NASA CR-99334

Rec'd 2/3/69

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A Technical Report to the

National Aeronautics and Space Administration
Grant No. NGR 05-010-020

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January 30, 1969

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ABSTRACT

The volume average of the strain tensor in a body moving in an inverse-square force field is evaluated. The calculation is carried out assuming the satellite to be an isotropic elastic body whose center of mass moves in a planar orbit. An approximate expression, in terms of its volume and elastic properties, is presented for the strain energy in the satellite. Using this expression the equation of planar librational motion is written explicity. This equation is discussed for both circular and elliptic orbits and is modified to include the effects of energy dissipation in the body. It is shown that the concept of Adiabatic Invariants allows one to analyze the influence of slow changes in the material volume and elasticity.

*This work was supported by NASA Grant No. NGR 05-010-020.

I. INTRODUCTION

This paper aims to study the effects of material elasticity on the librational motion of an arbitrary shaped satellite. In particular, the influence of the elastic behavior on the librational frequency is determined. The approach adopted is quite general in that the specific shape of the satellite is not prescribed other than to assume that the orbit plane of its center of mass coincides with a principal plane of the satellite. Thus the in-plane librational motion is considered to be uncoupled from the out-of-plane motions.

The elastic behavior is assumed to be describable within the context of the classical theory of infinitesimal elasticity. Accordingly the body forces, inertial and gravitational, to which the satellite material is subjected are computed as if the satellite were rigid. However since the material is actually deformable it contains strain energy of deformation which directly influences the libration frequency.

The desire to avoid the specification of the shape of the satellite removes the analysis of the elastic behavior from the normal class of boundary value problems in the mathematical theory of elasticity. Therefore an averaging method is used to compute an approximate strain energy density the knowledge of which enables one to write the differential equation for the librational motion. This equation, whose form depends upon whether or not allowance is made for energy dissipation, has been derived previously for perfectly elastic materials in a special case [1,2]. For materials with significant internal friction the decay time of the librational oscillation can be written in terms of a quality factor, Q, the elastic constants and the volume of the material. Slow changes in the physical properties of the body can be examined by constructing an Adiabatic Invariant of the motion.

II. THE ENERGY OF DEFORMATION

Assuming the orbital and librational motions to be uncoupled one can write the equation which determines the planar librational motion of a rigid body in the form

$$\frac{d^2\varphi}{dt^2} + \frac{3K}{R_c^3}\left(\frac{B-A}{C}\right) \text{Sin } \varphi \text{ Cos } \varphi = \frac{-d^2\theta}{dt^2} \tag{1}$$

The angle of libration is denoted by φ, the position of the center of mass by (R_c, θ) and the gravitational parameter by K. At the center of mass the principal moments of inertia are A, about the line from which φ is measured, B, and C about the axis normal to the orbit plane. This geometric configuration is illustrated in Fig. 1. To modify equation (1) in order to make allowance for the elastic deformation of the orbiting body it is necessary to determine the work done in this deformation by the body forces. It can be shown* that the body force, per unit mass, acting at the position (x,y) is

$$\underset{\sim}{f} = \underset{\sim}{i}(xP_1 - yP_2) + \underset{\sim}{j}(xP_3 + yP_4) \tag{2}$$

in which

$$P_1 = (\dot{\varphi}+\dot{\theta})^2 + \frac{K}{R_c^3}(3 \text{ Cos}^2 \varphi - 1) \tag{3}$$

$$P_2 = \frac{3K}{R_c^3}(\alpha + 1) \text{ Sin } \varphi \text{ Cos } \varphi \tag{4}$$

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*The details are presented in the Appendix

$$P_3 = \frac{3K}{R_c^3} (\alpha - 1) \operatorname{Sin} \varphi \operatorname{Cos} \varphi \tag{5}$$

$$P_4 = (\dot{\varphi} + \dot{\theta})^2 + \frac{K}{R_c^3} (3 \operatorname{Sin}^2 \varphi + 1) \tag{6}$$

and

$$\alpha = (B - A)/C \tag{7}$$

It follows from the theory of infinitesimal elasticity [3] that the strain energy density, W, in a homogeneous isotropic body is

