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Attitute Stability of Deformable Satellites.

by

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Abstract.

In this paper a general formulation for the dynamics of deformable bodies is derived. The attitute stability of deformable earth-pointing satellite is investigated in a general form and simple stability criteria are obtained.

Résumé.

Cet article propose une formulation générale de la dynamique des corps déformables. La stabilité d'attitude de satellites déformables orientés sur orbite circulaire est étudiée sous forme générale et des critères simples de stabilité sont obtenus.

1. Introduction.

The dynamics of non-rigid bodies will be considered here in order to investigate the stability of attitude of space vehicles.

A spacecraft may contain some instrumentation moving relatively to the main structure and having then an influence on the dynamics of the whole body. Also, as it is imperative to maintain the weight of the payload as low as possible, the rigidity will be affected. Elastic and, sometimes, plastic deformations may occur and the structure is subject to vibrations. With plastic or viscoelastic deformation is associated a dissipation of energy which is a critical point of stability investigation.

Most of the ideas of this paper have been introduced in Buckens' papers [1][2]. The formulation, being rather general, does not depend on a particular configuration. As deformations are expressed in terms of normal modes, the applicability of this theory is generally limited to small deformations of elastic systems or viscoelastic systems with "classical damping".

The attitude stability of earth pointing satellites will be investigated in the sense of Liapounov.

2. Normal Modes of Vibration.

The modes of vibration will be defined relatively to a state of minimal internal potential energy, the total momentum and total angular momentum of the system being equal to zero. The deformation of the system may be described by a number of independent parameters equal to the number of degrees of freedom.

Only the quadratic terms in the independent parameters will be kept in the expansion of the potential energy in order to get the equations of deformation in the form of a conservative system of linear differential equations for which eigenvalues may be determined.

The axes of reference centered at the center of mass are coinciding with the principal axes of inertia of the body before deformation (Fig. 1).

The position vector $\underline{\rho}$ of the element of mass dm relative to the center of mass is equal to the sum of the position vector $\underline{x}$ of dm in the undeformed body and the displacement vector $\underline{u}$

$$\underline{\rho} = \underline{x} + \underline{u} \tag{2.1}$$

From the conditions of zero total momentum and angular momentum the integrals over the mass of the whole body (m) of $\dot{\underline{\rho}}$ and $\underline{\rho} \times \dot{\underline{\rho}}$ are equal to zero :

$$\int_m \dot{\underline{\rho}}\, dm = \int_m (\dot{\underline{x}} + \dot{\underline{u}}) dm = \int_m \dot{\underline{u}}\, dm = 0 \tag{2.2}$$

and

$$\int_m \underline{\rho} \times \underline{\rho}\, dm = \int_m (\underline{x} + \underline{u}) \times \dot{\underline{u}}\, dm = 0 \tag{2.3}$$

The center of mass does not move. This obvious result may be obtained by integration of (2.2).

If the components of the vector deformation $\underline{u}$ are linear functions of the modes the relation (2.3) reduces, for every mode ν with frequency ω_ν, to :

$$\int_m (\underline{x} \times \underline{\phi}_\nu) dm = 0 \tag{2.4}$$

where ϕ_ν is a vector relating the shape of the displacement in the ν^{th} mode and is a function of position only.

The displacement for this mode is equal to the product of ϕ_ν by a harmonic function of time β_ν.

When the components of $\underline{u}$ are not linear functions of the modes but may be linearized, the relation (2.4) will hold in linear approximation.

3. Refernce Axes for Dynamic Investigation.

As stated before, the axes of reference are centered at the center of mass and are coinciding with the principal axes of inertia of the body before deformation. After deformation they will be directed in such a manner that the following conditions are satisfied :

$$\int_m \underline{u}\, dm = 0 \qquad\qquad (3.1)$$
$$\int_m \underline{x} \times \underline{u}\, dm = 0$$

Under these conditiohs the deformations may be expressed as a linear combination of normal modes and in linear approximation †

$$\underline{u} = \beta_\nu \, \underline{\phi}_\nu \qquad\qquad (3.2)$$

The relation (2.4) being, a condition of orthogonality of the mode ν with the rigid modes of translation and rotation, the above defined system of axes will follow the motions of translation and general rotation of the body.

Note that these axes of reference are not necessarily the principal axes of the deformed body, nor those for which the relative angular momentum vanishes, these latter axes being generally used in the study of deformable planets [3].

This frame will be called the D frame and will have a vector basis $\underline{\hat{d}}_1$ $\underline{\hat{d}}_2$ $\underline{\hat{d}}_3$.

4. Equations of motion.

The rotational equations are derived from the vectorial relation

$$\dot{\underline{H}} = \underline{L} \qquad\qquad (4.1)$$

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† From here, the repetition of one index implies summation on all the values of this index.

where $\underline{H}$ is the angular momentum vector with respect to the center of mass ,

$\underline{L}$ is the applied torque acting about the center of mass, the dot (·) meaning time derivative in an inertial space, the index (o) being reserved for the time derivative with respect to some moving reference frame, here to the D frame. By definition H is equal to

$$\underline{H} = \int_m \underline{\rho} \times \dot{\underline{\rho}} \, dm \tag{4.2}$$

or

$$\underline{H} = \int_m \underline{\rho} \times \overset{\circ}{\underline{\rho}} + \int_m \underline{\rho} \times \underline{\omega} \times \underline{\rho} \, dm \tag{4.3}$$

where ω is the angular velocity of the D frame. Defining the inertia dyadic J as

$$\mathsf{J} = \int_m (\underline{\rho} \cdot \underline{\rho} \; \mathsf{E} - \underline{\rho}\underline{\rho}) dm \tag{4.4}$$

where E is the unit dyadic :

$$\mathsf{E} = \delta_{\alpha\beta} \; \underline{\hat{a}}_\alpha \; \underline{\hat{a}}_\beta \tag{4.5}$$

$$\delta_{\alpha\beta} \begin{cases} = 1 & \alpha = \beta \\ = 0 & \alpha \neq \beta \end{cases} \tag{4.6}$$

the second integral of (4.3) is equal to

$$\int_m \underline{\rho} \times \underline{\omega} \times \underline{\rho} \, dm = \int_m [(\underline{\rho} \cdot \underline{\rho})\underline{\omega} - \underline{\rho}(\underline{\rho} \cdot \underline{\omega})] \, dm \tag{4.7}$$

$$= \mathsf{J} \cdot \underline{\omega}$$

then

$$\underline{H} = \mathsf{J} \cdot \underline{\omega} + \int_m \underline{\rho} \times \overset{\circ}{\underline{\rho}} \, dm \tag{4.8}$$

Replacing $\dot{\underline{H}}$ by its value obtained from the time derivation of (4.8) and taking into consideration the fact that for every dyadic T

$$\dot{\mathsf{T}} = \overset{\circ}{\mathsf{T}} + \underline{\omega} \times \mathsf{T} - \mathsf{T} \times \underline{\omega} \qquad (4.9)$$

the relation (4.1) may be written as :

$$\underline{L} = \mathsf{J} \cdot \dot{\underline{\omega}} + \underline{\omega} \times \mathsf{J} \cdot \underline{\omega} + \overset{\circ}{\mathsf{J}} \cdot \underline{\omega} \qquad (4.10)$$

$$+ \int_m \underline{u} \times \overset{\circ\circ}{\underline{u}}\, dm + \underline{\omega} \times \int_m \underline{u} \times \overset{\circ}{\underline{u}}\, dm$$

All the vectors will be expressed in the D frame. For operational purposes let define the d "vector array" [4].

$$d = \begin{bmatrix} \underline{\hat{d}}_1 \\ \underline{\hat{d}}_2 \\ \underline{\hat{d}}_3 \end{bmatrix} \qquad (4.11)$$

a matrix with vector elements.

