element, was determined by fitting the curves to the Saari-Shorthill[1] measurements. The best fit was obtained for

$$r = 0.13 \pm 0.02 \tag{11.1}$$

We have also tried to estimate the order of magnitude of the parameter t contained in (3.1) and find

$$t = 0.00 \pm 0.03 \qquad (11.2)$$

The calculations contained in Sections 2-4, which included s^2 terms, permit us, furthermore, to estimate the effect of the omission of these on our results. It turns out that they tend to increase the predicted temperature by 2-3°K. This is therefore the limit of accuracy of our calculations.

Numerical results for several typical situations are shown in Figs. 10-12.

12. Conclusion

We have been able to explain the directional characteristics of the lunar infrared radiation in terms of a non-Lambertian parameter r = 0.13 and a Φ = 30°rms lunar surface slope. Included in this surface slope are all surface features that are not in thermal equilibrium. Surface features of an average size $\leq$ 10cm have a rather uniform temperature due to the finite thermal conductivity of the moon.[6]

The main novel features of our work as compared to previous theories[3,4] are:

(a) The inclusion of a non-Lambertian term in the basic radiation law (3.1).

(b) The use of tensor calculus and group theory, which enabled us to carry out the rather elaborate mathematical operations necessary in this work.

Table 1

All energies in cal/cm^2 min

i	$S\cos\Theta$	Reflected Energy	Conducted Energy	Remaining for infrared radiation = $S_o\cos\Theta$
0	1.99	0.096	0.010	1.88
5	1.98	0.096	0.010	1.87
10	1.96	0.095	0.011	1.85
15	1.92	0.093	0.011	1.82
20	1.87	0.091	0.011	1.77
25	1.80	0.088	0.012	1.70
30	1.72	0.085	0.012	1.62
35	1.63	0.081	0.012	1.54
40	1.52	0.076	0.012	1.43
45	1.40	0.071	0.012	1.32
50	1.28	0.065	0.011	1.20
55	1.14	0.059	0.010	1.07
60	1.00	0.0516	0.009	0.937
65	0.840	0.0437	0.007	0.786
70	0.681	0.0352	0.006	0.630
75	0.515	0.0260	0.004	0.485
80	0.346	0.0162	0.001	0.329
85	0.173	0.0063	-0.001	0.168

Table 2

The Shadowing Function u

s	10°	20	30	40	50	60	70	80	90
0.02								0.27	1
0.04							0.06	0.48	1
0.06							0.19	0.58	1
0.08						0.03	0.30	0.64	1
0.10						0.10	0.37	0.69	1
0.12						0.17	0.44	0.72	1
0.14					0.03	0.23	0.49	0.75	1
0.16					0.08	0.29	0.53	0.78	1
0.18					0.13	0.34	0.56	0.80	1
0.20				0.02	0.17	0.38	0.60	0.82	1
0.22				0.05	0.22	0.42	0.63	0.83	1
0.24				0.09	0.26	0.46	0.66	0.85	1
0.26				0.12	0.30	0.50	0.69	0.86	1
0.28			0.02	0.15	0.34	0.53	0.71	0.87	1
0.30			0.05	0.20	0.38	0.56	0.74	0.88	1
0.32			0.08	0.23	0.41	0.59	0.76	0.90	1
0.34			0.11	0.27	0.46	0.62	0.78	0.90	1
0.36		0.02	0.14	0.30	0.48	0.65	0.80	0.92	1
0.38		0.04	0.18	0.34	0.51	0.68	0.82	0.93	1
0.40		0.07	0.21	0.38	0.55	0.70	0.83	0.94	1
0.42		0.10	0.24	0.41	0.58	0.73	0.85	0.94	1
0.44	0.02	0.13	0.28	0.45	0.61	0.75	0.87	0.95	1
0.46	0.03	0.16	0.32	0.48	0.64	0.77	0.89	0.96	1
0.48	0.06	0.19	0.35	0.52	0.67	0.80	0.91	0.97	1
0.50	0.08	0.23	0.39	0.56	0.70	0.83	0.92	0.98	1

Table 3

The Shadowing Function v

s	10°	20°	30°	40°	50°	60°	70°	80°	90°
0.02								0.04	1
0.04								0.15	1
0.06							0.02	0.25	1
0.08							0.05	0.32	1
0.10							0.08	0.38	1
0.12						0.01	0.12	0.43	1
0.14						0.03	0.16	0.48	1
0.16						0.04	0.20	0.52	1
0.18					0.01	0.06	0.24	0.55	1
0.20					0.01	0.09	0.27	0.59	1
0.22					0.02	0.11	0.31	0.62	1
0.24					0.03	0.14	0.34	0.64	1
0.26				0.01	0.05	0.17	0.38	0.67	1
0.28				0.01	0.07	0.20	0.41	0.70	1
0.30				0.02	0.08	0.23	0.45	0.72	1
0.32				0.03	0.10	0.26	0.48	0.75	1
0.34				0.04	0.13	0.29	0.52	0.77	1
0.36			0.01	0.05	0.15	0.33	0.55	0.79	1
0.38			0.01	0.07	0.18	0.36	0.59	0.82	1
0.40			0.02	0.08	0.21	0.40	0.62	0.84	1
0.42			0.03	0.10	0.24	0.44	0.66	0.86	1
0.44			0.04	0.12	0.28	0.48	0.69	0.88	1
0.46		0.01	0.06	0.16	0.32	0.52	0.73	0.90	1
0.48		0.02	0.07	0.19	0.36	0.57	0.77	0.93	1
0.50		0.02	0.09	0.22	0.40	0.61	0.81	0.95	1

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Figure Captions

Figure 1: Geometry of Spherical Craters.

Figure 2: The Stereographic Projection.

Figure 3: Reradiation Corrections.

Figure 4: Overlap of Actual and Apparent Shadowing.

Figure 5: Relative Positions of Shadows.

Figure 6: Calculation of b.

Figure 7: The Shadowed Area.

Figure 8: Coordinate System Used to Calculate the Influence of Shadowing.

Figure 9: The Shadowing Functions u and v.

Figure 10: Brightness temperatures at full moon (α=0°) and quarter moon (α=90°) as functions of the center-lunar latitude ψ. The full moon data have been obtained from the Saari-Shorthill values by averaging over center-lunar longitude.

Figure 11: Brightness temperatures at various angles of incidence (i = 30°, 60°, 80°) as functions of the angle of observation e. The sign convention for e is that the phase angle is given by $\alpha = |i+e|$. The direction of incidence and emission are coplanar with the surface normal.

Figure 12: The brightness temperatures of the subsolar (i = 0) and subterrestrial (e = 0) points as functions of the phase angle α.

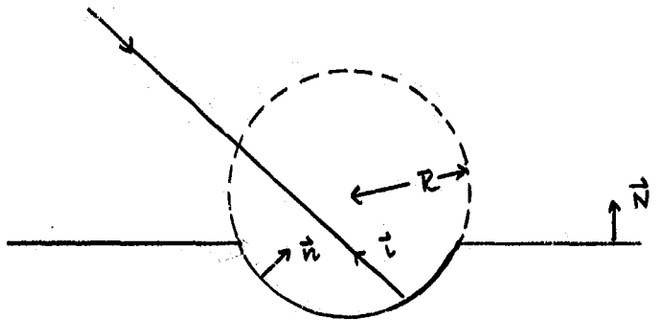


Fig. 1

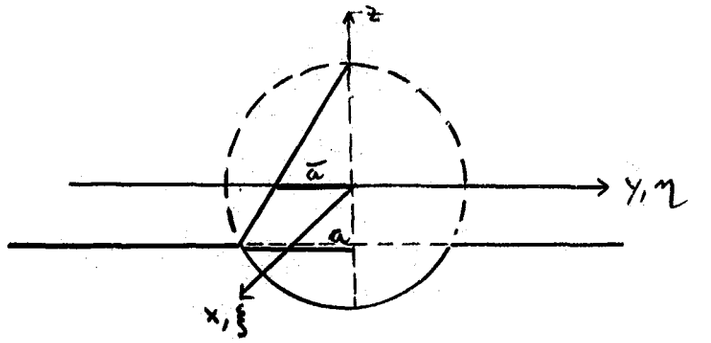


Fig. 2

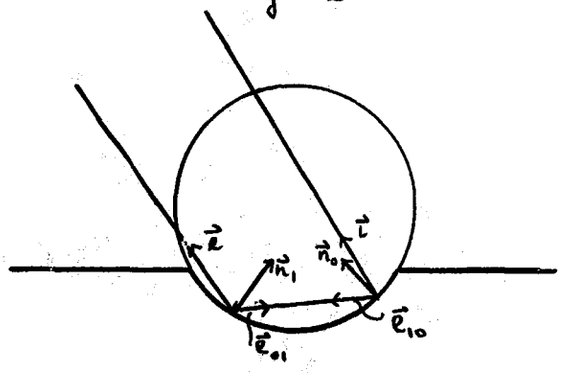
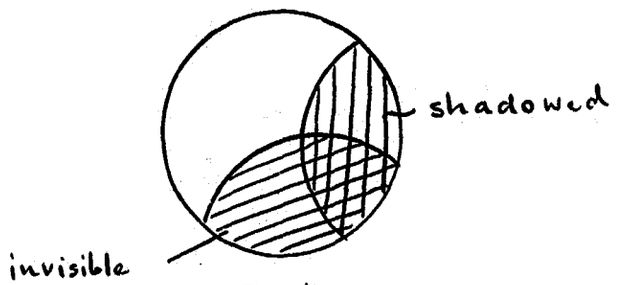


Fig. 3



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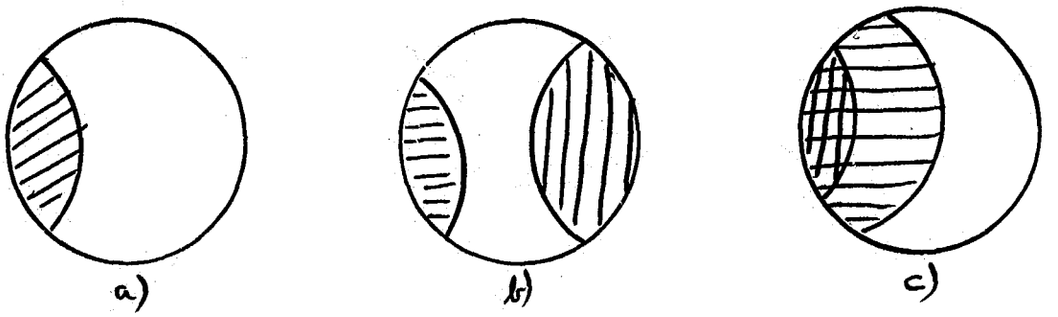

Fig. 5

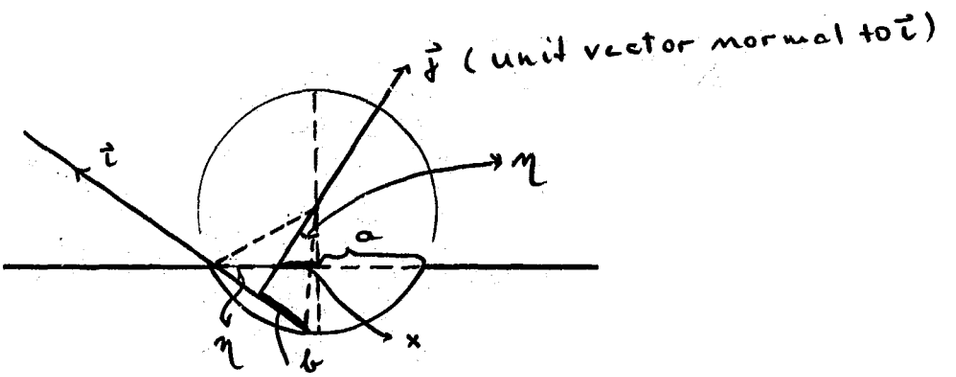

Fig. 6

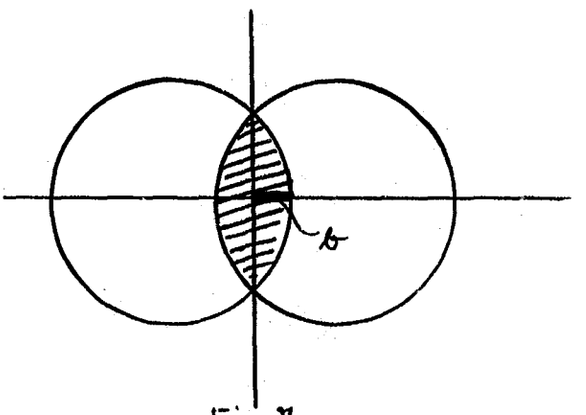

Fig 7

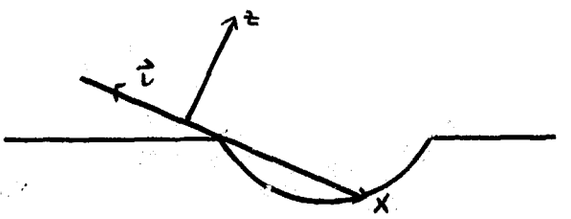

Fig. 8

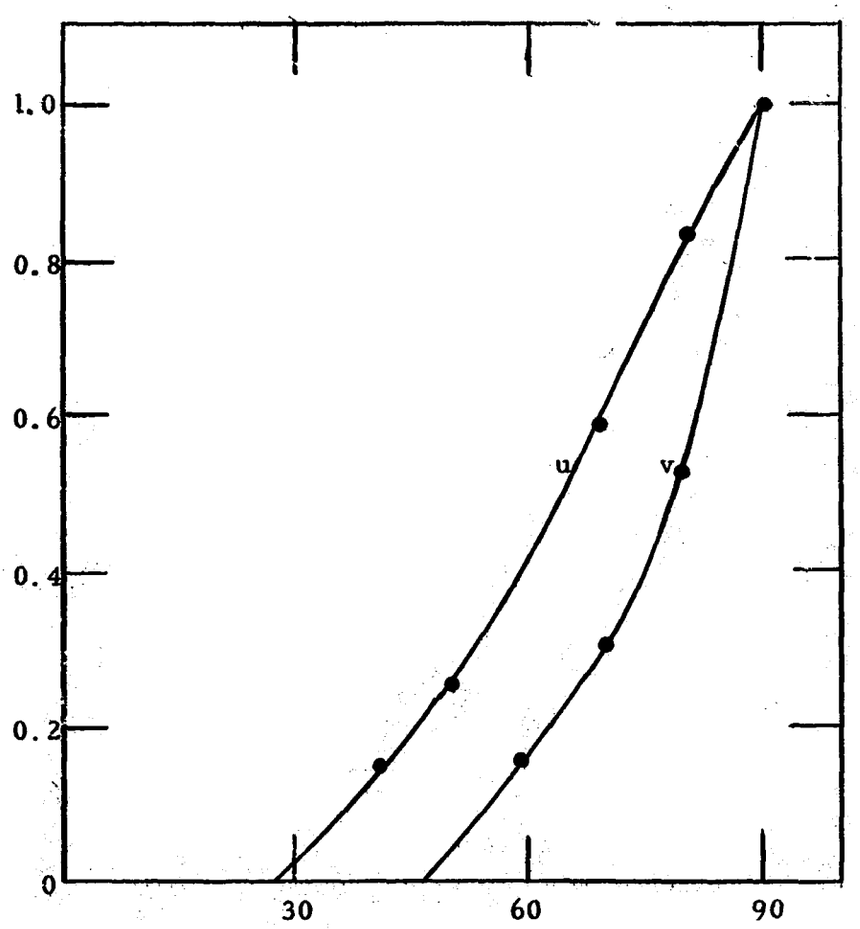

Fig. 9

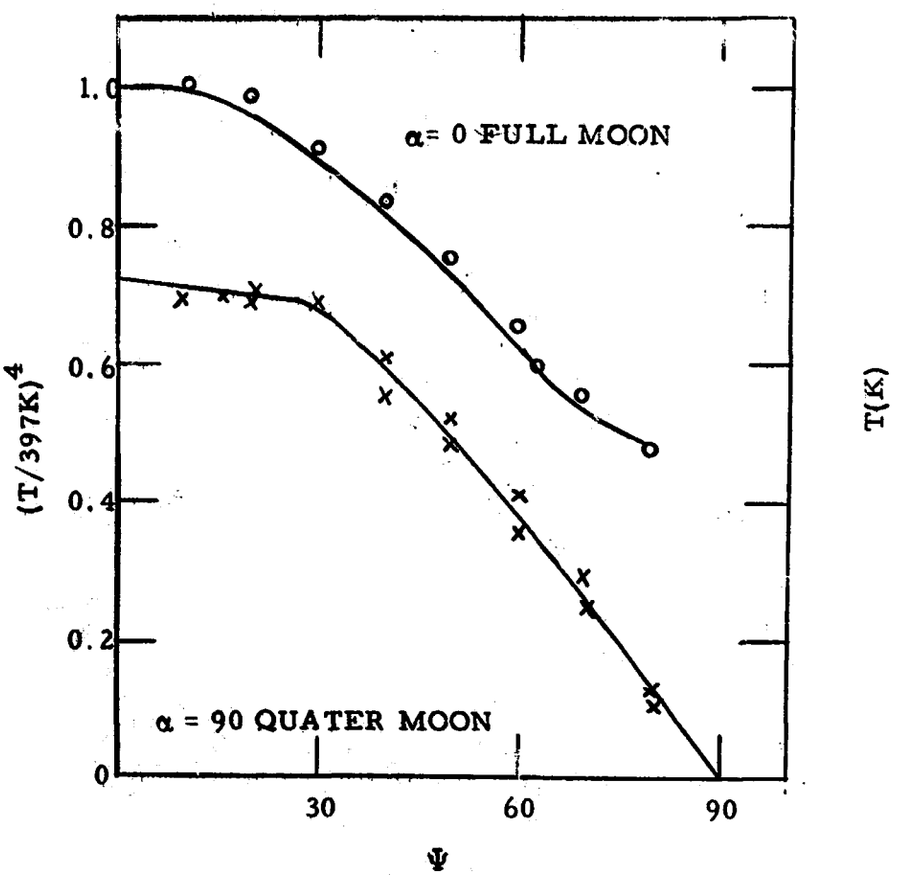
α= 0 FULL MOON
α = 90 QUATER MOON
(T/397K)⁴
T(K)
1.0
0.8
0.6
0.4
0.2
0
30
60
90
Ψ

