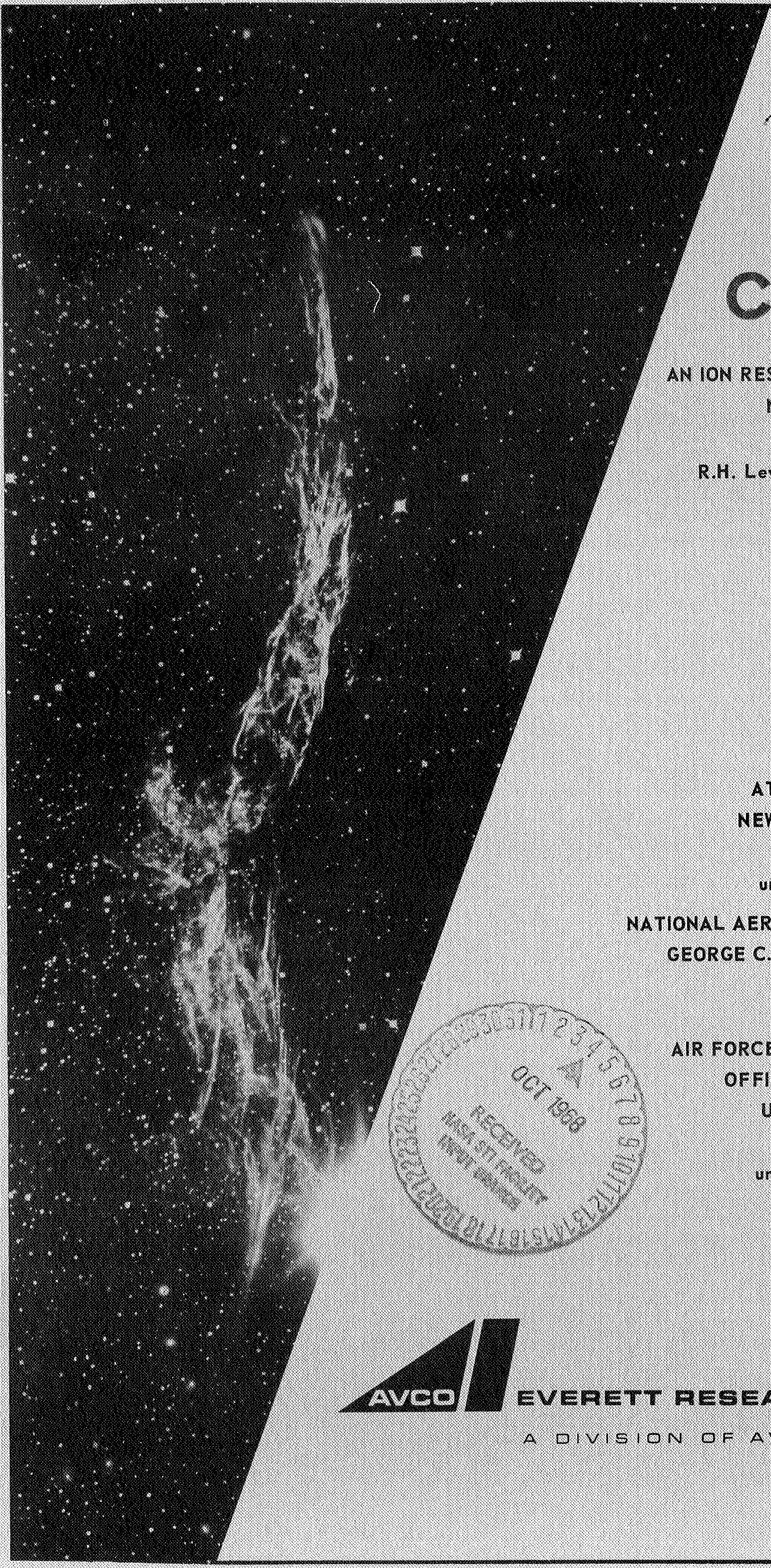

AN ION RESONANCE INSTABILITY IN GROSSLY NON-NEUTRAL PLASMAS

R.H. Levy, J.D. Daugherty and O. Buneman

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ABSTRACT

The addition of a certain fraction of ions to an otherwise stable cloud of electrons can in certain circumstances result in an instability. The case of a low density ($q = \omega_p^2/\omega_c^2 \ll 1$) electron cloud in which the electron motions can be described by $\underset{\sim}{E} + \underset{\sim}{v}_e \times \underset{\sim}{B} = 0$ is considered in detail. The condition for onset of the instability is that the unperturbed ion orbits (which characteristically traverse the entire electron cloud) should involve frequencies close to that of the diocotron wave that can propagate on the electron cloud. This condition is equivalent to having the ratio $Zm_e/m_i q$ of the order of unity and is independent of the ion density. When instability is present, the growth rate is of the order of the ion plasma frequency.

TABLE OF CONTENTS

I. INTRODUCTION

There are several devices presently being studied in which it is proposed to contain a grossly non-neutral plasma in a magnetic field. In particular, the containment of a relatively small number of ions in a background of electrons is attractive in various applications[1,2,3] because the assemblage of charged particles as a whole has a charge to mass ratio much larger than the value appropriate to the ions alone. As a result, it should in principle be possible to control such ensembles with magnetic fields far weaker than would be required for control of the ions alone. A basic question relevant to each of these applications concerns the number of ions that can be contained by a given electron cloud. As the ratio of ion to electron charge density is generally small, it is natural to begin by examining the containment (equilibrium, stability, etc.) of the electron cloud by itself. When this is reasonably well understood, one can imagine the ions to be slowly added to the electron cloud and seek to define the point at which an instability can occur.

A very basic difference between neutral and non-neutral plasmas is the presence of a strong (zeroth order) electric field, which is related to the charge imbalance by Gauss' law. Since the gradient of the electric field is non-vanishing, it is evident that there is no such thing as an infinite homogeneous uniform non-neutral plasma. (An analagous situation arises in the theory of gravitation.) Another important difference distinguishing neutral and non-neutral plasmas is the implication of Earnshaw's theorem (for the non-neutral case) that there can be no static equilibrium. The equilibria of electron clouds are therefore

necessarily dynamic, and will involve certain characteristic frequencies (e. g., of rotation) of the electrons. The frequencies of importance can be different in different devices, so that in one case[3] the gyrofrequency is basic to the equilibrium while in another case[1,2] a frequency of the order of E/BL may be of greater importance. (E, B and L represent characteristic electric fields, magnetic fields, and dimensions.) In the latter case the mass of the electron is unimportant.

Let us now imagine a single ion to be added to an electron cloud contained in some way in a magnetic field. The ion is trapped by the potential well of the electrons, and is also influenced by the magnetic field. If the ion has charge Ze and mass m_i, and if n_e is a characteristic electron density, the ion motion will be characterized by two frequencies, namely the gyrofrequency ZeB/m_i, and a frequency associated with oscillation in the potential well which is roughly $(Z n_e e^2/2\epsilon_o m_i)^{1/2}$ These two frequencies are both independent of the number of ions present, and decrease with increasing values of m_i/Z. If the ion is very massive (m_i/Z large), the ion motion will involve frequencies far below those characterizing the electron cloud, and the two systems can be thought of as essentially decoupled. But as m_i/Z gets smaller, we will eventually arrive at a resonance, implying the possibility of an instability. Such an instability could clearly occur even if there is only a single ion, but in this case the growth rate would be small. Larger numbers of ions could be expected to have a fairly small effect on the stability limits of the parameters of the electron cloud, but the growth rate will depend in some way on the ion density. These general statements agree with the analytical results of the paper in suggesting that,

for non-neutral plasmas, the critical number of ions that can be stably contained depends more strongly on ratios independent of the ion density than on ratios in which the ion density figures importantly.

The outline of the paper is as follows: the basic model is described in Section II, along with the character of the equilibrium, and the zeroth order electron and ion orbits. In this section, all the important frequencies are defined. In Section III the drift approximation is discussed. Sections IV and V respectively treat the equilibrium ion distribution, the perturbed ion density in any mode number. In Section VI these results are applied to the fundamental mode of oscillation. Explicit dispersion relations for two special cases are derived and discussed in Sections VII and VIII. The remaining Sections are devoted to a general discussion of the results.

II. THE MODEL

The model used in this paper is intended to be representative of some [1,2] of the non-neutral plasma devices currently of interest. Its basic features are sketched in Fig. 1. It consists of an infinite perfectly conducting circular cylinder, of radius a, having a uniform magnetic field B along its axis, and containing a non-neutral cloud of electrons and ions. We assume that the effect of currents in the plasma on the applied magnetic field is negligible (quasi-static approximation), and that all quantities are at all times independent of the axial (z) coordinate.

A steady state equilibrium is assumed in which all quantities are independent of the azimuthal (θ) coordinate. Thus, for the zero-order quantities (distinguished throughout by the subscript o):

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi_o}{dr}\right) = -\frac{1}{r}\frac{d}{dr}\left(r E_{ro}\right) = \frac{e}{\epsilon_o}\left(n_{eo} - Z n_{io}\right)$$

For simplicity only one species of ion is considered, and all the ions are Z-fold ionized. A useful ratio is

$$\alpha = Z n_{io}/n_{eo} \qquad (2.1)$$

representing the fractional neutralization of the electron cloud. All the zero-order quantities listed so far are functions only of the radial (r) coordinate. The motions of each charged particle will conserve both energy and canonical angular momentum; it is assumed that there is no motion in the z-direction.

