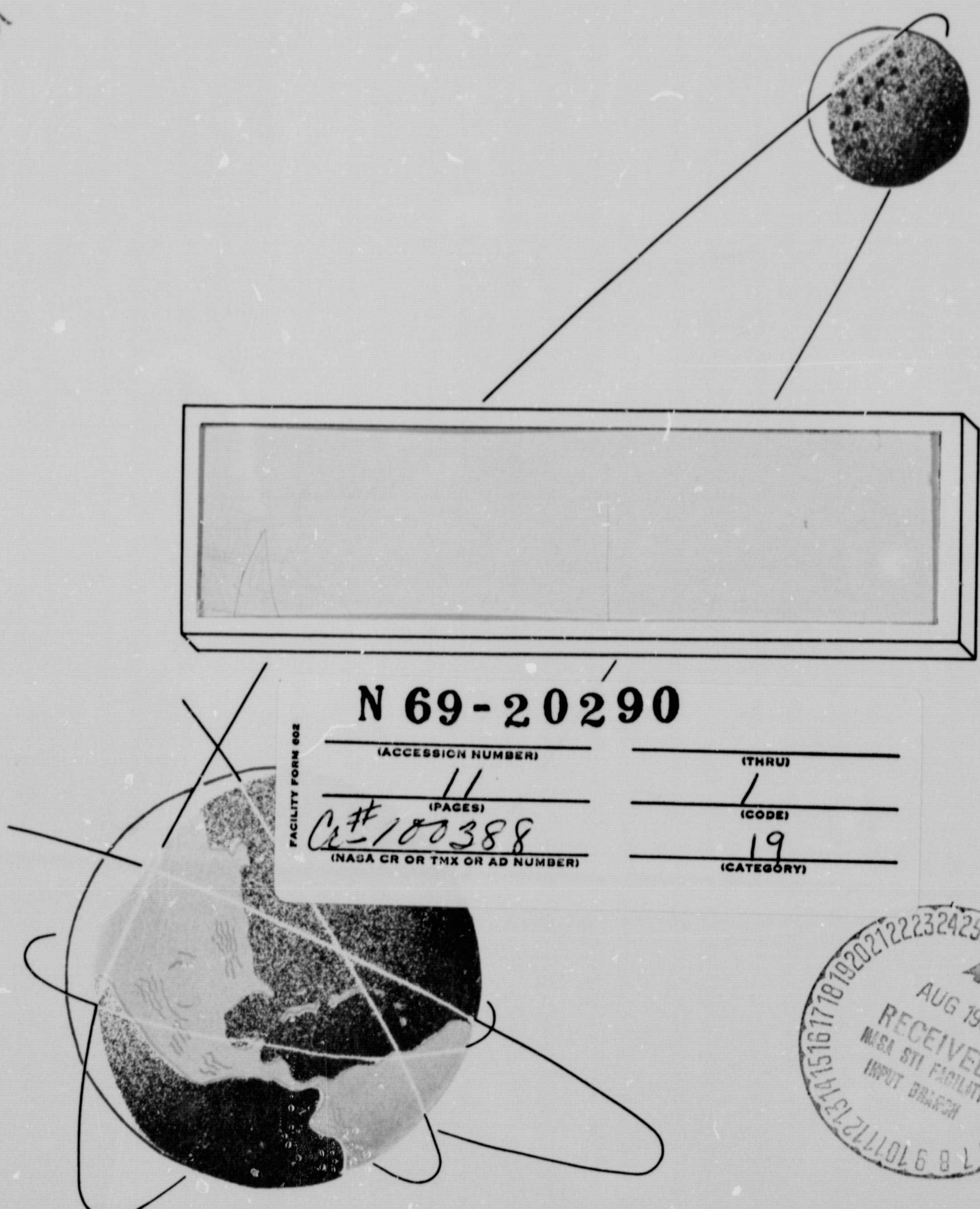

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ON THE STABILITY OF INERTIAL SYSTEMS