$$W = \frac{1}{2} \lambda (e_{ii})^2 + \mu \, e_{ij} \, e_{ij} \tag{8}$$

where the Lamé Constants are represented by λ and μ and e_{ij} is the strain tensor created by the body force distribution given by equation (2). An exact evaluation of W would necessitate the specification of the geometric shape of the body and the solution of the pertinent boundary value problem. Since this specification is precisely what it is sought to avoid in the present analysis one has to be content with an estimate of the density W. This estimate can be achieved by using the Betti Reciprocal Theorm [3] to obtain the volume average of the strain tensor. A potentially awkward feature of the application of the theorem - the evaluation of surface integrals - does not arise here because the surface of the body is stress free. The volume averages of the strain tensor components are found for symmetrical bodies to be

$$\bar{e}_{xx} = \left[(B+C-A)P_1 - \sigma \, (C+A-B)P_4 \right] /2EV \tag{9}$$

$$\bar{e}_{yy} = \left[(C+A-B)P_4 - \sigma \, (B+C-A)P_1 \right] /2EV \tag{10}$$

$$\bar{e}_{zz} = -\sigma \left[(B+C-A)P_1 + (C+A-B)P_4 \right] /2EV \tag{11}$$

$$\bar{e}_{xy} = \left[(B+C-A)P_3 - (C+A-B)P_2 \right] /4\mu V \tag{12}$$

and

$$\bar{e}_{yz} = \bar{e}_{zx} = 0 \tag{13}$$

where V is the volume of the body. The well-known relationships

$$E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$$

and

$$\sigma = \lambda/2(\lambda + \mu)$$

have been used in the derivation of the averages in order to simplify their final form. Direct substitution in equation (8) and considerable algebraic reduction produces

$$W = A_1(\dot{\theta}+\dot{\varphi})^4 + A_2(\dot{\theta}+\dot{\varphi})^2 + A_3 \text{ Sin}^4\varphi + A_4 \text{ Sin}^2\varphi + A_5(\dot{\theta}+\dot{\varphi})^2 \text{ Sin}^2\varphi \tag{14}$$

in which

$$A_1 = (\beta^2 + \gamma^2 - 2\beta\gamma\sigma)/8EV^2 \tag{15}$$

$$A_2 = K(2\beta^2 - \gamma^2 - 2\beta\gamma\sigma)/4EV^2R_c^3 \tag{16}$$

$$A_3 = 9K^2 \left[\beta^2 + \gamma^2 - \beta\gamma\sigma - 8(1+\sigma)\, \beta^2\gamma^2/C^2 \right] /8EV^2R_c^6 \tag{17}$$

$$A_4 = K^2 \left[2\beta^2 + 3\gamma^2 - 3\beta\gamma\sigma + 36(1+\sigma)\beta^2\gamma^2/C^2 \right] /4EV^2R_c^6 \tag{18}$$

$$A_5 = 3K(\gamma^2 - \beta^2)/4EV^2R_c^3 \tag{19}$$

$$\beta = B + C - A \tag{20}$$

and

$$\gamma = C + A - B \tag{21}$$

If the center of mass of the satellite is moving in an elliptic orbit of semi-major axis a and eccentricity e, the parameter R_c entering in equations (16), (17), (18) and (19) may be expressed in the form

$$R_c = \frac{a(1 - e^2)}{1 + e \,\mathrm{Cos}\theta} \tag{22}$$

III. THE EQUATION OF LIBRATION

Adopting the form (14) for the strain energy density one can write the Lagrangian of the motion as

$$L = \frac{m}{2}(\dot{R}_c^2 + R_c^2 \,\dot{\theta}^2) + \frac{C}{2}(\dot{\theta} + \dot{\varphi})^2 + \frac{Km}{R_c} - \frac{K}{2R_c^3}(B - 2A)$$

$$+ \frac{3K}{2R_c^3}(B - A)\,\mathrm{Cos}^2\varphi + VW \tag{23}$$

which leads, restricting the analysis to $O(\varphi)$, to the basic libration equation

$$\frac{d}{dt}\left[(C + \frac{2K}{R_c^3} A_6)(\dot{\theta} + \dot{\varphi})\right] + \left[\frac{3K}{R_c^3}(B-A) - \frac{2K^2}{R_c^6} A_7\right]\varphi = 0 \tag{24}$$

where

$$A_6 = (2\beta^2 - \gamma^2 - 2\beta\gamma\sigma)/4EV \tag{25}$$

and

$$A_7 = \left[2\beta^2 + 3\gamma^2 - 3\beta\gamma\sigma + 36(1 + \sigma)\,\beta^2\gamma^2/c^2\right]/4EV \tag{26}$$