Any vector v is related to the 3 × 1 matrix

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

of its components in the $\underline{\hat{d}}_1\underline{\hat{d}}_2\underline{\hat{d}}_3$- basis by the relation

$$\underline{v} = d^T v \qquad (4.12)$$

where the superscript T means that the matrix d is transposed.

Similarly any dyadic $\mathsf{T} = t_{ij}\, \underline{\hat{d}}_i\, \underline{\hat{d}}_j$ may be expressed in terms of the matrix $T = (t_{ij})$ as follows

$$\mathsf{T} = d^T T d \qquad (4.13)$$

Further with any 3 × 1 matrix u is associated a 3 × 3 skewsymmetric matrix $\tilde{u}$ defined as

$$\tilde{u} = [\tilde{u}_{ij}] = [\varepsilon_{ikj} u_k] \qquad (4.14)$$

where ε_{ikj} is the Levi-Civita density defined to be zero if some indices are repeated and equal to +1 or -1 according as ikj is a cyclic permutation of 123 or 132.

Using the above conventions the equation (4.10) may be written in matrix form as :

$$L = J\dot{\omega} + \tilde{\omega}J\omega + \overset{\circ}{J}\omega + \int_m \tilde{u}\,\overset{\circ\circ}{u}\,dm + \tilde{\omega}\int \tilde{u}\,\overset{\circ}{u}\,dm \qquad (4.15)$$

In order to have a complete set of equations in (3+n) unknows n equations of deformation have to be derived.

The simplest way to do this is to derive the Lagrange's equations in the variables β_ν :

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\beta}_\nu} - \frac{\partial T}{\partial \beta_\nu} + \frac{\partial U}{\partial \beta_\nu} + \frac{\partial W}{\partial \dot{\beta}_\nu} = N_\nu \qquad (4.16)$$

where T : the kinetic energy

U : the potential energy

W : the Rayleighs dissipation function

N_ν : the generalized force in the ν^{th} mode of deformation.

By definition the kinetic energy is equal to :

$$T = \frac{1}{2}\int_m (\underline{v}.\underline{v})dm \qquad (4.17)$$

where $\underline{v}$ is the velocity of the element dm relative to an inertial frame

$$\underline{v} = \dot{\underline{r}} + \overset{\circ}{\underline{\rho}} + \underline{\omega} \times \underline{\rho} \qquad (4.18)$$

$\underline{r}$ being the vector from the origin of the inertial frame to the center of mass of the body. Then the vector $\underline{x}$ being constant in the D frame, the kinetic energy may be written :

$$T = \frac{1}{2}\int_m (\dot{\underline{r}} + \underline{\omega} \times \underline{\rho} + \overset{\circ}{\underline{u}}).(\dot{\underline{r}} + \underline{\omega} \times \underline{\rho} + \overset{\circ}{\underline{u}})\, dm \tag{4.19}$$

Noting from (4.7) that

$$\begin{aligned}&\int_m (\underline{\omega} \times \underline{\rho}).(\underline{\omega} \times \underline{\rho})\, dm \\ &= \underline{\omega} \cdot \int_m \underline{\rho} \times \underline{\omega} \times \underline{\rho}\, dm \\ &= \underline{\omega} \cdot (\mathcal{J} \cdot \underline{\omega})\end{aligned} \tag{4.20}$$

T may be written

$$\begin{aligned}T = \frac{1}{2}m\, (\dot{\underline{r}} \cdot \dot{\underline{r}}) + \underline{\omega} \cdot \mathcal{J}\, \dot{\underline{\omega}} \\ + \frac{1}{2}\int_m \overset{\circ}{\underline{u}} \cdot \overset{\circ}{\underline{u}}\, dm \\ + \underline{\omega} \cdot \int_m \underline{u} \times \overset{\circ}{\underline{u}}\, dm\end{aligned} \tag{4.21}$$

where $m = \int_m dm$ (4.22)

In matrix form

$$\begin{aligned}T = \frac{1}{2}m\, (\dot{r}^T r) + \frac{1}{2}\, \omega^T J\, \omega \\ + \frac{1}{2}\int_m \overset{\circ}{u}{}^T\, \overset{\circ}{u}\, dm \\ + \omega^T \int_m \tilde{u}\, \overset{\circ\circ}{u}\, dm\end{aligned} \tag{4.23}$$

The potential energy of deformation U_d is, for purely linear and "elastic" deformation, given by

$$U_d = \frac{1}{2}m_\nu \beta_\nu^2\, \omega_\nu^2 \tag{4.24}$$

where m_ν is the generalized mass of the ν^{th} mode, i.e.,

$$m_\nu = \int_m |\underline{\phi}_\nu|^2 \; dm \tag{4.25}$$

ω_ν is the natural frequency of the ν^{th} mode.

The equations will now be written explicitly in terms of the deformation variables.

The formulation presented here is exact only when the displacements one linear functions in the variables β_ν and is exact in linear approximation when the development in power of β_ν of the displacement does not, contain terms in β_ν^2. In both cases the displacement will be equal to

$$\underline{u} = \beta_\nu \underline{\phi}_\nu = \beta_\nu \phi_{\nu_i} \, \underline{a}_i \tag{4.26}$$

In terms of the matrices I, Δ, and U defined by

$$\begin{aligned} I &= \int_m (x^T x E - x x^T) \; dm \\ \Delta &= \int_m (x^T u E - x u^T) \; dm \\ U &= \int_m (u^T u E - u u^T) \; dm \end{aligned} \tag{4.27}$$

the J matrix is given by :

$$J = I + 2\Delta + U \tag{4.28}$$

From the definitions (4.26) and (4.27), elements of Δ and U are given by

$$\Delta_{ij} = \int_m (\beta_\nu x_k \phi_{\nu_k} \delta_{ij} - \beta_\nu x_i \phi_{\nu_j}) \; dm \tag{4.29}$$

$$U_{ij} = \int_m (\beta_\nu \beta_\mu \phi_{\nu_k} \phi_{\mu_k} \delta_{ij} - \beta_\nu \beta_\mu \phi_{\nu_i} \phi_{\mu_j}) \; dm \tag{4.30}$$

The nxl matrix β is defined as :

$$\beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \tag{4.31}$$

and the nxl matrix ϕ_α is the matrix with elements equal to the components along the α-axis of the shape of the different modes.

$$\phi_\alpha = \begin{bmatrix} \phi_{1_\alpha} \\ \vdots \\ \phi_{n_\alpha} \end{bmatrix} \tag{4.32}$$

