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Numerical Study of Weakly Unstable Electron Plasma Oscillations*

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ABSTRACT

The initial value problem for an unstable electron plasma has been solved by numerically integrating the Vlasov equation in one dimension. The situation chosen is the familiar "bump-on-the-tail" situation of quasi-linear theory. The solution is followed well beyond the point at which the electrostatic field energy has reached its maximum value. The electric field spectrum is eventually dominated by the single most linearly unstable wave number, which lies in the middle of the allowed range of wave numbers; it undergoes what appear to be the beginnings of gentle long-period oscillations characteristic of trapped-particle periodicities. It is argued that differences from quasi-linear predictions may be explained in terms of the level of initial excitations, or "noise", from which the instability is assumed to proceed.

I. INTRODUCTION

No collisionless plasma phenomenon has been more extensively examined than the one-dimensional electrostatic two-stream instability. A large literature has been generated, which we will not attempt to survey exhaustively here (cf., however, Refs. 1). This literature has not converged, however, to very many universally accepted interpretations of observable or numerically calculable phenomena. A case in point is the "bump-on-the-tail" limit of the two-stream instability in the collisionless electron plasma: the situation which is considered in quasi-linear theory. A recent numerical investigation by Dawson and Shanny[2] treated a sheet model of the "bump-on-the-tail" electron plasma instability and found sharp disagreement with the analytical predictions of quasi-linear theory.

Several processes were suggested by Dawson and Shanny for this disagreement, all of which rely in the last analysis on particle discreteness. In their computations, the number of sheets per Debye length was on the order of 20, which raises some question as to how applicable a Vlasov description should be to the dynamics of the sheets. This has led us to compute a similar bump-on-the-tail situation directly from the Vlasov equation, so that there are no discrete-particle processes of any kind. We expected to find the predictions of quasi-linear theory confirmed, and have been surprised to find that in many respects they were not. The departures from quasi-linear predictions we have observed are in large measure different from those seen by Dawson and Shanny and are believed to have a different origin; they are, however, perhaps of some interest in their own right.

Section II describes the computation and the results. In Sec. III, we offer some physical interpretations of these results.

II. RESULTS OF THE COMPUTATION

The method of computation has been discussed extensively elsewhere,[3] and will only be summarized briefly here. The one-dimensional electric field $E(x,t)$ is represented by

$$E(x,t) = \sum_{n=-\infty}^{\infty} E_n(t) \exp(ink_o x) , \tag{1}$$

and the electron distribution function $f(x,v,t)$ by

$$f(x,v,t) = \sum_{n=-\infty}^{\infty} f_n(v,t) \exp(ink_o x) , \tag{2}$$

$$f_n(v,t) = \sum_{m=0}^{\infty} Z_{mn}(t)\, h_m(v) \exp(-v^2/2) , \tag{3}$$

where $h_m(v)$ is the orthonormal Hermite polynomial of degree m. Substituting Eqs. (1), (2), (3) into the Vlasov and Poisson equations, making use of the orthogonality and recursion properties of the basis functions, gives ordinary first-order differential equations[3] in the time for the matrix elements $Z_{mn}(t)$. These are then advanced forward in time from arbitrary initial values by a Runge-Kutta-Gill technique. Because of the reality of f, we need compute only for non-negative n, and the electric field is always recoverable from the m = 0 row of the matrix.

Initial conditions for this present computation are:

$$f(x,v,o) = f_o(v)\ \{1 + \epsilon \sum_{j=1}^{N} \cos(jk_o x)\}\ , \tag{4}$$

where $N = 8$, $\epsilon = 0.006$, $k_o = 0.15$, and

$$f_o(v) = (2\pi)^{-\frac{1}{4}}\ \{h_o(v) + \sqrt{2/3}\ h_4(v)\}\ e^{-v^2/2}\ . \tag{5}$$

All lengths are in units of the electron Debye length, all velocities in units of the electron thermal speed, all times in units of the inverse electron plasma frequency ω_p^{-1}, and both f and E are dimensionless.