In writing the Lagrangian given by equation (23) it is assumed that the kinetic energy of the elastic modes of deformation is negligible in comparison with the orbital and librational energies. Furthermore it should be remarked that in the rigid body limit, $E \to \infty$, both A_6 and A_7 tend to zero and hence equation (24) becomes equation (1) to $O(\varphi)$. It is convenient in some cases to write equation (24) with θ rather than t in the role of independent variable. This transformation which can be made by invoking Kepler's Second Law

$$R_c^2 \dot{\theta} = \left[Ka(1 - e^2)\right]^{\frac{1}{2}} \tag{27}$$

results in the differential equation

$$(1+e\cos\theta)\left[C + 2A_6\,\Omega^2(1+e\cos\theta)^3\right]\frac{d^2\varphi}{d\theta^2} - 2e\sin\theta\left[C + 5A_6\Omega^2(1+e\cos\theta)^3\right]\frac{d\varphi}{d\theta}$$
$$+\left[3(B-A) - 2A_7\Omega^2(1+e\cos\theta)^3\right]\varphi = 2e\sin\theta\left[C+5A_6\Omega^2(1+e\cos\theta)^3\right] \tag{28}$$

with

$$\Omega^2 = K\left[a(1 - e^2)\right]^{-3} \tag{29}$$

Equations (24) and (28) will be discussed separately for circular and elliptic orbits respectively

IV. CIRCULAR ORBITS

In the case $e = 0$ equation (24) simplifies to

$$(C + 2A_6\Omega^2)\frac{d^2\varphi}{dt^2} + \Omega^2\left[3(B-A) - 2A_7\Omega^2\right]\varphi = 0 \tag{30}$$

which corresponds to a simple harmonic oscillator with a frequency, ω, determined by

$$\omega^2 = \left[\frac{3(B-A) - 2A_7\Omega^2}{C + 2A_6\Omega^2} \right] \Omega^2 \tag{31}$$

Equation (31) provides a simple easily applied formula for estimating the frequency of planar libration of a homogeneous isotropic body of <u>arbitrary</u> shape, volume V, principal moments of inertia A, B, and C, Lamé Constants λ, μ and whose center of mass is moving in a circular orbit with orbital angular velocity Ω. The derivation of this formula was the primary purpose for undertaking the present analysis. That ω should be less than the rigid body frequency $\Omega\left[3(B-A)/C\right]^{\frac{1}{2}}$ is to be expected. In fact, by utilizing equations (20), (21), (25) and (26) one can show that if $B > A > C$ then A_6 and A_7 are both necessarily positive. Accordingly, it is clear that the effect of the material's elasticity is to decrease the frequency of libration below the frequency attributed to the same body were it rigid. Two further aspects of equation (30) may be remarked upon.

Firstly, if the body possesses appreciable internal friction the decay time of the libration can immediately be written, in terms of a material quality factor Q, in the form $2Q/\omega$. Here quality factor is defined, as usual, to be

$$Q = 2\pi \frac{\text{energy stored}}{\text{energy lost per period}}$$

and ω is given by equation (31). Estimates of Q for natural satellites in the Solar System are available [4] . Secondly, if the analysis is to be applied to natural satellites whose physical properties may change over long time periods one can write that

$$\overline{E}/\omega = \text{constant} \tag{32}$$

where $\overline{E}$ represents the average energy in the libration. Equation (32) reflects the fact that the action is an Adiabatic Invariant of the librational motion for slow changes in the basic parameters. Explicitly, in the present case one has

$$\varphi_o^4 \Omega^2 \left[3(B-A) - 2A_7\Omega^2\right] (C + 2A_6\Omega^2) = \text{Constant} \tag{33}$$

with φ_o representing the amplitude of the oscillation.

V. ELLIPTIC ORBITS

The form of equation (28) as it stands is not particulary suited to a discussion of the libration of a body in an elliptic orbit. The equation can be cast into a more convenient form by introducing the transformation

$$\Psi = (1 + e \text{ Cos } \theta)\, \varphi \tag{34}$$

In terms of the new variable Ψ equation (28) becomes

$$\left[C + 2A_6\Omega^2(1 + e \text{ Cos}\theta)^3\right] \frac{d^2\Psi}{d\theta^2} + 6eA_6\Omega^2(1 + e \text{ Cos}\theta)^2 \text{ Sin } \theta \frac{d\Psi}{d\theta}$$

$$+ \left\{\left[C+2A_6\Omega^2(1+e\text{Cos}\theta)^3\right] \left(\frac{e\text{Cos}\theta}{1+e\text{Cos}\theta}\right) + 6e^2A_6\Omega^2(1+e\text{Cos}\theta) \text{ Sin}^2\theta\right.$$

$$\left.+ \left(\frac{1}{1+e\text{Cos}\theta}\right) \left[3(B-A) - 2A_7\Omega^2(1+e\text{Cos}\theta)^3\right]\right\} \Psi$$

$$= 2e\left[C + 5A_6\Omega^2(1 + e\text{Cos}\theta)^3\right] \text{Sin}\theta \tag{35}$$