In the devices to which this model is supposed to apply, the electrons

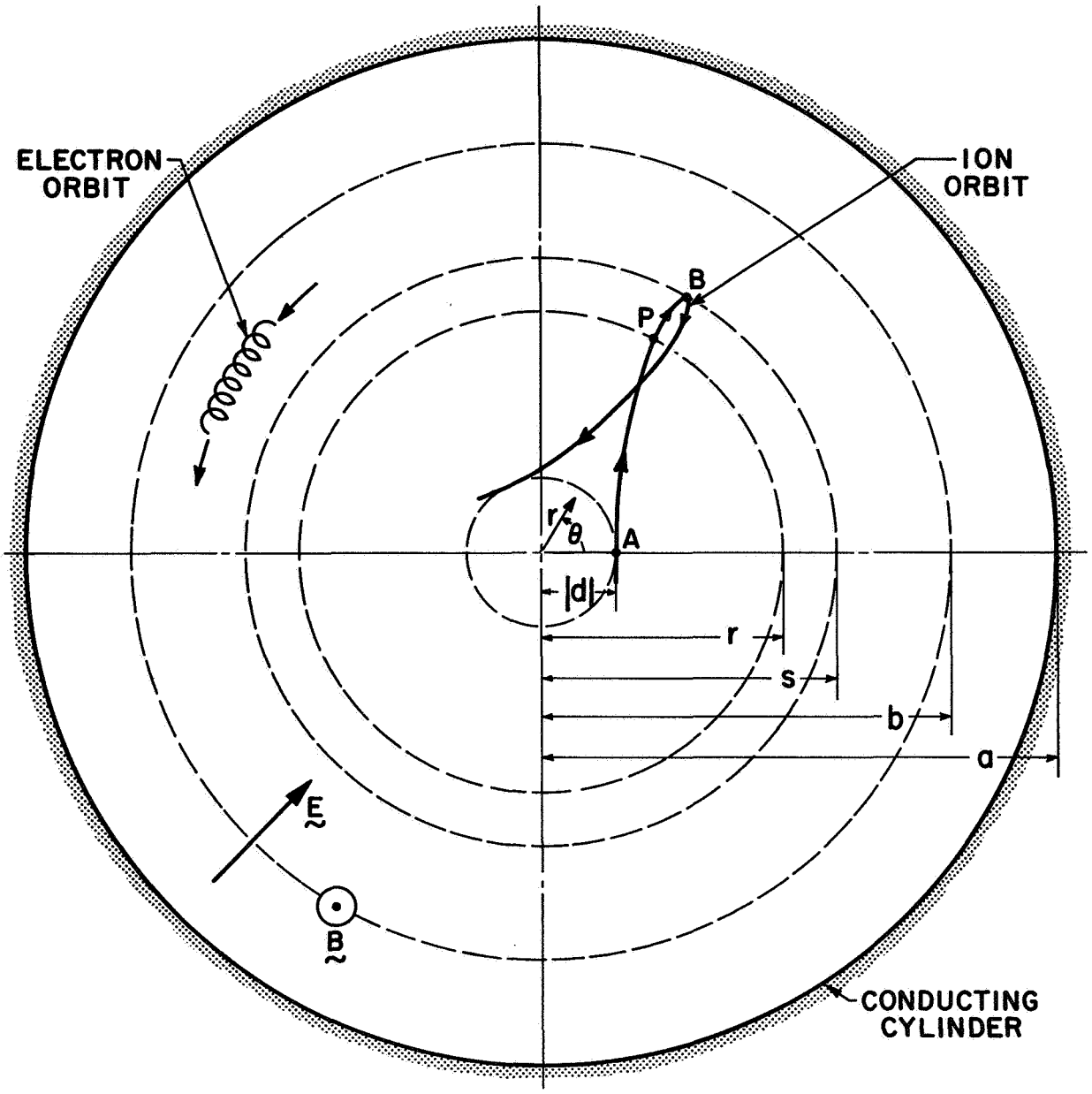

Fig. 1 Basic features of the model studied, showing typical electron and ion orbits. The electrons and ions occupy the region $0 \leq r \leq b$; the region $b \leq r \leq a$ is empty. The orbit of each ion is fixed by the numbers d and s corresponding to the minimum and maximum values of r . The size of the loops in the electron orbit is exaggerated.

satisfy the condition that

$$q = \omega_p^2 / \omega_c^2 = n_{eo} m_e / \epsilon_o B^2 << 1 \qquad (2.2)$$

Furthermore, the kinetic energy of each electron is characteristically much less than its potential energy. It follows that the electrons can be considered as "cold", their motions being governed by the equation $\underset{\sim}{E} + \underset{\sim}{v}_e \times \underset{\sim}{B} = 0$. The electrons being cold, there is no need to introduce directly an electronic distribution function. In equilibrium, $\underset{\sim}{E}_0$ is radial and $\underset{\sim}{B}$ is axial, so that $\underset{\sim}{v}_{eo}$ is azimuthal, and $v_{\theta\, eo} = - E_{ro}/B$. The electrons at radius r therefore rotate around the z-axis with the angular velocity $- E_{ro}/rB$. As indicated in the Introduction, this quantity is independent of the electron mass. A typical electron orbit having a small amount of thermal energy in addition to its drift energy is shown in Fig. 1.

We now fix the equilibrium radial distributions of the electrons and the ions in a very special manner. We choose them in such a way that:

$$\left.\begin{aligned} n_{eo}(r) &= n_{eo} = \text{const.} \\ n_{io}(r) &= n_{io} = \text{const.} \end{aligned}\right\} \ (0 \leq r \leq b)$$
$$n_{eo}(r) = n_{io}(r) = 0 \quad (b \leq r \leq a) \qquad (2.3)$$

A narrow radial range will be allowed near r = b for continuous transition of the density.

The analysis of this paper could be carried through using somewhat less special electron and ion densities. This point is discussed in Section IX. The choice (2.3) of the electron and ion densities makes

α a constant (for $r \leq b$) and determines the electric field as:

$$E_{ro} = \begin{cases} -n_{eo}\, e r\, (1-\alpha)/2\epsilon_o & (0 \leq r \leq b) \\ -n_{eo}\, e b^2\, (1-\alpha)/2 r \epsilon_o & (b \leq r \leq a) \end{cases}$$

and the potential as

$$\phi_o(r) = n_{eo}\, e\, r^2\, (1-\alpha)/4\epsilon_o \quad (0 \leq r \leq b) \tag{2.4a}$$

$$\phi_o(b)\ \left[1 + \ell n\left(r^2/b^2\right)\right] \quad (b \leq r \leq a) \tag{2.4b}$$

The electrons rotate about the axis as a solid body with the unique angular velocity

$$\omega_e = n_{eo}\, e\, (1-\alpha)/2\epsilon_o B \tag{2.5}$$

The ion distribution fuction is (for the present) not constrained beyond the condition (2.3).

This choice of electron and ion density profiles will be seen to simplify the instability calculation to the point where useful results can be obtained, but it also carries with it certain rather far reaching implications. These follow from the special quality of the ion orbits in a parabolic potential well. We shall first derive these orbits, and then comment on the implications. We can define two significant frequencies for the ions, as follows:

$$\left.\begin{aligned} \Omega_E &= \left[Z n_{eo}\, e^2\, (1-\alpha)/2\epsilon_o m_i\right]^{1/2} \\ \Omega_c &= Z e B/m_i = \Omega_E^2/\omega_e \end{aligned}\right\} \tag{2.6}$$

The first of these is the frequency with which an ion would oscillate in the electric potential well defined by (2.4a) if there were no magnetic field. The second, Ω_c, is just the ion gyrofrequency. The combination of the electric and magnetic fields results in a frequency $(\Omega_E^2 + \Omega_c^2/4)^{1/2}$ for the ion motion. An important ratio is Ω_c^2/Ω_E^2 which may be related to q by:

$$\lambda \equiv \Omega_c^2/\Omega_E^2 = 2Zm_e/m_i\, q\,(1 - \alpha) \qquad (2.7)$$

where q is defined in (2.2). Although the electron mass appears explicitly in this expression, q is proportional to m_e so that λ is independent of it. The form (2.7) is convenient, however, because it compares the electron-ion mass ratio directly with q. In particular, it is helpful to see the effect of varying the ion mass, all other quantities being held fixed. The various frequencies introduced so far satisfy the proportionality relations:

$$\Omega_c : \Omega_E : \omega_e : \omega_p : \omega_c = \lambda : \lambda^{1/2} : 1 : 2\,(1-\alpha)^{-1} q^{-1/2} : 2\,(1-\alpha)^{-1} q^{-1} \qquad (2.8)$$

Since $q \ll 1$, it is clear that for massive ions ($\lambda \ll 1$), the ion frequencies are far below all the frequencies associated with the electrons, amongst which the lowest are of the order ω_e, a characteristic frequency for "diocotron waves".

These waves are described in references 4, 5, 6. In the drift approximation ($\underset{\sim}{v}_e = \underset{\sim}{E} \times \underset{\sim}{B}/B^2$) they are pure surface waves. Purely azimuthal modes have the lowest frequencies, given by

$$\omega/\omega_e = \ell - 1 + (b/a)^{2\ell} \qquad (2.9)$$

where ℓ is the azimuthal mode number.

For very massive ions no electron-ion resonance is possible. As progressively lighter ions are considered, λ increases until, when λ is of the order of unity, a resonance can occur between the ion motions and the electron rotation frequency ω_e. It is this resonance which we shall examine in detail.

The motions of the ions can be understood most readily by viewing them in the Larmor frame of reference which rotates with angular velocity $\Omega_c/2$ and in which the Lorentz force is just cancelled by the Coriolis force. A centrifugal potential energy $m(\Omega_c/2)^2 r^2/2$, however, must be added to the electrostatic potential energy $m\Omega_E^2 r^2/2$ (see eq. 2.6). The result is an effective parabolic potential well $mr^2\Omega^2/2$ with

$$\Omega = (\Omega_E^2 + \Omega_c^2/4)^{1/2} \tag{2.10}$$

the frequency of oscillation of each Cartesian co-ordinate in this frame. The orbits are ellipses, and with the appropriate choice of axes:

$$x = s \cos \Omega t \qquad y = d \sin \Omega t \tag{2.11}$$

We shall assume s to be the major semi-axis and take s always positive while d, never larger than s in magnitude, can be positive or negative so as to describe clockwise and anticlockwise orbits. Figure 1 illustrates an elliptical orbit transformed back to the lab frame, ranging between $r = d$ and $r = s$. Unlike the electrons, the ions are not prevented by the magnetic field from crossing a substantial part of the potential well.