The equilibrium $f_o(v)$ in Eq. (5) is an even function of v which has its maximum at $v = 0$; it has minimum value of zero at $v = \sqrt{3}$ and local maximum value of about 0.032 at $v = \sqrt{7}$. It appears later as the dashed line in Fig. 4. (Note that there are really two electron beams, one right travelling and one left travelling. Since the unstably growing electrostatic waves are known to have phase velocities about equal to these beam velocities, and to interact strongly mainly with particles in nearby portions of velocity space, we can to a good approximation ignore the effect of the left-travelling waves when talking about right-travelling particles and vice versa. The right- and left-travelling waves will contribute equally to the $E_n(t)$, however.)

There are several differences between these initial conditions and those we have used previously.[3] The most apparent is that there are eight harmonics of the fundamental wave number kept, whereas other problems were carefully tailored to require at most two, with $|E_1| \gg |E_2| \gg |E_3|$, etc., as the convergence condition for truncation of the Fourier series. These eight wave numbers correspond to waves which in the linear limit completely span the bump with their phase velocities, ranging from $\omega/k \cong 7.0$ for E_1 to $\omega/k \cong 1.54$ for E_8. It was established by trial runs with very small ϵ that in the linear limit, E_2, E_3, E_4 are exponentially growing waves, with growth rates less than 0.1 ω_p, E_7 and E_8 are heavily Landau damped, and E_1, E_5, E_6 are either marginally stable or weakly Landau damped. Thus in both the phase velocity spectrum and wave number spectrum, there is a middle range of allowed linearly unstable waves, flanked on both sides by stable ones which in the linear limit would never grow above their initial amplitudes. All satisfy $|\gamma/\omega_p| \ll 1$.

A second difference from our previous computations lies in the large size of the matrix required to follow the dynamics of this situation. This is because most of the activity (in velocity space) occurs at velocities well above a thermal velocity, requiring us to use large values of m in Eq. (3). A single run, for example, consumed the equivalent of about ten hours of time on an IBM 360/65. At the time we begin to throw away rows,[3] the Z_{mn} matrix contains about 8000 elements.

In grossest outline, the development of the instability proceeded as in earlier calculations.[3] There was an initial ($\leqslant 10\ \omega_p^{-1}$) regime in which little systematic evolution occurred, this being interpreted here (as before) as the time necessary for the Landau-damped modes present in the initial conditions to go away. There is then a period of fairly calm exponential growth until the growing waves have reached large enough amplitudes for nonlinear effects to become important. This happens at around $t \cong 40\ \omega_p^{-1}$. The growth then turns itself off, and the envelope of the maxima of the electrostatic field energy appears to be going into long period oscillations about a finite value which is on the order of a very few percent of the total energy (see Fig. 1).

In contrast to the expectations aroused by quasi-linear theory, however, the electrostatic field energy does <u>not</u> become shared by adjacent wave numbers; rather E_3 almost completely dominates the spectrum, containing for example about four times as much field energy at $t \cong 42.5$ as all of the other waves put together. E_3 versus time appears in Fig. 2. A plot of E_3 and its two nearest competitors, E_2 and E_4, versus time is shown in Fig. 3. The point seems to be that the dominant wave number in the limiting spectrum is the fastest growing wave in the linear theory. Further discussion of this point is deferred until Sec. III.

We turn finally to the velocity distribution function. For reasons discussed in detail elsewhere,[3] we can compute the electric field amplitudes accurately for up to twice as long as we can the distribution. Therefore we have less information about the long-time form of the distribution function than we would like. In Fig. 4, we show $f_o(v,42.5)$. This has the following similarities and differences from the quasi-linear final state and from the velocity distribution computed by Dawson and Shanny.

First, the hole in the distribution has filled in. However, there is little similarity to the "plateau" predicted by quasi-linear theory. Nor, however, does the high energy tail of supra-thermal particles seen by Dawson and Shanny appear. For all practical purposes $f_o(v,t)$ remains equal to its initial value above $v \cong 4.2$. The picture of $f_o(v,t)$ we have obtained is more nearly similar to that resulting from a particle-in-cell simulation of the two-stream instability recently done by Morse and Nielson.[4] However, their calculation also includes some particle discreteness effects.