Fig. 10

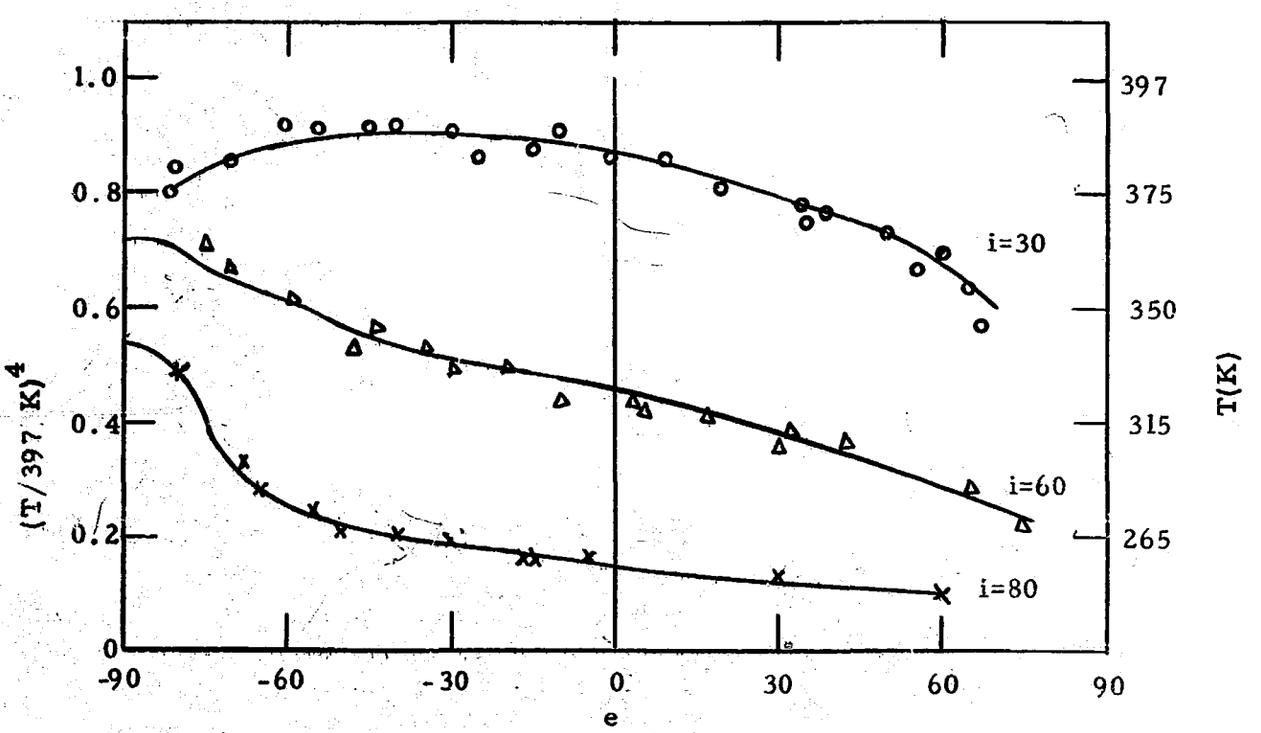
(T/397. K)⁴
T(K)
i=30
i=60
i=80
1.0
0.8
0.6
0.4
0.2
0
397
375
350
315
265
-90
-60
-30
0
30
60
90
e

Fig. 11

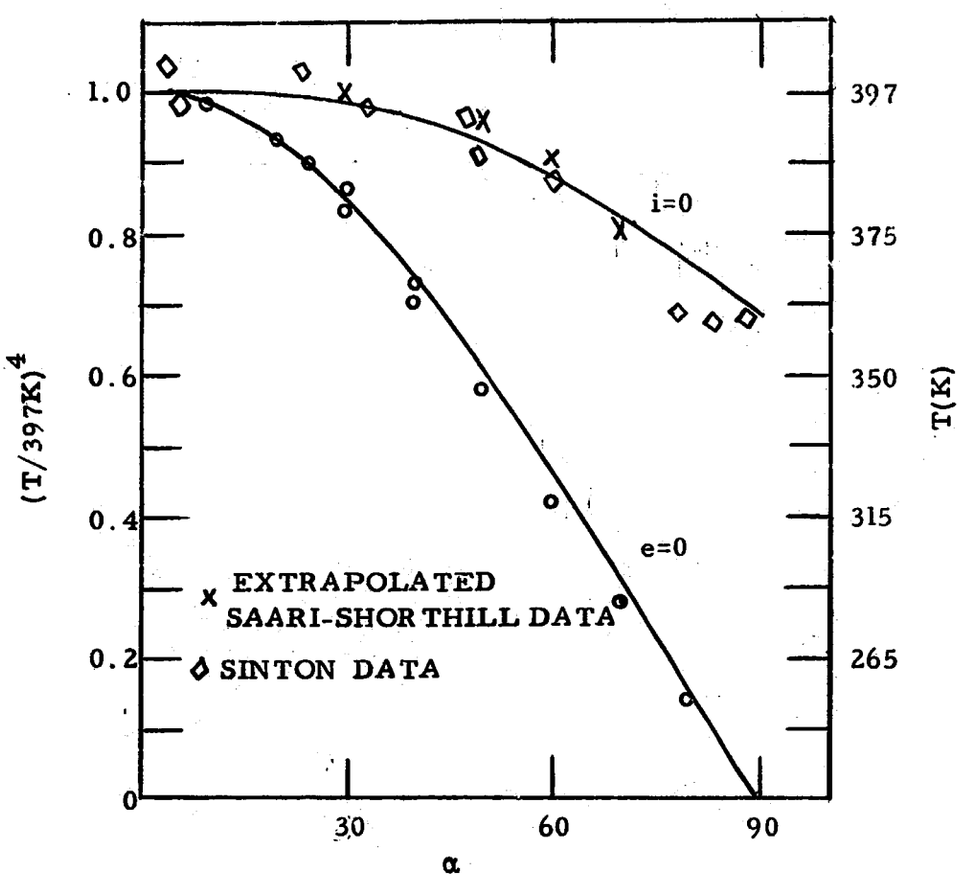
1.0
0.8
0.6
0.4
0.2
0
(T/397K)^4
397
375
350
315
265
T(K)
i=0
e=0
EXTRAPOLATED
SAARI-SHORTHILL DATA
SINTON DATA
30
60
90
α

Fig. 12

CHAPTER II

ANGULAR DISTRIBUTION OF LUNAR INFRARED RADIATION

by

Neil Ashby

Experimental data on the angular distribution of lunar infrared radiation (as observed through the 10-12 micron window in the atmosphere), is now quite extensive.[1,2,3] Examination of this data reveals distinct non-diffuse emittance characteristics. At smaller angles of incidence of the solar illumination, there is a gentle peak whose maximum is directed back toward the sun. As the angle of incidence increases this peak sharpens and acquires some structure. For fixed incidence and phase angles, the brightness temperature depends on the angle of observation. This dependence quickly becomes pronounced as the magnitude of the phase angle increases from zero, bringing its character at a phase approximately 90° in magnitude.

We have found it possible to fit most of the experimental data by means of a relatively simple mathematical expression. In writing this expression the following notation is used: i and ϵ are the angles of incidence and observation, respectively, measured from the outward surface normal; α is the magnitude of the phase angle. The thermal longitude is denoted by λ; $\lambda = 0$ corresponds to the great circle directed from the sub-solar point to the sub-earth point. In terms of i, ϵ, and α, λ is given by

$$\cos\epsilon = \cos\alpha\cos i+\sin\alpha\sin i\cos\lambda . \qquad (1)$$

Then the infrared radiance, in cal/cm^{2}min steradian, is given by

$$I(i,\epsilon,\alpha) = C(i,\lambda,\alpha)[(a_1\cos i+a_2S(\alpha))/(1+a_4[1-S(\alpha)])+a_3S(\alpha)] \qquad (2)$$

where $S(\alpha)$ is the Schönberg shadowing function,

$$S(\alpha) = \frac{1}{\pi}[(\pi-\alpha)\cos\alpha+\sin\alpha] \qquad (3)$$

and the a = constants a_i are given by

$$a_1 = .458, \quad a_2 = .149, \quad a_3 = .0607, \quad a_4 = .23.$$

The factor $C(i,\lambda,\alpha)$ is constructed so that along the thermal meridian $\lambda=0$, or at the subsolar point, or at zero phase, $C(i,\lambda,\alpha) = 1$. Then $C(i,\lambda,\alpha)$ is given by:

$$C(i,\lambda,\alpha) = [B(\alpha,i)+(1-B(\alpha,i))\cos\lambda]^{-1} \qquad (4)$$

where

$$B(\alpha,i) = A(\alpha)+(1-A(\alpha))\cos i \qquad (5)$$

$$A(\alpha) = 1- .42\sin 2\alpha/(1+\alpha). \qquad (6)$$

The brightness temperature T(°K) may be obtained from $B(i,\epsilon,\alpha)$ by

$$T = [\pi B(i,\epsilon,\alpha)/\sigma]^{\frac{1}{4}} \qquad (7)$$

where $\sigma = 0.8127\times10^{-10}$ cal/cm^{2}min°K^4 is the Stefan-Boltzmann constant.

The infrared radiance given by Equation (3) represents an average for the lunar surface; no distinction is made between maria and highlands, or between positive and negative phases, or between positive and negative thermal latitudes. The expression is not valid within about 5 degrees of the terminator or on the dark side.

The rms derivation between the expression (3) and the data of Saari and Shorthill[4] is, in terms of radiance, .036 cal/cm^{2}min ster which at a temperature of 375°K would correspond to an error of ±7°K.

The emitted energy, integrated over all angles, obtained from (3) can be represented to within 1% by 1.88cosi(cal/cm^{2}min) out to within a few degrees of the terminator. This result is in excellent agreement with considerations of energy balance.[5]

It is interesting to note that the distribution across the lunar disc at full moon, obtained from (3) by setting $\alpha=0$, is $[a_1\cos i+a_1+a_3]$. The angular form of this result is identical with the $\cos^{2/3} i$ law proposed by Pettit and Nicholson,[3] to within 4%, for angles i less than about 60°.

The r.m.s. derivation between (3) and the data of Sinton[2] is .031 cal/cm^{2}min ster, or about 5% on the average. Expression (3) is perhaps a little too sharply peaked at ϵ=0 in this case.

The deviations from Lambertian behavior in Eq. (3) are for the most part small--by far the largest contribution arises from the term $a_1 \cos i$. These deviations can probably be explained mainly as the result of surface roughness and shadowing effects. We shall make no attempt here to present a model on the basis of which (3) may be derived. Our intent is rather to present the expression as an aid in machine computation of energy fluxes or view factors for objects near the lunar surface. Harrison[6] has performed such calculations using an earlier version of Eq. (3). The errors obtained using the above expression in such calculations should be no more than about 2%, as the main errors arise in situations involving only very small contributions to the energy (e.g., i near 90°, ϵ near 90°, α near 180°).

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APPENDIX TO CHAPTER II

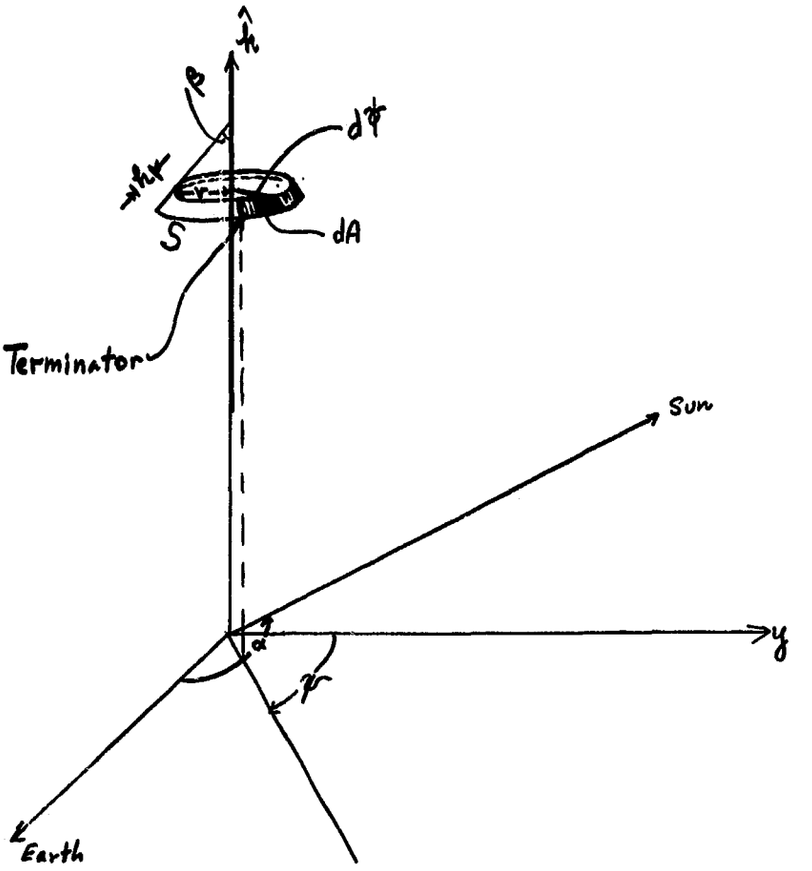
EXTENSION OF SCHÖNBERG SHADOWING FUNCTION

In the semi-empirical expression obtained for the radiant energy leaving the lunar surface, presented above, the Shönberg function

$$S(\alpha) = \frac{1}{\pi}\{(\pi-\alpha)\cos\alpha+\sin\alpha\} \qquad (1)$$

appeared. This function represents the energy received from a smooth, diffusely reflecting (or radiating) sphere, at phase angle α between source of illumination and observer, if the energy received at zero phase angle is unity. This is a well-known result for a spherical geometry. We wish to prove here that the expression (1) is of considerably wider validity. It holds for any object which possesses rotational symmetry about an axis which lies at right angles to the directions of the illuminating rays and the line from observer to center of object. Consider in the diagram the sun as the source of illumination and the earth as the observer's position. Let the element of surface S represent a portion of the surface of some object which is symmetric with respect to rotations about the z-axis (i.e., along $\hat{k}$). The object is illuminated by the sun; a portion of the illuminated surface is viewed from the earth. We wish to calculate the fraction of energy received by an observer at the earth

at a general phase angle α, as compared to the amount received when $\alpha = 0$.



Let dA be an element of surface area at angle Ψ. If the surface S has a radius r, a slant height dh and lies at an angle of β with the vertical, then

$$dA = dh\,r\,d\Psi .$$

CHAPTER III

DERIVATION OF PHOTOMETRIC FUNCTION

by

D. G. Burkhard

1. Introduction

In this chapter we derive an improved photometric function for the lunar surface by including more detail in the kind of model proposed by Hapke.[1] The model on which Hapke's derivation is based consists of a semi-infinite, porous or loosely packed medium containing channels of various depths down which light can penetrate. Incident light will penetrate various depths before being reflected from an interfering object or reflected from the bottom of an open channel. Based on this model the expression for the reflected energy, $I^{(re)}(i,\varepsilon,\alpha)$ per unit solid angle per unit time from a unit area of lunar surface, when the incident solar flux is incident at angle i from the mean surface normal and the reflected radiation is observed at angle ε from the mean surface normal, with α the angle between incident and observation directions is given by

$$I^{(re)}(i,\varepsilon,\alpha) = S_o f b(\alpha) \frac{\cos\varepsilon \cos i}{\cos i + \cos\varepsilon} B(\alpha,g) \qquad (1)$$

where S_0 is the solar constant, and f is the total reflectivity of an object, i.e., the fraction of light reflected by the object into all directions. The function $b(\alpha)$ is the scattering law of an individual object. Hapke uses the backscattering expression

$$b(\alpha) = [(\pi-\alpha)\cos\alpha+\sin\alpha]/\pi \qquad (2)$$

which satisfies the normalization condition $\int b(\alpha)d\alpha = 1$ and corresponds to diffuse reflection from an opaque spherical particle. The term $\cos\epsilon\cos i(\cos i+\cos\epsilon)^{-1}$ is the Lommel-Seeliger factor and gives the amount of light which reaches the surface after the incident and scattered beams experience absorption in the medium. The larger ϵ and i are, the greater is the path length in the medium and therefore the smaller is the amount of light eventually reaching the surface. In a later paper,[2] Hapke introduced a wrinkled surface (sinusoidal) and obtained an averaged Lommel-Seeliger factor containing three independent parameters. With the photometric function so modified Hapke obtains somewhat better agreement with experimental data than the simpler expression (1).

The function $B(\alpha,g)$ expresses the shadowing effect of the channels down which the light penetrates. Hapke gives the following expression for this so-called "retro-directive function":

$$B(\alpha,g) = 2 - \frac{\tan\alpha}{2g}(1-e^{-g/\tan\alpha})(3-e^{-g/\tan\alpha}) \qquad \alpha \leq \pi/2,$$
$$= 1 \qquad \alpha > \pi/2 \qquad (3)$$

where g is a parameter related to the degree of porosity of the lunar surface material, or to the fraction of void space at the surface. The parameter g may vary from place to place on the surface; Hapke obtains many of his good fits with experimental data by choosing $g = 0.6$. However, g may vary from 0.4 to 0.8. The best average value for g using Equation (3) for the whole moon seems to be $g \simeq 0.7$.

The parameter f may also vary from place to place; it is determined from the albedo at the particular location of interest.

These expressions, with proper choice of multiplication factor b, represent reasonably well the observed photometric properties of the moon. At large angles of incidence, however, and at large phase angles, the observed reflected light is greater than that predicted by Equation (1).

We now ask, how can we improve the expression (1)? We note that basically the model gives rise to a result which is the product of three factors; there are first the Lommel-Seeliger factor $(\cos\varepsilon\cos i)/(\cos i+\cos\varepsilon)$ which insures that $I^{(re)} = 0$ when $i = 90°$; secondly, the reflectivity factor f and the scattering law $b(\alpha)$; and, thirdly, the shadowing function $B(\alpha,g)$. We shall

now consider possible modifications of Hapke's argument which will still allow $I^{(re)}$ to be written as a product of the same three factors.

If one looks at the detailed physical model implied in Hapke's derivation, one sees that the square tubes (square, since the reflecting objects are assumed to be square for simplicity in calculating the unshadowed (seen) and shadowed (unseen) reflecting areas at the base of the tubes) representing the incident light are truncated at right angles to the beam at the location of the mean lunar surface. These tubes are then oriented so that one edge is perpendicular to the plane formed by the incident ray and the direction of observation. This simplification is made in order that the area seen at the base of the cylinder is a rectangle whose area is easy to compute as a function of α, and not a triangle or a five-sided figure as it would be if the incident square cylinder of light were not rotated. In our treatment we shall continue to rotate reflecting particles about the axis of the incident light cylinder in order to simplify the area of the reflecting base. Hapke introduces the mean lunar surface as a reference plane when actually one should treat the face of each reflecting unit of the irregular lunar surface itself as a reference plane. We shall refer to such an area, large or small, which makes a definite angle with respect to the mean lunar surface as a facet. It would be a good approximation to use the mean lunar surface as a reference

plane if the penetration of the light into the material were large compared with the dimensions of the reflecting particles or elements in the medium. This is not the case, however, since Hapke found that the penetration depth is approximately equal to the diameter of the reflecting particles in the medium. He obtained best fit with experiment when the ratio $g = (a/\tau) \simeq 1$. a is the diameter of the particles and τ the attenuation distance, that is, the distance over which radiation decreases to 1/e of its initial value. A literal interpretation of Hapke's model would be one in which all facets were normal to the direction of the incident beam. If this were the case, then the distance over which the light is attenuated on its way out should be measured to the surface of the facet. Such a surface would be simply an extension of the surface end of the square cylinder. Hapke, however, extends the path of the reflected rays to the mean lunar surface. A literal interpretation of the model, however, may not ignore the details of the geometry of the light cylinders at the surface and the change in path length in the absorbing medium associated with those details. Therefore, it is not proper to write $z\cos i = z'\cos\epsilon = y$ as does Hapke (Figure 2), where y is the vertical depth from mean lunar surface to a reflecting particle and z and z' are incident and reflecting path lengths from mean lunar surface to reflecting object. Measuring these lengths to the lunar surface enables Hapke to introduce the angles i and ϵ through the above

equation and thereby obtain the Lommel-Seeliger laws as part of his final result.

Since the surface of the facets do not coincide with the mean lunar surface i and ϵ should be replaced by i' and ϵ' where the latter are measured with respect to a normal to the facet itself rather than the normal to the lunar surface. Allowing the facets arbitrary orientation by introducing i' and ϵ' one may generalize the problem by letting the facet upon which the light is incident truncate the incident cylinder in an arbitrary manner. One of the difficulties that enters the details of this model is deriving a simple expression for the area of the bottom of the cylinder as seen by an observer. The visible area, the non-visible area, the incident path length, and exiting path length would be expressed in terms of angles i' and ϵ'. After an expression for the intensity of the emergent beam is obtained for a particular facet, one must introduce a reference line normal to the mean lunar surface and average final results over all orientations of the facets. The algebra required to relate results for an arbitrary facet and involving angles i' and ϵ' to a final expression involving i and ϵ plus an additional angle required to uniquely specify facet orientation and then average over all values of this angle is quite complicated. Although in this paper we treat the geometry of the light cylinders more realistically we avoid the problem of averaging over all facet

orientations by introducing the Lommel-Seeliger law as a separate factor (effectively by assuming this law to be the result of averaging) as will be seen later. The Lommel-Seeliger law is separately modified. The purpose of pursuing at least some generalization of Hapke's model is to establish a reasonable range of flexibility in the final results. One will then know how much he can tamper with the theoretical curves in order to obtain best agreement with experiment.