A critical feature of these ion orbits is the fact that the frequencies involved are independent of the amplitude of the motion. It follows that

after a time $2\pi/\Omega$ the ions all have the same relative positions as they had initially, although in the lab frame the picture as a whole has rotated through the angle $-\pi\Omega_c/\Omega$. A vitally important consequence of this effect is that no Landau damping can occur as a result of synchronism of some of the ions with any wave. This result has been noted in other circumstances,[7] and represents a generalization of the well-known fact that Landau damping does not occur in a cold plasma. Of course, the ions are not cold in the sense that all the ions at any place share a common velocity. It is therefore necessary to introduce a microscopic distribution function to describe the ions. But the mutual synchronism of the ions suggests that there is a sense in which the meaning of the word "cold" can be extended to cover our case. In this sense, our analysis aims at results analagous to cold plasma theory. As in the cold plasma case, our results will definitely need modifying for application to warm cases - "warm" in this context meaning a non-parabolic potential well. Although no analysis of warm cases is presented, a discussion of some of the possibilities is offered in Section IX. Theoretically, the situation is precisely analagous to the use of plasma models with discrete structure, as treated, for instance, by Stix[8] and Dawson[9]. Such models cannot exhibit Landau damping, but can and do exhibit gross instabilities associated with resonances of one kind or another. The best known of these is the two-stream instability, and it therefore comes as no surprise that our dispersion relations (Sections VII and VIII) are very similar in structure to the dispersion relation describing the two stream instability.

III. THE DRIFT APPROXIMATION

For the electrons, it is adequate to describe the motion by the drift approximation:

$$\left.\begin{aligned} v_r &= E_\theta / B \quad \text{(a perturbation only)} \\ v_\theta &= -E_r/B \quad (= r\omega_e \text{ for unperturbed conditions}) \end{aligned}\right\} \tag{3.1}$$

This is the basis of some previous studies of the "diocotron" modes (references 4, 5, 6). For the ions, (3.1) is a poor approximation, but we shall find it a useful guide to what actually happens when a full distribution function analysis is carried out.

Since $\underset{\sim}{E}$ is curl-free and the velocity field is the $\underset{\sim}{E}/B$-field turned through a right angle, the velocity is divergence-free and the continuity condition states that the particle density n is unchanged along the flow lines:

$$\frac{\partial n_1}{\partial t} + \frac{E_\theta}{B}\frac{dn_o}{dr} + \omega_e \frac{\partial n_1}{\partial \theta} = 0 \tag{3.2}$$

The subscripts "0" (unperturbed values) and "1" (perturbations) may be placed as shown since $\partial n/\partial t$ and $\partial n/\partial\theta$ are only perturbations. With $\exp(i\ell\theta - i\omega t)$ describing the ℓ^{th} order mode, (3.2) leads to

$$n_1 = \frac{iE_\theta}{B(\ell\omega_e - \omega)}\frac{dn_o}{dr} \tag{3.3}$$

a formula which is almost literally confirmed by the elaborate distribution function analysis to be carried out in Sections IV - VI - at least for the lowest order mode, $\ell = 1$, and the case where the steady-

state potential well is effectively parabolic.

From (3. 3) one sees that:

(i) over the interior range where n_o is assumed constant (see eq. 2. 3) one has no density perturbation and the potential will be Laplacian, proportional to r^{ℓ}.

(ii) in the "skin" where n_o drops to zero one gets a surface charge density wave

$$- \frac{i E_\theta \rho_o}{B (\ell \omega_e - \omega)} \tag{3.4}$$

Each species, electrons and ions, will create a surface charge layer proportional to its unperturbed density in accordance with (3. 4), assuming the drift approximation to be valid for both. In that case (3. 4) can also be taken to give the net surface charge density due to both species, provided one interprets ρ_o as the net charge density $- e n_{eo} + Z e n_{io}$, or $(1 - \alpha)\, \rho_{eo}$, identifiable as $- 2 \epsilon_o B \omega_e$ in accordance with eq. (2. 5).

On this basis one can now proceed to match the ratio iE_r/E_θ across the boundary, using $\phi_1 \propto r^{\ell} - (a^2/r)^{\ell}$ outside. The ratio jumps from 1 to $(b^{2\ell} + a^{2\ell})/(b^{2\ell} - a^{2\ell})$, and the difference must be equated to $i/\epsilon_o E_\theta$ times the surface charge density. This leads to the "diocotron" dispersion formula (2. 9) which had previously been derived for electrons only.

The ions seem to have no influence on dispersion. This is not surprising since in the drift approximation the particle mass remains irrelevant and the ions are just "charges", like the electrons, scaling all charge densities down to the factor $1 - \alpha$. To get significant ion

effects, one must improve on the drift approximation, at least for ion dynamics. If one goes to the opposite extreme of the ions being infinitely heavy rather than so light as to make the drift approximation valid, then the surface wave is only that due to the electrons - meaning that it is not partially neutralized. It is, in fact, a factor $1/(1-\alpha)$ stronger than in the estimate above. One then finds, instead of (2.9), the dispersion formula:

$$\omega/\omega_e = \ell - (1-(b/a)^{2\ell})/(1-\alpha) \qquad (3.5)$$

IV. EQUILIBRIUM ION DISTRIBUTION

In calculating ion dynamics, ion orbits, the ion distribution function and its perturbations, we follow the principles laid down in references 10, 11, 12, where cylindrically symmetrical configurations and their stability analyses are discussed.

One classifies particles by their canonical momenta. One of these is the total energy W. Two others, the canonical axial momemtum and the canonical angular momentum are, like the energy, constants of motion in the steady state. We have already pointed out that the most competitive electron modes are those with no variation along the axis and hence we restrict all further studies to axial uniformity. The axial co-ordinate remains completely ignorable and axial momenta remain irrelevant. In any distribution function to be recorded, we shall imply that any axial momenta have been integrated over.

Regarding the canonical angular momentum, a simplifying feature is the constancy and uniformity of the magnetic field. For one species (namely the ions), one can transform away this constant magnetic field by doing the entire analysis in the Larmor frame in which the canonical angular momentum becomes the ordinary angular momentum. Let us now consider the manifold of all possible ion orbits of the type (2.11). The total energy W,

$$W = \frac{1}{2} m (v_r^2 + v_\theta^2) + Ze\Phi' \qquad (4.1)$$

contains, in its effective electrostatic potential energy $Ze\Phi'$, the Larmor centrifugal part $mr^2\Omega_{ci}^2/8$, bringing the total potential energy to $mr^2\Omega^2/2$

for the steady state. Substitution then gives:

$$W = \frac{1}{2} m\Omega^2 (s^2 + d^2) \tag{4.2}$$

The angular momentum L is readily calculated to be

$$L \equiv mrv_\theta = m\Omega sd \tag{4.3}$$

Throughout this and the two subsequent sections, an unsubscripted "m" will denote the species under study - primarily the ions in our case. However, putting Z = -1, and going to the appropriate Larmor frame, makes the analysis immediately applicable to electrons also.

One can use s and d as alternative variables to W and L for classifying particles in a distribution. The most common variables used in a distribution function would be v_r and v_θ or v_x and v_y. The Jacobian for passing from these to W and L is:

$$\frac{\partial (W, L)}{\partial (v_r, v_\theta)} = m^2 r v_r = m [2mr^2 (W - Ze\Phi') - L^2]^{1/2} \tag{4.4}$$

In cylindrical polars, with θ now measured in the Larmor frame, the orbits (2.11) are given by:

$$r^2 = \frac{1}{2} (s^2 + d^2) + \frac{1}{2} (s^2 - d^2) \cos 2\Omega t$$

$$= \frac{W + (W^2 - \Omega^2 L^2)^{1/2} \cos 2\Omega t}{m\Omega^2} \tag{4.5}$$

and $\tan\theta = (d/s)\tan\Omega t$, or

$$\cos 2\theta = \frac{(W^2 - \Omega^2 L^2)^{1/2} + W\cos 2\Omega t}{W + (W^2 - \Omega^2 L^2)^{1/2}\cos 2\Omega t} = \frac{W - L^2/mr^2}{(W^2 - \Omega^2 L^2)^{1/2}} \tag{4.6}$$

The steady-state distribution function, $f_o(r, v_r, v_\theta)$, should depend only upon the constants of motion, W and L, or s and d:

$$f_o(r, v_r, v_\theta) = g_o(W, L) \tag{4.7}$$

which leads to an ion density:

$$n_{io}(r) = 2\iint \frac{g_o(W, L)\, dW dL}{m(2mr^2 W - m^2 r^4 \Omega^2 - L^2)^{1/2}} \tag{4.8}$$

the range of integration being the parabolic domain

$$2W \geq mr^2\Omega^2 + L^2/mr^2 \tag{4.9}$$

which insures a real v_r.

The factor 2 in (4.8) is due to the ambiguity of sign of v_r after specifying only W and v_θ, i.e. whether g_o counts ingoing or outgoing particles. In the steady state these must be equally plentiful, but under perturbed conditions we shall have to distinguish between two functions, g^+ and g^-.