If further the nxl matrix Λ_{ij} is defined as :

$$\Lambda_{ij} = \int_m (x_k \phi_k \delta_{ij} - x_j \phi_i) \, dm \tag{4.33}$$

Δ_{ij} is equal to :

$$\Delta_{ij} = \beta^T \Lambda_{ij} \tag{4.34}$$

Finally defining the 3nx3 matrix B by

$$B = \begin{bmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix} \tag{4.35}$$

and the 3nx3 matrix Λ by

$$\Lambda = \left[\Lambda_{ij}\right] \quad \text{the matrix } \Delta \text{ is equal to}$$
$$\Delta = B^T \Lambda \tag{4.36}$$

Similarly

$$\begin{aligned} U_{ij} &= \int_m (\beta^T \phi_k \phi_k^T \beta \delta_{ij} - \beta^T \phi_i \phi_j^T \beta) \, dm \\ &= \beta^T \int_m (\phi_k \phi_k^T \delta_{ij} - \phi_i \phi_j^T) \, dm \end{aligned} \tag{4.37}$$

Defining Γ_{ij} as

$$\Gamma_{ij} = \int_m (\phi_k \phi_k^T \delta_{ij} - \phi_i \phi_j^T)\, dm \quad . \tag{4.38}$$

U_{ij} is equal to

$$U_{ij} = \beta^T \Gamma_{ij} \beta \tag{4.39}$$

In terms of the matrix B and the 3nx3n matrix Γ is defined by

$$\Gamma = (\Gamma_{ij}) \tag{4.40}$$

the matrix U is finally equal to

$$U = B^T \Gamma B \tag{4.41}$$

The 3x1 matrix

$$R = \int_m \tilde{u}\, \overset{\circ}{u}\, dm \tag{4.42}$$

may be expressed in terms of the elements of the Γ matrix, as follows :

$$R = B^T C \dot{\beta} \tag{4.43}$$

where the 3nxn matrix C is given by :

$$C = \begin{bmatrix} \varepsilon_{\alpha 1 \beta} & \Gamma_{\alpha\beta} \\ \varepsilon_{\alpha 2 \beta} & \Gamma_{\alpha\beta} \\ \varepsilon_{\alpha 3 \beta} & \Gamma_{\alpha\beta} \end{bmatrix} \tag{4.44}$$

with definitions given by (4.28), (4.36), (4.41), (4.43), the equation (4.15) may be written

$$\begin{aligned} L = {} & I\dot{\omega} + \tilde{\omega} I \omega \\ & + 2B^T \Lambda \dot{\omega} + 2\tilde{\omega} B^T \Lambda \omega + 2\dot{B}^T \Lambda \omega \\ & + B^T \Gamma B \dot{\omega} + \tilde{\omega} B^T \Gamma B \omega + \dot{B}^T \Gamma B \omega \\ & + B^T \Gamma \dot{B} \omega + B^T C \ddot{\beta} + \tilde{\omega} B^T C \dot{\beta} \end{aligned} \tag{4.45}$$

In order to express the kinetic energy in β_ν explicitly we need the expression for $\int_m u^T u \, dm$. This integral is equal to

$$\int_m u^T u \, dm = \beta^T \int_m \phi_k \phi_k^T \, dm \, \beta = \beta^T M_d \beta \tag{4.46}$$

The matrix M_d is a diagonal matrix because of the normality of the modes, namely by use of expression (4.25)

$$M_d = \begin{bmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_n \end{bmatrix} \tag{4.47}$$

For elastic and viscoelastic material the potential energy of deformation may be written

$$U_d = \frac{1}{2} \beta^T M_d \Omega^2 \beta \tag{4.48}$$

where the matrix Ω is a diagonal matrix given by

$$\Omega = \begin{bmatrix} \omega_1 & & 0 \\ & \ddots & \\ 0 & & \omega_n \end{bmatrix} \tag{4.49}$$

The dissipation function is

$$U = \frac{1}{2} \dot{\beta}^T Z \beta \tag{4.50}$$

and the kinetic energy is given by

$$\begin{aligned} T = \frac{1}{2} m (\dot{r}^T \dot{r}) + \frac{1}{2} \omega^T I \omega + \omega^T B^T \Lambda \omega \\ + \frac{1}{2} \omega^T B^T \Gamma B \omega + \omega^T B^T C \dot{\beta} + \frac{1}{2} \dot{\beta}^T M_d \dot{\beta} \end{aligned} \tag{4.51}$$

The matrices $B_{,k}$ and $\beta_{,k}$ are defined by

$$\begin{aligned} B_{,k} &= \frac{dB}{d\beta_k} \\ \beta_{,k} &= \frac{d\beta}{d\beta_k} \end{aligned} \tag{4.52}$$

That is to say, $B_{,k}$ $(\beta_{,k})$ is equal to the $B(\beta)$ matrix in which all the β_i, with $i \neq k$, are equal to zero and β_k is equal to 1.

With these definitions the Lagrange's equation in β_ν is

$$\begin{aligned} & m_\nu \ddot{\beta}_\nu + m_\nu \omega_\nu^2 \beta_\nu + \omega^T \dot{B}^T C \beta_{,\nu} + \dot{\omega} B^T C \beta_{,\nu} \\ & - \omega^T B_{,\nu}^T A \omega - \frac{1}{2} \omega^T B_{,\nu}^T \Gamma B \omega - \frac{1}{2} \omega^T B^T \Gamma B_{,\nu} \omega \\ & - \omega^T B_{,\nu}^T C \dot{\beta} + \frac{\partial V}{\partial \beta_\nu} + m_\nu \xi_{\nu\mu} \dot{\beta}_\mu = N_\nu \\ & \qquad (\nu = 1 \ldots n) \end{aligned} \tag{4.53}$$

where V is the potential energy due to an external field.

When the development of the displacement contains quadratic terms in the variable β_ν, the J matrix may be written, up to the second order in β_ν, as :

$$J = I + 2B^T A + B^T \Pi B \tag{4.54}$$

where the matrices I and A are defined in (4.27) and (4.35), respectively. The 3nx3n matrix Π may now differ from the matrix Γ.

The equations of the motion linearized in β_ν are now given by

$$\begin{aligned} L = {} & I\dot{\omega} + \tilde{\omega} I \omega \\ & + 2B^T A \dot{\omega} + 2\tilde{\omega} B^T A \omega \\ & + 2\dot{B}^T A \omega \end{aligned} \tag{4.55}$$

and

$$\begin{aligned} & m_\nu \ddot{\beta}_\nu + m_\nu \omega_\nu^2 \beta_\nu + \omega^T \dot{B}^T C \beta_{,\nu} + \dot{\omega} B^T C \beta_{,\nu} \\ & - \omega^T B_{,\nu}^T A \omega - \frac{1}{2} \omega^T B_{,\nu}^T \Pi B \omega - \frac{1}{2} \omega^T B^T \Pi B_{,\nu} \omega \\ & - \omega^T B_{,\nu}^T C \dot{\beta} + \frac{\partial V}{\partial \beta_\nu} + m_\nu \xi_{\nu\mu} \beta\mu = N_\nu \\ & \qquad (\nu = 1 \ldots n) \end{aligned} \tag{4.56}$$

The equations (4.55) and (4.56) form a complete set of (3+n) linear equation.