The solution of this equation can be found as follows as a power series in the eccentricity e ($o < e < 1$). Let

$$\Psi = \sum_{n=o}^{\infty} e^n \Psi_n \tag{36}$$

and equate like powers of e on both sides of equation (35). This process leads to a set of differential equations which can be solved in sequence. The present discussion will be confined to the first two of these equations merely to demonstrate the method. One finds

$$\frac{d^2\Psi_o}{d\theta^2} + (a_1 - a_2)\,\Psi_o = 0 \tag{37}$$

and

$$\frac{d^2\Psi_1}{d\theta^2} + (a_1 - a_2)\Psi_1 = -3a_3\left[\text{Cos}\,\theta\,\frac{d^2\Psi_o}{d\theta^2} + \text{Sin}\,\theta\,\frac{d\Psi_o}{d\theta}\right]$$

$$+ (a_1 + 2a_2 - 1)\,\Psi_o\,\text{Cos}\,\theta + 2a_4\,\text{Sin}\,\theta \tag{38}$$

where

$$a_1 = 3(B-A)/(C + 2A_6\Omega^2) \tag{39}$$

$$a_2 = 2A_7\Omega^2/(C + 2A_6\Omega^2) \tag{40}$$

$$a_3 = 2A_6\Omega^2/(C + 2A_6\Omega^2) \tag{41}$$

and

$$a_4 = 2(C + 5A_6\Omega^2)/(C + 2A_6\Omega^2) \tag{42}$$

The solution of equation (37) is

$$\Psi_o = C_o \text{ Sin } (a_1 - a_2)^{\frac{1}{2}} \theta + D_o \text{ Cos } (a_1 - a_2)^{\frac{1}{2}} \theta \tag{43}$$

The homogeneous solution of equation (38) is of similar form and its particular solution, which may be found by the method of variation of parameters, is

$$\Psi_1 = \left[\frac{-(a_1+2a_2-1)}{4u} + \frac{3}{4}a_3 u\right] x \left\{(1+p)C_o\text{Sin}r\theta - (1-q)C_o\text{Sin}s\theta - (1+p)D_o\text{Cos}r\theta + (1-q)D_o\text{Sin}s\theta\right\}$$
$$+ \frac{3}{4}a_3 \left\{(1-p)C_o\text{Sin}r\theta + (1+q)C_o\text{Sin}s\theta + (1-p)D_o\text{Cos}r\theta + (1+q)D_o\text{Cos } s\theta\right\}$$
$$+ \frac{2a_4}{a_1-a_2-1} \text{ Sin } \theta \tag{44}$$

in which

$$2(a_1 - a_2)^{\frac{1}{2}} + 1 = p^{-1}$$

$$2(a_1 - a_2)^{\frac{1}{2}} - 1 = q^{-1}$$

$$r = (1 - p)/2p; \; r - 1 = u$$

and

$$s = (1 - q)/2q$$

This solution process may be continued step by step to the required degree of accuracy and the influence of the orbital eccentricity on the libration subsequently analyzed. This further development will not be presented here however.

IV. CONCLUSIONS

The librational frequency of arbitrarily shaped elastic bodies can be represented approximately in a relatively simple way. This representation, equation (31), is based upon a generalization of the classic McCullagh Formula, for the gravitational potential of a rigid body in a Newtonian Force Field, to include deformation energy. The approximate nature of the generalization arises because the non-specification of the body's exact shape precludes an exact calculation of its elastic strain energy. The modification necessary to allow for inelastic behavior is immediate if a Quality Factor Q is available. Consequently both natural and artificial bodies may be treated. In the former case the effects of slow changes in the physical parameters may be examined by constructing an Adiabatic Invariant. In the latter, where conceptual satellite designs vary widely in geometric shape, the generality of the formulas for the libration frequency and the strain energy, though approximate, should prove useful. The extension of the analysis to cover librational motion in the field of two centers of force proceeds in a manner directly similar to that for a single center of force presented in this paper.