We modify Hapke's treatment in two ways. First we recognize that the reflected light from the shadowed area is not attentuated over a distance measured from the reflecting surface to the mean lunar surface or even to the surface of the reflecting facet but is attentuated only after it reaches the wall of the incident light cylinder. Secondly, we recognize that by truncating the incident light by the surface facet the incident light attenuation path varies from ray to ray in the incident light cylinder. When such details are important the exponential attenuation factor itself may not strictly be applicable. In all calculations the cylinder is rotated so the edge is perpendicular to the plane formed by the incident ray and the reflected ray. To calculate the distances travelled by both penetrating and scattered light rays, the rays are regarded as entering and exiting from a facet at right angles to the light cylinder. In the latter case the surface will first be assumed to coincide with the mean lunar surface. The result

depends only on α and gives Hapke's retrodirective function.

The angles i and ϵ enter Hapke's expression through the Lommel-Seeliger law which appears as a separate factor multiplying the function $B(\alpha)$. In order to make formulas satisfy optical reciprocity, Hapke makes the substitution $\sec i \simeq (\sec i + \sec \epsilon)/2$. i and ϵ do not enter our description when deriving the retrodirective function. We introduce them by supposing that the average slope of the facets coincides with the mean lunar surface. Instead of attempting to average our retrodirective function over all facet orientations we assume that such averaging can be replaced by writing $z\cos i = z'\cos\epsilon = y$ (Figure 2). This statement recognizes that a large ϵ or a large i will introduce a longer attenuation path to such a given vertical distance below the surface of the absorbing medium that would a small ϵ or i. Using the above expression one derives the Lommel-Seeliger law in the usual manner independently of any considerations about the retrodirective function. The Lommel-Seeliger function then modulates the function $B(\alpha)$.

2. Lommel-Seeliger Law

Consider a plane wave of light incident upon a medium consisting of randomly distributed scattering elements. Suspended particles of dust would satisfy this model if the particles were kept separated from each other by electrostatic forces. When light encounters a particle, it will be scattered in the angle range $d\alpha$ as specified

by a scattering function $b(\alpha)$. Above the particle there is effectively an empty tube extending to the surface of the medium out of which some light can escape with zero probability of a collision with another particle. The decrease in the area of the beam B (that is, the portion scattered) in traveling distance dz is:

$$dB = -n_o a_o B dz$$

$$B = Ce^{-n_o a_o z} ,$$

where $a_o = a^2$ is the area of the intercepting elements and n_o is the number of elements per unit volume. Area of beam at surface is B_o so that:

$$B = B_o e^{-n_o a_o z} .$$

Thus energy scattered out of beam in distance dz (layer dy) (Figure 3) is

$$B_o S_o n_o a_o e^{-n_o a_o z} dz = B_o S_o n_o a_o e^{-\frac{n_o a_o y}{\cos i}} dy/\cos i$$

since $dz = dy/\cos i$. The beam is incident at angle i with respect to the normal to the surface. S_o is the energy flux per unit area of the beam. This energy is incident over an area of lunar surface $B_o/\cos i$. So energy incident over unit surface area of the medium

and scattered out in distance z is $S_o n_o a_o e^{-n_o a_o y/\cos i} dy$. The energy scattered per unit volume (since volume of scattering region is $(B_o/\cos i)dy$) is $S_o n_o a_o e^{-n_o a_o y/\cos i}$. Therefore $S_o n_o a_o e^{-n_o a_o y/\cos i}$ is the energy produced per unit volume of scattering material per unit solid angle. If there is some pure absorption then only a fraction (say f) of this energy will be scattered.

If we consider the light which hits a scattering center as again traveling through the same medium and therefore having a probability of being scattered again before reaching the surface, we can write:

$$dB = -n_o a_o B dz'$$

$$B = B' e^{-n_o a_o z'}$$

where $B = B'$ when $z' = 0$. Thus the fraction of light scattered at $z' = 0$ which gets out after traveling distance z' is $e^{-n_o a_o z'} = e^{-n_o a_o y/\cos\varepsilon}$.

We will now let $n_o a_o = 1/\tau$ where τ is the mean attenuation length of a beam of light rays in the medium. If the particles are far apart (average distance apart large compared with diameter of particles) then $\tau = 1/n_o a_o$ exactly. If the objects are close together, shielding of one object by another will be common so that

each object is less efficient at blocking light and hence $\tau > 1/n_0 a$.

The energy leaving a volume of thickness dy and unit area is $S_0(\exp -y/\tau\cos i)/\tau$. Multiplying this by the fraction $\exp -y/\tau\cos\epsilon$ which reaches the surface, we obtain for the energy E, leaving a unit area of surface per unit solid angle per second!

$$E = \frac{S_0 \cos i \cos\epsilon}{\cos i + \cos\epsilon}$$

This is the Lommel-Seeliger law. This treatment helps one understand why arbitrary symmetrization in i and ϵ (strictly valid only for small i and ϵ) to achieve Hapke's result does not seriously restrict application to large values of i and ϵ. We shall later independently modify the Lommel-Seeliger law by averaging it over all surface facets.

In terms of the brightness this energy is expressed as $B(i,\epsilon)\cos\epsilon$. Thus:

$$B(i,\epsilon) = \frac{S_0 \cos i}{\cos i + \cos\epsilon} . \qquad (4)$$

The brightness is the energy through a unit area perpendicular to the direction of observation. Equation (4) must be multiplied by the projected lunar area seen by the detector and also multiplied by the solid angle subtended by the detector as seen from the lunar surface.

3. Derivation of Hapke's Results

The general features of Hapke's model have already been described. Results are derived here in order to examine the assumptions. An absorbing, scattering medium will attenuate a light beam exponentially. If the scattering centers are of dimensions comparable to the attenuation distance, then one will wish to allow for the fact that some of the light will be scattered back in the direction of the incident beam and can escape from the medium without additional scattering or absorption taking place. This is because there is an open channel from the surface to the first scattering center. Light will be scattered back without further absorption or scattering in directions from which the bottom of the light cylinder can be "seen" from a position outside of the light cylinder. An observer looking along line O'P (see Figure 4) will only see part of the area of the bottom of the cylinder. Referring to Figure 4 the viewed area is given by $A = a(a-x)$ where $x = z \tan\alpha$. As mentioned earlier, Hapke now writes

$$z\cos i = z'\cos\varepsilon = y \qquad (5)$$

so that the exiting ray is measured to the lunar surface. Z is the path of incident ray and z' the path of the exiting ray. If one goes along with this, we can now write

$$A = a(a-z\tan\alpha) = a^2 - ay\sec i\tan\alpha.$$

From this the seen fraction $F = A/a^2$ of the total area A is:

$$F = A/A^2 = 1-y(\sec i\tan\alpha)/a = 1-x/a \qquad (6)$$

The fraction of the area which cannot be seen directly is: $1-F = x/a$. The expression for F is not symmetrical in i and ϵ. In order to make it symmetrical in i and ϵ Hapke then makes the substitution $\sec i \simeq (\sec i+\sec\epsilon)/2$. With this substitution Eq. (6) for F becomes

$$(A/a^2) = F = 1-y(\sec i+\sec\epsilon)(\tan\alpha)/2a$$

$$= 1-z'[(\cos\epsilon/\cos i)+1]\tan\alpha/2a \qquad (7)$$

This expression for F is the same as Hapke's except that in his notation, his x is our x/a and his y is our a/2. For a given a, A becomes zero for $z > z_o$ where $z_o = a\cot\alpha$.

Energy scattered out of beam in distance dz is $S_o Ae^{-z/\tau}dz/\tau$ for seen area; $S_o(a^2-A)e^{-z/\tau}dz/\tau$ for unseen area; $S_o a^2 e^{-z/\tau}dz/\tau$ is energy scattered out in dz for unseen area after depth is such that no part of base area is visible. The proceeding energies are incident over an area of lunar surface $a^2/\cos i$. Therefore energy incident

over unit area of surface area and scattered out in distance z is sum of above divided by $a^2/\cos i$. Since Hapke uses the relation $z\cos i = z'\cos\varepsilon = y$ one can say that energy scattered per unit volume is the sum of the above factor divided by $a^2 dz = a^2 dy/\cos i$. To obtain net energy out of lunar surface one must multiply sum of energy from seen and unseen area by fraction f_o, which is scattered multiply; by the scattering function $b(\alpha)$, and by the attenuation factor $e^{-z/\tau}/\tau$. After the critical depth z_o is reached energy from unseen is multipled by these factors.

Thus energy $I^{(re)}$ leaving per unit angle per second from a unit area of surface is:

$$I = \frac{S_o f b(\alpha)\cos i}{a^2\tau}\{\int_0^{z_o} Ae^{-z/\tau}dz+\int_0^{z_o}(a^2-A)e^{-(z+z')/\tau}dz+a^2\int_{z_o}^{\infty} e^{-(z+z')/\tau}dz\}$$

where $z_o = a\cot\alpha$.

Expressing results in terms of y one has

$$I = [S_o f b(\alpha)\cos i/a^2\tau]\{\int_0^{y_o}(1-\sec i\tan\alpha e^{-y/\tau\cos i}dy)/a\cos i$$
$$+a^2\int_0^{y_o} y\sec i\tan\alpha e^{-[(1/\cos i)+1/\cos\varepsilon]y/\tau}dy/a\cos i$$
$$+a^2\int_0^{\infty} e^{-[(1/\cos i)+1/\cos\varepsilon]y/\tau}dy/\cos i \qquad (8)$$

where $y_o = a\cot\alpha\cos\varepsilon$.

Examining these integrals we note that neither the exponential factor nor the multiplicative factor in the first integral are symmetrical in i and ϵ. To satisfy optical reversibility Hapke symmetrizes these by arbitrarily making the substitution $\sec i = (\sec i + \sec\epsilon)/2$ and $1/\cos i = [(1/\cos i) + 1/\cos\epsilon]/2$. Note that the latter substitution in the exponential is equivalent to introducing an attenuation of the reflected light on the way out from the seen area. The extra path length so introduced is only half the exiting path length ; at the same time the incoming path length is reduced by half as a result of the substitution.

The area factors ($\sec i$) of the first two integrals must be replaced by $(\sec i + \sec\epsilon)/2$ in order to completely symmetrize the expressions. Literally, the symmetrization implies $\cos\epsilon = \cos i$ and would therefore be a good approximation only for small angles. The formula gives reasonable results however even for large angles. The first integral is proportional to light reflected from the "seen" area at the base of the channel. For deep channels i and ϵ will be small so $\cos\epsilon \simeq \cos i$ is satisfied. For shallow channels the area factor will be largest (more area seen) for small i and ϵ. As i and/or ϵ increase the seen area becomes small and the effect of the symmetry assumption is less important. As far as the exponential factor is concerned the symmetrizing approximation will be largest when ϵ and/or i are large, however, the corresponding small-seen-area multiplicative factor will reduce

the overall contribution to the net result. The second integral describes reflected light from unseen area when part of the total base area is still visible. The exponential in this term is already symmetrical. Some error is introduced in the multiplicative area factor as a result of replacing seci but is not as important as a change would be in the exponential. These observations are made in order to understand why such artificial symmetrization may not have a profound effect on the results. The third integral represents contributions to total light when the channel depth and the angle of observation is such that the base is not "seen". This integral is already symmetric in i and ϵ. The limit $y_o = a\cot\alpha\cos\epsilon$ is not symmetric in i and ϵ. Adjustment of this term through the replacement $\cos\epsilon = (\cos\epsilon+\cos i)/2$ probably is the source of greatest error when symmetrizing.

Integrating and collecting terms (without symmetrizing the limits):

$$I = \frac{S_o f b(\alpha)}{h}\{(2-3\tan\alpha)/2g+[-2+h\ell+(2+\tan\alpha)/g]e^{-gh\cos\epsilon/2\tan\alpha}$$

$$+[1-h\ell/2-(\tan\alpha)/2g]e^{-gh\cos\epsilon/\tan\alpha}\} \qquad (9)$$

where $g = a/\tau$ (Hapke's y is our a/2), $h = 1/\cos\epsilon+1/\cos i$ and $\ell = (\cos\epsilon+\cos i)/2$ so $h\ell = (1+\cos i/\cos\epsilon)(1+\cos\epsilon/\cos i)/2$. $h\ell$ can be replaced by 2 since it changes only slowly with i and ϵ. For

example, when i = 0 and ε = 0, bℓ = 2. When i = 0 and ε = 45°, bℓ = 2.05. In the exponential hcosε is replaced by 2, the largest source of error associated with symmetrization. The error in this substitution is small when cosε≃cosi.

$$I = \frac{S_o f b(\alpha)\tau}{bh}\{2-\frac{3\tan\alpha}{2g}+\frac{2\tan\alpha}{g}e^{-g/\tan\alpha}-\frac{\tan\alpha}{2g}e^{-2g/\tan\alpha}\} \quad . \qquad (10)$$

The expression in brackets is the same as Hapke's retrodirective function B(α,g) (formula 4-81 reference [1]). 1/h is the Lommel-Seeliger factor.

4. Modification of Hapke's Model

We now modify Hapke's model by explicitly truncating the square incident light cylinders at right angles to the incident light. The surface of the cylinder is not the lunar surface. Therefore, the angles i and ε do not enter the geometry and the relationship zcosi = z'cosε which leads to the Lommel-Seeliger law does not appear.

Referring to Figure 4, the incident energy per unit area scattered in a unit solid angle per second is:

$$I^{(re)} = S_o f b(\alpha)\{\int_0^{z_o} a(a-x)e^{-z/\tau}dz+\int_0^{z_o} axe^{-(z+z')/\tau}dz+0+\int_{z_o}^{\infty} a^2e^{-(z+z')/\tau}dz\} \qquad (11)$$

The first integral is the seen area times the incoming attenuation factor; the second integral is the unseen area times incoming attenuation factor times outgoing attenuation factor. Both integrals are integrated between $z = 0$ and $z = z_0$. z_0 is the maximum value for z for which the bottom area can be seen. $z_0 = a/\tan\alpha$. Both z and z' are measured to the common edge of the seen and unseen area. We will express the integrands in terms of z using the fact that $z' = z/\cos\alpha$. The integrand in the first integral is zero after $z = z_0$ since after that $x = a$. x in the second integral is replaced by a and the integral extended from $z = z_0$ to ∞ as shown in the third integral. In terms of z, (11) becomes

$$= \int_0^{a/\tan\alpha} (a^2 - az\tan\alpha) e^{-z/\tau} dz + a\tan\alpha \int_0^{a/\tan\alpha} z e^{-(z+z/\cos\alpha)/\tau} dz + a^2 \int_{a/\tan\alpha}^{\infty} e^{-(z+z/\cos\alpha)/\tau} dz \qquad (12)$$

Carrying out the integrations, dividing by a^2 and multiplying by $S_o \rho b(\alpha)$ one obtains:

$$I^{(re)} = S_o b(\alpha) \{ 1 - \frac{\tau}{a}\tan\alpha + \frac{\tau}{a} \frac{\sin\alpha\cos\alpha}{(1+\cos\alpha)^2} + (\frac{\tau}{a}\tan\alpha) e^{-a/\tau \tan\alpha} - \frac{\tau}{a} \frac{(\sin\alpha\cos\alpha)}{(1+\cos\alpha)^2} e^{-a(1+\cos\alpha)/\tau \sin\alpha} \} a^2 \tau. \qquad (13)$$

It is interesting to note that if we replace the integrands $e^{-(z+z/\cos\alpha)/\tau}$ by $e^{-2z/\tau}$ then (12) integrates to

$$B(\alpha) = \frac{a^2\tau}{2}\{2-\frac{3\tau}{2a}\tan\alpha+\frac{2\tau}{a}\tan\alpha e^{-a/\tau\tan\alpha}-\frac{\tau}{2a}\tan\alpha e^{-2a/\tau\tan\alpha}\}. \quad (14)$$

The expression in brackets in (14) is Hapke's retrodirective function $B(\alpha)$.

Neglecting the $1/\cos\alpha$ factor in the exponential in the preceding integrands is equivalent to attenuation of the light on the way out of the medium (after being scattered) by the same amount (same path length) as it was attenuated in reaching the scattering center. If the exponential factors are regarded as measures of probability, then the probability of penetration to a distance z is taken as the same as a probability of escaping after being scattered at z. There is no physical basis for this assumption but it is interesting that by making it, we arrive at Hapke's retrodirective function. The preceding discussion has served to bring out the approximations involved in obtaining the retrodirection function (14).

We now derive another retrodirective function paying closer attention to the physical details which should be taken into consideration in this model. Calculations will be made to compare the result with Hapke's retrodirective function. Specifically, we will take into account the fact that the light scattered from the unseen area will not attenuate until it escapes from the cylinder.

The outgoing attenuation distance will be measured from the center of the unseen area. This means that before z reaches z_o, $z = 2z'\cos\alpha$. When $z > z_o$, the attenuation path is AP as shown in Figure 5. The integrals then become:

$$\int_0^{a/\tan\alpha} (a^2 - az\tan\alpha) e^{-z/\tau} dz + a\tan\alpha \int_0^{a/\tan\alpha} z e^{-(z+z/2\cos\alpha)/\tau} dz$$

$$+ a^2 \int_{a/\tan\alpha}^{\infty} e^{-(z+AP)/\tau} dz \qquad (15)$$

where upon using the following relations:

$$\ell = x/2\tan\alpha \qquad \ell' = z - \ell$$

$$AP\cos\alpha = \ell' = z - a/2\tan\alpha$$

$$AP = \ell'/\cos\alpha = (z-\ell)/\cos\alpha = z/\cos\alpha - a/2\tan\alpha\cos\alpha$$

$$= z/\cos\alpha - a/2\sin\alpha.$$

Integration of (15) yields:

$$\int_0^{a/\tan\alpha} a^2 e^{-z/\tau} dz = a^2\tau(1 - e^{-a/\tau\tan\alpha}),$$

$$-a\tan\alpha \int_0^{a/\tan\alpha} dz\, z e^{-z/\tau} = -a\tan\alpha \left[-\frac{a\tau}{\tan\alpha} e^{-a/\tau\tan\alpha} + \tau^2 (1 - e^{-a/\tau\tan\alpha}) \right],$$

$$+a\tan\alpha \int_0^{a/\tan\alpha} z\, dz\, e^{-a(1+1/2\cos\alpha)/\tau}$$

$$= a\tan\alpha \left[-\frac{a\tau e^{-\frac{a}{\tau\tan\alpha}(1+1/2\cos\alpha)}}{\tan\alpha(1+1/2\cos\alpha)} + \frac{\tau^2 \left(1 - e^{\frac{a}{\tau\tan\alpha}(1+1/2\cos\alpha)}\right)}{(1+1/2\cos\alpha)^2} \right]$$

$$a^2 e^{\frac{a}{2\tau \sin\alpha}} \int_{a/\tan\alpha}^{\infty} e^{-z(1+1/\cos\alpha)/\tau} dz$$

$$= a^2 e^{a/2\tau\sin\alpha} \frac{\tau}{(1+1/\cos\alpha)} e^{-(1+1/\cos\alpha)\frac{a}{\tau\tan\alpha}}$$

Adding,

$$a^2\tau\{1-\frac{\tau\tan\alpha}{a}+\frac{\tau 4\sin\alpha\cos\alpha}{a(1+2\cos\alpha)^2}+e^{-a/\tau\tan\alpha}+\frac{\tau\tan\alpha}{a}e^{-a/\tau\tan\alpha}-\frac{e^{-a/\tau\tan\alpha(1+\frac{1}{2\cos\alpha})}}{1+\frac{1}{2\cos\alpha}}$$

$$-\frac{\tau\tan\alpha e^{-\frac{a}{\tau\tan\alpha}(1+\frac{1}{2\cos\alpha})}}{a(1+\frac{1}{2\cos\alpha})^2}+\frac{e^{-a/\tau(1+\frac{1}{2\sin\alpha}+1/\cos\alpha)}}{(1+1/\cos\alpha)}\}$$

$$= a^2\tau\{\frac{1-\tau\tan\alpha}{a}+\frac{\tau 4\sin\alpha\cos\alpha}{a(1+2\cos\alpha)^2}+\frac{(1+\tan\alpha)}{a}e^{-a/\tau\tan\alpha}-e^{-a/\tau\tan\alpha(1+1/\cos\alpha)}$$

$$-\frac{e^{-\frac{a(1+\frac{1}{2\cos\alpha})}{\tau\tan\alpha}}}{1+\frac{1}{2\cos\alpha}}(1+\frac{\tau\tan\alpha}{a(1+\frac{1}{2\cos\alpha})})+\frac{e^{-a/2\tau\sin\alpha}}{(1+1/\cos\alpha)}e^{-a/\tau\tan\alpha}\} \qquad (16)$$

Results of calculations with this expression have been presented in the final report of our preceding contract.