The function $g_o(W, L)$ could be determined from a knowledge of the ionization process. One would have comparatively low initial kinetic energies in the lab frame and hence $W \approx \frac{1}{2} mr_o^2 [\Omega^2 + (\Omega_c/2)^2]$ and $L \approx mr_o^2 \Omega_c/2$ in the Larmor frame where r_o is the radius of creation, with small spreads about these values. However, our analysis is

restricted by condition 2.3, namely constant n_{io}. To comply with this condition, we ought to determine g_o (W, L) by inverting the integral equation (4.8). Such an inversion is possible in principle, but it is not unique since the kernel only depends on the square of L and makes no distinction between clockwise and anticlockwise orbits. One can distribute a given total number of particles arbitrarily among these.

Fortunately, in most of the analysis that follows it will be unnecessary to specify precisely the solutions g_o (W, L) of the integral equation which results from the condition (2.3) on the total density. But in Section VIII, we make use of an especially simple form for g_o. If we express W and L in terms of s and d by means of (4.2, 4.3) and and choose

$$g_o(W, L) = \frac{n_{io}}{\pi\Omega^2} \lim_{\epsilon \to 0} \frac{\delta(s - |d| - \epsilon)}{\epsilon} \qquad s \leq b$$

$$= 0 \qquad s \geq b \qquad (4.10)$$

all ions have effectively $s = |d|$, that is, they are moving in circles around the origin. The ion cloud has constant density n_{io} for $0 \leq r \leq b$; for $b \leq r \leq a$ there are no ions. The ions are also divided into two classes; and each class rotates as a solid body. The angular velocities of the clockwise and anticlockwise classes are $-\frac{1}{2}\Omega_c \pm \Omega$ in the lab. frame. For the plus sign, this angular speed is in the sense which makes the Lorentz force $Ze\underset{\sim}{v} \times \underset{\sim}{B}$ point radially outward, opposing the electric force. For the minus sign, the vector $\underset{\sim}{v} \times \underset{\sim}{B}$ points radially inward, adding to the electric force, and requiring a larger angular velocity to provide the necessary outward centrifugal force. Any division of the ions

between these two classes may be chosen.

It may be seen that if we had retained the full equations of motion for the electrons, as well as for the ions, we would have developed essentially the same formalism for each case allowing only for the differences in charge and mass. Since in our model the electrons are actually moving in circles, the distribution function (4.10) would be appropriate for these as for the ions, with all the electrons in the class for which the Lorentz force $-e\,(\underset{\sim}{v} \times \underset{\sim}{B})$ opposes the electric force $-e\underset{\sim}{E}$. The angular speed for this class is

$$\omega_c/2 - [\omega_c^2/4 - (1-\alpha)\,\omega_p^2/2]^{1/2} \tag{4.11}$$

But we have assumed (2.2) that $\omega_p^2 \ll \omega_c^2$, so that this angular velocity is approximately

$$(1-\alpha)\,\omega_p^2/2\omega_c = n_{eo}\,(1-\alpha)/2\epsilon_o B = \omega_e \tag{4.12}$$

as stated above. It will appear later that all our results are susceptible of generalization by allowing the electrons to have a velocity distribution restricted by just those conditions which are imposed on the ions. However, since it adds little to our understanding of the situation, this permissible generalization is not carried through our analysis.

The other angular speed available to the electrons corresponds to choosing the + sign for the square root in (4.11), and, for $q \ll 1$, is about ω_c. These two angular speeds have been described as the slow and fast solutions.[13]

Since both in the constant density requirement (2.3) and in any comparison of the full theory with the drift approximation (3.3) the radial

density gradient is important, we develop here an expression for dn_{io}/dr in terms of the distribution function g_o from (4.8). To remove the variable r from the boundary condition 4.9, we introduce dimensionless energy and momentum variables:

$$\mu \equiv \frac{W}{mr^2\Omega^2/2} \quad , \quad \xi = \frac{L}{mr^2\Omega} \tag{4.13}$$

giving n_{io} in the form:

$$n_{io}(r) = r^2\Omega^2 \iint g_o(\mu mr^2\Omega^2/2,\ \xi\, mr^2\Omega)\, d\xi\, d\mu/(\mu - 1 - \xi^2)^{1/2}, \tag{4.14}$$

$$\mu > 1 + \xi^2$$

now readily differentiated to give:

$$\frac{dn_{io}}{dr} = r\Omega^2 \iint 2\left[g_o + \xi \frac{\partial g_o}{\partial \xi} + \mu \frac{\partial g_o}{\partial \mu} \right] \frac{d\xi\, d\mu}{(\mu - 1 - \xi^2)^{1/2}} \tag{4.15}$$

The vanishing of the double integral constitutes condition (2.3).

V. THE PERTURBED ION DENSITY

We follow the principles outlined in references 10 and 11 for calculating distribution perturbations for systems with planar or cylindrical symmetry. The perturbed outward and inward distribution functions, of the form

$$g_o\,(W,\,L)\;+\;g_1^{+}\;(r,\,W,\,L)\;e^{i\ell\theta\;-\;i\omega t} \tag{5.1}$$

must be constant along the (perturbed) orbits in phase-space. θ and ω are implied to be measured in the Larmor frame, throughout this section and section VI. Unperturbed orbits can be used in the second (small) term of 5.1, but orbital changes ΔW and ΔL due to an electrostatic potential perturbation $\Delta\Phi$ of the form

$$\Delta\Phi \;=\; \phi_1\,(r)\;e^{i\ell\theta\;-\;i\omega t} \tag{5.2}$$

must be taken into account in the first term. We have the dynamical equations:

$$\begin{aligned} d\Delta L &= -Ze\,\frac{\partial\,\Delta\Phi}{\partial\,\theta}\;dt \;= i\ell\,Ze\,\Delta\Phi\;dt \\ d\Delta W &= \;Ze\,\frac{\partial\,\Delta\Phi}{\partial\,t}\;dt \;=\; i\omega Ze\,\Delta\Phi\;dt \end{aligned} \tag{5.3}$$

which we integrate along the (unperturbed) orbits between an inner turning point, A, (see Fig. 1), at radius $|d|$, and a point P at arbitrary radius r. Because our aim is to determine the (yet unknown) function $\phi_1\,(r)$ of (5.2), we use r rather than t as a parameter of integration, with $dt = dr/v_r$. The phase of the perturbations (ℓ times the angle in the

rotating wave frame),

$$\psi\,(r) \equiv \ell\,\theta\,(r) - \omega t\,(r) \tag{5.4}$$

can be evaluated as a function of r by means of eqs. (4.5, 4.6).

Only the combination $\Delta L\,\partial g_o/\partial L + \Delta W \partial g_o/\partial W$ of angular momentum and energy changes is of interest; its orbital change is given by:

$$d\,(\Delta L\,\frac{\partial g_o}{\partial L} + \Delta W\,\frac{\partial g_o}{\partial W}) = -\,i g_o'\,Ze\,\Delta\Phi\,dr/v_r \tag{5.5}$$

where

$$g_o' \equiv \ell\,\frac{\partial g_o}{\partial L} + \omega\,\frac{\partial g_o}{\partial W} \tag{5.6}$$

The orbital constancy of the distribution function 5.1 yields:

$$g_1^+\,(r)\,e^{i\psi\,(r)} = g_1\,(|d|)\,e^{i\psi\,(|d|)} + i\,g_o' \int_{|d|}^{r} Ze\,\phi_1\,(r')\,e^{i\psi(r')}\,dr'/v_r\,(r') \tag{5.7}$$

We have omitted the "+" superscript deliberately from $g_1\,(|d|)$ since at the turning points $r = |d|$ and $r = s$ the inward and outward functions g merge into each other and must be identical. The turning point conditions are discussed in some detail in reference 10, as well as by Ehrmann[14] in studies of oscillations of a plasma in a potential well.

Let us indicate phase changes between the inner turning point A at $r = d$ and a general point P, or between any two positions P, P' on an outward orbit, as follows:

$$\psi_{PA} = \psi\,(r) - \psi\,(|d|), \quad \psi_{PP'} = \psi\,(r) - \psi\,(r'). \tag{5.8}$$

Then (5.7) becomes:

$$g_1^+(r) = g_1(|d|)e^{-i\psi_{PA}} + i g_o' \int_{|d|}^{r} Ze\,\phi_1(r')\, e^{-i\psi_{PP'}}\, dr'/v_r(r') \qquad (5.9)$$

and for the inward function g_1^- one obtains a similar formula by integration of 5.5 between $r' = r$ and $r' = |d|$. The following differences between what we have done for g_1^+ and what should be done for g_1^- will be noted:

(i) the integration runs backward in radius

(ii) $v_r = -|v_r|$ on this leg of the orbit

(iii) the phase differences on an inward orbit are just opposite to those on an outward orbit.

Thus one obtains, using ψ_{PA} and $\psi_{PP'}$ of (5.8) as calculated along an outward orbit and the definition $v_r \equiv |v_r|$ which applies on an outward orbit,

$$g_1^-(r) = g_1(|d|)\, e^{i\psi_{PA}} - i g_o' \int_{|d|}^{r} Ze\,\phi_1(r')e^{i\psi_{PP'}}\, dr'/v_r(r') \qquad (5.10)$$

One obtains a condition on $g_1(|d|)$ by identifying r with the outer turning point B at $r = s$ in both (5.9) and (5.10) and by equating g_1^+ with g_1^- there:

$$g_1(|d|) = \frac{g_o'}{\sin\psi_{BA}} \int_{|d|}^{s} Ze\,\phi_1(r')\cos\psi_{BP'}\, dr'/v_r(r') \qquad (5.11)$$

This value, inserted back into (5.9) and (5.10), leads to a combined inward and outward density perturbation:

$$g_1^+ + g_1^- = \frac{2 g_o'}{\sin\psi_{BA}} \left[\int_{|d|}^{r} Ze\,\phi_1(r')\cos\psi_{P'A}\cos\psi_{BP}\, dr'/v_r(r') + \int_{r}^{s} Ze\,\phi_1(r')\cos\psi_{PA}\cos\psi_{BP'}\, dr'/v_r(r') \right], \qquad (5.12)$$

after some elementary trigonometrical manupulation for the first term in the square bracket. The two integrals can be combined:

$$g_1^+ + g_1^- = \frac{2\, g_o'}{\sin\psi_{BA}} \int_{|d|}^{s} Ze\,\phi_1(r')\cos\psi_{P_< A}\cos\psi_{BP_>}\, dr'/v_r(r') \qquad (5.13)$$

where $P_<$ is P or P' and $P_>$ is P' or P according as $r < r'$ or $r > r'$.