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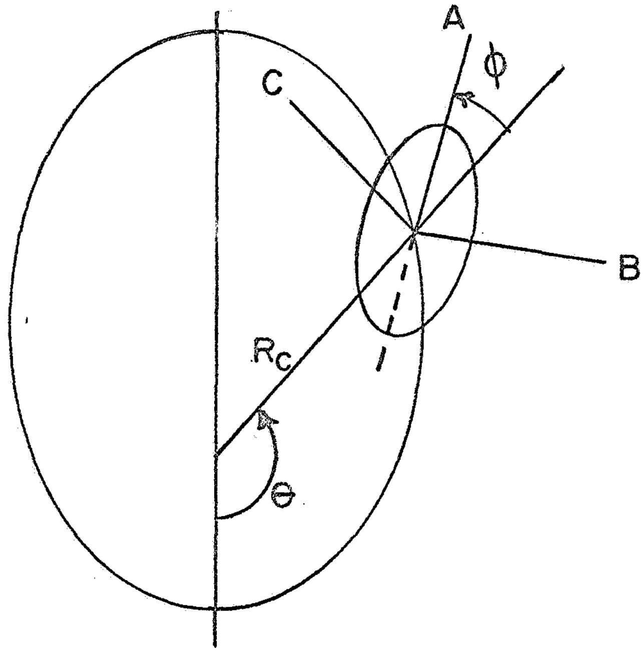

Fig. 1 Angle of Libration and Principal Axes.

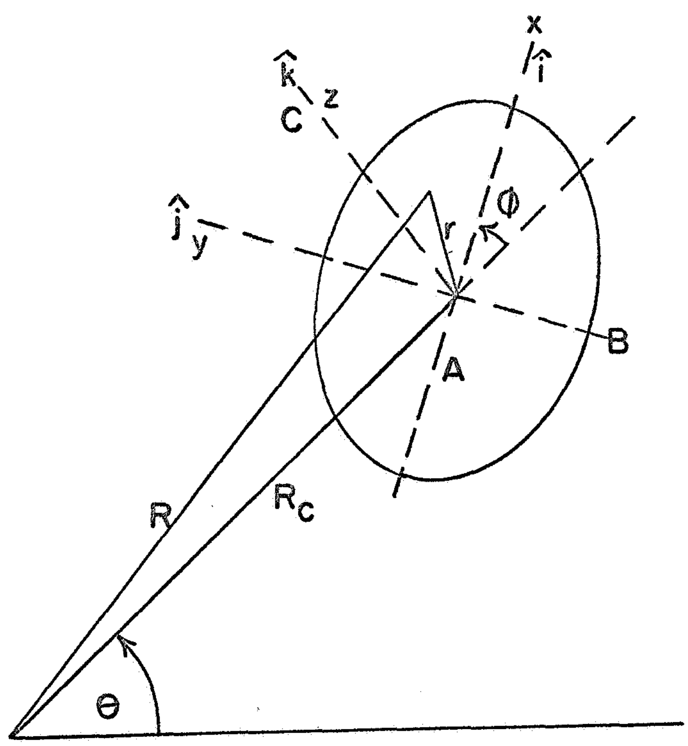

Fig. 2 Coordinate Systems to Determine Body Force.

APPENDIX

This appendix is devoted to outlining the steps in the calculation of the body force per unit mass acting on the satellite. The force is composed of the gravitational force, and the reaction forces associated with the accelerations of the satellite. Referring to Fig. 2, let XYZ be an inertial coordinate system with origin at the focus of the orbit, X-axis directed toward the pericenter and Z-axis normal to the orbit plane. The xyz system of axes is fixed in the satellite with origin at the center of mass and axes coinciding with the principal axes of inertia. The z-axis is parallel to the Z - axis. Then

$$\ddot{\underset{\sim}{R}} = \ddot{\underset{\sim}{R}}_c + \ddot{\underset{\sim}{r}} \tag{A1}$$

with

$$\ddot{\underset{\sim}{R}}_c = (\ddot{R}_c - R_c\dot{\theta}^2)\,\underset{\sim}{e}_{R_c} + (2\dot{R}_c\dot{\theta} + R_c\ddot{\theta})\,\underset{\sim}{e}_\theta \tag{A2}$$

and

$$\ddot{\underset{\sim}{r}} = \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r}) + (\dot{\underset{\sim}{\omega}} \times \underset{\sim}{r}) + \ddot{\underset{\sim}{r}}_r + 2\underset{\sim}{\omega} \times \dot{\underset{\sim}{r}}_r \tag{A3}$$

in which $\ddot{\underset{\sim}{R}}_c$ and $\ddot{\underset{\sim}{r}}$ are calculated relative to the XYZ system, $\ddot{\underset{\sim}{r}}_r$ and $\dot{\underset{\sim}{r}}_r$ relative to the xyz system and ω is the relative angular velocity of the two systems. Thus

$$\underset{\sim}{\omega} = (\dot{\theta} + \dot{\varphi})\,\underset{\sim}{k} \tag{A4}$$