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ON THE STABILITY OF INERTIAL SYSTEMS

Summary:

The system described by the differential equation

$$\ddot{x} + f(\sigma) = 0$$

where $\sigma = a_0 x + a_1 \dot{x}$, a_0, $a_1 > 0$ is stable provided $\sigma f(\sigma) > 0$, $\sigma \neq 0$, $\int_0^\sigma f(u)\, du \to \infty$ as $|\sigma| \to \infty$. The present paper discusses some properties of the class of higher order systems which are obtained by coupling second order systems of the above form. Sufficient conditions are derived for such a coupled system to be stable. The controllability of the coupled system when each subsystem is forced, as well as the oscillatory behavior of the system is discussed.

1. Introduction

The stability of systems described by the differential equations

$$\dot{y}_i = Y_i(y_1, y_2, \ldots, y_m) \qquad (1)$$

$$i = 1,2,\ldots, m$$

has been extensively investigated by Liapunov's Direct Method. An extensive bibliography to this subject may be found in the recent survey paper by Brockett [1]. The present paper discusses the stability properties of inertial systems described by the differential equations

$$\ddot{x}_i = X_i(x_1, x_2, \ldots, x_n) \qquad (2)$$

$$i = 1,2,\ldots, n\ .$$

Obviously, the system (2) may be reduced to the form (1) but there are certain classes of systems which may be investigated readily in the form (2). Aggarwal and Richie [2] have investigated the stability and oscillatory behavior for the systems described by differential equations

$$\ddot{x}_i + f_i(x_i, \dot{x}_i) + g_i(x_1, x_2, \ldots, x_n) = 0$$

$$i = 1,2,\ldots, n \ .$$

This study is a continuation of the above investigation and the system under discussion here are obtained by coupling second order systems described by the differential equation

$$\ddot{x} + f(\sigma) = 0 \qquad (3)$$

where $\sigma = a_o + a_1\dot{x}$ and

(i) $f(0) = 0,\ \sigma f(\sigma)>0,\ \sigma \neq 0$

(ii) $\int_0^\sigma f(u)du \to \infty$ as $|\sigma| \to \infty$ (*)

(iii) a_o, a_1 are real .

The coupled system is of the form

$$\ddot{x}_i + f_i(\sigma_i) + k_i(\sigma_1, \sigma_2, \ldots, \sigma_n) = 0 \qquad (4)$$

where

$$\sigma_i = a_{oi}x_i + a_{1i}\dot{x}_i \quad \text{and} \quad i = 1,2,\ldots, n \ .$$

Here again the conditions (*) hold for the function $f_i(\sigma_i)$ and the constants a_{oi} and a_{ii}, are real, $i = 1,2,\ldots, n$. It is found that the properties of the higher order system may be based upon the corresponding properties of the second order systems and the coupling, and with this in view, the properties of the second order system will be discussed and these properties will be used in the study of the coupled system.

Two results due to LaSalle [3] and Cetaev [4] are given below for the sake of completeness since these results will be used extensively.

<u>LaSalle's Theorem:</u> Given the system

$$\dot{\underline{\xi}} = \underline{X}(\underline{\xi})$$

Let $V(\underline{\xi})$ be a scalar function with continuous partial derivatives satisfying

(i) $V(\underline{\xi}) > 0$ for all $\underline{\xi} \neq 0$, $V(0) = 0$

(ii) $\dot{V}(\underline{\xi}) \leq 0$ for all $\underline{\xi}$

(iii) $V(\underline{\xi}) \to \infty$ as $||\underline{\xi}|| \to \infty$

where $||\underline{\xi}|| = (\xi_1^2 + \ldots + \xi_n)^{1/2}$ and if $\dot{V}$ is not identically zero along any solution other than the origin, then the system is asymptotically stable in the large.