Before a really basic improvement of the photometric function can be made, the random orientation of the individual reflecting facets on the lunar surface must be taken into account. As described in Chapter I of this report we are now able to do this. We hope that the next phase of our work will result in an improved and more realistic treatment of the lunar photometric properties.

REFERENCES FOR CHAPTER III

1. Hapke, B. W. A Theoretical Photometric Function for the Lunar Surface. Journal of Geophysical Research, 68, 4571 (1963).

2. Hapke, B. W. An Improved Theoretical Lunar Photometric Function. Astron. J. 71, 333 (1966).

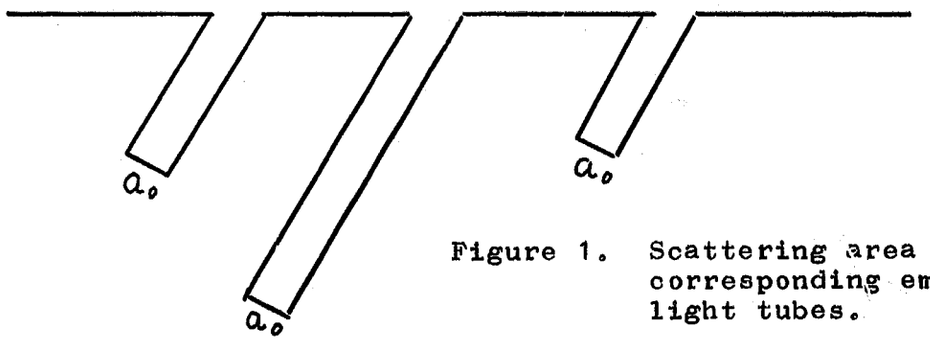


Figure 1. Scattering area a_o and corresponding empty light tubes.

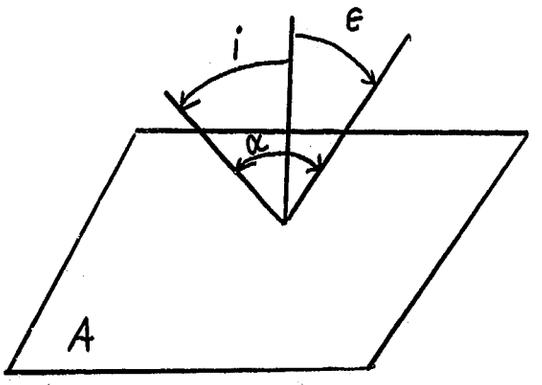


Figure 2. Reflecting area A showing angle of incidence i, angle of reflection ε, angle α.

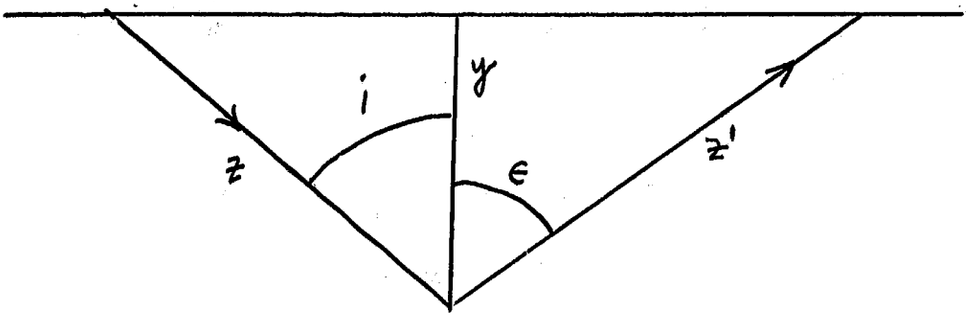


Figure 3. Incoming path z and outgoing z' after scattering of light from a point vertical distance y in medium.

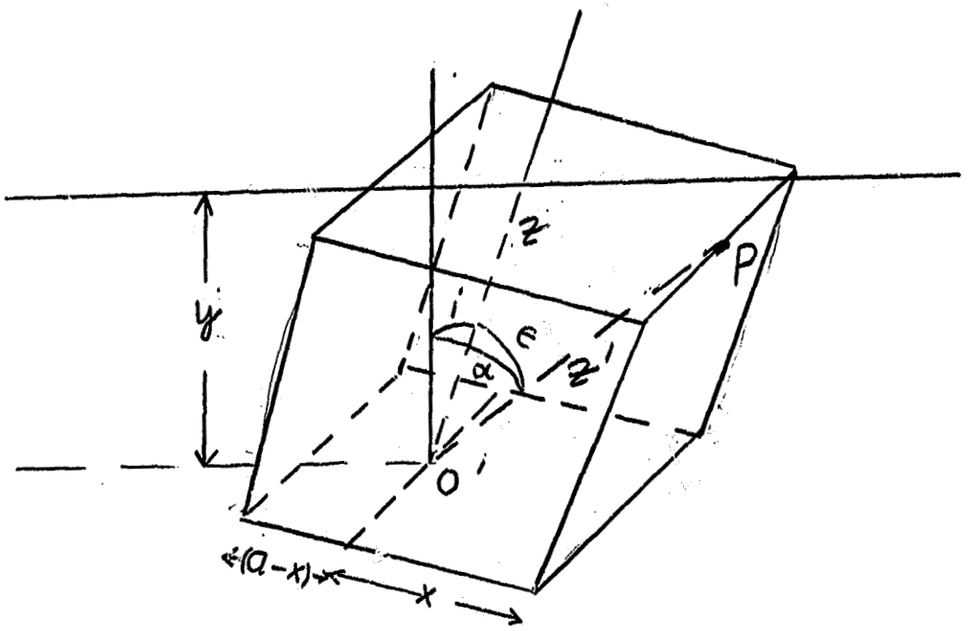


Figure 4. Figure for deriving Hapke's photometric function.

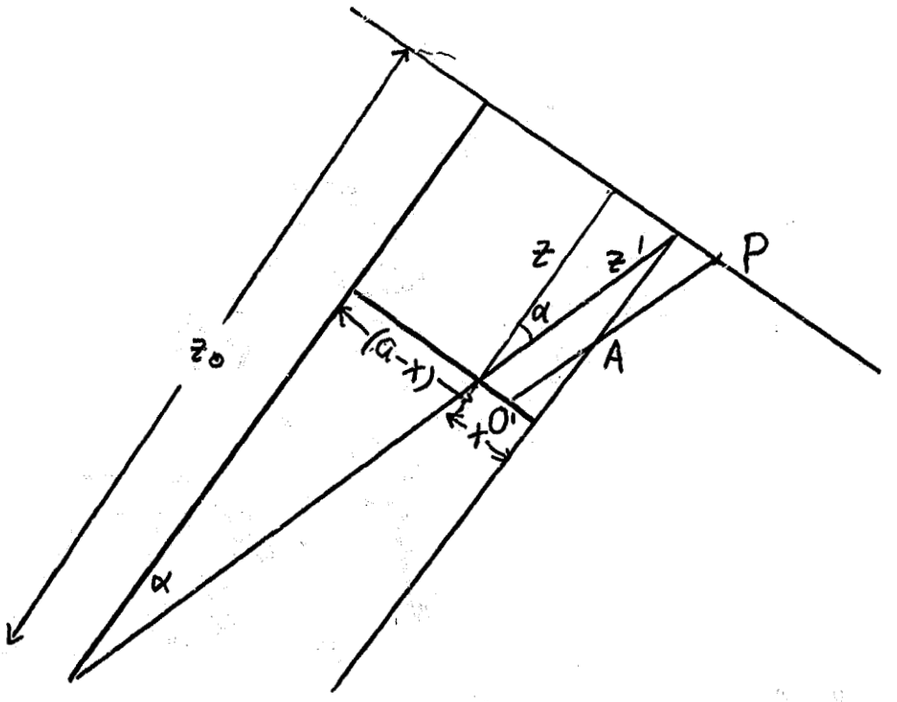


Figure 5. Figure for deriving modified retrodirective function.

CHAPTER IV

ON THE TREATMENT OF LUNAR SCATTERING AND RELATED PROBLEMS BY STOCHASTIC OPERATOR EQUATIONS (PART I)

by

G. Adomian

Introduction:

The field reflected by the lunar surface when illuminated by electromagnetic waves fluctuates in time with slow fluctuation attributed to ionospheric effects and much more rapid variations associated with the irregularities of the moon's surface. In order to determine the effect of these irregularities on the reflected radiation, it is convenient to suppose that the surface is represented (as done by Beckman and others) by a stationary random function or stochastic process $\zeta(x,\omega)$, (or simply $\zeta(x)$, suppressing ω), where x represents position (e.g. latitude and longitude); x is an element of the index space of the process; ω is an element of the probability space. Thus ζ can represent the surface height which is unknown explicityly; if the variance of ζ is zero, we have a smooth surface. The the problem at hand is essentially one of solving a stochastic differential equation, i.e., a wave equation involving random functions in the boundary conditions or elsewhere (coefficients or source function) depending upon the particular formulation.

Random boundary conditions have been used e.g. by D. E. Barrick (A More Exact Theory of Backscattering from Statistically Rough Surfaces, Ph.D. Dissertation, Department of Electrical Engineering, Ohio State University, 1965) to study backscattering of electromagnetic waves from rough surfaces. Starting with the Chu-Stratton vector integral equation, he obtains backscattering cross sections for rough surfaces with various statistical models.

To allow sufficient flexibility for a mathematically correct and general treatment of lunar scattering as well as a number of important related problems (e.g. scattering in turbulent layers of the atmosphere, absorption of infra-red in aerosols, etc.) we will consider the proper solution of stochastic operator equations with stochastic processes involved anywhere--in the boundary conditions, in the forcing function, or in the coefficients. (Generalizations to vector equations and partial differential equations have caused no essential difficulty). Thus the objective is to make available mathematical models for consideration of the stochastic properties of the solution process whether the randomness is due to a surface function scattering incoming wave, a deep dust layer on the surface which the waves penetrate, or effects of the ionosphere. The effects of ionosphere are usually separated, however the presence of several layers of plasma with densities and positions varying with height and time makes the problem one of propagation in a random medium which has not been treated

adequately for the case of appreciable randomness. Setting aside this problem, we must solve at least a reduced wave equation with random boundary conditions. In many treatments in the literature, one finds the random functions handled as if they were explicit deterministic functions then an ensemble average thrown in at the end. It is supposed that the average solution of stochastic equations is the solution of averaged equations, which is generally not the case, and operations are made on the random functions which cannot be made except in some carefully defined sense on restricted processes. We will allow the random behavior to be in the boundary conditions as mentioned above, or in the forcing functions, if e.g. we consider the illuminated random surface as a source; or in the coefficients if our model includes a random index of refraction, or a random k for the wave vector of the scattered wave in $(\nabla^2+k^2)\psi = 0$, or if we employ the method of the pseudo potential (see e.g. Y. M. Chen, J. Math. Phys. 5, 11, Nov. 1964, p. 1541-6) to incorporate the effect of boundary conditions in our equation.

The scattering from a volume or three-dimensional configuration of discrete scatterers is also of interest here comparing the lunar problem to one of radar scintillation from a complex aircraft target where as the aircraft rotates with respect to the observer, various areas change in their contribution to the echo. Similar phenomena occur in stellar images in telescopes due to the random medium of the atmosphere leading to a "twinkling"

(irregular fluctuation of the light intensity) and a "quivering" (irregular fluctuation of the angle of arrival), also called amplitude and angular scintillation. The problem of multiple scattering by randomly positioned objects has, of course, been studied by a large number of authors, however, the present approach offers an opportunity to avoid restriction to small random effects, averaging methods and the approximations generally made, and further allows the calculation of spectral densities, mutual coherence functions, and correlations instead of simple expectations. We seek to find distributions or at least second order statistics, not to assume them or the solutions and to avoid mathematical restrictions not physically necessary and to see what errors are involved in the usual approximations. Hence for stochastic differential and integral equations arising in physical problems, the objectives, limitations, and restrictive assumptions of the various methods will first be studied. Some promising new methods have been developed which eliminate the various limitations and allow treatment of a wide class of applications as well as the lunar scattering itself.

Of course, considerable literature exists on random equations, most of it however applies to first order differential equations or to equations with constant coefficients with the element of randomness arising from either a random forcing function or random boundary conditions. These cases are fundamentally simpler

than the case of stochastic coefficients because of the deterministic relationship of the probabilistic or statistical properties of the random quantity. These cases are well covered in a survey by Syski[1] and will not be discussed here except as special cases of a more general approach. Studies of random operators on Banach spaces have usually been inapplicable to stochastic differential operators for reasons to be discussed. Much of the related work in applications (propagation and scattering) has minimized the probabilistic aspects using methods whose validity is open to question and which are generally incorrect, (averaging methods, sample function approach, etc.). Finally much interesting work has been restricted to special processes (white noise, etc.) and it would be desirable to consider the problem in greater generality. This paper will be concerned particularly with the linear random operator equations

$$y = \mathcal{H}\{x\}$$

$$\mathcal{L}\{y\} = x$$

where

$$\mathcal{L} = \sum_{\nu=0}^{n} a_\nu(t) \frac{d^\nu}{dt^\nu}$$

and the $a_\nu(t)$ and $x(t)$ are stochastic processes. The $x(t)$, or $x(t,\omega)$, is to be taken as statistically independent of the $a_\nu(t)$, or $a_\nu(t,\omega)$, (although the $a_\nu(t)$ may be correlated with each other). This, too has been studied by a number of workers in the last few years; we refer particularly to the work of J. C. Samuels,[2] and J. C. Samuels and A. C. Eringen[3] (cf. also U. Grenander[4]). Some important matters will not be discussed here, e.g., the stability of systems described by the stochastic differential equations, and we will make no attempt to discuss the numerous possible applications. (See G. Adomian[5] or a forthcoming monograph by A. T. Bharucha-Reid[6] and most recently the work of L. Sibal.[7])

Keller has distinguished two methods of solving random differential equations of the form $\mathcal{L}y = x$ where the operator $\mathcal{L}$ involves a random parameter α (the "honest" and "dishonest" methods). In the first method, the solution $y(r,t,\alpha)$ is determined for each value of the random parameter α. Thus the randomness plays no real role in the process of finding y. Then the random nature of the solution is considered and the expectation

$$\langle y \rangle = \int_A y(r,t,\alpha)P(\alpha)d\alpha$$

is computed as a weighting of each specific solution with its probability. Higher order statistics are similarly found. This is a sample function approach and is generally not valid.

In the second method, a direct determination is made of the equations satisfied by the various moments of the solution. To do this, the equation, and a hierarchy derived from it (in a manner to be discussed) are averaged. This is generally incorrect also. The solution of the averaged equation is not always (or even usually) the average solution of the random equation. Furthermore, a so-called "closure approximation" is involved which cannot be justified and, in actuality, is equivalent to the use of perturbation theory to some order in the first place. The first method is right in principle but asks too much. It involves obtaining a solution first and then obtaining the statistics. This is not always possible; in fact, it is essentially limited to the case of a first order equation (considered by Tikhonov,[8] Adomian,[5] Astrom,[9] and others) since it involves the inversion of a stochastic matrix, and even then only when the iterative method (of Adomian or its formulation by Schul) are used. Since only statistical information or ensemble averages are to be the end result, then elimination of the intermediate step which requires more than this should result in a cleaner and more useful method. The ultimate goal in such problems is the complete statistical description of the output y from the statistical knowledge of the coefficients and the input. Often, however, it is sufficient to determine less complete "statistical measures"* in the form of expectation, correlation

*The term will be rigorously defined elsewhere; here, it will be "defined" by example and will simply stand for the useful quantities which can be calculated for the random process y.

function, or spectral density. Of course, by the use of various approximations based on assumptions of slow variation of parameters, small fluctuations, and restricted classes of processes (white noise), etc., solutions can be found to a wider class of differential equations than the first order equation so the statistical properties can be determined (Samuels,[2] Chelpanov,[10] Keller,[7]...); however, we wish to eliminate these restrictions. To summarize, then, the first method requires the iterative approach; as used, it is often involved. The second method involves unsupportable assumptions of separability and works only in highly restricted (Dirac measure) spaces or within a perturbation theory framework. Our objective is a method for determining desired statistical measures of the dependent variable directly without averaging and unjustifiable assumptions. There are two possible ways of doing this. The first is a modification of Samuels' method which avoids the difficulty in separability. The second, a stochastic expansion method suggested by Adomian,[11] appears promising and some interesting results have been obtained. The question asked is whether one can find a stochastic Green's function for the desired statistical measure of the dependent variable analogous to finding a Green's function for an ordinary equation.

Most equations of mathematical physics and engineering science are assumed to be in the form

$$L\, f(x) = s(x)$$

where L is a deterministic linear operator. (L may be a differential operator of n^{th} order, or a Hamiltonian operator, or the (self-adjoint) Sturm-Liouville operator $-\frac{d}{dx}[p(x)\frac{d}{dx}]-q(x)$. It is usually assumed that to L corresponds an integral kernal or Green's function $k(x,x')$ such that

$$f(x) = L^{-1}\, s(x) = \int_{-\infty}^{\infty} k(x,x')\, s(x')\, dx'$$

is the solution.

The analogous equations for a stochastic operator can be written

$$\mathcal{L} f(x) = s(x) \qquad 1(a)$$

or

$$f(x) = \mathcal{H} s(x) \qquad 1(b)$$

Both forms simply show a function being transformed into a new function by an operator but it is convenient for both mathematical and physical reasons to consider the two equations separately. Equation 1(a) involves a differential operator $\mathcal{L}$*

*Generalization to random fields and partial differential operators follows but will not be discussed specifically here. See referance 5 also.

and a function s(x) which may be an ordinary function and specifically given, or a random function whose statistics are known. Then the solution f(x) is a random function (stochastic process) whose statistics must be found. In the second form 1(b), s(x) is again given, either as an ordinary function if deterministic, or through its statistics if random. $\mathcal{H}$ is a given random transformation which transforms s(x) into another random function f(x). $\mathcal{H}$ is a known operator, a measurement, or a processing of the signal s(x) (by a known "black box" $\mathcal{H}$). $\mathcal{H}$ is stochastic by virtue of parameters, one or more of which are random, but whose statistics are known to the necessary degree. We can consider $\mathcal{H}$ to be a"stochastic filter" and the resulting theory a generalization of ordinary filter theory.

Ability to treat these two possibilities would give us a two-pronged attack on most physical problems, in which we would either determine the characteristics of the solution to a (stochastic) differential equation, or the (statistical) characteristics of the random function resulting from a stochastic transformation. Since the result in either case is a stochastic process, our knowledge of the process will be specified in the form of expectations, spectral densities, higher moments, or at most, in a distribution function. We will lump these terms together under the term "statistical measures" for the process (and use it rather than the unsatisfactory term "statistics"

used above).* Thus, our objective will be to specify desired statistical measures for an output process in terms of the known statistical measures of the input process and an integral kernel depending upon the operator which we then call a "stochastic Green's function."[11]

*Statistical measures, statistical operators, stochastic operators and the relationship of stochastic mappings on Cartesean product spaces and the random operators of space are discussed in a forthcoming book.