5. Torque and Potential Due to an Inverse Square Gravitational Field.

The force due to this field on the element of mass dm is

$$\underline{dF} = -k\frac{\underline{r}+\underline{\rho}}{|\underline{r}+\underline{\rho}|^3}\,dm \qquad (5.1)$$

where k is the constant equal to the product of the gravitation constant G and the mass M of the attracting body (here the Earth).

The total torque around the center of mass of the body will be

$$\underline{M} = \int_m \underline{\rho} \times \underline{dF} \qquad (5.2)$$

Neglecting the terms in $(\rho/r)^2$ relative to ρ/r one has

$$\frac{1}{|r+\rho|^3} = \frac{1}{r^3}\left(1 - 3\,\frac{\underline{\rho}\cdot\underline{r}}{r^2}\right) \qquad (5.3)$$

where r is the norm of $\underline{r}$ and

ρ the norm of $\underline{\rho}$.

and

$$\underline{M} = -\frac{k}{r^3}\int_m \left(1 - 3\,\frac{\underline{\rho}\cdot\underline{r}}{r^2}\right)[\underline{\rho}\times(\underline{r}+\underline{\rho})]\,dm \qquad (5.4)$$

or

$$\underline{M} = -\frac{3k}{r^5}\int_m \underline{r}\times\underline{\rho\rho}\cdot\underline{r}\,dm \qquad (5.5)$$

Defining the rotating frame $\underline{\hat{a}}_1\ \underline{\hat{a}}_2\ \underline{\hat{a}}_3$, such that $\underline{r} = r\underline{\hat{a}}_1$ and with $\underline{\hat{a}}_3$ perpendicular to the plane of the orbit and aligned along the angular momentum vector, the torque $\underline{M}$ is

$$\underline{M} = -\frac{3k}{r^3}\,\underline{\hat{a}}_1\times\int_m \underline{\rho\rho}\,dm\cdot\underline{\hat{a}}_1 \qquad (5.6)$$

By virtue of the definition of $\mathcal{J}$ and as :

$$\underline{\hat{a}}_1 \times \; E \; \cdot \underline{\hat{a}}_1 = 0 \tag{5.7}$$

$$\underline{M} = \frac{3k}{r^3} \underline{\hat{a}}_1 \times \; J \; \cdot \underline{\hat{a}}_1 \tag{5.8}$$

In order to express the equation (5.8) in matrix notation, the components of the vector $\underline{\hat{a}}_1$ in the D frame have to be determined. The orientation of the body-frame D relative to the orbiting reference frame A will be defined by the three angles of rotation θ_1, θ_2, θ_3. Intermediate frames B and C will then be determined. A rotation θ_1 about the axis $\hat{a}_1$ of the frame A brings this frame into coincidence with the frame B. Similarly a rotation θ_2 about the axis $\underline{\hat{b}}_2$ brings the frame B in coincidence with frame C and a rotation θ_3 about the axis $\underline{\hat{c}}_3$ brings the C frame in coincidence with the body frame D (fig. 2).

The frame A is related to the frame D by the kinematical relation

$$\begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix} = \theta \begin{bmatrix} \hat{d}_1 \\ \hat{d}_2 \\ \hat{d}_3 \end{bmatrix} \tag{5.9}$$

where the 3x3 matrix H is a function of θ_1, θ_2 and θ_3, given by

$$\theta = \begin{bmatrix} c\theta_2 c\theta_3 & -c\theta_2 s\theta_3 & s\theta_2 \\ c\theta_1 s\theta_3 + s\theta_1 s\theta_2 c\theta_3 & c\theta_1 c\theta_3 - s\theta_1 s\theta_2 s\theta_3 & -s\theta_1 c\theta_2 \\ s\theta_1 s\theta_3 - c\theta_1 s\theta_2 c\theta_3 & s\theta_1 c\theta_3 + c\theta_1 s\theta_2 s\theta_3 & c\theta_1 c\theta_2 \end{bmatrix} \tag{5.10}$$

where c stands for cosine, and
s stands for sine.

Then,

$$\underline{\hat{a}}_1 = \theta_{11} \underline{\hat{d}}_1 + \theta_{12} \underline{\hat{d}}_2 + \theta_{13} \underline{\hat{d}}_3 \tag{5.11}$$

With the matrix θ_1

$$\theta_1 = \begin{bmatrix} \theta_{11} \\ \theta_{12} \\ \theta_{13} \end{bmatrix} \tag{5.12}$$

the vector $\underline{\hat{a}}_1$ is written as

$$\underline{\hat{a}}_1 = d^T\theta_1 \tag{5.13}$$

Equation (5.8) becomes in matrix form :

$$M = \frac{3k}{r^3}\tilde{\theta}_1 J\theta_1 \tag{5.14}$$

or

$$M = \frac{3k}{r^3}\tilde{\theta}_1 \,(I + 2B^T A + B^T HB)\theta_1$$

The gravity potential V is :

$$V = -k\int_m \frac{dm}{|r+\rho|} \tag{5.15}$$

Developing $\frac{1}{\underline{r}+\underline{\rho}}$ up to the second order in $\frac{\rho}{r}$ provides :

$$\begin{aligned} V &= -\frac{k}{r}\int_m\left(1 - \frac{\underline{\rho}\cdot\underline{r}}{r^2} - \frac{\rho^2}{2r^2} + \frac{3}{2}\frac{(\underline{r}\cdot\underline{\rho})^2}{r^4}\right) dm \\ &= -\frac{k}{r}m + \frac{k}{r^3}\int_m \frac{\rho^2}{2}\, dm - \frac{3k}{2r^3}\,\underline{\hat{a}}_1 \cdot \int_m \rho\underline{\rho}\, dm \cdot \underline{\hat{a}}_1 \end{aligned} \tag{5.16}$$

By using the definition of J and noting the relation

$$\underline{\hat{a}}_1 \cdot E \cdot \underline{\hat{a}}_1 = 1 \tag{5.17}$$

V is finally given by :

$$V = \frac{k}{r}m - \frac{k}{r^3}\int_m \underline{\rho}\cdot\underline{\rho}\, dm + \frac{3}{2}\frac{k}{r^3}\,\underline{\hat{a}}_1 \cdot J \; \underline{\hat{a}}_1 \tag{5.18}$$

$\int \underline{\rho}\cdot\underline{\rho}\ dm$ may be written

$$\int_m \underline{\rho}\cdot\underline{\rho}\ dm = \frac{1}{2}\ trJ = \frac{1}{2}\ J_{kk} \tag{5.19}$$

where the symbol tr means trace of the following matrix. From (5.18), the potential may be written in matrix form as :

$$V = -\frac{k}{r}\ m - \frac{k}{2r^3}\ trJ + \frac{3k}{2r^3}\ \theta_1^T J \theta_1 \tag{5.20}$$

With the previous definitions, potential energy in the inverse square field may be written

$$\begin{aligned} V = \frac{k}{r}\ m - \frac{k}{2r^3}\ trI + \frac{3k}{2r^3}\ \theta_1^T I \theta_1 \\ - \frac{k}{r^3}\ tr(B^T\Lambda) + \frac{3k}{r^3}\ \theta_1^T B^T \Lambda \theta_1 \\ - \frac{k}{2r^3}\ tr(B^T\Gamma B) + \frac{3k}{2r^3}\ \theta_1^T B^T \Gamma B \theta_1 \end{aligned} \tag{5.21}$$

and

$$\begin{aligned} \frac{\partial V}{\partial \beta_\nu} = -\frac{k}{r^3}\ tr(B^T_{,\nu}\Lambda) - \frac{k}{2r^3}\ tr(B^T_{,\nu}\Gamma B + B^T\Gamma B_{,\nu}) \\ + \frac{3k}{r^3}\ \theta_1^T B^T_{,\nu} \Lambda \theta_1 \\ + \frac{3k}{2r^3}\ \theta_1^T\ (B^T_{,\nu}\Gamma B + B^T\Gamma B_{,\nu})\theta_1 \end{aligned} \tag{5.22}$$

Equation (5.22) gives the influence of the gravity potential or the ν^{th} mode.