When calculating the total ion density perturbation at radius r,

$$\Delta n_i = n_{il}(r)\, e^{i\ell\theta - i\omega t} \qquad (5.14)$$

one has, in addition to the integral of $g_1^+(r) + g_1^-(r)$ over W and L, a contribution from the perturbation of the Jacobian (4.4) which is used in the integral over g_o and which is susceptible to potential perturbations. This contribution is, for the outward class,

$$\phi_1 e^{i\ell\theta - i\omega t} \iint g_o \frac{\partial}{\partial \Phi'}\left(\frac{1}{m^2 r v_r}\right) dW dL \qquad (5.15)$$

Since differentiation of the Jacobian with respect to $(Ze\Phi')$ is the negative of differentiation with respect to W, one can convert this contribution by an integration by parts. Taking into account both inward and outward streams, as well as the natural requirement $g_o \to 0$ as $W \to \infty$, one gets no terminal contributions in this process and the total perturbation n_{il}, including what arises from $g_1^+ + g_1^-$, becomes:

$$n_{il}(r) = 2 \iint \left[\frac{\partial g_o}{\partial W} Ze\phi_1(r) + \frac{g_o'}{\sin\psi_{BA}} \int_{|d|}^{s} Ze\,\phi_1(r')\cos\psi_{P_< A}\cos\psi_{BP_>} \frac{dr'}{v_r(r')}\right] \frac{dW dL}{m^2 r v_r} \qquad (5.16)$$

The first term in the integrand - the one due to the perturbation in the Jacobian - exemplifies the general occurrence of perturbations in the displacement between ordinary and canonical momentum space, as emphasized in reference 10. In pinches, one has found significant marginally unstable modes which owe their existence purely to this effect and in which the canonical momentum distribution g remains unperturbed (references 15, 16 and 17).

Formula (5.16) has quite general validity and is, as such, not restricted to the parabolic well. The features of this well are contained only in the nature of the unperturbed orbits, which enter into the relation between the phase, ψ, and the radius, r, (see eqs. 4.5, 4.6, 5.4) as well as into the dependence of the turning points s, d on W, L (see eqs. 4.2, 4.3). Formula 5.16 can even be used when there are axial variations in the perturbations - it only means that in the definition 5.4 of the phase one has to include a term kz (r). Also, if the steady-state distribution function g_o depends on axial momentum P (the case of axial drifts), one would add a term $k\partial g_o/\partial P$ in 5.6 and, of course, integrate in 5.16 over P as well as W and L.

Perhaps the most important feature in the application of 5.16 to the parabolic well lies in the universality of the resonances between the particle motion and the wave, shown by the zeros of the denominator $\sin\psi_{BA}$. For the parabolic well, these resonances,

$$\frac{\omega}{\Omega} = \text{integer of same parity as } \ell$$

(since $\theta_{BA} = (\text{sgn L})\,\pi/2$, $t_{BA} = \pi/2\Omega$)-do not depend upon W or L -

they are universal for all ions. As explained in Section II, the resonant demoninator is of the type associated with plasma models having a discrete structure in which the ions are either all resonant or all non-resonant. If the unperturbed potential well had been non-parabolic, the frequency Ω would have depended on the orbit energy W and/or the angular momentum L and, in general, only a small fraction of the ions would have been at or near to resonance. In that case, the poles $\sin \psi_{BA} = 0$ inside the integral would have to be treated in the usual way for Landau damping. But it is the limited object of this paper to consider the "quasi-cold" case in order to establish a boundary for gross instability. For this purpose the discrete resonance model is adequate. However, we do offer some general comments on the Landau damping questions in Section IX.

VI. DENSITY PERTURBATIONS IN THE FUNDAMENTAL MODE

A complete perturbation theory requires the insertion of the ion charge density perturbation Zen_{il}, with that of the electrons - en_{el} (derived from ϕ_1 by appropriate methods, perhaps even in full analogy with 5.16), into Poisson's equation. This results in an integral equation for ϕ_1. A common method of solution is iteration, starting with some inspired guess.

In our case, one might start such an iteration with the known results for the drift approximation, or for the case of ions which are unresponsive to the potential wave. With constant unperturbed densities in the interior (condition 2.3), the potential ϕ_1 is then a solution of Laplace's equation and $\phi_1 \propto r$ in the fundamental mode $\ell = 1$. It turns out - fortunately - that with such a ϕ_1 the integrals in (5.16) can be evaluated and that an equation like (3.3) results for the ion density perturbations. Using, again, the constancy of n_{io}, as postulated by 2.3, the ions still behave incompressibly so that the Laplacian nature of the ϕ_1 in the interior is confirmed and no further iteration is needed.

It cannot be claimed that this method of analysis - iteration with a preconceived start - constitutes as exhaustive search for all electron-ion interaction modes, but the most likely resonance, and the one first encountered as the proportions in (2.8) close up, is the one where the ions just begin to respond to the diocotron mode, and this case is correctly covered by our verification that the Laplacian potential is self-consistent.

To evaluate n_{i1} from (5.16) we first calculate the integrals in (5.12). For the fundamental, ψ (s) = 0 for the outward leg which begins at $t = -\pi/2\Omega$ and finishes at t = 0. Hence ψ ($|d|$) = $-\psi_{BA}$ and with ϕ_1 (r) $\propto$ r, one has to integrate such expressions as

$$r\cos(\theta - \omega t + \psi_{BA}) = s\cos\Omega t\cos(\omega t - \psi_{BA}) + d\sin\Omega t\sin(\omega t - \psi_{BA})$$

and

$$r\cos(-\theta + \omega t) = s\cos\Omega t\cos\omega t + d\sin\Omega t\sin\omega t$$

with respect to t in order to get $g_1^+ + g_1^-$ by equation 5.12. (Reverting from r to t as variable of integration is expedient here.) After some helpful cancellations, one finds

$$g_1^+(r) + g_1^-(r) = \frac{2g_o' Ze\phi_1(r)}{\Omega^2 - \omega^2}\left(\omega + \frac{L}{mr^2}\right) \qquad (6.1)$$

and hence

$$n_{i1}(r) = \frac{2Ze\phi_1(r)}{\Omega^2 - \omega^2}\iint\left\{\frac{\partial g_o}{\partial L}\left(\omega + \frac{L}{mr^2}\right) + \frac{\partial g_o}{\partial W}\left(\Omega^2 + \frac{\omega L}{mr^2}\right)\right\}\frac{dW\,dL}{m^2 r v_r} \qquad (6.2)$$

To facilitate the integration, we introduce the dimensionless energy and angular momentum-variables μ and ξ defined in (4.13), giving:

$$n_{i1}(r) = \frac{Ze\phi_1(r)\,\Omega^2}{m(\Omega^2 - \omega^2)}\iint_{\mu>1+\xi^2}\left[\frac{\partial g_o}{\partial \xi}\left(\frac{\omega}{\Omega} + \xi\right) + 2\frac{\partial g_o}{\partial \mu}\left(1 + \frac{\omega}{\Omega}\xi\right)\right]\frac{d\xi\,d\mu}{\sqrt{\mu - 1 - \xi^2}} \qquad (6.3)$$

which we can compare with the drift approximation (3.3). Consider, first, the factor in front of the integral. ϕ_1 (r) can be replaced by $iE_\theta r$. In the denominator we use

$$\Omega^2 = \left(\frac{\Omega_c}{2}\right)^2 + \frac{\Omega_E^2}{\Omega_c}\,\Omega_c = \left(\frac{\Omega_c}{2}\right)^2 + \omega_e\,\Omega_c, \qquad (6.4)$$

employing the definitions (2.5), (2.6) and (2.10). Regarding ω, we have to remember the distinction between the lab frame and the Larmor frame:

$$\omega_{Larmor} = \omega_{lab} + \Omega_c/2 \qquad (6.5)$$

for the fundamental mode. Combining (6.4) and (6.5) the factor in front of the integral in (6.3) becomes:

$$\frac{iE_\theta\, r\Omega^2}{B\,(\omega_e - \omega_{lab} - \omega_{lab}^2/\Omega_c)} \qquad (6.6)$$

and for the drift approximation, which implies $\omega_{lab} \ll \Omega_c$, we shall have achieved confirmation of (3.3), with substitution of dn_o/dr from (4.15), provided we can establish identity of the two integrals - that in (4.15) and that in (6.3).

This identity is readily checked by observing that the integrands differ only by a perfect divergence in ξ, μ-space, namely of the vector with

$$\xi \text{ - component } = 2\sqrt{\mu - 1 - \xi^2}\;\partial\left[\left(\frac{\omega}{\Omega} - \xi\right) g_o\right]/\partial\mu \qquad (6.7)$$

$$\mu \text{ - component } = 2\sqrt{\mu - 1 - \xi^2}\;\left(g_o + \partial\left[\left(\xi - \frac{\omega}{\Omega}\right) g_o\right]/\partial\xi\right) \qquad (6.8)$$

both vanishing on the boundaries $\mu = 1 + \xi^2$ and $\mu \to \infty$.