I. Operator Formulation:

Suppose a linear stochastic operator $\mathcal{H}_{\alpha\beta\ldots}$, or $\mathcal{H}$ for brevity, acts upon a random function $\{x(\xi,\omega),\omega\in\Omega\}$, which we usually write simply as $x(\xi)$ and assume to be statistically known. Assume $\mathcal{H}$ is statistically independent of x. Physically, $\mathcal{H}$ may represent a filter, a scattering medium, a communication channel, or a measurement. $\alpha(\xi),\beta(\xi),\ldots$ represent system parameters, or parameters of the operator $\mathcal{H}$, at least one of which we suppose to be random. $\mathcal{H}$ is then determined by the distributions of its parameters and represents an ensemble of operators H collectively represented by $\{H\}$ or $\mathcal{H}$. Writing the stochastic equation

$$y = \mathcal{H}_{\alpha}[x]$$

we see this represents an ensemble of equations depending upon the parameter α which ranges over an appropriate space A, in this case one in which a probability density $p(\alpha)$ is defined. $p(\alpha)$ determines the probability of a given value for α and therefore of the corresponding equation of the ensemble. We view $\mathcal{H}$ as mapping the random function x into the random function y. Conceptually, each H of $\{H\}$ has an associated Green's function $h(\xi,\eta)$ which represents the response at ξ of the

representative system H to a unit impulse $\delta(\xi-\eta)$.* Then the response of the H system to an input x is $y(\xi)$ where

$$y(\xi) = Hx(\xi) = \int_{-\infty}^{\infty} h(\xi,\eta)x(\eta)d\eta$$

if x is a defined continuous member of the x process (i.e., a realization), and H is a causal (physically realizable) system. For example, if ξ and η are the time coordinates t and τ, $h(t,\tau) = 0$ for $t < \tau$.

We define now the autocorrelation of the transformed process $y = \mathcal{H}[x]$ by:

$$R_y(\xi_1,\xi_2) = E\{y(\xi_1)\overset{*}{y}(\xi_2)\}_{H,x}$$

or

$$R_y(\xi_1,\xi_2) = \langle y(\xi_1)\overset{*}{y}(\xi_2)\rangle_{H,x}$$

when the notation indicates an averaging over both ensembles involved. Quite generally the expectation value symbols can be moved inside the integral. (Fubini's Theorem). Hence $R_y(\xi_1,\xi_2)$

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*Of course one need never talk of delta functions. We can equally well use Stieltjes integrations or distributions in the sense of Schwartz. The latter approach is particularly interesting and leads to stochastic generalized functions or distributions related to the work of Gelfand, Urbanik and Ullrich.

is now

$$= \int_{-\infty}^{\infty} \int \langle h(\xi_1,\eta_1)\overset{*}{h}(\xi_2,\eta_2)x(\eta_1)\overset{*}{x}(\eta_2)\rangle d\eta_1 d\eta_2$$

Recognizing $\langle x(\eta_1)\overset{*}{x}(\eta_2)\rangle$ as the autocorrelation function $R_x(\eta_1,\eta_2)$ of the original process, the integrand can be separated into the product of ensemble averages for the system and the input by virtue of the statistical independence of $\mathcal{H}$ and s. Thus,

$$R_y(\xi_1,\xi_2) = \int_{-\infty}^{\infty}\int \langle h(\xi_1,\eta_1)\overset{*}{h}(\xi_2,\eta_2)\rangle R_x(\eta_1,\eta_2)d\eta_1 d\eta_2$$

or as a convenient general form:

$$R_{\mathcal{H}x}(\xi_1,\xi_2) = \int_{-\infty}^{\infty}\int G_{\mathcal{H}}(\xi_1,\xi_2,\eta_1,\eta_2)R_x(\eta_1,\eta_2)d\eta_1 d\eta_2 \qquad (2)$$

i.e., the autocorrelation of the transformed process $\mathcal{H}[x]$ is given in the terms of the autocorrelation of the original process $x(\xi)$ and a kernel function-in this case, $G_{\mathcal{H}}$, (which is itself an autocorrelation-depending on the stochastic system $\mathcal{H}$). This kernel function is called the "stochastic Green's function." In general x might be a function of time, position, or frequency and the Green's functions are appropriate to the operator. We will discuss the representative case where $x = x(t)$ and $\mathcal{H}$ is a

randomly time-varying operator which may represent a medium of transmission, or of scattering, or an observation, experiment, or measurement.

$$R_{\mathcal{H}x}(t_1,t_2) = \int_{-\infty}^{\infty}\int G_{\mathcal{H}}(t_1,t_2,\tau_1,\tau_2)R_x(\tau_1,\tau_2)d\tau_1 d\tau_2$$

The limits of integration may be finite in a particular problem and depend on the range of definition of the processes involved and on the existence of causalty and memory.

We do not expect, of course, to find Green's functions for each and every realization although this may be possible in the analysis of assumed mathematical models where we may generalize a Green's function for a deterministic problem to a stochastic Green's function for the corresponding stochastic problem.* Rather, we hope to calculate the stochastic Green's function from statistical knowledge which would ordinarily be available. We emphasize we do not make a sample function approach. This is a formal relation which occurs generally and which suggests an approach directly through the stochastic Green's function. If instead of simple functions we consider a matrix or state space formulation, the equation $y = \mathcal{H}x$ relates the s.p. y to the s.p. x and the $h(\xi,y)$ becomes a random Green's function. Taking the correlation of the s.p. y we immediately obtain the same relationship between correlation function of "output" and "input" and the identification of the stochastic Green's function for

*See Ch. 4 of dissertation referenced in (5).

that particular statistical measure of input and output processes. Details will be discussed in the section on the iterative procedure.

Let us now examine some forms of (2). Suppose x(t) is a stationary stochastic process. Then

$$R_x(\tau_1,\tau_2) = R_x(\tau_1-\tau_2) \triangleq R_x(\sigma)$$

Suppose the transformation by $\mathcal{H}$ preserves stationarity in the transformed process. (In general, it does not but it can under special conditions; we defer these questions for now. Precise conditions have been derived and will be published in a dissertation by the author's student, W. W. Walker, Jr.). Then

$$R_{\mathcal{H}x}(t_1,t_2) = R_{\mathcal{H}x}(t_1-t_2) = R_{\mathcal{H}x}(\beta) \qquad (3)$$

Consequently,

$$R_{\mathcal{H}x}(\beta) = \int g_{\mathcal{H}}(\beta,\sigma)R_x(\sigma)\,d\sigma$$

where

$$g_{\mathcal{H}}(\beta,\sigma) = \int d\tau\langle h(t,\tau)h(t+\beta,\tau+\sigma)\rangle \qquad (4)$$

is the stochastic Green's function for the autocorrelation for the

stationary case. By appropriate transformation, the spectral density can be found.

$$\Phi_y(f) = \int K_{\mathcal{H}}(s,f)\Phi_x(s)ds \qquad (5)$$

where*

$$K_{\mathcal{H}}(s,f) = \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} d\beta e^{2\pi i f\beta} e^{-2\pi i s\sigma} g_{\mathcal{H}}(\beta,\sigma)$$

Expression (4) must be satisfied for stationarity to hold, i.e., the kernel must not depend on time, so the ensemble average must be time independent, which requires that the parameters of $\mathcal{H}$ must at least be stationary. If $\mathcal{H}$ is ergodic as well, each member of the ensemble gives the same time average as every other member so it may be possible to find a single Green's function, and from it, the stochastic Green's function by suitable averaging. Thus

$$\langle h(t,\tau)h(t+\beta,\tau+\sigma)\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} du\ h(t+u,\tau)\overset{*}{h}(t+\beta+u,\tau+\sigma)$$

Since the $\alpha,\beta\ldots$ enter into the h's, we have more explicitly

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*An example in this form appears in a detailed consideration of the random sampling of a random process in the author's dissertation. (5).

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} du \int_0^\infty d\alpha \int_0^\infty d\beta \ldots p(\alpha)p(\beta)\ldots h(\alpha,\beta,\ldots t+u,\tau)\cdot \overset{*}{h}(\alpha,\beta,\ldots t+\beta+u,\tau+\sigma).$$

II. Perturbation Theory Approach:*

Let us suppose first that the random variations are sufficiently small so corrections to the deterministic solution are of low order. Then perturbation theory is useful. Consider a linear operator $\mathcal{L}$ depending upon a parameter α which ranges over a measure space or set A. Suppose $p(\alpha)$ is a probability density defined over A (or appropriate subsets of A). Then $\mathcal{L}_\alpha$ is a stochastic operator and we can consider the stochastic equation.

$$\mathcal{L}_\alpha u = g$$

where (at first) g is a given non-random element of the linear space. Assuming a unique solution $u(\alpha)$ for each α, the solution depends upon α. Thus $u(\alpha)$ is a random solution and $p(\alpha)$ determines the probability density of $u(\alpha)$. The statistical measure of u which is of interest to us is specified to be the expectation $\langle u \rangle$ $= \int_A u(\alpha)p(\alpha)d\alpha$.

Assume $\mathcal{L}_\alpha = \mathcal{L}(\alpha,\epsilon)$ depends upon a small parameter ϵ, and that for $\epsilon=0$, $\mathcal{L}$ reduces to a deterministic operator L. Expanding

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*See reference 7.

$\mathcal{L}_\alpha$ in powers of ϵ, we may write

$$\mathcal{L} = L+\epsilon\mathcal{L}_1(\alpha)+\epsilon^2\mathcal{L}_2(\alpha)+O(\epsilon^3)$$

and

$$\mathcal{L}u(\alpha,\epsilon) = g$$

Thus $\mathcal{L}$ is given by the sum of the operator L which is deterministic, and $\mathcal{L}_1$ and $\mathcal{L}_2$ which are random and represent stochastic perturbations of the deterministic operator L. Suppose $\langle\mathcal{L}_1\rangle = \langle\mathcal{L}_2\rangle = 0$ for simplicity. If $\epsilon = 0$, the solution of the resulting determinstric equation $Lu = g$ is $u_o = L^{-1}g$ assuming L^{-1} is defined. Now

$$u = u_o-\epsilon L^{-1}\mathcal{L}_1u-\epsilon^2L^{-1}\mathcal{L}_2u+O(\epsilon^3)$$

If we average now to get $\langle u\rangle$, we have quantities like $\langle\mathcal{L}_1u\rangle$ or $\langle\mathcal{L}_2u\rangle$ which ordinarily would not separate further since they involve the operator and the dependent variable or solution process. However they do now because we have assumed perturbation theory to be applicable. Thus by iteration or successive substitution $u = u_o+\epsilon u_1+\epsilon^2u_2+\ldots.$ becomes

$$u = u_o-\epsilon L^{-1}\mathcal{L}_1u_o+\epsilon^2L^{-1}(\mathcal{L}_1L^{-1}\mathcal{L}_1-\mathcal{L}_2)u_o+O(\epsilon^3)$$

Taking the expectation now,

$$\langle u\rangle = u_0+\epsilon^2 L^{-1}\langle \mathcal{L}_1 L^{-1}\mathcal{L}_1\rangle u_0+O(\epsilon^3)$$

$$= L^{-1}g(1+\epsilon^2 L^{-1}\langle \mathcal{L}_1 L^{-1}\mathcal{L}_1\rangle)+O(\epsilon^3)$$

Equivalently, since $u_0 = \langle u\rangle+O(\epsilon^2)$

$$\langle u\rangle = L^{-1}g+\epsilon^2 L^{-1}\langle \mathcal{L}_1 L^{-1}\mathcal{L}_1\rangle\langle u\rangle+O(\epsilon^3)$$

$$[L-\epsilon^2\langle \mathcal{L}_1 L^{-1}\mathcal{L}_1\rangle]\langle u\rangle = g+O(\epsilon^3)$$

which is given by Keller (reference 7) (and also found by Adomian (reference 17)).

We note it is not necessary that g be deterministic. If g is stochastic, $L^{-1}g$ is stochastic, and in taking the expectation we get $L^{-1}\langle g\rangle$. We find that averages involving $\mathcal{L}_1$ or $\mathcal{L}_2$ and g separate by only the statistical independence of $\mathcal{L}$ and g. Thus we need make no further assumptions of an artificial nature in an attempt to separate $\langle \mathcal{L}u\rangle$ into $\langle \mathcal{L}\rangle\langle u\rangle$. In general $\mathcal{L}$ cannot be assumed to be statistically independent of u but $u_0 = L^{-1}g$ is deterministic in the first case so it separates out. And in the second case $L^{-1}g$ is stochastic but the statistical independence of $\mathcal{L}$ and g is sufficient for the separations. To see this we proceed as follows:

$$[L+\epsilon\mathcal{L}_1+\epsilon^2\mathcal{L}_2+O(\epsilon^3)]u = g_o+\epsilon g_1+\epsilon^2 g_2+O(\epsilon^3)$$

Let $u_o = L^{-1}g_o$. Then

$$Lu = g_o+\epsilon g_1+\epsilon^2 g_2-\epsilon\mathcal{L}_1 u-\epsilon^2\mathcal{L}_2 u+O(\epsilon^3)$$

$$u = u_o+\epsilon L^{-1}(g_1-\mathcal{L}_1 u_o)+\epsilon^2 L^{-1}(g_2-\mathcal{L}_2 u_o-\mathcal{L}_1 u_1)$$

Thus the coefficient of the ϵ term is u_1; the coefficient of the ϵ^2 term is u_2.

$$\langle u\rangle = u_o+\epsilon L^{-1}[\langle g_1\rangle-\langle\mathcal{L}_1\rangle u_o]+\epsilon^2 L^{-1}[\langle g_2\rangle-\langle\mathcal{L}_2\rangle u_o-\langle\mathcal{L}_1 L^{-1}g_1\rangle+\langle\mathcal{L}_1 L^{-1}\mathcal{L}_1\rangle u_o]$$

$$= L^{-1}\langle g\rangle-\epsilon L^{-1}\langle\mathcal{L}_1\rangle L^{-1}g_o-\epsilon^2 L^{-1}[\langle\mathcal{L}_2\rangle L^{-1}g_o+\langle\mathcal{L}_1 L^{-1}g_1\rangle-\langle\mathcal{L}_1 L^{-1}\mathcal{L}_1\rangle L^{-1}g_o]$$

$$= L^{-1}\langle g\rangle-\epsilon L^{-1}\langle\mathcal{L}_1\rangle L^{-1}g_o-\epsilon^2 L^{-1}\langle\mathcal{L}_2\rangle L^{-1}g_o-\epsilon^2 L^{-1}\langle\mathcal{L}_1\rangle L^{-1}\langle g_1\rangle$$
$$-\epsilon^2 L^{-1}\langle\mathcal{L}_1 L^{-1}\mathcal{L}_1\rangle L^{-1}g_o$$

Combining the second and the fourth terms

$$\langle u\rangle = L^{-1}\langle g\rangle-\epsilon L^{-1}\langle\mathcal{L}_1\rangle L^{-1}[g_o+\epsilon\langle g_1\rangle+\ldots]-\epsilon^2 L^{-1}[\langle\mathcal{L}_2\rangle+\langle\mathcal{L}_1 L^{-1}\mathcal{L}_1\rangle]L^{-1}g_o$$

$$= L^{-1}\langle g\rangle-\epsilon L^{-1}\langle\mathcal{L}_1\rangle L^{-1}\langle g\rangle-\epsilon^2 L^{-1}[\langle\mathcal{L}_2\rangle+\langle\mathcal{L}_1 L^{-1}\mathcal{L}_1\rangle]L^{-1}g_o$$

$$\langle u\rangle = L^{-1}\langle g\rangle - \epsilon^2 L^{-1}\langle \mathcal{L}_1 L^{-1}\mathcal{L}_1\rangle L^{-1}\langle g\rangle$$

if $\langle \mathcal{L}_1\rangle = \langle \mathcal{L}_2\rangle = 0$, this can be simplified further, thus

$$\langle u\rangle = L^{-1}\langle g\rangle[1-\epsilon^2 L^{-1}\langle \mathcal{L}_1 L^{-1}\mathcal{L}_1\rangle]$$

or

$$[L+\epsilon^2\langle \mathcal{L}_1 L^{-1}\mathcal{L}_1\rangle]\langle u\rangle = \langle g\rangle .$$

Without the assumption of statistical independence of $\mathcal{L}$ and g, additional terms appear as Chen points out[18] which disappear only if either $\mathcal{L}_1$ and $\mathcal{L}_2$ or g_1 and g_2 are zero, which is physically reasonable. However, in the case of statistical independence, it is not true that there is a coupling term $\epsilon^2 L^{-1}\langle \mathcal{L}_1\rangle L^{-1}\langle g_1\rangle$. This term and the $\mathcal{L}u_o$ part of $u_1 = L^{-1}(g_1 - \mathcal{L}_1 u_o)$ combine to give $\epsilon L^{-1}\langle \mathcal{L}_1\rangle L^{-1}\langle g\rangle$ just as $u_o + \epsilon L^{-1}\langle g_1\rangle + \epsilon^2 L^{-1}\langle g_2\rangle$ becomes $L^{-1}\langle g\rangle$.

The use of the perturbation theory follows from the requirement that the random part of $\mathcal{L}$ be small, which in turn arises from the need to make only u_o be involved in the $\langle \mathcal{L}u\rangle$ so it will separate. Thus we have seen that the desired "statistical measure" of u, in this case, the mean or expectation value, is given in terms of the same statistical measure of g (when g is random) and a functional involving only certain averages over L.[17]

Green's Function Method:[7] Let $u = u(x)$ be a n-component vector function of a vector variable x. Then L, $\mathcal{L}_1, \mathcal{L}_2, \ldots$ are represented by n^{th} order matrices, each element of which is an operator (differential or integral operator). The inverse operator L^{-1} is also a n^{th} order matrix which we shall represent as an integral operator. The kernel $G(x,x')$ is the Green's matrix defined by

$$LG(x,x') = I\delta(x-x')$$

where I is the unit matrix and δ is the Dirac δ. Now in general

$$L^{-1}f(x) = \int G(x,x')f(x')dx'$$

We had $\mathcal{L}u(x) = g(x)$ so $\mathcal{L} = \mathcal{L}(x)$. Now the equation for $\langle u \rangle$ derived earlier is written

$$L(x)\langle u(x)\rangle + \epsilon^2\langle \mathcal{L}_1(x)\int G(x,x')\mathcal{L}_1(x')\langle u(x')\rangle dx'\rangle = g(x) + O(\epsilon^3)$$

$$\langle u(x)\rangle = \int G(x,x')g(x')dx' - \epsilon^2\int G(x,x')\langle \mathcal{L}_1(x')\int G(x',x'')\mathcal{L}_1(x'')\langle u(x'')\rangle dx''$$

Suppose we now do not immediately assume the random part is small--and simply assume we can separate the operator into the sum of two operators L + R where L is deterministic and R is random. Then

$$\mathcal{L}u = g$$

becomes

$$(L+R)u = g$$

or

$$u = L^{-1}g - L^{-1}Ru \tag{6}$$

where we assume L^{-1} is defined. Now we proceed as before but assume for simplicity that $g_1 = g_2 = 0$ and $\langle \mathcal{L}_2 \rangle = 0$, i.e., we assume "small randomness" by letting $u = u_0 + \epsilon u_1 + \epsilon^2 u_2$. Then we get

$$\langle u \rangle = L^{-1}g(1 - \epsilon L^{-1}\langle \mathcal{L}_1 \rangle + \epsilon^2 L^{-1}\langle \mathcal{L}_1 L^{-1}\mathcal{L}_1 \rangle)$$

the perturbation theory result. However suppose we simply call $L^{-1}g = u_0$ and let $u = u_0 - u_1 + u_2 \ldots$. i.e., we say nothing about smallness of R. Now

$$u = u_0 - L^{-1}Ru_0 + L^{-1}Ru_1 - L^{-1}Ru_2 \ldots$$

$$= L^{-1}g - L^{-1}RL^{-1}g + L^{-1}RL^{-1}RL^{-1}g \ldots$$

$$\langle u \rangle = L^{-1}g - L^{-1}\langle R \rangle L^{-1}g + L^{-1}\langle RL^{-1}R \rangle L^{-1}g \ldots$$

$$\langle u \rangle = L^{-1}g(1 - L^{-1}\langle R \rangle + L^{-1}\langle RL^{-1}R \rangle - \ldots$$

we see that if R is small this is the same as perturbation theory. However if it isn't, and the convergence can be established for cases of interest, the results may be of interest. Consequently we will examine this is much more detail presently.