In the gravitational field the left-hand side of (4.15) is given by (5.14) and the value of U in expression (4.16) is equal to the sum of U_d and V.

6. Attitude stability of deformable earth-pointing satellites.

6.1 Determination of the angular velocity.

The influence of the rotation of the body on the translational motion will be neglected and the earth will be considered as a homogeneous sphere.

The rotating reference axes are the A-axes defined in part 5 and represented in Figure 2. The body "fixed"-axes are the D-axes defined in part 3 by the relations (3.1). These axes are directed in such a manner that the deformations are expressed as a sum of normal modes and are coinciding with the principal axes of the undeformed body. The angular position of the D-frame with respect to the reference frame A is given by the three angles of rotation θ_1, θ_2, θ_3.

The total angular velocity of the D-frame, $\underline{\omega}$, is given by the relation

$$\underline{\omega} = \dot{\nu}\, \underline{\hat{a}}_3 + \dot{\theta}_1 \underline{\hat{a}}_1 + \dot{\theta}_2 \underline{\hat{b}}_2 + \dot{\theta}_3 \underline{\hat{c}}_3 \qquad (6.1)$$

where ν is the orbital angular velocity or in the D-frame

$$\underline{\omega} = \sum_{\alpha=1}^{3} \omega_\alpha \underline{\hat{d}}_\alpha \qquad (6.2)$$

where the components ω_α are given by :

$$\begin{aligned} \omega_1 &= (\dot{\theta}_2 + \dot{\nu} \sin\theta_1)\sin\theta_3 + (\dot{\theta}_1\cos\theta_2 - \dot{\nu} \cos\theta_1\sin\theta_2)\cos\theta_3 \\ \omega_2 &= (\dot{\theta}_2 + \dot{\nu} \sin\theta_1)\cos\theta_2 - (\dot{\theta}_1\cos\theta_2 - \dot{\nu} \cos\theta_1\sin\theta_2)\sin\theta_3 \qquad (6.3) \\ \omega_3 &= \dot{\theta}_3 + \dot{\theta}_1\sin\theta_2 + \dot{\nu} \cos\theta_1\cos\theta_2 \end{aligned}$$

For earth-pointing satellites the three angles θ_1, θ_2, θ_3 will be considered as small angles and the equations will be linearized.

The linearized components of $\underline{\omega}$ are then :

$$\begin{aligned}\omega_1 &= \dot\theta_1 - \dot\nu\,\theta_2\\ \omega_2 &= \dot\theta_2 + \dot\nu\,\theta_1\\ \omega_3 &= \dot\nu + \dot\theta_3\end{aligned} \qquad (6.4)$$

The vector $\dot{\underline{\omega}}$ is also expressed in the D frame

$$\dot{\underline{\omega}} = \sum_{\alpha=1}^{3} \dot\omega_\alpha\, \underline{\hat{a}}_\alpha$$

where the linearized components are

$$\begin{aligned}\dot\omega_1 &= \ddot\theta_1 - \dot\nu\,\dot\theta_2 - \ddot\nu\,\theta_2\\ \dot\omega_2 &= \ddot\theta_2 + \dot\nu\,\dot\theta_1 + \ddot\nu\,\theta_1\\ \dot\omega_3 &= \ddot\theta_3 + \ddot\nu\end{aligned} \qquad (6.5)$$

The matrix is then :

$$\omega = \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} \qquad (6.6)$$

and its derivative $\dot\omega$ is simply given by

$$\dot\omega = \begin{bmatrix}\dot\omega_1\\ \dot\omega_2\\ \dot\omega_3\end{bmatrix} \qquad (6.7)$$

For convenience, the mean anomaly τ will be taken as new independent variable and the dot ($\dot{}$) will mean, from now, derivative with respect to τ.

6.2 Equilibrium Position on a circular Orbit.

This paragraph is developed from the concepts of ref. [4] where the equilibrium orientation of rigid bodies on circular orbit was investigated. A deformable satellite is in equilibrium with respect to the rotating frame A when the following conditions are satisfied :

1) the generalized coordinates of deformation, β_ν, are equal to some constants ;
2) the body "fixed" frame D has the same angular velocity in inertial space as the reference frame A.

The normalized angular velocity of the frame A is a unit vector aligned with the axis $\underline{\hat{a}}_3$:

$$\underline{\omega} = \underline{\hat{a}}_3 \qquad (6.9)$$

At equilibrium the rotational equations (4.10) reduce to

$$\underline{\omega} \times \mathcal{J} \cdot \underline{\omega} = \underline{L} \qquad (6.10)$$

or

$$\underline{\hat{a}}_3 \times \mathcal{J} \cdot \underline{\hat{a}}_3 = \underline{L} \qquad (6.11)$$

The torque due to the inverse-square gravitational field is given by (5.6). On circular orbit the normalized equation (6.10) is then :

$$\underline{\hat{a}}_3 \times \mathcal{J} \cdot \underline{\hat{a}}_3 = 3\underline{\hat{a}}_1 \times \mathcal{J} \cdot \underline{\hat{a}}_1 \qquad (6.12)$$

The vectors $\mathcal{J} \cdot \underline{\hat{a}}_3$ and $\mathcal{J} \cdot \underline{\hat{a}}_1$ may be expressed in the A-basis :

$$\begin{aligned} \mathcal{J} \cdot \underline{\hat{a}}_3 &= a\,\underline{\hat{a}}_1 + b\,\underline{\hat{a}}_2 + \lambda\,\underline{\hat{a}}_3 \\ \mathcal{J} \cdot \underline{\hat{a}}_1 &= c\,\underline{\hat{a}}_3 + d\,\underline{\hat{a}}_2 + \mu\,\underline{\hat{a}}_1 \end{aligned} \qquad (6.13)$$

Substituting the relations (6.13) in (6.12) provides

$$b = d = 0$$
$$a = -3c$$

Dot multiplying $\mathcal{J} \cdot \underline{\hat{a}}_3$ by $\underline{\hat{a}}_1$ and $\mathcal{J} \cdot \underline{\hat{a}}_1$ by $\underline{\hat{a}}_3$ provides the relations :

$$\underline{\hat{a}}_1 \cdot \mathcal{J} \cdot \underline{\hat{a}}_3 = -3c$$
$$\underline{\hat{a}}_3 \cdot \mathcal{J} \cdot \underline{\hat{a}}_1 = c \tag{6.14}$$

From the definitions of the inertia dyadic these two quantities have to be equal and then c = 0. Finally, one has :

$$\mathcal{J} \cdot \underline{\hat{a}}_3 = \lambda \, \underline{\hat{a}}_3$$
$$\mathcal{J} \cdot \underline{\hat{a}}_1 = \mu \, \underline{\hat{a}}_1 \tag{6.15}$$

The two relations (6.15) imply that the axes $\underline{\hat{a}}_1$ and $\underline{\hat{a}}_3$ are principal axes of inertia. Then in equilibrium the A-axes are the principal axes of the body.