Not only have we now checked the drift approximation as the large gyrofrequency limit of the more general analysis, we have also found how to correct it, namely by filling in the full denominator of 6.6, to

replace the simple denominator $B(\omega_e - \omega)$ in 3.3. The identity of the integrals, in (6.3) and (4.15) does not depend upon any approximation and one deduces that, with ω as measured in the lab frame,

$$n_{il}(r) = \frac{iE_\theta}{B(\omega_e - \omega - \omega^2/\Omega_c)} \frac{dn_{io}}{dr} \qquad (6.9)$$

is the correct generalization of (3.3) to cases where the drift approximation is inadequate and where a full dynamical analysis is called for. This is a remarkably simple result in view of the wide variety of conditions under which it applies. It does depend on the potential well being parabolic, and therefore on the constancy of $n_{eo} - Zn_{io}$ over the range $0 \leq r \leq b$, but not on the independent constancy of n_{eo} and n_{io}. However, n_{eo} and n_{io} were assumed in (2.3) to be independently constant over the range $0 \leq r \leq b$. It then follows that the perturbed ion density vanishes identically in the interior and the Laplacian nature of $\phi_1(r) \exp(i\theta - i\omega t)$ is confirmed.

In the neighborhood of $r = b$, for the ions as for the electrons, there is a surface charge layer. The electron surface charge density is determined by "cold" laminar flow theory in the diocotron analysis (references 4, 5, 6) or by the drift approximation (3.4). For the ions, laminar flow theory is inadequate. Instead, we apply a limiting process to our theory developed in sections IV - VI. We allow for a thin radial range (thickness δ) in which the total unperturbed charge density $Zen_{io} - en_{eo}$ slopes off to zero continuously (and differentiably) from its full uniform interior value. Within this range the field that makes the ions turn round and roll back into the interior is only changed by an amount of the order δ^2. There will thus be phase delays of the order δ,

but over most of their orbits the ions that penetrate into the boundary layer will oscillate harmonically in the perfect parabolic part of the well.

The phase integrals preceeding the result (6.1) are therefore substantially unchanged and any deviations from 6.1 will go to zero with δ. The fact that the ion flow is not laminar and not purely peripheral, but that most ions dip down into the unspoilt part of the well is an advantage in this argument. The transport of perturbing field effects by the particles does not occur in the skin itself, but mainly in the body of the plasma.

The energy-momentum integrations 6.2 - 6.3 are, again, concerned with the distribution function overall and not with particles that are restricted to the "skin". The nature of the function g_o (W, L) which produces the sloping ion density in the skin need not be known, and the final result (6.9) based on a Laplacian interior potential ϕ_1 (r) would appear to be correct to the order "δ" even within the skin. This allows us to integrate 6.3 through the thickness of the skin, to give the ion surface charge density:

$$\frac{-iE_\theta \ \rho_{io}}{B\ (\omega_e - \omega - \omega^2/\Omega_c)} \qquad (6.10)$$

as the valid generalization of (3.4).

VII. THE DISPERSION RELATION

In this section we put together our previous results and find the dispersion relation corresponding to the fundamental ($\ell = 1$) wave which has a surface charge layer of ions given by (6.10) and a surface charge layer of electrons given by (3.4). Matching iE_r/E_θ across the combined layer leads to:

$$\frac{1}{1-b^2/a^2} = \frac{\omega_e}{(1-a)(\omega_e - \omega)} - \frac{a\,\omega_e}{(1-a)(\omega_e - \omega - \omega^2/\Omega_c)} \qquad (7.1)$$

This is the dispersion relation for the wave being examined. Before studying the behavior of the dispersion relation in detail, we can simplify its form by defining

$$x = \omega/\omega_e$$

so that x is an unknown non-dimensional frequency. Next let

$$x_D = 1 - (1-b^2/a^2)\ (1-a)^{-1}$$

be the dimensionless frequency of the diocotron wave as it would be for unresponsive ions (see Eq. 3.5). Recalling the definition (2.7) of λ, the dispersion relation (7.1) reduces to:

$$(x_D - 1)^{-1} = (x - 1)^{-1} - a\lambda\ (x^2 + x\lambda - \lambda)^{-1} \qquad (7.2)$$

This dispersion relation is (for the reasons explained in Section II) similar in character to the dispersion relation describing the two-stream instability. We see that for given values of x_D, a, and λ there will be

three values of x. The condition for stability is that all three values for x be real, and in the marginal case two of these will coincide. The geometry factor b^2/a^2 enters only into x_D.

Preliminary conclusions that may be drawn from (7.2) are that if either α or λ is sufficiently small, there are in general three real roots which approach x_D and the roots of $x^2 + x\lambda - \lambda = 0$. But this statement will not be true if $(x_D^2 + x_D\lambda - \lambda)$ is also small, or in other words, if the diocotron wave is closely in resonance with one or the other characteristic ion frequencies. In the absence of such a resonance then, stability will follow if the ions are either sufficiently few ($\alpha << 1$) or sufficiently massive ($\lambda << 1$). In either of these cases, the two species are effectively uncoupled.

More precisely, suppose that x_D is a fixed positive number, and that α is small. If we let

$$x_1 = -\lambda/2 - (\lambda + \lambda^2/4)^{1/2}$$
$$x_2 = -\lambda/2 + (\lambda + \lambda^2/4)^{1/2}$$

there will be a real negative root of (7.2) close to x_1. This root can be found as a power series in α, the term in α^o being just x_1. It is then possible to use this root to factor the cubic equation; we are left with a quadratic equation of the form

$$x^2 + 2C_1 x + C_2 = 0 \qquad (7.3)$$

the coefficients C_1 and C_2 can be found as lower series in α; to lowest order $C_1 = -(x_2 + x_D)/2$, $C_2 = x_2 x_D$. The stability condition is now

simply $C_1^2 - C_2 = 0$. On evaluation of the terms in a in the expansions for C_1 and C_2, we find the condition for instability is, approximately:

$$|x_2 - x_D| < 2 \left[a \lambda (1 - x_2)^2 / (x_2 - x_1) \right]^{1/2}$$

Hence instability is present only when the diocotron frequency x_D differs from the ion frequency x_2 by a quantity of order $a^{1/2}$. Furthermore, when instability is present, the growth rate is of the order $\omega_e a^{1/2}$, and this is proportional to $n_{io}^{1/2}$. This suggests that although ω_e is the natural unit for measurement of the real part of the frequency ω, it is more appropriate to refer the imaginary part of ω, representing the growth rate, to the ion plasma frequency Ω_p. ω_e and Ω_p are connected by the relation

$$\Omega_p / \omega_e = \left[2 a \lambda / (1 - a) \right]^{1/2}$$

Using the standard theory of cubic equations, the stability boundary associated with (7.2) can be found. The equation for this boundary has the form of a cubic in λ, the coefficients depending on a and x_D. The boundary is shown in Fig. 2 (solid lines) for three values of the geometrical parameter b/a. For any b/a, points in (λ, a) parameter space above the curves are unstable, and vice versa. We note that 10% neutralization of the electron cloud can be achieved without instability for values of λ up to 0.2 for b/a = 0.75 and up to 1.4 for b/a = 0.9. Extending the electron cloud closer to the wall brings the charge perturbation closer to its image and hence speeds the wave up. This in turn makes it possible to contain lighter ions without approaching the condition of resonance. The stable region below the solid curves but to

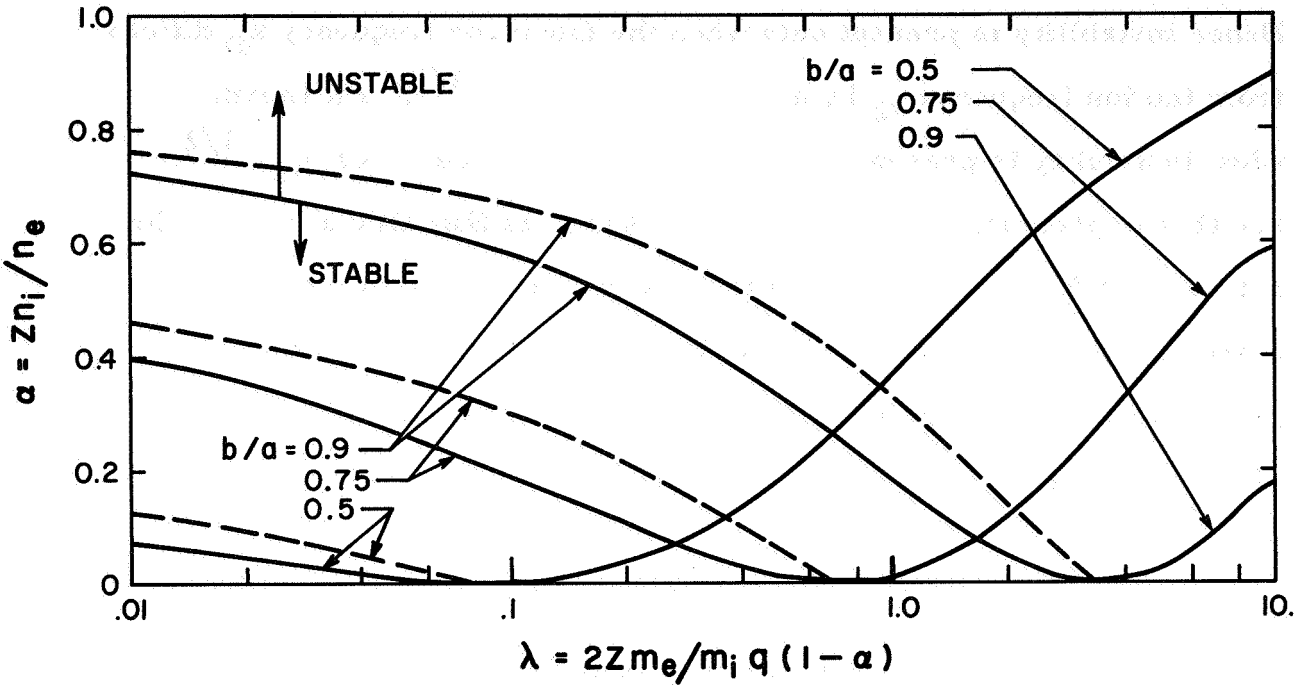

Fig. 2 Stability boundaries in λ , α parameter space associated with the dispersion relation (7.2). For three values of the geometrical parameter b/a , the stability boundary is shown as a solid line. Points above the boundary are unstable, points below it are stable. The dashed lines connect points which, for each α , maximize the ratio of the instability growth rate to the ion plasma frequency. Absolute values of this ratio are shown in Fig. 3.

the right of the resonant point is of less physical interest.