III. DISCUSSION

The unexpected dominance of the final-state electric field spectrum by a single wave number ($|E_3| \gg$ all other $|E_n|$) is the most noteworthy result uncovered by the computation. Viewed from a coordinate system in which the right-travelling half of E_3 appears time-independent, the "trapping width" of the wave, $2\sqrt{|E_3|/3\,k_o}$, occupies at $t \cong 42$ a slice of velocity space of about $\pm$ 1.15 on either side of the phase velocity. This is sufficiently large to "trap" most of the particles in the beam itself, and we believe this, crudely speaking, to be the mechanism by which the instability turns itself off. (Similar mechanisms may in fact be invoked sometimes to explain cold plasma results.) The oscillations in maximum amplitude into which $|E_3|$ appears to have gone by the end of the run seem to have a period of about $30\ \omega_p^{-1}$. This is consistent with a mean value for the amplitude $|E_3|$ of 0.1 (trapping period $\sim 2\pi/\sqrt{3\,k_o\,|E_3|_{\text{avg.}}}$), and suggests an explanation parallel to that of our earlier case: oscillations in the steady state are due to the periodicities of the trapped particle part of the electron distribution.

This is a consistent picture of the limiting state of the instability, but we must still answer the question of how it happened that E_3 grew to its dominant position; in particular we need to reconcile this with the prediction of quasi-linear theory which, superficially at least, would appear to imply in its "H-like theorem" a sharing of the energy among several adjacent values of k, and a time-independent limiting state for the electric field spectrum. We now address ourselves to that question.

The answer may be found in the period between $t \cong 10\ \omega_p^{-1}$ and about $t \cong 30\ \omega_p^{-1}$ which is characterized by nearly pure exponential growth of the linearly unstable waves, without, however, reaching large enough amplitudes for nonlinear effects to appear. E_3 has a significantly larger growth rate than the other unstable waves, and because the initial amplitude is so small, it has an opportunity to acquire almost an order of magnitude more energy than its competitors before the nonlinear effects set in.

This is the point at which the quasi-linear equations[1] would ordinarily be expected to begin to have something to say about the evolution of $f_o(v,t)$. However, implicit in their derivation is the assumption that the initial electrostatic field energy is smoothly distributed over several adjacent k's; unless this is the case, it does not make sense to pass to the continuum limit and write down the differential equation for the evolution of the spectral density at all. In our case, the "initial condition" which has developed by the time of the onset of the nonlinear effects is essentially a delta-function spike in k-space: not close to what would be required for the invocation of quasi-linear theory.

In summary, then, we have determined a limiting state for a bump-on-the-tail electron plasma instability. The result has not confirmed many of the predictions of quasi-linear theory for Vlasov plasmas. Neither can it be honestly said to have "disproved" quasi-linear theory: only to have made explicit a limitation on allowable initial conditions which was perhaps not obvious before. This limitation may be given in capsule form by saying that the "noise" level from which the quasi-linear final state must grow cannot be too much less than that of the final state spectrum itself. Otherwise the nonlinear regime may effectively proceed from an "initial" state which is dominated by a single wave number, as we have seen here. How restrictive this may be in actual plasmas we do not know. It may be that the question cannot be answered independently of specific experiments.

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FIGURE CAPTIONS

Figure 1. Development of the total electrostatic field energy, $\Sigma_i |E_i|^2$, as a function of time. Note that quasi-linear theory predicts a monotonic approach to a maximum value for this quantity.

Figure 2. The dominant field component $E_3(t)$ as a function of time. At maximum value, the trapping width of this wave alone extends across the bump.

Figure 3. Envelopes of the maxima of $|E_3(t)|$ and its two nearest competitors. The other $|E_n(t)|$ all remain smaller than these. By the time the amplitudes have grown large enough for nonlinear effects to set in, $E_3(t)$ has monopolized the electrostatic field energy.

Figure 4. The electron velocity distribution for $t = 0$ and $t = 42.5\ \omega_p^{-1}$ (time of maximum electrostatic field energy). The phase velocity of the dominant wave E_3 is about 2.25, and the "trapping width" extends about 1.15 units on either side of this. Note the absence of distortion above $v \cong 4.1$.