The determination of the orientation of the body fixed axes in the frame turns out to be an eigenvalue problem as shown in [4].

6.3 Equations of Motion.

The rotational equations are given by (4.55) where the components of the vectors ω and $\dot{\omega}$ are given by (6.4) and (6.5) As only circular orbit is treated the normalized orbital angular velocity $\dot{\nu}$ is equal to one. The gravitational torque is given by (5.14), where the normalized value of $3k/r^3$ is 3, $\dot{\nu}$ being equal to k/r^3.

The linearized rotational equations are then :

$$I_1\ddot{\theta}_1 - (I_1 + I_2 - I_3)\dot{\theta}_2 - (I_2 - I_3)\theta_1 - 2\Lambda^T_{23}\beta + 2\Lambda^T_{13}\dot{\beta} = 0$$

$$I_2\ddot{\theta}_2 + (I_1 + I_2 - I_3)\dot{\theta}_1 - 4(I_1 - I_3)\theta_2 + 8\Lambda^T_{31}\beta + 2\Lambda^T_{23}\dot{\beta} = 0 \tag{6.16}$$

$$I_3\ddot{\theta}_3 - 3(I_1 - I_2)\theta_3 + 2\Lambda^T_{33}\dot{\beta} - 6\Lambda^T_{21}\beta = 0$$

The modal equations are given by (4.56) and in the inverse square gravitational field, these linearized equations are in matrix form :

$$
\begin{aligned}
& M_d\ddot{\beta} + M_d Z\dot{\beta} + M_d\Omega^2\beta \\
& + (\Pi_{11} + \Pi_{11}^T - \Pi_{33} - \Pi_{33}^T - \tfrac{1}{2}\Pi_{22} - \tfrac{1}{2}\Pi_{22}^T)\,\beta \\
& + (\Pi_{21}^T - \Pi_{12}^T + \Pi_{21} - \Pi_{12})\,\dot{\beta} \\
& - 2A_{23}\theta_1 + 8A_{13}\theta_2 - 6A_{12}\theta_3 \qquad\qquad (6.17) \\
& - 2A_{13}\dot{\theta}_1 - 2A_{23}\dot{\theta}_2 - 2A_{33}\dot{\theta}_3 \\
& = - 2A_{11} + A_{22} + 2A_{33}
\end{aligned}
$$

It is seen that when the matrix Π is equal to Γ, or more generally when Π is a symmetric matrix, the gyroscopic coupling between the modes disappears.

The right-hand side of (6.17) reflects the effect of centrifugal force on the deformation. In equilibrium, the coordinates θ_1, θ_2, θ_3 and β_ν will not be necessarily equal to zero but it is assured that the new principal axes are oriented along the A-axes.

The generalized coordinates are then

$$
\begin{aligned}
\theta_i &= \theta_{io} + \delta\theta_i \\
\beta_\nu &= \beta_{\nu o} + \delta\beta_\nu
\end{aligned} \qquad (6.18)
$$

where θ_{io} and $\beta_{\nu o}$ are respectively the value of θ_i and β_ν at equilibrium.

The (n+3) vector n is defined as

$$
x = \begin{bmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\theta_3 \\ \delta\beta \end{bmatrix} \qquad (6.19)
$$

The system formed of (6.16) and (6.17) may then be written in matrix form as :

$$M\ddot{x} + G\dot{x} + Kx = - D\dot{x} \tag{6.20}$$

where the (3+n) × (3+n) matrices M, D, G, and K are given by

$$M = \begin{bmatrix} I_1 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & I_3 & 0 \\ 0 & 0 & 0 & M_d \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Z \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & -(I_1+I_2-I_3) & 0 & 2\Lambda_{13}^T \\ I_1+I_2-I_3 & 0 & 0 & 2\Lambda_{23}^T \\ 0 & 0 & 0 & 2\Lambda_{33}^T \\ -2\Lambda_{13} & -2\Lambda_{23} & -2\Lambda_{33} & \Pi_{21}^T-\Pi_{12}^T+\Pi_{21}-\Pi_{12} \end{bmatrix} \tag{6.21}$$

$$K = \begin{bmatrix} -(I_2-I_3) & 0 & 0 & -2\Lambda_{23}^T \\ 0 & -4(I_1-I_3) & 0 & 8\Lambda_{13}^T \\ 0 & 0 & -3(I_1-I_2) & -6\Lambda_{21}^T \\ -2\Lambda_{23} & 8\Lambda_{13} & -6\Lambda_{12} & \begin{matrix}\Pi_{11}+\Pi_{11}^T-\Pi_{33}-\Pi_{33}^T \\ \frac{\Pi_{22}}{2}-\frac{\Pi_{22}^T}{2}+M_d\Omega^2\end{matrix} \end{bmatrix}$$

The matrices Λ_{12}, Λ_{23} and Λ_{13} are equal to Λ_{21}, Λ_{32} and Λ_{31}, respectively ; it is seen that the matrices M, D and K are symmetric and that the matrix G is skew-symmetric.

The Hamiltonian of the system (6.20) is

$$H = \dot{x}^T M\dot{x} + x^T K x \tag{6.22}$$

and its time derivative is

$$\dot{H} = - \dot{x}^T D \dot{x} \tag{6.23}$$

From (6.20) it is seen that $\dot{H}$ is a negative semi-infinite function.

The Hamiltonian is taken as Liapunov function, and its positiveness is a necessary and sufficient condition for stability when the damping is complete [5], [6], [7], which occurs when all the variables are coupled.

The Hamiltonian is positive definite when the matrix K is positive definite. From Sylvester's criterion the determinants of all the principal minors of K have to be positive.

It is seen immediately that the relations

$$I_3 \geq I_2 \geq I_1 \tag{6.24}$$

must hold to have stability.

This shows that the presence of energy dissipation destabilizes the so-called "Delp"-region [8] in the K_1K_2-plane, the parameters $K_1K_2K_3$ being defined as

$$K_1 = \frac{I_2 - I_3}{I_1}$$
$$K_2 = \frac{I_3 - I_1}{I_2} \tag{6.25}$$
$$K_3 = \frac{I_1 - I_2}{I_3}$$

In this Delp-region for which

$$I_1 \geq I_2 \geq I_3$$

rigid orbiting satellites are stable.

Further, the relation (6.24) is a necessary condition for stability and the requirement of positiveness for other minors of K may only decrease the stability in the Lagrange region.