III. Hierarchy Equations

This approach which has received considerable use in theoetical physics* involves obtaining a hierarchy of equations and the use of some truncation procedure (closure approximation) to terminate the hierarchy. Thus, given a scalar function $u(x)$ satisfying the stochastic equation

$$\mathcal{L}u(x) = g(x) \qquad (7)$$

where the stochastic operator $\mathcal{L} = L_x + n(x)$, where we assume that L_x is a deterministic operator, and $g(x)$ is a deterministic function, but $n(x)$ is random (so, of course, $u(x)$ is random). If (1) is averaged in an attempt to find $\langle u(x)\rangle$, we find that $L_x\langle u(x)\rangle + \langle n(x)u(x)\rangle = g(x)$. But in the second term on the left side, $n(x)$ and $u(x)$are not statistically independent and cannot

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*Tamm-Dancoff method (I. Tamm, Relativistic Interaction of Elementary Particles, J. Phys. (USSR), 9, 449, 1945; Green's function method of Martin and Schwinger (P. C. Martin and J. Schwinger, Theory of Many Particle Systems I, Phys. Rev. (2), 115, 1342-1373, 1959; BBGKY Hierarchy (N.N. Bogoliubov, Problems of a Dynamical Theory in Statistical Physics, - Studies in Statistical Mechanics, de Boer and G. E. Uhlenbeck, eds., p. 5-116, North Holland, Amsterdam, 1962).

be separated for the determination of $\langle u(x)\rangle$. If, to determine $\langle n(x)u(x)\rangle$ we multiply (7) by $n(x)$ and average again, we get the term $\langle n(x)L_x u(x)\rangle$, but L_x does not commute in general with $n(x)$. However, $n(x_1)$ can commute with L_x, therefore we multiply by $n(x_1)$ and average to get

$$L_x\langle n(x_1)u(x)\rangle+\langle n(x_1)n(x)u(x)\rangle = \langle n(x_1)\rangle g(x). \qquad (8)$$

If we could solve this for the first term, evaluate the result as it approaches $x_1=x$ and substitute back, we get $\langle u(x)\rangle$. Aside from the generally ignored question of the validity of letting $x_1=x$, we find we need now the new moment $\langle n(x_1)n(x)u(x)\rangle$ so it is necessary to repeat the procedure multiplying by $n(x_2)$. We are led to an infinite set of equations called the hierarchy equations for the hierarchy of moments. If we want also higher moments of $u(x)$, we proceed in the same manner multiplying by $u(x_1)$, etc. Thus, the solution involves averaging first which will later be shown to be wrong and an arbitrary truncation procedure which has been justified only by its arbitrary assumption.

If at any level in the hierarchy the "closure approximation" can be made, then the hierarchy can be terminated. Thus, if $\langle n(x_1)n(x)u(x)\rangle = \langle n(x_1)n(x)\rangle\langle u(x)\rangle$ in (8), we can solve for $\langle n(x_1)u(x)\rangle$, evaluate at $x_1 = x$ and substitute into (7) to "solve" for $\langle u(x)\rangle$, assuming L_x is a known operator and the initial conditions are specified. One other possibility exists

but appears to be of little physical interest. If there exists a very small correlation length for $n(x)$ compared to u, i.e., if n is varying very rapidly compared to u, the truncation may be justified. However, since a in a sense is the solution to be found, a priori assumptions on its behavior are undesirable.

The same procedure has been used to get a correlation function for us or its higher moments. The general case gives the infinite set of equations[19,20] (hierarchy equations)

$$L_x\langle u(x)u(x_1)\ldots u(x_j)n(x_{j+1})\ldots n(x_{j+k})\rangle$$
$$+\langle n(x)u(x)u(x_1)\ldots u(x_j)n(x_{j+1})\ldots n(x_{j+k})\rangle$$
$$= g(x)\langle u(x_1)\ldots u(x_j)n(x_{j+k})\rangle \qquad j,k = 0,1,2,\ldots.$$

The source term $g(x)$ is assumed statistically independent of the parameter $n(x)$. We see that an infinite set of equations are needed to find all the moments, i.e., any moment involves all the moments of higher order and a closure procedure is necessary to get a cutoff. Thus, in the k^{th} member of the hierarchy, we set

$$\langle n(x_1)\ldots n(x_{k-1})n(x)u(x)\rangle \cong \langle n(x_1)\ldots n(x_{k-1})n(x)\rangle\langle u(x)\rangle.$$

Connection to Perturbation Theory:

Set $\mathcal{L}_2 = 0$ for simplicity and also assume $\langle \mathcal{L}_1 \rangle = 0$. Take g as non-random. We have

$$\mathcal{L}u = g$$

$$[L+\epsilon\mathcal{L}_1(\alpha)]u(\alpha,\epsilon) = g+O(\epsilon^3)$$

The expectation (dropping $O(\epsilon^3)$) is

$$L\langle u\rangle+\epsilon\langle\mathcal{L}_1 u\rangle = g \qquad (9)$$

But $\langle\mathcal{L}_1 u\rangle$ is unknown and it cannot be separated without perturbation theory. To see if perturbation theory can be avoided, we write before averaging again

$$Lu+\epsilon\mathcal{L}_1 u = g$$

Multiply by L^{-1}

$$u+\epsilon L^{-1}\mathcal{L}_1 u = L^{-1}g$$

Now multiply by $\mathcal{L}_1$ and average to get an expression for $\langle\mathcal{L}_1 u\rangle$. Thus,

$$\langle\mathcal{L}_1 u\rangle+\epsilon\langle\mathcal{L}_1 L^{-1}\mathcal{L}_1 u\rangle = \langle\mathcal{L}_1\rangle L^{-1}g = 0$$

To avoid going on forever, we assume blindly* that

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*Closure approximation or so-called "local independence."

$$\langle \mathcal{L}_1 L^{-1} \mathcal{L}_1 u \rangle \cong \langle \mathcal{L}_1 L^{-1} \mathcal{L}_1 \rangle \langle u \rangle,$$

Now,

$$\langle \mathcal{L}_1 u \rangle = -\epsilon \langle \mathcal{L}_1 L^{-1} \mathcal{L}_1 \rangle \langle u \rangle. \tag{10}$$

Substituting (10) in (9),

$$L\langle u \rangle - \epsilon^2 \langle \mathcal{L}_1 L^{-1} \mathcal{L}_1 \rangle \langle u \rangle = g$$

which is the same answer as derived by perturbation theory. The significance will be obvious shortly. We could go further in the hierarchy of course before invoking "closure." We can return to the expression for u, multiply by $\mathcal{L}_1 L^{-1} \mathcal{L}_1$ and average to get

$$\langle \mathcal{L}_1 L^{-1} \mathcal{L}_1 u \rangle + \epsilon \langle \mathcal{L}_1 L^{-1} \mathcal{L}_1 L^{-1} \mathcal{L}_1 u \rangle = 0$$

and separate the second term by assumption again into $\langle \mathcal{L}_1 L^{-1} \mathcal{L}_1 L^{-1} \mathcal{L}_1 \rangle \langle u \rangle$.

Now let g be stochastic

$$\mathcal{L} u = g$$

$$[L + \epsilon \mathcal{L}_1 + \epsilon^2 \mathcal{L}_2 + O(\epsilon^3)] u = g$$

$$L\langle u \rangle + \epsilon \langle \mathcal{L}_1 u \rangle = \langle g \rangle.$$

In the perturbation approach, we had

$$u = u_0 - \epsilon L^{-1}\mathcal{L}_1 u_0 + O(\epsilon^2).$$

If we use this and multiply by $\mathcal{L}_1$ we have

$$\mathcal{L}_1 u = \mathcal{L}_1 u_0 - \epsilon \mathcal{L}_1 L^{-1}\mathcal{L}_1 u_0 + O(\epsilon^2)$$

$$\langle \mathcal{L}_1 u\rangle = \langle \mathcal{L}_1\rangle\langle u_0\rangle - \epsilon\langle \mathcal{L}_1 L^{-1}\mathcal{L}_1\rangle\langle u_0\rangle + O(\epsilon^2)$$

$$= -\epsilon\langle \mathcal{L}_1 L^{-1}\mathcal{L}_1\rangle\langle u_0\rangle$$

$$= -\epsilon\langle \mathcal{L}_1 L^{-1}\mathcal{L}_1\rangle\langle u\rangle.$$

Therefore,

$$L\langle u\rangle - \epsilon^2\langle \mathcal{L}_1 L^{-1}\mathcal{L}_1\rangle\langle u\rangle = \langle g\rangle + O(\epsilon^3)$$

which is the same as the previous result if $\mathcal{L}_2 = 0$ and $\langle \mathcal{L}_1\rangle = 0$. Thus as Keller has pointed out, when the randomness is small, i.e., when the perturbation theory approach is useful, the average $\langle \mathcal{L}_1 L^{-1}\mathcal{L}_1 u\rangle$ can be written $\langle \mathcal{L}_1 L^{-1}\mathcal{L}_1\rangle\langle u\rangle$, and it is not necessary to assume it. It works for perturbation theory but then it isn't needed. If $\mathcal{L}$ is a differential operator, it is obviously wrong to assume statistical independence of $\mathcal{L}$ and its operand u although it is reasonable to assume statistical independence of $\mathcal{L}$ and g.

The hierarchy method has been widely used in statistical field physics, both for the linear and nonlinear case. In the quantum

mechanical many body problem or in the theory of turbulence,[20] non-linear stochastic equations arise. The dynamical equations lead to an infinite hierarchy of coupled equations in which given ensemble averages are related to successively higher order terms. This difficulty occurs, as we have seen, in the linear case as well. The closure approximation (a truncation and closure of the hierarchy) is always assumed to find an approximation for the desired statistical quantities but the validity or error involved has not been adequately discussed. It is interesting to note that if we consider

$$[L+\alpha(t)]u(t) = g$$

average to get

$$L\langle u(t)\rangle+\langle\alpha(t)u(t)\rangle = \langle g\rangle \qquad (11)$$

difficulties arise even in truncating at the next level. Thus, if we write (assuming the separation)

$$L\langle\alpha(t_1)u(t)\rangle+\langle\alpha(t_1)\alpha(t)\rangle\langle u(t)\rangle = \langle\alpha(t_1)\rangle\langle g\rangle = 0$$

$$L\langle\alpha(t_1)u(t)\rangle = -\langle\alpha(t_1)\alpha(t)\rangle\langle u(t)\rangle,$$

it has been assumed L (or L_t) commutes with $\alpha(t_1)$ which is the reason for not multiplying by $\alpha(t)$). However, if we then evaluate

at $t_1=t$, a commutator will have been neglected. (Furthermore the separation is not valid when $t_1=t$). Let us first solve the above for $\langle\alpha(t_1)u(t)\rangle$ by multiplying by L^{-1}, i.e.,

$$\langle\alpha(t_1)u(t)\rangle = -L_t^{-1}\ \langle\alpha(t_1)\alpha(t)\rangle\langle u(t)\rangle$$

$$= -\langle\alpha(t_1)L_t^{-1}\alpha(t)\rangle\langle u(t)\rangle.$$

Now

$$\langle\alpha(t)u(t)\rangle = \langle\alpha(t_1)u(t)\rangle_{t_1=t} = -\langle\alpha(t)L_t^{-1}\alpha(t)\rangle\langle u(t)\rangle, \quad (12)$$

Substituting (12) into (11), the equation can be rearranged into

$$\langle u(t)\rangle = L^{-1}\langle g\rangle[1+\langle\alpha(t)L_t^{-1}\alpha(t)\rangle]$$

$$= \int_0^t \ell(t-\tau)\langle g(\tau)\rangle[1+\langle\alpha(t)\int_0^t \ell(t-\tau)\alpha(\tau)d\tau\rangle]d\tau$$

$$= \int_0^t \ell(t-\tau)\langle g(\tau)\rangle[1+\int_0^t \ell(t-\tau)R_\alpha(t-\tau)d\tau]d\tau$$

where R_α is the correlation function for α, We observe we have $\langle u\rangle$ in terms of $\langle g\rangle$ and an integral kernel or stochastic Green's function[5,11,22] $\ell(t-\tau)[1+\int_0^t \ell(t-\tau)R_\alpha(t-\tau)d\tau$ where ℓ and R_α are given.

Suppose instead of multiplying by $\alpha(t_1)$, we solve for $u(t)$ and multiply by $\alpha(t)$, thus

$$Lu(t)+\alpha(t)u(t) = g$$

$$u(t) = L^{-1}g-L^{-1}\alpha(t)u(t)$$

Multiplying by u(t),

$$\alpha(t)u(t) = \alpha(t)L^{-1}g-\alpha(t)L^{-1}\alpha(t)u(t)$$

averaging,

$$\langle\alpha(t)u(t)\rangle = -\langle\alpha(t)L^{-1}\alpha(t)\rangle\langle u(t)\rangle$$

if we assume the separability on the right and a zero mean process for u(t). This obviously gives the same result as the previous procedure if we substitute back into the first averaged equation (11). This avoids the commutator problem but makes it still less clear why the separation can be assumed since $\alpha(t)$ cannot be said to be statistically independent of u(t).

The basic error here is in the averaging. Averaging the equation $\mathcal{L}u = \xi$ results in $\langle\mathcal{L}u\rangle = \langle g\rangle$. If $\mathcal{L} = L + R$ we have $L\langle u\rangle+\langle Ru\rangle = \langle g\rangle$. If $\mathcal{L}$ is replaced by L which clearly means throwing away the R so we have a simple deterministic equation, then $\langle\mathcal{L}u\rangle$ becomes $\langle Lu\rangle = L\langle u\rangle$ which can be rewritten $\langle\mathcal{L}\rangle\langle u\rangle$. The misconception has appeared in the literature that the solution of the averaged equation is the expected value of the solution of the random equation which we see is true if $\mathcal{L}$ is replaced by L, i.e., if R=0. At the next level of the hierarchy we write

$$Lu+Ru = g \tag{13}$$

$$u = L^{-1}g-L^{-1}Ru \tag{14}$$

$$\langle u\rangle = L^{-1}\langle g\rangle-L^{-1}\langle Ru\rangle \tag{15}$$

To evaluate $\langle Ru\rangle$ in (15) we return to (14) and multiply by R to get

$$Ru = RL^{-1}g-RL^{-1}Ru \tag{16}$$

averaging

$$\langle Ru\rangle = \langle R\rangle L^{-1}\langle g\rangle-\langle RL^{-1}Ru\rangle. \tag{17}$$

If $\langle R\rangle = 0$

$$\langle Ru\rangle = -\langle RL^{-1}Ru\rangle \tag{18}$$

Now just as we replaced $\mathcal{L}$ by L in order to separate $\langle \mathcal{L}u\rangle$ we now replace $RL^{-1}R$ by $\langle RL^{-1}R\rangle$ in the right hand side of (18). Then $\langle Ru\rangle = -\langle RL^{-1}R\rangle\langle u\rangle$.

Thus $$\langle u\rangle = L^{-1}\langle g\rangle+L^{-1}\langle RL^{-1}Ru\rangle$$

becomes

$$\langle u\rangle = L^{-1}\langle g\rangle+L^{-1}\langle RL^{-1}R\rangle\langle u\rangle$$

$\langle u\rangle$ is the solution of the averaged equation with the operator $RL^{-1}R$ replaced by its average. We have of course thrown away the random part of $RL^{-1}R$ just as we threw away the random part of $\mathcal{L}$ if we did this at the first stage of the hierarchy.

It clearly is not true that the averaged equation gives the expectation of the solution of the stochastic equation since at both levels of the hierarchy discussed, we have replaced a random operator by a deterministic one and therefore have lost something.

[In reference 22 (page 14) we see that the replacement of the operator by its average should not give the expectation $\langle y \rangle$ since all the terms involving correlation of α would have been lost.]

Many treatments of random equations in applications--particularly for waves propagating in a random medium, while differing in detail, are essentially variations of the hierarchy or perturbation methods and, further, involve many questionable and restrictive assumptions. The methods which follow remove many of the restrictions and applications based on the present discussion will be separately reported.

IV. Iterative Method:[22]

Suppose we consider the stochastic differential equation $\mathcal{L}y = x$ where

$$\mathcal{L} = \sum_{\nu=0}^{n} a_\nu(t) \frac{d^\nu}{dt^\nu}$$

and $x(t)$ and the $a_\nu(t)$ are random functions (or stochastic processes) whose statistics (statistical measures) are known. It is assumed that $x(t;\omega)$ is statistically independent of the $a_\nu(t;\omega)$. The objective is to determine the stochastic Green's

function for the desired statistical measure of y, expressed directly in terms of the statistics of the coefficients $a_\nu(t)$. Then the solution in a statistical sense will have been obtained and we will have avoided asking for a "solution" of the stochastic equation which is then to be used to find the statistics. The following is based on the earlier work of Samuels[1,2] and on a suggestion of Adomian[11] for an approximation method eliminating the unjustifiable assumptions of apriori spectral separation or the closure approximation. The method is elementary in principle since it is based on a Born approximation from scattering theory, or on successive approximations. Suppose that $\mathcal{L} = L+R$ where L is a deterministic operator and R is a random operator. Then we can write

$$Ly = x-Ry$$

To do this, assume, for example, that

$$a_\nu(t) = \langle a_\nu(t)\rangle + \alpha_\nu(t)$$

Thus

$$L = \sum_{\nu=o}^{n} \langle a_\nu(t)\rangle \frac{d^\nu}{dt^\nu}$$

and

$$R = \sum_{\nu=o}^{n} \alpha_\nu(t) \frac{d^\nu}{dt^\nu} .$$

Now writing $\mathcal{L}(t,\tau)$ as the ordinary Green's function* corresponding to the deterministic operator L, we write**

$$y = L^{-1}x - L^{-1}Ry$$

$$= \int_0^t \ell(t,\tau)x(\tau)d\tau - \int_0^t \ell(t,\tau)\sum_{\nu=0}^{n}\alpha_\nu(\tau)\frac{d^\nu y(\tau)}{d\tau^\nu}d\tau + \sum_{\nu=0}^{n} c_\nu \phi_\nu(t).$$

The ϕ_ν are a set of independent solutions of the homogeneous equation Ly = 0 and the c_ν are arbitrary constants. This can be written as

$$y(t) = F(t) - \int k(t,\tau)y(\tau)d\tau ***$$

where

$$F(t) = \int \ell(t,\tau)x(\tau)d\tau + \sum_{\nu=0}^{n} c_\nu \phi_\nu(t)$$

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*For simplicity we will take coefficients in L to be constants. The $\ell(t,\tau)$ is actually $\ell(t-\tau)$.

**The limits are a matter of choice. The upper limit can be written ∞ as well since the Green's function is zero if $t-\tau<0$ for a realizable system. The lower limit can be $-\infty$ or t_0 depending on the memory involved. Using t_0 implies that the earlier values are not significant.

***It is interesting that this expression is similar to that for a closed system with feedback. In the first term $\ell(t,\tau)$ is the impulse function of the forward loop, and in the second term, $k(t,\tau)$ is the feedback impulse function.

Integration by parts of the remaining integral or use of Green's formula in terms of the adjoint operator yields

$$k(t,\tau) = \sum_{\nu=0}^{n} (-1)^{\nu} \frac{d^{\nu}}{d\tau^{\nu}}[\alpha_{\nu}(\tau)\ell(t,\tau)]$$

The expectation of y is

$$\langle y(t)\rangle = \langle F(t)\rangle - \int \langle k(t,\tau)y(\tau)\rangle d\tau \qquad (19)$$

The difficulties arise in the last term in sttempting to separate the ensemble average or expectation into the product of averages over k and over y, since y is not statistically independent of the α involved in the k. (i.e. in R). It is desirable at this point to avoid the use of apriori spectral separation* or either explicit or implicit assumptions of small amount of randomness in $\mathcal{L}$.