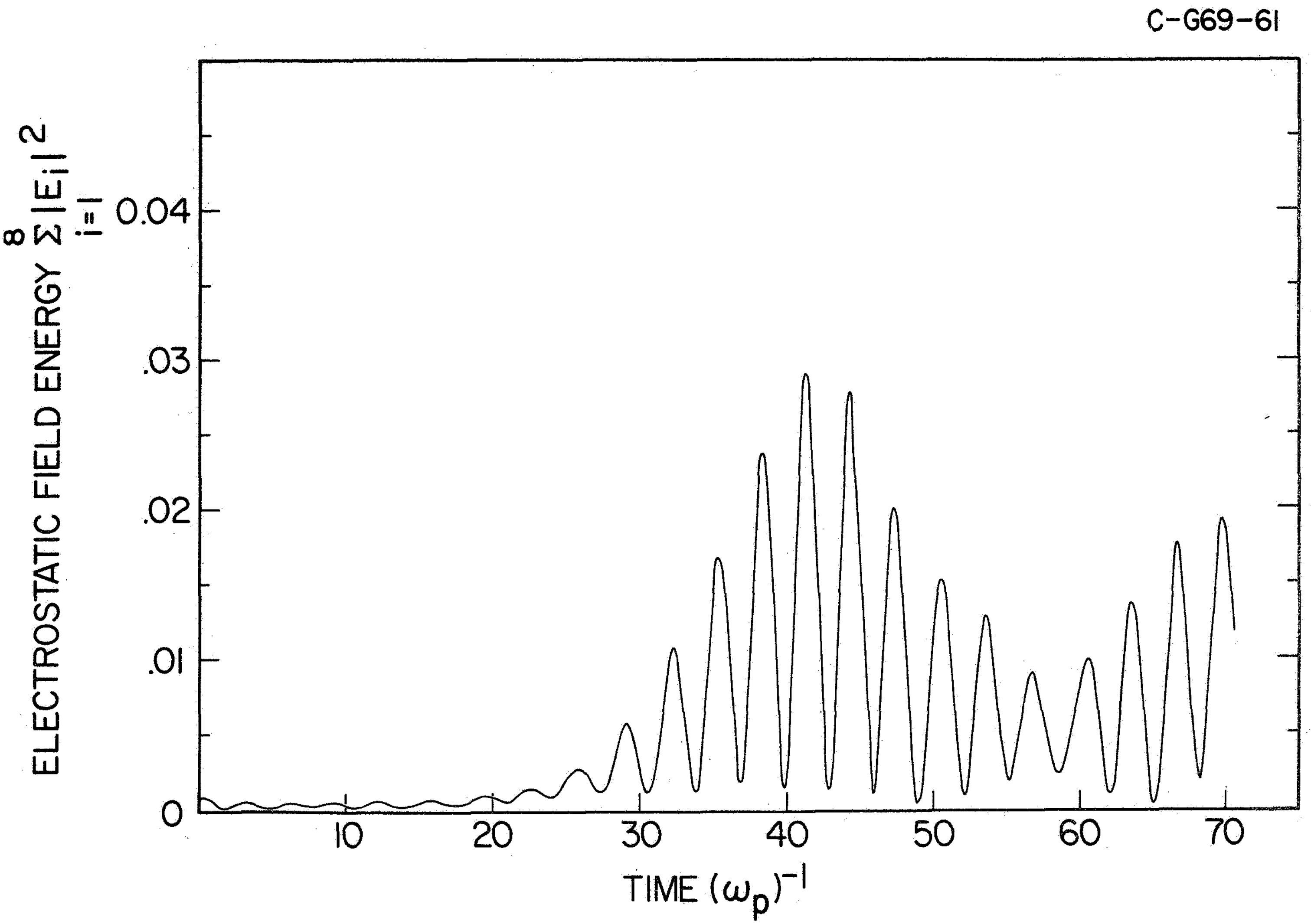
C-G69-61
ELECTROSTATIC FIELD ENERGY $\sum_{i=1}^{8}|E_i|^2$
0.04
.03
.02
.01
0
10
20
30
40
50
60
70
TIME $(\omega_p)^{-1}$