For completely damped systems the stability of eq.(6.20) is the same as the stability of

$$M\ddot{x} + Kx = 0 \tag{6.26}$$

as seen in [5]. The eigenfrequencies of the system (6.26) are given by the roots of

$$|K - \omega^2 M| = 0 \tag{6.27}$$

There are (3+n) real eigenvalues ω^2 which satisfy (6.27). If these 3+n eigenvalues are ordered in the manner such that

$$\omega_1^2 \leq \omega_2^2 \leq \omega_3^2 \ldots \leq \omega_{3+n}^2 \tag{6.28}$$

it is seen that if there exist n linearly independent constraints between the coordinates of the systems, the three eigenvalues $\omega_1'^2$ $\omega_2'^2$ and $\omega_3'^2$ of the restricted system are neither smaller than ω_1^2 ω_2^2 and ω_3^2, respectively, nor larger than ω_{n+1}^2, ω_{n+2}^2 and ω_{n+3}^2. This theorem given in [9] by Courant and Hilbert was already available from Cauchy's work [10].

Then, certainly

$$\omega_1^2 \leq \omega_1'^2 \tag{6.29}$$

When there exist n linearly independent constraints between the n coordinates of deformation, all the β_ν are identically equal to zero and the square of the frequencies $\omega_i'^2$ are the eigenvalues of the system (6.22) for rigid bodies.

To the boundaries of the Lagrange's region in the K_1-K_2 plane correspond frequencies ω' equal to zero. Then when deformations described by n independent coordinates occur, the system (6.26) may have up to n negative eigenvalues ω_i^2 which are leading to instability for this system, and then also for the system (6.20). Further, from a theorem by Liapunov [11], it is seen that the stability of the linear system is not modified when nonlinear terms are considered, all the conditions of this theorem being satisfied.

6.4 Particular Example.

A particular example with one mode of deformation will now be investigated.

The satellite under consideration is composed of two rigid bodies B_1 and B_2 attached at their center of mass (fig. 3 and 4). One hinge permits a relative motion between the two bodies. The axis of the hinge is directed along a common axis of principal moment of inertia, the $\hat{a}_2$-axis. When the system is in equilibrium in free space, the axis of principal moments of inertia of both bodies are coinciding with the axes $\hat{a}_1$, $\hat{a}_2$ and $\hat{a}_3$. These moments of inertia are, respectively, I_1^1, I_2^1 and I_3^1 for the body B_1 and I_1^2, I_2^2 and I_3^2 for the body B_2. A linear torsional spring with constant k_d and a viscous damper with constant c resist the relative motion of the two bodies.

The rotations of B_1 and B_2 about the $\hat{a}_2$-axis are described by the angle β and γ, respectively. These angles are defined with their sign in fig. 4.

When the composite body is freely vibrating in inertial space, the modal equation is :

$$\ddot{\beta} + \omega_\nu^{\,2}\beta \tag{6.30}$$

where

$$\omega_\nu^{\,2} = k_d \left(\frac{1}{I_2^{\,1}} + \frac{1}{I_2^{\,2}}\right) \tag{6.31}$$

and the angle γ is then related to β by

$$\gamma = \rho\beta \tag{6.32}$$

where

$$\rho = \frac{I_2^{\,1}}{I_2^{\,2}} \tag{6.33}$$

In the D-frame the components of the position vector $\underline{\rho}$ of the element of mass dm relatively to the center of mass are expressed in terms of the components of the position vector dm in the undeformed body, x, and the generalized coordinate β as :

For B_1
$$\begin{aligned} \rho_1^1 &= x_1^1\cos\beta + x_3^1\sin\beta \\ \rho_2^1 &= x_2^1 \\ \rho_3^1 &= x_3^1\cos\beta - x_1^1\sin\beta \end{aligned} \tag{6.34}$$

For B_2
$$\begin{aligned} \rho_1^2 &= x_1^2\cos\rho\beta - x_3^2\sin\rho\beta \\ \rho_2^2 &= x_2^2 \\ \rho_3^2 &= x_3^2\cos\rho\beta + x_1^2\sin\rho\beta \end{aligned} \tag{6.35}$$

Up to the second power in β the matrix J is then given by

$$J = I + 2A\beta + \Pi\beta^2 \tag{6.36}$$

where the nonzero elements of I, A and Π are :

$$\begin{aligned} I_i &= I_i^1 + I_i^2 \qquad i = 1,2,3 \\ A_{31} &= A_{13} = \frac{1}{2}\,[(I_3^1-I_1^1) + \rho(I_1^2-I_3^2)] \\ \Pi_{11} &= I_3^1 - I_1^1 + \rho^2(I_3^2-I_1^2) \\ \Pi_{22} &= I_1^1 - I_3^1 + \rho^2(I_1^2-I_3^2) \end{aligned} \tag{6.37}$$

Further the generalized mass of the mode of vibration, $m_\nu = 1/2\Gamma_{\mu|}$ is given by

$$m_\nu = \rho I_2$$

On circular orbit one equilibrium is obtained when the frame D coincides with the A frame, the compositive body is stable when the K matrix defined in (6.21) is positive definite. The matrix M being positive definite, the matrix $M^{-1}K$ has also to be positive definite to have stability.

From (6.21) and (6.32) this matrix is

$$M^{-1}K = \begin{vmatrix} -K_1 & 0 & 0 & 0 \\ 0 & 4K_2 & 0 & 4\frac{\rho}{\rho+1}(K_2^1-K_2^2) \\ 0 & 0 & -3K_3 & 0 \\ 0 & \frac{4}{\rho+1}(K_2^1-K_2^1) & 0 & \omega_\nu^2+\frac{4}{\rho+1}(K_2^1+\rho K_2^2) \end{vmatrix} \tag{6.39}$$

where K_1, K_2, K_3 are defined in (6.26) and

$$K_2^i = \frac{I_3^i - I_1^i}{I_2^i} \qquad (i = 1,2) \tag{6.40}$$

It must be noted that K_2 is related to K_2^1 and K_2^2 by the relation

$$K_2 = \frac{\rho}{\rho+1} K_2^1 + \frac{1}{\rho+1} K_2^2 \tag{6.41}$$

The system is stable if the following conditions are satisfied

$$\begin{aligned} K_1 &\leq 0 \\ K_2 &\geq 0 \\ K_3 &\leq 0 \end{aligned} \tag{6.42}$$

and

$$\begin{vmatrix} 4K_2 & 4\frac{\rho}{\rho+1}(K_2^1-K_2^2) \\ \frac{4}{\rho+1}(K_2^1-K_2^2) & \omega_\nu^2 + \frac{4}{\rho+1}(K_2^1+\rho K_2^2) \end{vmatrix} \geq 0 \tag{6.43}$$

The first three conditions are satisfied in the so-called Lagrange's regions of the K_1-K_2 plane, the Delp-region [8] being Liapunov unstable.

When $\omega_\nu = 0$ the condition (6.43) reduces to

$$K_2^1 K_2^2 \geq 0 \tag{6.44}$$

Then, from the conditions (6.42) and the relation (6.41) this latter condition requires :

$$K_2^1 \geqslant 0$$

and

$$K_2^2 \geqslant 0$$

When k_d is zero to have stability, the two bodies B_1 and B_2 have to be stable separately. This is an obvious conclusion.

7. Conclusions.

The formulation presented for the dynamics of deformable bodies is very powerful mainly when the deformations are small, in other words, when the equations may be linearised. The equations are presented in matrix form which has some advantages for numerical computation. Further the elements of the matrices are easily determined by modal analysis of the system. This may be done theoretically or realised experimentally in the laboratory.