If the random part were assumed to be small (e.g. let each coefficient $a_i = \gamma_i + \epsilon\alpha_i$ and use the same ϵ in a perturbation series for y) we can then achieve a separation. But without assuming a small random part, or slow variations, or white noise, we use in the place of a perturbation series a simple alternating series $y(t) = y_0 - y_1 + y_2 - \cdots$ i.e.,

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*The Born-Oppenheimer approximation is not always justified, (in considering electrons in a metal but neglecting electon-phonon interactions - see H. B. Callen in the Handbook of Physics.)

$$y(t) = \sum_{i=0}^{\infty} (-1)^i y_i(t)$$

This is suggested by the Born expansion and Born approximation used in a most important class of problems involving continuous eigenvalues--that of scattering theory. The state function for an elastic scattering problem satisfies the partial differential equation

$$(\nabla^2+k^2)\psi(r) = U(r)\psi(r)$$

or in operator form

$$F[\psi(r)] = g(r) \equiv U(r)\psi(r).$$

By the Green's function method we write

$$\psi(r) = F^{-1}[g(r)]$$

$$= \psi_0(r)+\int G(r,r')U(r')\psi(r')dr'$$

where $\psi_0(r)$ is the solution of the homogeneous equation $F[\psi(r)] = 0$ or $(\nabla^2+k^2)\psi_0 = 0$. The general problem is a difficult one involving the solution of the integral equation subject to the boundary conditions and the proper asymptotic form for ψ at large distances from the scattering center. One solves by iteration taking ψ_0 as a zeroth approximation and the n^{th} approximation from ψ_{n-1} in the integrand. If the scattering is small

the first approximation is sufficient. We will now proceed in the same manner taking $F(t)$ as y_0. Thus y_0 is the solution neglecting R in the original equation. For simplicity, write $\langle a_\nu(t)\rangle = a_\nu$. If the a_ν are constants $\ell(t,\tau) = \ell(t-\tau)$. Assume the $a_\nu(t)$ are zero mean random processes so $\langle \alpha_\nu(t)\rangle = 0$. If the α_ν are zero, the solution is just $F(t)$. We will neglect the homogeneous solution ϕ_ν for the purposes of this paper. We will not consider stability questions here but only stable solutions will be of interest. We have now

$$y_0 = F(t) = \int \ell(t,\tau)x(\tau)d\tau$$

and each successive y_i is given in terms of the preceding y_{i-1} thus

$$y_i = \int k(t,\tau)y_{i-1}(\tau)d\tau \qquad i = 1,2,\ldots$$

The convergence question is crucial to get more than a formal solution and we will return to it. However, the immediately interesting point is that the identification of $F(t)$ as the first term and the iterative procedure result in the desired separation of ensemble averages of the α's and y. This is because each term of y_i can be worked backward through y_{i-1}, y_{i-2}, to y_0, or $F(t)$, which depends only on $\ell(t,\tau)$ and on $x(t)$ but not on the α's. Thus the assumption of statistical independence of $\mathcal{L}$ and x is sufficient and one does not need to invoke separation

of y and the α's by assuming it apriori, or using a "closure" approximation. Further we get a reversed hierarchy where each term involves the one before it rather than the one after it. If the randomness is small, this gives the same result as perturbation theory*. If it is not, the number of terms can go to infinity. Before considering the matter further, let us see if the procedure appears useful for higher moments. We can, e.g., find the correlation function $R_y(t_1,t_2) = \langle y(t_1)\overset{*}{y}(t_2)\rangle$ by multiplying $y(t_1)$ and $\overset{*}{y}(t_2)$ and averaging. Thus

$$R_y(t_1,t_2) = \langle y(t_1)\overset{*}{y}(t_2)\rangle$$

$$= \langle [F(r_1)-\int k(t_1,\tau_1)y(\tau_1)d\tau_1][\overset{*}{F}(t_2)-\int \overset{*}{k}(t_2,\tau_2)\overset{*}{y}(\tau_2)d\tau_2]\rangle$$

$$= \langle F(t_1)\overset{*}{F}(t_2)\rangle - \int \langle k(t_1,\tau_1)\overset{*}{F}(t_2)y(\tau_1)\rangle d\tau_1$$

$$-\int \langle \overset{*}{k}(t_2,\tau_2)F(t_1)\overset{*}{y}(\tau_2)\rangle d\tau_2 + \int\int \langle k(t_1,\tau_1)\overset{*}{k}(t_2,\tau_2)y(\tau_1)\overset{*}{y}(\tau_2)\rangle d\tau_1 d\tau_2$$

Examination of this expression term by term will show that all the averages will separate in the manner we have discussed when the iteration series is substituted. This is because each time

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*i.e. the perturbation expansion accomplishes the same result but is only valid if the expansion parameter is small (small randomness).

a y_k is worked backword to y_{k-1}, another k and another integration appear. When we get to y_0, statistical independence of x and the α's allows us to separate the entire average into a product of two averages, one over the product of all the k's and the other involving only the x, which is identified as the auto-correlation of x. Examining the expression for $R_y(t_1,t_2)$ term by term shows the separation always occurs. Thus the first term is

$$\langle F(t_1)\overset{*}{F}(t_2)\rangle = \int d\tau \int d\sigma \ell(t_1,\tau)\ell(t_2,\sigma)\langle x(\tau)x(\sigma)\rangle$$

$$= \int d\tau \int d\sigma \ell(t_1,\tau)\ell(t_2,\sigma)R_x(t,\sigma)$$

The integrand of the second term R_y with the series substitution is

$$\langle k(t_1,\tau_1)\overset{*}{F}(t_2)y(\tau_1)\rangle = \langle k(t_1,\tau_1)\overset{*}{F}(t_2)[y_o(\tau_1)+\ldots]\rangle$$

The first term of this is zero since

$$\langle k(t_1,\tau_1)\overset{*}{F}(t_2)y_o(\tau_1)\rangle = \langle k(t_1,\tau_1)\overset{*}{F}(t_2)F(\tau_1)\rangle$$

$$= \langle k(t_1,\tau_1)\rangle\langle\overset{*}{F}(t_2)F(\tau_1)$$

$$= 0$$

since $\langle\alpha_\nu\rangle = 0$. The second term of the same expression (second term of R_y) is

$$\langle k(t_1,\tau_1)\overset{*}{F}(t_2)y_1(\tau_1)\rangle$$

$$= \langle k(t_1,\tau_1)\overset{*}{F}(t_2)\int k(\tau_1,\sigma)F(\sigma)d\sigma\rangle$$

$$= \int d\sigma \langle k(t_1,\tau_1)k(\tau_1,\sigma)\rangle\langle\overset{*}{F}(t_2)F(\sigma)\rangle$$

The third term of the same expression is

$$\langle k(t_2,\tau_1)\overset{*}{F}(t_2)y_2(\tau_1)\rangle$$

$$= \int\int \langle k(t_1,\tau_1)k(\tau_2,\beta)k(\beta,\gamma)\rangle\langle\overset{*}{F}(t_2)F(\gamma)\rangle d\beta d\gamma.$$

Thus it is clear that the separations always occur but successive terms of the kernel will go to higher and higher moments even though only the correlation of the output is required. Remembering that the terms we discussed and the higher moments not explicitly shown were for the second term of R_y, similar expressions can be written down by inspection for the third term of R_y, etc. Finally we consider the last term of the expression:

$$\int\int d\tau_1 d\tau_2 \langle k(t_1,\tau_1)\overset{*}{k}(t_2,\tau_2)y(\tau_1)\overset{*}{y}(\tau_2)\rangle$$

$$= \int\int d\tau_1 d\tau_2 \langle k(t_1,\tau_1)\overset{*}{k}(t_2,\tau_2)[y_o(\tau_1)-y_1(\tau_1)+y_2(\tau_1)- \ldots]$$

$$\cdot[\overset{*}{y}_o(\tau_2)-\overset{*}{y}_1(\tau_2)+\overset{*}{y}_2(\tau_2)- \ldots]\rangle$$

To shorten the writing, let $\int d\tau_1 \int d\tau_2$ be denoted by $\int d\tau$ and $k(t_1,\tau_1)\overset{*}{k}(t_2,\tau_2) = k_1 k_2$ which results in

$$\int d\tau \langle k_1 k_2 y_o(\tau_1) \overset{*}{y}_o(\tau_2)\rangle - \int d\tau \langle k_1 k_2 y_o(\tau_1) \overset{*}{y}_1(\tau_2)\rangle + \int d\tau \langle k_1 k_2 y_o(\tau_1) \overset{*}{y}_2(\tau_2)\rangle \ldots$$

$$- \int d\tau \langle k_1 k_2 y_1(\tau_1) \overset{*}{y}_o(\tau_2)\rangle + \int d\tau \langle k_1 k_2 y_1(\tau_1) \overset{*}{y}_1(\tau_2)\rangle - \ldots$$

all of which separate. Thus in the general expression

$$R_y(t_1, t_2) = \iint G_H(t_1, t_2, \tau_1, \tau_2) R_x(\tau_1, \tau_2) d\tau_1 d\tau_2$$

we can identify and calculate the stochastic Green's function G_H as a series of terms involving moments of α. Assumption of stationarity simplifies the expressions. If the α_ν are assumed to be gaussian, products involving an odd number of k's, like $\langle k_1 k_2 k_3 \rangle$ are zero. The even product $\langle k_1 k_2 k_3 k_4 \rangle$ involves three averages by pairs thus

$$\langle k_1 k_2 k_3 k_4 \rangle = \langle k_1 k_2 \rangle \langle k_3 k_4 \rangle + \langle k_2 k_3 \rangle \langle k_1 k_4 \rangle + \langle k_1 k_3 \rangle \langle k_2 k_4 \rangle$$

The term with 6 k's involves 15 averages by pairs, etc. However all of these are given in terms of the autocorrelation of the α which is known. Thus the output autocorrelation (statistical measure) is given in terms of the input autocorrelation and a stochastic Green's function depending only on the known autocorrelation of $\alpha_\nu(t)$.*,**

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*Of course, the input and the operator need not be time dependent. The Green's function can be the response for an impulse in time or frequency or position depending on the quantities of interest in a physical problem.

**If an explicit expression for G_H is written for the Gaussian case, the results should correlate with the stochastic Green's function for the first order stochastic differential equation (Ch. 4 of reference 5).

Further results for the iterative approach will deal with the case of a differential operator involving a single random coefficient.

Differential Operator With One Random Coefficient:

In the stochastic differential operator

$$\mathfrak{L} \cong \sum_{\nu=0}^{n} a_\nu(t) \frac{d^\nu}{dt^\nu}$$

let all the $a_\nu(t)$ be deterministic except one. We will suppose the one random coefficient is $a_0(t) = \langle a_0(t)\rangle + \alpha(t)$ and assume that the non-random coefficients are actually constants, so $\ell(t,\tau) = \ell(t-\tau)$. The random coefficient need not be a_0; it can be $a_i(t)$ where $i<n$ and the same treatment will apply since in $(L+R)y = x$

$$\begin{aligned} y &= L^{-1}x - L^{-1}Ry \\ &= \int_0^t \ell(t-\tau)x(\tau)d\tau - \int_0^t \ell(t-\tau)R_\tau[y(\tau)]d\tau \\ &= \int_0^t \ell(t-\tau)x(\tau)d\tau - \int_0^t R_\tau^\dagger[\ell(t-\tau)]y(\tau)d\tau * \\ &= \int_0^t \ell(t-\tau)x(\tau)d\tau - \int_0^t (-1)^i \frac{d^i}{d\tau^i}[\ell(t-\tau)\alpha_i(\tau)]y(\tau)d\tau \\ &= y_0(t) - \int_0^t k(t,\tau)y(\tau)d\tau \end{aligned}$$

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*By use of Green's formula. The bilinear concomitant is zero.

where only the form of k changes if $i \neq 0$. However we consider the case where a_0 is random as stated. The equation to be considered is therefore of the form

$$[L+\alpha(t)]y(t) = x(t)$$

where L is the deterministic operator and $\alpha(t)$ and $x(t)$ are stochastic processes. We write

$$y(t) = F(t)-\int_0^t k(t,\tau)y(\tau)d\tau$$

where

$$F(t) = \int_0^t \ell(t,\tau)x(\tau)d\tau \ ^*$$

$\ell(t,\tau)$ is the Green's function for L and $k(t,\tau) = \ell(t,\tau)\alpha(\tau)$. Let $y(t)$ be given by

$$y(t) = \sum_{i=0}^{\infty} (-1)^i y_i(t)$$

and identify $F(t)$ as y_0. Then

$$y(t) = F(t)-\int_0^t k(t,\tau)y_0(\tau)d\tau+\int_0^t k(t,\tau)y_1(\tau)d\tau-\int_0^t k(t,\tau)y_2(\tau)d\tau+....$$

$$y(t) = F(t)-\int_0^t k(t,\tau)F(\tau)d\tau+\int_0^t d\tau\int_0^\tau d\gamma\, k(t,\tau)k(\tau,\gamma)F(\gamma)$$

$$-\int_0^t d\tau\int_0^\tau d\gamma\int_0^\gamma d\sigma\, k(t,\tau)k(\tau,\gamma)k(\gamma,\sigma)F(\sigma)$$

$$+\int_0^t d\tau\int_0^\tau d\gamma\int_0^\gamma d\sigma\int_0^\sigma d\xi\, k(t,\tau)k(\tau,\gamma)k(\gamma,\sigma)k(\sigma,\xi)F(\xi)- ....$$

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*F(t) properly includes the solutions of the homogeneous equation which we have neglected for simplicity.

Taking the expectation

$$\langle y(t)\rangle = \langle F(t)\rangle - \int_0^t d\tau \langle k(t,\tau)\rangle \langle F(\tau)\rangle + \int_0^t d\tau \int_0^\tau d\gamma \langle k(t,\tau)k(t,\gamma)\rangle \langle F(\gamma)\rangle$$

$$- \int_0^t d\tau \int_0^\tau d\gamma \int_0^\gamma d\sigma \langle k(t,\tau)k(\tau,\gamma)k(\gamma,\sigma)\rangle \langle F(\sigma)\rangle$$

$$+ \int_0^t d\tau \int_0^\tau d\gamma \int_0^\gamma d\sigma \int_0^\sigma d\xi \langle k(t,\tau)k(\tau,\gamma)k(\gamma,\sigma)k(\sigma,\xi)\rangle \langle F(\xi)\rangle \ldots$$

where the ensemble averages all separate assuming only the statistical independence of α and x, since each y_i is expressed in terms of the preceding y_{i-1} until we get to y_0 which is F(t) and which depends only on x. We assume $F(t) \neq 0$ and have implicitly assumed the operator $\mathcal{L}$ can be written as the sum of L and R and the Green's function for L exists. Since the Central Limit Theorem is so often involved in physical applications, it is of interest now to investigate the result if $\alpha(t)$ is assumed to be Gaussian. Also, for simplicity, let both $\alpha(t)$ and x(t) be stationary.

The assumption that $\alpha(t)$ is a (zero-mean) Gaussian process causes the terms involving the expectation of an odd number of k's (or α's) in (19) vanish. Thus

$$\langle y(t)\rangle = \int_0^t d\tau\, \ell(t,\tau)\langle x(\tau)\rangle$$

$$+ \int_0^t d\tau \int_0^\tau d\gamma \int_0^\gamma d\delta\, \ell(t,\tau)\ell(\tau,\gamma)\ell(\gamma,\delta)\langle \alpha(\tau)\alpha(\gamma)\rangle\langle x(\delta)\rangle$$

$$+ \int_0^t d\tau \int_0^\tau d\gamma \int_0^\gamma d\delta \int_0^\delta d\sigma \int_0^\sigma d\eta\, \ell(t,\tau)\ell(\tau,\gamma)\ell(\gamma,\delta)\ell(\delta,\sigma)\ell(\sigma,\eta)$$

$$\cdot\langle \alpha(\tau)\alpha(\gamma)\alpha(\delta)\alpha(\sigma)\rangle\langle x(\eta)\rangle + \ldots . \qquad (20)$$

Let $\langle \alpha(\tau)\alpha(\gamma)\rangle = R_\alpha(\gamma-\tau)$ and $\langle \alpha(\tau)\alpha(\gamma)\alpha(\delta)\alpha(\sigma)\rangle = R_\alpha(\gamma-\tau)R_\alpha(\sigma-\delta)+R_\alpha(\delta-\tau)R_\sigma(\sigma-\gamma)+R_\alpha(\sigma-\tau)R_\alpha(\delta-\gamma)$ because of the Gaussian property. Now

$$\langle y(t)\rangle = \int_0^t d\tau\, \ell(t,\tau)\langle x(\tau)\rangle + \int_0^t d\tau \int_0^\tau d\gamma \int_0^\gamma d\delta\, \ell(t,\tau)\ell(\tau,\gamma)\ell(\gamma,\delta)R_\alpha(\gamma-\tau)\langle x(\delta)\rangle$$

$$+ \int_0^t d\tau \int_0^\tau d\gamma \int_0^\gamma d\delta \int_0^\delta d\sigma \int_0^\sigma d\eta\, \ell(t,\tau)\ell(\tau,\gamma)\ell(\gamma,\delta)\ell(\delta,\sigma)\ell(\sigma,\eta)\left[R_\alpha(\gamma-\tau)R_\alpha(\sigma-\delta)\right.$$

$$\left. + R_\alpha(\delta-\tau)R_\alpha(\sigma-\gamma) + R_\alpha(\sigma-\tau)R_\alpha(\delta-\gamma)\right]\langle x(\eta)\rangle + \ldots .$$

With some changes of variables,* we have

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*In the second term let $\gamma-\tau = \xi$, $d\gamma = d\xi$. In the third term, let $\gamma-\tau = \xi$, $d\gamma = d\xi$. Also $\sigma-\delta = \rho$. Now $\delta-\tau$ is unchanged. $\sigma-\gamma = \delta+\rho-\tau-\xi$, $\sigma-\tau = \delta+\rho-\tau$, and $\delta-\gamma = \delta-\tau-\xi$.

$$\langle y(t)\rangle = \int_0^t d\tau \ell(t,\tau)\langle x(\tau)\rangle + \int_0^t d\tau \int_{-\tau}^{0} d\xi \int_0^{\xi+\tau} d\delta \ell(t,\tau)\ell(\tau,\tau+\xi)\ell(\tau+\xi,\delta)R(\xi)\langle x(\delta)\rangle$$

$$+\int_0^t d\tau \int_{-\tau}^{0} d\xi \int_0^{\tau+\xi} d\delta \int_{-\delta}^{0} d\rho \int_0^{\delta+\rho} d\eta \ell(t,\tau)\ell(\tau,\tau+\xi)\ell(\tau+\xi,\delta)\ell(\delta,\delta+\rho)$$

$$\cdot\ell(\delta+\rho,\eta)[R(\xi)R(\rho)$$

$$+R(\delta-\tau)R(\delta+\rho-\tau-\xi)+R(\delta+\rho-\tau)R(\delta-\tau-\xi)\langle x(\eta)\rangle + \ldots$$

we note that the first term, $\int_0^t d\tau \ell(t,\tau)\langle x(\tau)\rangle = L^{-1}(t)\langle x(t)\rangle$ by definition of the Green's function $\ell(t,\tau)$.