On using the fact that the center of mass is moving in a Keplerian Orbit and that the problem is being considered within Infinitesimal Elasticity Theory one finds

$$\ddot{\underset{\sim}{R}} = \left[- \frac{K}{R_c^2} \operatorname{Cos} \varphi - (\dot{\theta} + \dot{\varphi})^2 r \operatorname{Cos} \xi - (\ddot{\theta} + \ddot{\varphi}) r \operatorname{Sin} \xi \right] \underset{\sim}{i}$$

$$+ \left[- \frac{K}{R_c^2} \operatorname{Sin} \varphi - (\dot{\theta} + \dot{\varphi})^2 r \operatorname{Sin} \xi + (\ddot{\theta} + \ddot{\varphi}) r \operatorname{Cos} \xi \right] \underset{\sim}{j} \qquad \text{(A5)}$$

where $\tan \xi = y/x$. Substitution from the planar libration equation

$$\ddot{\theta} + \ddot{\varphi} = - \frac{3K}{R_c^3} \left(\frac{B-A}{C} \right) \operatorname{Sin} \varphi \operatorname{Cos} \varphi$$

produces the form

$$\ddot{\underset{\sim}{R}} = \left[- \frac{K}{R_c^2} \operatorname{Cos}\varphi - (\dot{\varphi} + \dot{\theta})^2 r \operatorname{Cos} \xi + \frac{3K}{R_c^3} \left(\frac{B-A}{C} \right) r \operatorname{Sin}\xi \operatorname{Sin}\varphi \operatorname{Cos}\varphi \right] \underset{\sim}{i}$$

$$+ \left[- \frac{K}{R_c^2} \operatorname{Sin}\varphi - (\dot{\varphi} + \dot{\theta})^2 r \operatorname{Sin} \xi - \frac{3K}{R_c^3} \left(\frac{B-A}{C} \right) r \operatorname{Cos}\xi \operatorname{Sin}\varphi \operatorname{Cos}\varphi \right] \underset{\sim}{j} \qquad \text{(A6)}$$

The gravitational force per unit mass is

$$\underset{\sim}{f}_g = \frac{-K}{\left| \underset{\sim}{R}_c + \underset{\sim}{r} \right|^3} \left(\underset{\sim}{R}_c + \underset{\sim}{r} \right)$$

$$= \left[- \frac{K}{R_c^2} \operatorname{Cos}\varphi - \frac{Kr}{R_c^3} \operatorname{Cos}\xi + \frac{3Kr}{R_c^3} \operatorname{Cos} \varphi \operatorname{Cos} (\xi + \varphi) \right] \underset{\sim}{i}$$

$$+ \left[\frac{K}{R_c^2} \operatorname{Sin}\varphi - \frac{Kr}{R_c^3} \operatorname{Sin}\xi - \frac{3Kr}{R_c^3} \operatorname{Sin}\varphi \operatorname{Cos} (\xi + \varphi) \right] \underset{\sim}{j}$$

$$+ O\left(\frac{r^2}{R_c^2} \right) \qquad \text{(A7)}$$

The total body force per unit mass is accordingly

$$\underset{\sim}{f} = \underset{\sim}{i} \left[xP_1 - yP_2 \right] + \underset{\sim}{j} \left[xP_3 + yP_4 \right] \tag{A8}$$

where the notation

$$P_1 = (\dot{\theta} + \dot{\varphi})^2 + \frac{K}{R_c^3} (3 \operatorname{Cos}^2\varphi - 1)$$

$$P_2 = \frac{3K}{R_c^3} (\alpha + 1) \operatorname{Sin}\varphi \operatorname{Cos}\varphi$$

$$P_3 = \frac{3K}{R_c^3} (\alpha - 1) \operatorname{Sin}\varphi \operatorname{Cos}\varphi$$

$$P_4 = (\dot{\varphi} + \dot{\theta})^2 + \frac{K}{R_c^3} (3 \operatorname{Sin}^2\varphi + 1)$$

and

$$\alpha = (B - A)/C$$

is used. Equation (A8) coincides with equation (2) of the main text.