The Liapunov stability of earth-pointing satellites is determined for any equilibrium orientation. It is seen that, in equilibrium, on circular orbit the principal axes of the satellites are coinciding with the orbital reference axes. When there is some energy dissipation, equilibrium may be obtained only when the axes of larger and smaller moment of inertia are respectively perpendicular to the plane of the orbit and directed towards the center of the earth.

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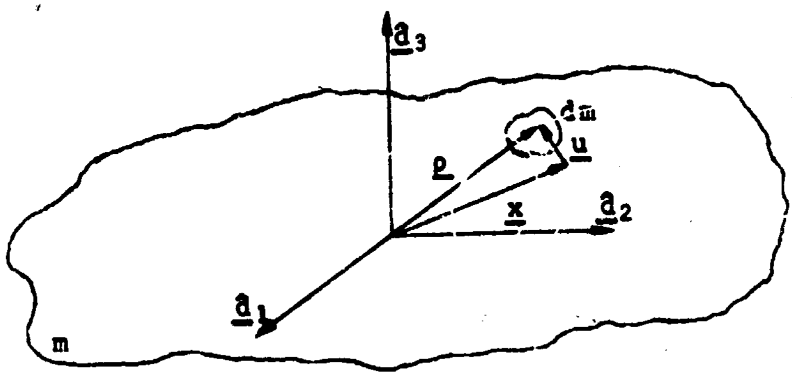


Fig. 1

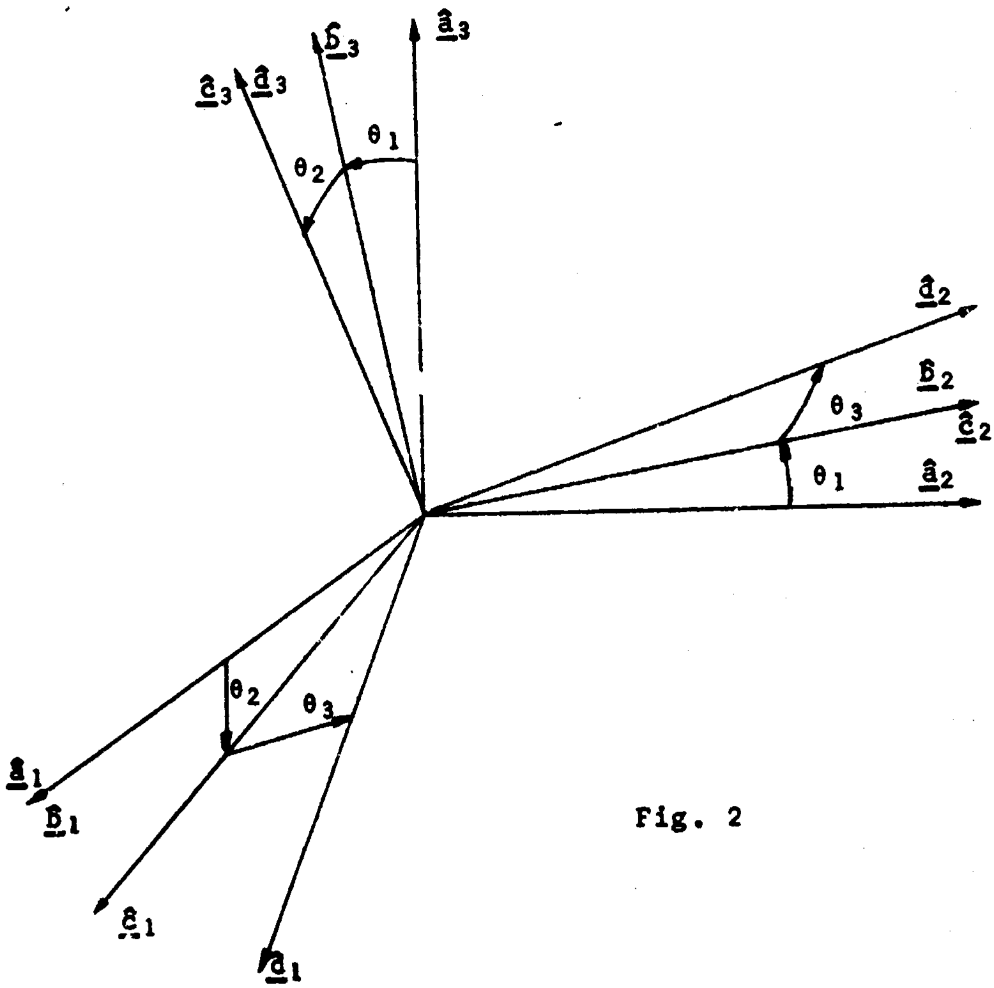
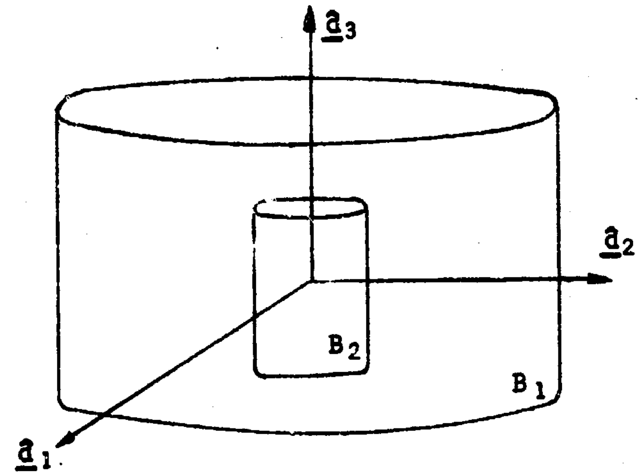


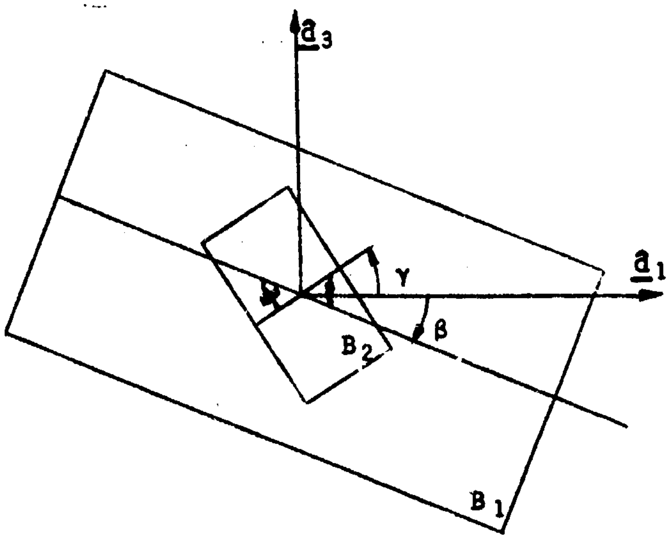
Fig. 2

Fig. 3



$\underline{a}_3$
$\underline{a}_2$
B_2
B_1
$\underline{a}_1$

Fig. 4



$\underline{a}_3$
γ
$\underline{a}_1$
β
B_2
B_1