In the second term, the similar integral

$$\int_0^{\xi+\tau} d\delta \ell(\tau+\xi,\delta)x(\delta)$$

becomes $L^{-1}(\tau+\xi)\langle x(\tau+\xi)\rangle$ or $L^{-1}(\tau+\xi)\langle x(t)\rangle$ since x is stationary. The second term becomes

$$\int_0^t d\tau \int_{-\tau}^{0} d\xi L^{-1}(\xi+\tau)\ell(t,\tau)\ell(\tau,\tau+\xi)R(\xi)\langle x(t)\rangle$$

$$= \int_0^t d\tau \ell(t,\tau)f(\tau) = L^{-1}(t)f(t)$$

if we let $f(\tau) = \int_{-\tau}^{0} d\xi L^{-1}(\xi+\tau)\ell(\tau,\tau+\xi)R(\xi)\langle x(t)\rangle$

Therefore the second term is

$$L^{-1}(t)\int_{-\tau}^{o} d\xi L^{-1}(\xi+t)\ell(t,t+\xi)R(\xi)\langle x(t)\rangle$$

$$= L^{-1}(t)\int_{o}^{t} d\tau L^{-1}(\tau)\ell(t,\tau)R(\tau-t)\langle x(t)\rangle$$

$$= L^{-1}(t)\int_{o}^{t} d\tau \ell(t,\tau)f(\tau)$$

$$= L^{-1}(t)L^{-1}(t)L^{-1}(t)R(o)\langle x(t)\rangle.$$

In the third term,

$$\int_{o}^{\delta+\rho} d\eta \ell(\delta+\rho,\eta)g(\eta) = L^{-1}(\delta+\rho)g(\delta+\rho)$$

so the entire term can be written

$$\int_{o}^{t} d\tau \int_{-\tau}^{o} d\xi \int_{o}^{\tau+\xi} d\delta \int_{-\delta}^{o} d\rho L^{-1}(\delta+\rho)\ell(t,\tau)\ell(\tau,\tau+\xi)\ell(\tau+\xi,\delta)\ell(\delta,\delta+\rho)[R(\xi)R(\rho)$$

$$+R(\delta-\tau)R(\delta+\rho-\tau-\xi)+R(\delta+\rho-\tau)R(\delta-\tau-\xi)]\langle x(t)\rangle.$$

Now the ρ integration can be written as

$$\int_{o}^{\delta} d\rho L^{-1}(\rho)\ell(t,\tau)\ell(\tau,\tau+\xi)\ell(\tau+\xi,\delta)\ell(\delta,\rho)[R(\xi)R(\rho-\delta)+R(\delta-\tau)R(\rho-\tau-\xi)$$

$$+R(\rho-\tau)R(\delta-\tau-\xi)]\langle x(t)\rangle *$$

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*Let $\rho' = \rho+\delta$, $d\rho' = d\rho$ and then drop primes.

$$= \int_0^{\delta} d\rho\, \ell(\delta,\rho) f(\rho) = L^{-1}(\delta) f(\delta)$$

$$= L^{-1}(\delta) L^{-1}(\delta) \ell(t,\tau) \ell(\tau,\tau+\xi) \ell(\tau+\xi,\delta) [R(\xi)R(o)$$

$$+R(\delta-\tau)R(\delta-\tau-\xi)+R(\delta-\tau)R(\delta-\tau-\xi)]\langle x(t)\rangle$$

The entire term is

$$\int_0^t d\tau \int_{-\tau}^{o} d\xi \int_0^{\tau+\xi} d\delta \text{ (above expression)}$$

$$= \int_0^t d\tau \int_{-\tau}^{o} d\xi \int_0^{\tau+\xi} d\delta \ell(\tau+\xi,\delta) f(\delta)$$

$$= \int_0^t d\tau \int_{-\tau}^{o} d\xi L^{-1}(\tau+\xi) L^{-1}(\tau+\xi) L^{-1}(\tau+\xi) \ell(t,\tau) \ell(\tau,\tau+\xi) [R(\xi)R(0)$$

$$+R(\xi)R(0)+R(\xi)R(0)]\langle x(t)\rangle$$

Let $\tau+\xi = \xi'$, $d\xi = d\xi'$ (and then drop the prime). The above becomes

$$\int_0^t d\tau \int_0^{\tau} d\xi L^{-1}(\xi) L^{-1}(\xi) L^{-1}(\xi) \ell(t,\tau) \ell(\tau,\xi) [R(\xi-\tau)R(0)$$

$$+R(\xi-\tau)R(0)+R(\xi-\tau)R(0)]\langle x(t)\rangle$$

$$= \int_0^t d\tau \ell(t,\tau) f(\tau) = L^{-1}(t) f(t)$$

$$= L^{-1}(t) \int_0^t d\xi L^{-1}(\xi) L^{-1}(\xi) L^{-1}(\xi) \ell(t,t) \ell(t,\xi) [R(\xi-t)R(0)$$

$$+R(\xi-t)R(0)+R(\xi-t)R(0)]\langle x(t)\rangle$$

$$= L^{-1}(t) \int_0^t d\xi \ell(t,\xi) f(\xi) = L^{-1}(t) L^{-1}(t) f(t)$$

$$= L^{-1}(t)L^{-1}(t)L^{-1}(t)L^{-1}(t)L^{-1}(t)[R(0)R(0)+R(0)R(0)+R(0)R(0)]\langle x(t)\rangle$$

$$= 3[L^{-1}(t)]^5R^2(0)\langle x(t)\rangle$$

Therefore

$$\langle y(t)\rangle = L^{-1}(t)\langle x(t)\rangle+[L^{-1}(t)]^3R(0)\langle x(t)\rangle+3[L^{-1}(t)]^5R^2(0)\langle x(t)\rangle+\ldots$$

$$= \int_0^t d\tau G(t,\tau)\langle x(\tau)\rangle$$

where the "stochastic Green's function" for this particular "statistical measure" i.e., the expectation or mean, is

$$G(t,\tau) = \ell(t,\tau)[1+[L^{-1}(\tau)]^2R_\alpha(0)+3[L^{-1}(\tau)]^4R_\alpha^2(0)+\ldots.]$$

Additional terms can be constructed without difficulty since in (20) it is easy to write the following terms by inspection and use the expression in products of the correlation function for the expectation of the product of six, eight, or more α's. Then the procedure followed above is repeated over two additional integrations each time. Actually, the general term* can be found and the problem is easily adapted to machine computation. We see the

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*e.g. P. 83 Laning and Battin "Random Processes in Automatic Control," McGraw Hill (1956), or p. 71, Miller, K.S., "Multidimensional Gaussian Processes," Wiley, 1964.

stochastic Green's function is an infinite series beginning with $\ell(t,\tau)$ with the next term involving a two-fold integration, the next a four-fold integration, etc. If α is small enough so $R_\alpha(o)$ is small enough $G(t,\tau)\to\ell(t,\tau)$. The result here corresponds to the perturbation theory result for the first order correction since the gaussian assumption doesn't affect the first correction term. Thus $\langle y\rangle = L^{-1}\langle x\rangle+L^{-1}\langle\alpha L^{-1}\alpha\rangle L^{-1}\langle x\rangle$. However we are not limited to a perturbation result and no non-valid assumptions are incorporated in the separation of $\langle Ly\rangle$ into $\langle L\rangle\langle y\rangle$. In operator notation one can write immediately

$$[L+\alpha]y = x$$

$$y = L^{-1}x-L^{-1}\alpha y$$

$$= L^{-1}x-L^{-1}\alpha[y_o-y_1+y_2- \ldots]$$

$$= L^{-1}x-L^{-1}\alpha L^{-1}x+L^{-1}\alpha L^{-1}\alpha L^{-1}x- \ldots.$$

$$\langle y\rangle = L^{-1}\langle x\rangle+L^{-1}\langle\alpha L^{-1}\alpha\rangle L^{-1}\langle x\rangle+ \ldots.$$

where as before it is assumed that expectation and integration can be interchanged. Thus the problem has become completely elementary. The result for the gaussian case* can now be re-written in the following convenient form:

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*If we assume stationarity this will give us the same result as before with the stochastic Green's function $G(t,\tau)$. Also, if L is a constant coefficient operator, $\ell(t,\tau)$ is $\ell(t-\tau)$ however we will leave it in the above form.

$$\langle y(t)\rangle = \langle F(t)\rangle + \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ell(t,\tau_1)\ell(\tau_1,\tau_2)\langle\alpha(\tau_1)\alpha(\tau_2)\rangle\langle F(\tau_2)\rangle$$

$$+\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \int_0^{\tau_3} d\tau_4 \ell(t_1,\tau_1)\ell(\tau_1,\tau_2)\ell(\tau_2,\tau_3)\ell(\tau_3,\tau_4)$$

$$\cdot\langle\alpha(\tau_1)\alpha(\tau_2)\alpha(\tau_3)\alpha(\tau_4)\rangle\langle F(\tau_4)\rangle + \cdots$$

The general term is the n-fold integral

$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n \ell(t,\tau_1)\ell(\tau_1,\tau_2)\cdots\ell(\tau_{n-1},\tau_n)\langle\alpha(\tau_1)\cdots\alpha(\tau_n)\rangle\langle F(\tau_n)\rangle$$

where n = 0, 2, 4, 6, ...

Writing $\alpha(\tau_i) = \alpha_i$ for convenience we note

$$\langle\alpha_1\alpha_2\rangle = R_{12}$$

$$\langle\alpha_1\alpha_2\alpha_3\alpha_4\rangle = R_{12}R_{34}+R_{13}R_{24}+R_{14}R_{23}$$

Or three terms involving two pairs each of six averages. Similarly $\langle\alpha_1\alpha_2\cdots\alpha_6\rangle$ gives five terms of three pairs each or fifteen averages. $\langle\alpha_1\cdots\alpha_8\rangle$ gives seven terms of four pairs each or twenty-eight averages. $\langle\alpha_1\cdots\alpha_n\rangle$ for even n gives n-1 terms of n/2 pairs each so that the number of averages involved is always $\binom{n}{2}$.

For the general term, $\langle\alpha_1\cdots\alpha_n\rangle$ can be represented by $e_{ijk\cdots}R_{ij}\cdot R_{kl}\cdots$ where the number of indices involved is n, and each index varies from 1 to n. The symbol e is a number defined to be zero if any of the indices i, j, k, l, ... are

repeated, and zero unless the second symbol of each pair is greater than the first, thus j>i, l>k, etc., so they are always in the natural order. The symbol e has the value of one-half if none of the above conditions is violated. Thus $e_{123\ldots} = 1/2$. The summation rule applies if any index on e and an index on R are the same.* This amounts then to a commutation pairwise and throwing out of terms which are really the same since the internal order of the α's or the order of the R's doesn't matter. Now

$$\langle y(t)\rangle = \langle F(t)\rangle + \ldots + \int_0^t d\tau_1 \ldots \int_0^{\tau_{n-1}} d\tau_n \ell(t_1,\tau_1)\ell(\tau_1,\tau_2)\ldots\ell(\tau_{n-1},\tau_n)$$

$$\cdot e_{ijk\ldots} R_{ij} R_{kl} \ldots \langle F(\tau_n)\rangle + \ldots$$

with $n = 2, 4, \ldots\infty$.

We will assume a bounded input x(t) in an interval [0,T] and a time-limited** stationary stochastic process $\alpha(t)$ by which we mean

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*Equivalently e=1 and we add the condition that $\ell=0$ if the R's are reversed from the natural ascending order. Thus $R_{12}R_{34}$ is non-zero but $R_{34}R_{12}$ is zero.

**A time-limited stationary stochastic process is physically realizable. (See A.V. Balakrishnan theorem in IEEE Trans. on Information Theory, January 1965).

R vanishes for $|t| \geq T$. If $\ell(t,\tau)$ is continuous in the interval, it must be bounded. Similarly, $\alpha(t)$ is assumed continuous with probability one so thatalmost all sample functions are bounded. The absolute value of the general term is less than $M_1 M_2^n M_2^{n+1} t^n/n!$ if M_1 is the bound for $x(t)$, M_2 is the bound for α and M_3 is the bound for ℓ.*

For finite observation times, the series converges since the series for e^t converges. By placing the bound on F(t) rather than x(t) we can combine M_2 and M_3 to write not only the general term but the n^{th} partial sum as less than $Me^{M't}$. Actually since only even values of n occur here rather than all, we have

$$M\left[\frac{e^{M't}+e^{-M't}}{2} - 1\right] = M[\cosh(M't)-1]$$

as the bound. Thus at t=0, the bound is zero and $\langle y(t)\rangle = 0$. At t=T, the bound is $M[\cosh(M'T)-1]$ where M is the bound for x(t) in [0,T].

y is not necessarily stationary and we observe that for the non-gaussian case, the odd terms do not then drop out but its easy to see convergence holds. Of course the mean value and the correlation do not specify a general random function x(t)

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*Integration of the n-fold integral $\int_0^t \cdots \int_0^{\tau_{n-1}} d\tau_1 \cdots d\tau_n$ gives $t^n/n!$

uniquely. But random functions encountered in physical problems are often assumed to be gaussian so all their finite dimensional distribution functions can be assumed to be multi-dimensional gaussian. Then the mean value and the correlation completely specify $\alpha(t)$ since they determine all distribution functions. Stationarity in the strict sense and wide sense do not need to be distinguished for gaussian processes and both the mean and the correlation are finite. The higher moments $\langle\alpha(\tau_1)\ldots\alpha(\tau_n)\rangle$ depend only on elements of the covariance matrix (and the mean which was assumed zero.) That the limit as $\tau\to\infty$ of $R(\tau)$ is zero is usually clear from physical considerations.

In the correlation case we proceed as before

$$y(t) = L^{-1}x(t)-L^{-1}\alpha(t)y(t)$$

$$= F(t)-L^{-1}\alpha(t)y(t)$$

$$R_y = \langle y(t_1)y(t_2)\rangle = \langle[F(t_1)-L^{-1}(t_1)\alpha(t_1)y(t_1)]$$

$$\cdot[F(t_2)-L^{-1}(t_2)\alpha(t_2)y(t_2)]$$

$$= \langle F(t_1)F(t_2)\rangle-L^{-1}(t_1)\langle\alpha(t_1)F(t_2)y(t_1)\rangle-L^{-1}(t_2)\langle\alpha(t_2)F(t_1)y(t_2)\rangle$$

$$+L^{-1}(t_1)L^{-1}(t_2)\langle\alpha(t_1)\alpha(t_2)y(t_1)y(t_2)\rangle$$

Let $y(t) = y_0(t)-y_1(t)+y_2(t)-\ldots$ where $y_0(t) = F(t) = L^{-1}x(t)$

$$R_y = \langle F(t_1)F(t_2)\rangle-L^{-1}(t_1)\langle\alpha(t_1)F(t_2)[y_0(t_1)-y_1(t_1)+y_2(t_1)\ldots]\rangle$$

$$-L^{-1}(t_2)\langle\alpha(t_2)F(t_1)[y_0(t_2)-y_1(t_2)+y_2(t_2)\ldots]\rangle$$

$$+L^{-1}(t_1)L^{-1}(t_2)\langle\alpha(t_1)\alpha(t_2)[y_0(t_1)-y_1(t_1)\ldots][y_0(t_2)-y_1(t_2)+\ldots]\rangle$$

$$= \langle F(t_1)F(t_2)\rangle - L^{-1}(t_1)\langle \alpha(t_1 F(t_2) y_0(t_1)\rangle + L^{-1}(t_1)$$

$$\cdot \langle \alpha(t_1)F(t_2)y_1(t_1)\rangle - L^{-1}(t_1)\langle \alpha(t_1)F(t_2)y_2(t_1)\rangle \ldots$$

$$- L^{-1}(t_2)\langle \alpha(t_2)F(t_1)y_0(t_2)\rangle + L^{-1}(t_2)\langle \alpha(t_2)F(t_1)y_1(t_2)\rangle$$

$$- L^{-1}(t_2)\langle \alpha(t_2)F(t_1)y_2(t_2)\rangle \ldots$$

$$+ L^{-1}(t_1)L^{-1}(t_2)\langle \alpha(t_1)\alpha(t_2)y_0(t_1)y_0(t_2)\rangle$$

$$- L^{-1}(t_1)L^{-1}(t_2)\langle \alpha(t_1)\alpha(t_2)[y_0(t_1)y_1(t_2)+y_0(t_2)y_1(t_1)+\ldots.]\rangle$$

$$+ L^{-1}(t_1)L^{-1}(t_2)\langle \alpha(t_1)\alpha(t_2)y_0(t_1)y_2(t_2)+\ldots.\rangle$$

The first term is $\langle F(t_1)F(t_2)\rangle = \langle L^{-1}x(t_1)L^{-1}x(t_2)\rangle$

$$= L^{-1}(t_1)L^{-1}(t_2)\langle x(t_1)x(t_2)\rangle$$

The second term is $-L^{-1}(t_1)\langle \alpha(t_1)L^{-1}(t_2)x(t_2)L^{-1}(t_1)x(t_1)\rangle = 0$, since $\langle \alpha \rangle$ separates out and is equal to zero. The next term is $L^{-1}(t_1)\langle \alpha(t_1)F(t_2)L^{-1}(t_1)\alpha(t_1)F(t_1)\rangle$ or $L^{-1}(t_1)\langle \alpha(t_1)L^{-1}(t_1)\cdot\alpha(t_1)\rangle\langle F(t_2)F(t_1)\rangle$ etc. We see the separations occur but the labor of calculating terms becomes repellent and the convergence question becomes still more complicated. However the solution can be carried out as before.

V. General Stochastic Operator

For stochastic differential operators which are completely random rather than a sum of a deterministic and a random operator, or for calculating higher moments it is essential to find other methods of proceeding. The most feasible way of attacking the general problem appears to be that suggested by Adomian in 1961.* The method attempts to find a stochastic Green's function for the desired statistical measure of the dependent variable, and is a generalization of Green's function theory. Consideration of the example* of a first order stochastic differential equation as well as the form of the solutions for the "random sampling" operator** suggests that the approach is feasible for the general case. The reasons for this are clear even though more work is needed. An n^{th} order differential equation can be given as a system of first order equations and a first order equation can be solved in the desired form.**

The general approach has led to determination of the power spectral density for a random process which is randomly sampled in time according to a probability law and to the correlation function of the first order stochastic differential

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*Chapter 4, reference 5.

**Ibid.

equation (also obtained by Tikhonov.[8]) In principle it should lead to complete statistical determination and can be generalized to the nonlinear (stochastic) equations called "reducible to linear" by Pugachev.

The preliminary work has been discussed previously,[5,11] and its extensions will be reported later.

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ON THE TREATMENT OF LUNAR SCATTERING AND RELATED PROBLEMS BY STOCHASTIC OPERATOR EQUATIONS (PART II)

Summary of Recent Work:

After a preliminary report discussing perturbation, hierarchy method, and the author's iterative method of solving stochastic equations, substantial work was done on an integral equation formulation of the stochastic differential equation solution by iteration relating it to the Neumann series solution in terms of iterated kernels so that the stochastic Green's function becomes an ensemble average of a resolvent kernel. Problems of convergence have been studied and of construction of the stochastic Green's function and the mathematical properties of the operators. The results were extended to partial differential equations and the consideration of wave propagation equations. It is now a useful method of solution of a large class of stochastic differential and integral equations without resort to truncation procedures or perturbations.

The significance of the truncation or closure approximation has been clearly understood and errors defined. Further the theory of stochastic operators and the work of Spaček and others on random operators has been unified into a single coherent theory for dealing with physical problems. It has been proved that the average solution of the stochastic equation is not generally the solution of the averaged equation.

Better understanding of the iterative procedure and the convergence questions has led to a useful theory for attacking

multiplicative interference in statistical communication theory and control applications, change of polarization of a wave as it propagates through a stochastic medium, etc. The "quasi-monochromatic" assumption for propagation in a random medium (assuming the solution of the wave equation is a sinusoid at a single frequency if the source is a sinusoid at a single frequency) has been avoided and one can find the spreading of the power spectrum of the wave function by the randomly time varying medium. The scalar wave equation for a randomly space and time-varying medium is converted into a Fredholm integral equation for the spectral representation of the scalar wave function and solved for single scattering or multiple scattering by the iterative procedure.

A very considerable amount of work has been done on a study of analytic random functions and their application to stochastic equations and on the theory of mappings on Banach spaces. This has led to useful theorems stating conditions for the stationarity or nonstationarity of solutions of stochastic differential equations and a technique for finding the power series coefficients of solutions to stochastic differential equations.

These recent results are not yet available for publication.