Cetaev's Theorem: Given the system

$$\dot{\underline{\xi}} = \underline{X}\,(\underline{\xi})\ ,\quad \text{and}$$

let Ω be a neighborhood of the origin. Further, let there be given a scalar function $V(\underline{\xi})$ and a region Ω_1 in Ω with the following properties

(i) $V(\underline{\xi})$ has continuous first partials in Ω

(ii) $V(\underline{\xi})$ and $\dot{V}(\underline{\xi})$ are positive in Ω_1

(iii) At the boundary of points of Ω_1 inside Ω, $V(\underline{\xi}) = 0$

(iv) The origin is a boundary point of Ω_1, then the origin is an unstable singular point.

For compactness we shall use the notation

$$\vec{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$$

in the following.

2. Results on the Second Order System

(a) $a_o, a_1 > 0$

The system (3) is asymptotically stable in the large using the Liapunov function

$$V = \frac{1}{2} a_o \dot{x}^2 + \int_0^\sigma f(u)du \; . \tag{5}$$

In this case

$$\dot{V} = -a_1 \; f^2(\sigma) \; , \tag{6}$$

a negative semi-definite function, however applying the result due to LaSalle mentioned above gives asymptotic stability in the large.

(b) $a_o > 0, \; a_1 < 0$.

The system (3) is unstable and the result follows immediately from reversing time in case (a).

(c) $a_o, \; a_1 < 0$.

The function (5) is indefinite whereas its derivative (6) is positive semi-definite. The configuration of constant V curves in the $(\dot{x}, \sigma)$ plane is hyperbolic in nature and the solution point temporarily approaches the origin and then diverges again from the origin as the time elapses. The conditions of Cetaev's Theorem are satisfied and the system (3) unstable.

(d) $a_o < 0, \; a_1 > 0$.

The system (3) is unstable and this result follows immediately from reversing time in case (c).

The above results may be summarized as follows: The system (3) is stable if and only if $a_o, a_1 > 0$ and unstable otherwise. It may be further observed that the system (3) does not have any periodic solution (except of course, the trivial solution $x = \dot{x} = 0$).

3. Results on the Coupled System

(a) Stability. The system (4) is asymptotically stable in the large provided:

(i) $a_{oi}, a_{1i} > 0, \quad i = 1,2,\ldots, n$

(ii) $k_i(\vec{\sigma}) = U_{\sigma i}(\vec{\sigma})$ where $U(\vec{\sigma}) \geq 0, \vec{\sigma} \neq 0$, $U(\vec{0}) = 0$ for $i = 1,2,\ldots, n$, and

(iii) The set of equations $f_i(\sigma_i) + k_i(\vec{\sigma}) = 0$ $i = 1,2,\ldots, n$ has only the trivial solution $\vec{\sigma} = \vec{0}$. (Also each f_i satisfies the conditions (*)).

This result follows immediately by using the Liapunov function

$$V = \frac{1}{2} \sum_{i=1}^{n} a_{oi}\dot{x}_i^2 + \sum_{i=1}^{n} \int_o^{\sigma_i} f_i(u)du + U(\vec{\sigma}) . \qquad (6)$$

Here

$$\dot{V} = -\sum_{i=1}^{n} a_{ii} \{f_i(\sigma_i) + U_{\sigma_i}(\vec{\sigma})\}^2 \qquad (7)$$

and the conditions of LaSalle's Theorem are satisfied.