In addition to finding the stability boundary, we can also solve (7.2) directly for $x = \omega/\omega_e$. In the unstable region, the imaginary part of x represents the growth rate. If we traverse the unstable range of values of λ at a fixed value of α, the growth rate will start at zero, increase to a maximum, and then decrease to zero. The location and value of the maximum growth rate will both depend on the normalization adopted. But, as explained above, it is appropriate to normalize the growth rate to the ion plasma frequency. With this normalization, the dashed lines in Fig. 2 show the values of λ which, for a fixed α, give the largest growth rate. The absolute values of the maximum growth rates corresponding to points on these dashed lines are shown in Fig. 3. It may be seen that unless α is close to b^2/a^2, the peak growth rates are numerically smaller than the ion plasma frequency. When $\alpha \to b^2/a^2$, the peak growth rate becomes a large multiple of the ion plasma frequency, but this range is not of much physical interest since it corresponds to small values of $x_D = (b^2/a^2 - \alpha)/(1 - \alpha)$. Furthermore, for a given α, the peak growth rate occurs at a value of $\lambda \approx 4\, x_D^3/27\alpha$ which is extremely small. We conclude that in most unstable cases the growth rate will be comparable to or somewhat less than the ion plasma frequency. For very small α, and very close to resonance, the growth rate is approximately

$$\mathrm{Im}\,(\omega) = \Omega_p \left[(1 - b^2/a^2)^3 \Big/ \left\{ 2\,(2 - b^2/a^2)\, b^2/a^2 \right\} \right]^2 \qquad (7.4)$$

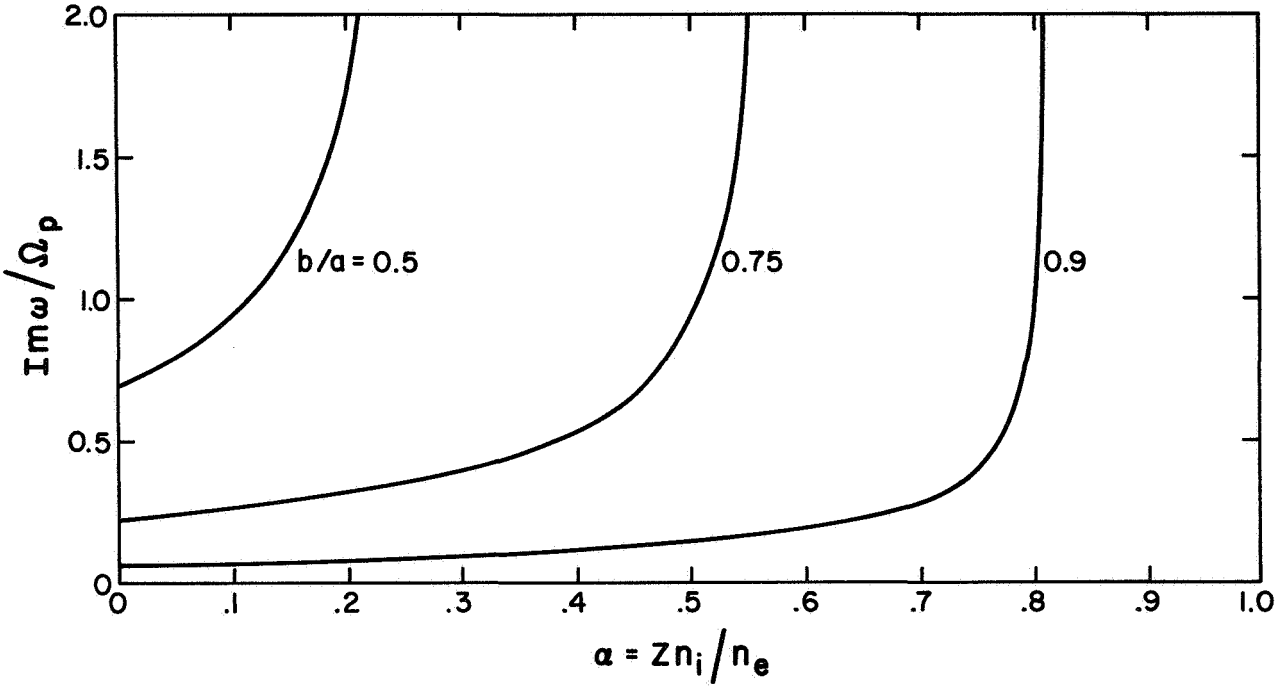

Fig. 3 Instability growth rate divided by the ion plasma frequency at points corresponding to the dashed lines in Fig. 2.

VIII. A SPECIAL ION DISTRIBUTION

The results given in the previous sections cover a wide variety of zeroth order ion distribution functions, but are restricted to the fundamental mode of oscillation. In this section, we shall discuss a special zeroth order ion distribution for which the dispersion relation corresponding to any value of ℓ may be found. This is the distribution of eq. (4.10) for which the ions are divided into two classes (+ and -) each of which is in simple solid body rotation. The angular velocities are, respectively, $(\pm\ \Omega\ -\ \frac{1}{2}\ \Omega_c)$. We could treat an arbitrary division of the ions into these classes, but prefer, for simplicity, to suppose that all the ions are in just one of these classes, having angular velocity Ω_{io}. The ion density is again uniform for $b \leq r \leq b$ and vanishes for $b \leq r \leq a$.

For this special choice of the ion distribution function the ions are truly cold in the sense that all the ions at any place have precisely the same velocity, and we can therefore treat the problem using the fluid (Euler) equations. Let v_r, v_θ be the radial and tangential components of the perturbed ion velocity. The first-order ion dynamics are then determined by:

$$v_r\,(\omega - \ell\ \Omega_{io}) - i v_\theta\ (2\Omega_{io} + \Omega_c) = -\,(\Omega_c/B)\ i\,d\phi/dr$$

$$v_r\,(2\Omega_{io} + \Omega_c) - i v_\theta\ (\omega - \ell\ \Omega_{io}) = -\,(\Omega_c/B)\ i\ell\phi/r \qquad (8.1)$$

As before, we seek a wave which involves no perturbation in the ion density except in the neighborhood of $r = b$. It will turn out that the choice

(3.4) of a Laplacian potential, $\phi \propto r^{\ell}$ is consistent with such a wave. For the region where there are ions (i.e., $r \leq b$), let $\phi(r) = (r/b)^{\ell} \phi(b)$. Solving (8.1) gives:

$$v_\theta = i v_r = (\Omega_c/B) [\ell \phi(b)/r] (r/b)^{\ell} \{(\omega - \ell \Omega_{io}) + (2\Omega_{io} + \Omega_c)\}^{-1} \qquad (8.2)$$

Note that the divergence of the perturbed ion velocity (8.2) vanishes. The perturbed ion continuity equation is

$$n_i (\omega - \ell \Omega_{io}) + i v_r \, dn_{io}/dr = 0$$

and this equation shows that, as expected, n_i vanishes except near $r = b$. For the surface charge density we obtain:

$$\phi_1 (b) (\kappa - \ell)^{-1} (\kappa - \ell + 2)^{-1} [2 \ell \epsilon_o a \Omega_E^2/b\Omega^2 (1 - a)] \qquad (8.3)$$

where

$$\kappa \equiv (\ell \Omega_c + 2\omega)/2\Omega$$

when $\Omega_{io} = (\Omega - \frac{1}{2} \Omega_c)$. When $\Omega_{io} = -(\Omega + \frac{1}{2} \Omega_c)$ the denominator of the first part of (8.3) becomes $(\kappa + \ell)(\kappa + \ell - 2)$. We can check that when $\ell = 1$ this formula reduces (in either case) to (6.10), as indeed it must. The dispersion relation for arbitrary ℓ is easily derived; its general properties are just the same as those of the dispersion relation (7.1) discussed in Section VII. If we let x_D be ω/ω_e from the diocotron relation (3.5) and also

$$x_1 = -\ell\lambda/2 + (\ell - 2)(\lambda + \lambda^2/4)^{1/2}$$

$$x_2 = -\ell\lambda/2 + \ell (\lambda + \lambda^2/4)^{1/2}$$

The form equivalent to (7.2) is

$$(x_D - \ell)^{-1} = (x - \ell)^{-1} - a\lambda\,(x - x_1)^{-1}\,(x - x_2)^{-1} \tag{8.4}$$

As for (7.2), when a and λ are both small, there are three real roots near $x = x_D$, x_1, and x_2. The last two correspond to $\kappa = \ell$ or $\ell - 2$ and in the other case occur at $\kappa = -\ell$ and $-\ell + 2$. Instability can occur for small a if λ is such that one of the ion frequencies x_1 or x_2 is close to the electron diocotron frequency x_D. In keeping with our general philosophy, we are principally concerned with conditions such that the electron diocotron frequency is positive and not too much smaller than ω_e, and such that the highest frequency available to the ions approaches the electron diocotron frequency. The highest frequency available to the ions is clearly x_2, and it is therefore the approach of x_2 to x_D that is the minimum condition for resonance. For the case in which Ω_{io} is negative and for $\ell \geq 2$, resonance is impossible for $x_D > 0$, which justifies our concentrating on the more dangerous case for which $\Omega_{io} > 0$.