The condition (iii) above insures that origin in the space $(x_1,\ldots, x_n, \dot{x}_1,\ldots, x_n)$ is the only singular point for the system (4). It may be observed that the function

$$U(\vec{\sigma}) + \sum_{i=1}^{n} \int_0^{\sigma_i} f_i(u)du \qquad (8)$$

is radially unbounded positive definite potential function. The above result may be stated in slightly different fashion as follows:

The system,

$$\ddot{x}_i + h_i(\vec{\sigma}) = 0, \quad i = 1,2,\ldots, n \quad \text{and} \qquad (9)$$

$$h_i(\vec{\sigma}) = U^*_{\sigma_i}(\vec{\sigma})$$

where $U^*(\vec{\sigma})$ is a positive definite radially unbounded function and $h_i(\vec{\sigma}) = 0$ $i = 1,\ldots, n$ has the trivial solution $\vec{\sigma} = \vec{0}$, is asymptotically stable in the large. It is interesting to compare this result with the properties of the system

$$\ddot{x}_i + h_i(\vec{x}) = 0, \quad i = 1,2,\ldots, n \quad \text{and} \qquad (10)$$

$$h_i(\vec{x}) = U^*_{x_i}(\vec{x})$$

where $U^*(\vec{x})$ is positive definite and radially unbounded function and $h_i(\vec{x}) = 0$, $i = 1,\ldots, n$ has the trivial solution $\vec{x} = \vec{0}$. This system (10) is conservative and has for its solution

$$\frac{1}{2}\sum_{i=1}^{n} \dot{x}_i^2 + U^*(\vec{x}) = \text{constant.}$$

Further it may be observed that the above conditions (**) for the stability of the system (4) are sufficient and not necessary as seen from the following example.

$$\ddot{x}_1 + \sigma_1 = -\alpha(\sigma_1-\sigma_2)$$
$$\ddot{x}_2 + \sigma_2 = \alpha(\sigma_1-\sigma_2) \quad . \qquad (11)$$

The characteristic equation for the above system is

$$\{s^2 + (a_{o1}+a_{11}s)(1+\alpha)\}\{s^2+(a_{02}+a_{12}s)(1+\alpha)\}$$
$$- \alpha^2(a_{02}+a_{12}s)(a_{o1}+a_{11}s) = 0 \ .$$

Now consider the case when the two systems are identical, the characteristic equations is

$$(s^2 + a_o + a_{11}s)\{s^2 +(a_{o1} + a_{11}s)(1 +2\alpha)\} = 0 \ .$$

The system (11) is stable provided $\alpha > -\frac{1}{2}$, however, $\alpha > 0$ is necessary for the same system to be stable via the conditions (**).

(b) Instability. In the case $a_{oi} > 0$, $a_{1i} < 0$, $i = 1,2,\ldots, n$ the system (4) is unstable and the result follows immediately from reversing time in the case (a) above.

In the case $a_{oi} < 0$, $a_{1i} < 0$ $i = 1,2,\ldots, n$, the V function (6) is indefinite and $\dot{V}$ is positive definite. By the application of Cetaev's Theorem, it follows that the system is unstable. The same holds for the case $a_{oi} < 0$, $a_{1i} > 0$, $i = 1,\ldots, n$.

Hence, for the above combination of a's, the system (4) does not have any periodic solutions.

(c) Controllability. It may be observed that

$$\ddot{x}_i = u'_i \,, \quad i = 1,2,\ldots, n \tag{12}$$

is a controllable system since it may be reduced to the form $\dot{\underline{y}} = A\underline{y} + B\underline{u}$ where

$$A = \begin{bmatrix} 0 & 1 & 0 & . & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & . & 0 \\ 0 & 0 & 0 & 1 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & . & 1 \\ 0 & 0 & 0 & . & . & . & . & 0 \end{bmatrix}_{2n\times 2n}$$

$$B = \begin{bmatrix} 0 & . & . & . & . & . & . & . & 0 \\ 1 & 0 & . & . & . & . & . & . & 0 \\ 0 & 0 & . & . & . & . & . & . & . \\ 0 & 1 & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & 0 \\ 0 & . & . & . & . & . & . & . & 1 \end{bmatrix}_{2n\times n} .$$

and, [B:AB] is of rank 2n. The system (4) may be put in the form (12) by the simple substitution $u_i = u'_i + f_i(\sigma_i) + k_i(\vec{\sigma})$.

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