For this case, then, the stability boundary is plotted in Fig. 4 as a function of a and λ, for a single value of the geometrical parameter b/a, and for $\ell = 1$, 2, 3 and 4. For the physically interesting range of values of λ (below the resonant value for the fundamental mode) we find as expected that the fundamental mode is the most likely to be unstable. This result therefore lends support to the claim that the stability boundary associated with (7.1), which is restricted to $\ell = 1$, is likely to be the true stability boundary associated with any mode number ℓ. However, no proof of this claim has been found, and the analysis of this

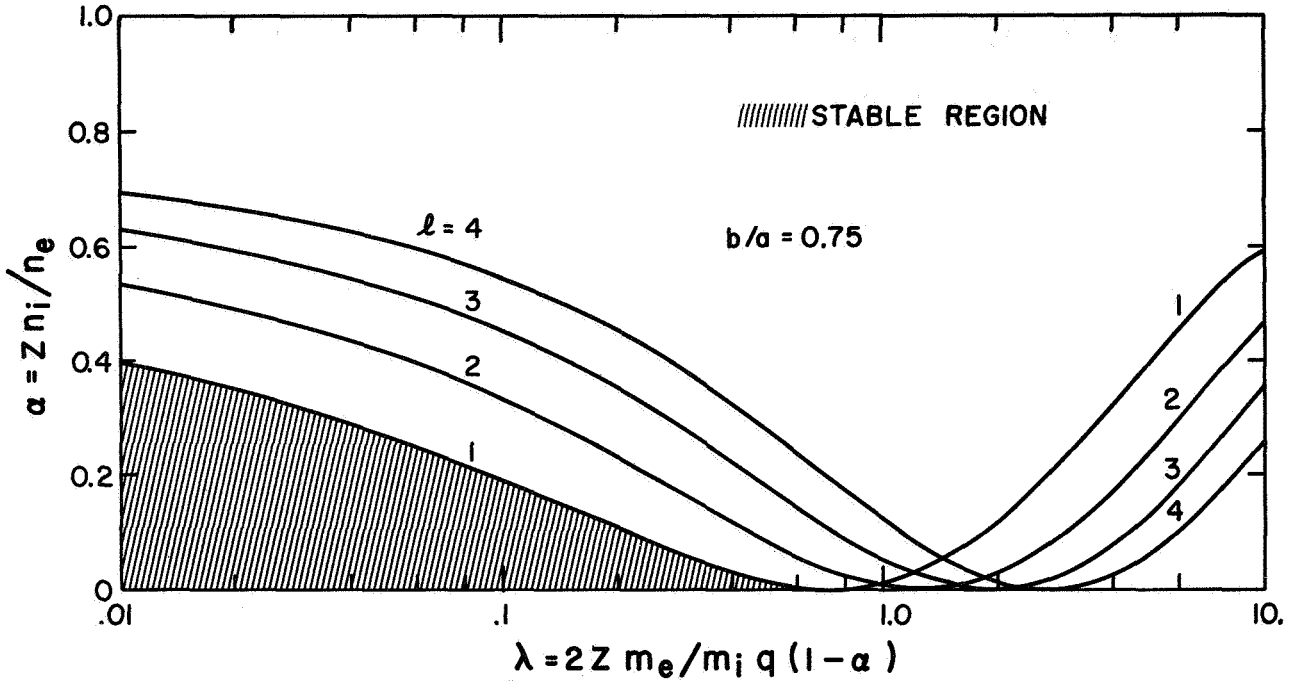

Fig. 4 Stability boundaries in λ , α parameter space associated with the dispersion relation (8.4). The geometrical parameter b/a is fixed at 0.75 and the different curves correspond to the mode numbers $\ell = 1, 2, 3$ and 4. Points in the cross-hatched region are stable against all mode numbers. For $\lambda \lesssim 1$, which is the range of physical interest, the fundamental mode is the most dangerous. For larger values of b/a , the entire system of curves shifts to the right, enhancing stability for moderate values of λ .

section is the only evidence in favor of its truth. For very small α, and very close to the resonance between x_D and x_2, the growth rate corresponding to (7.4) is

$$\mathrm{Im}\,(\omega) = \Omega_p \left[\ell \, (1 - b^{2\ell}/a^{2\ell})^3/2 \left\{ \ell^2 - (1 - b^{2\ell}/a^{2\ell})^2 \right\} \right]^{1/2}$$

For large ℓ this approaches $\Omega_p \, (2\ell)^{-1/2}$.

IX. DISCUSSION

We consider first various possible generalizations of the preceding analysis. The advantage of having a parabolic potential well is that it permits an analytical treatment similar to that used for plasma models with discrete structure. A non-parabolic well would greatly complicate the analysis, but should also throw some light on the question of Landau damping. This is discussed briefly below. The potential well will be parabolic provided that $n_{eo} - Zn_{io}$ is constant from the axis ($r = 0$) out to some radius; it is not necessary that n_{eo} and n_{io} be independently constant. The advantage of this further restriction is that it yields a wave in which the density perturbation is all concentrated at a single layer, which substantially simplifies the analysis. This approach is frequently used in problems of this type, and it is believed that results obtained for step-function density profiles are not importantly different from the results one would obtain from using somewhat smoother profiles. One generalization that has been tried is to have the steps in the electron and ion density profiles occur at different radii. As might be expected, the only effect of this is to reduce somewhat the coupling between the species. In this sense, the case studied is the most critical.

Other possible generalizations include the addition of different species of ions; as a special case, the electrons could also have been given a full dynamic treatment as opposed to the guiding center description actually used. This would have increased the complexity of the analysis with little compensating increase in clarity of the results. As a practical

matter, for small q , the electron plasma and gyro-frequencies are too high to interact with the ions, while, if more species of ions were included, one would inevitably concentrate on the species for which Z/m_i had the largest value.

It is interesting to note that the condition $q \ll 1$ is believed[5] to be important for the stability of the electron cloud. But we have shown that, with ions present, stability is enhanced if $\lambda \ll 1$. These two conditions can be written:

$$n_{eo}\ m_e\ c^2 \ll B^2/\mu_o \ll (1 - a)\ n_{eo}\ m_i\ c^2/2Z$$

Thus it is desirable to have the field strength B bracketed between two limits whose ratio is approximately $(m_i / 2Zm_e)^{1/2}$. There is no difficulty in satisfying these conditions in practical cases — in this instance the large value of the mass ratio can be used to advantage.

Turning next to the question of non-parabolic wells, we observe that such wells would produce an effective frequency spread in the ions. This is suggestive of Landau damping or growth, but only when the cold theory predicts a real (or nearly real) frequency. If the cold theory predicts a gross instability, it is not likely that moderate frequency spreads would be of much help. But stability for the type of dispersion relation studied here means purely real frequencies and it is therefore possible that spreads could produce small imaginary parts of the frequency — of either sign.

Consider an ion oscillating in a potential well characterized by $\phi \sim r^{2+\epsilon}$. For $\epsilon = 0$, the frequency is independent of the amplitude, and therefore of the energy. For $\epsilon < 0$, the frequency decreases with increasing amplitude or energy, and for $\epsilon > 0$ the frequency increases with increasing amplitude or energy. Now consider an ion in the likely situation of having an oscillation frequency just below that of a diocotron wave. If the ion becomes trapped by the wave, its frequency will be effectively raised, but, as we have seen, this can either increase or decrease the energy. For $\epsilon < 0$ (the reasonable case of the electron density decreasing with radius), trapping of a slightly slow ion will decrease its energy. Now in the usual Landau damping situation, loss of energy by a particle and absorption of the same energy by the wave results in growth, but in the diocotron case this familiar result is not valid. The diocotron wave has a negative energy in the sense that loss of wave energy (for example, because of slightly resistive walls) causes the wave to grow and vice versa. It therefore seems possible that if the trapping of slow ions transfers energy from the ions to the wave, damping will result.

It will be recognized that these assertions are highly speculative, and will remain so until a more appropriate analysis is forthcoming.

Finally, we have treated a cylindrical model, and would like to know how things differ if the infinite cylinder were bent round into a torus. The experiments reported by Daugherty et al[18] agree well with the theory of this paper, but were conducted in a torus. It is known[19] that the

electrons can be in equilibrium in a torus even without a rotational transform, but the ion effects are unknown. Although the ions must remain trapped since the potential well is topologically closed, their orbits are more complicated in that there is no known exact (or even approximate) integral of the motion corresponding to the angular momentum in a cylinder. Like the non-parabolic well, this is suggestive of frequency spreads, but nothing concrete is known.

X. CONCLUSIONS

Subject to a number of simplifying assumptions we have exhibited the possibility of an instability associated with resonance between the diocotron wave on a cylindrical crossed-field electron beam, and ion oscillations in the potential well of the electrons. Onset of the instability is associated with the occurrence of resonance which is largely independent of the number of ions present. The condition

$$2Z\epsilon_0 B^2 \ll m_i (n_{eo} - Zn_{io})$$

favors stability. If resonance occurs, instability is possible with growth rates of the order of the ion plasma frequency.

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13. ABSTRACT

The addition of a certain fraction of ions to an otherwise stable cloud of electrons can in certain circumstances result in an instability. The case of a low density ($q = \omega_p^2/\omega_c^2 \ll 1$) electron cloud in which the electron motions can be described by $\underset{\sim}{E} + \underset{\sim}{v}_e \times \underset{\sim}{B} = 0$ is considered in detail. The condition for onset of the instability is that the unperturbed ion orbits (which characteristically traverse the entire electron cloud) should involve frequencies close to that of the diocotron wave that can propagate on the electron cloud. This condition is equivalent to having the ratio $Zm_e/m_i q$ of the order of unity and is independent of the ion density. When instability is present, the growth rate is of the order of the ion plasma frequency.

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