Figure 1

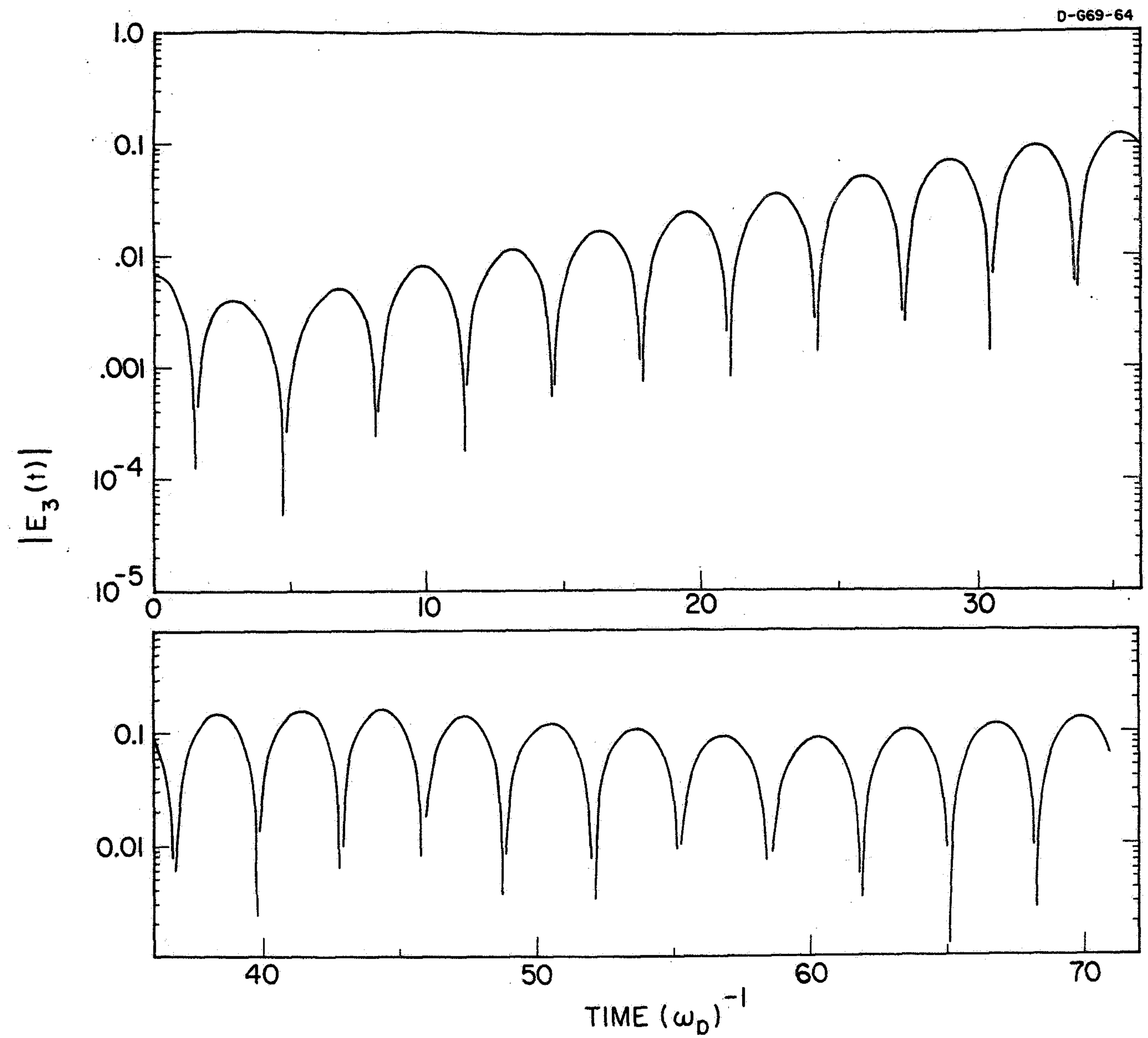
D-G69-64
1.0
0.1
.01
.001
10^{-4}
10^{-5}
$|E_3(t)|$
0
10
20
30
0.1
0.01
40
50
60
70
TIME $(\omega_D)^{-1}$

Figure 2

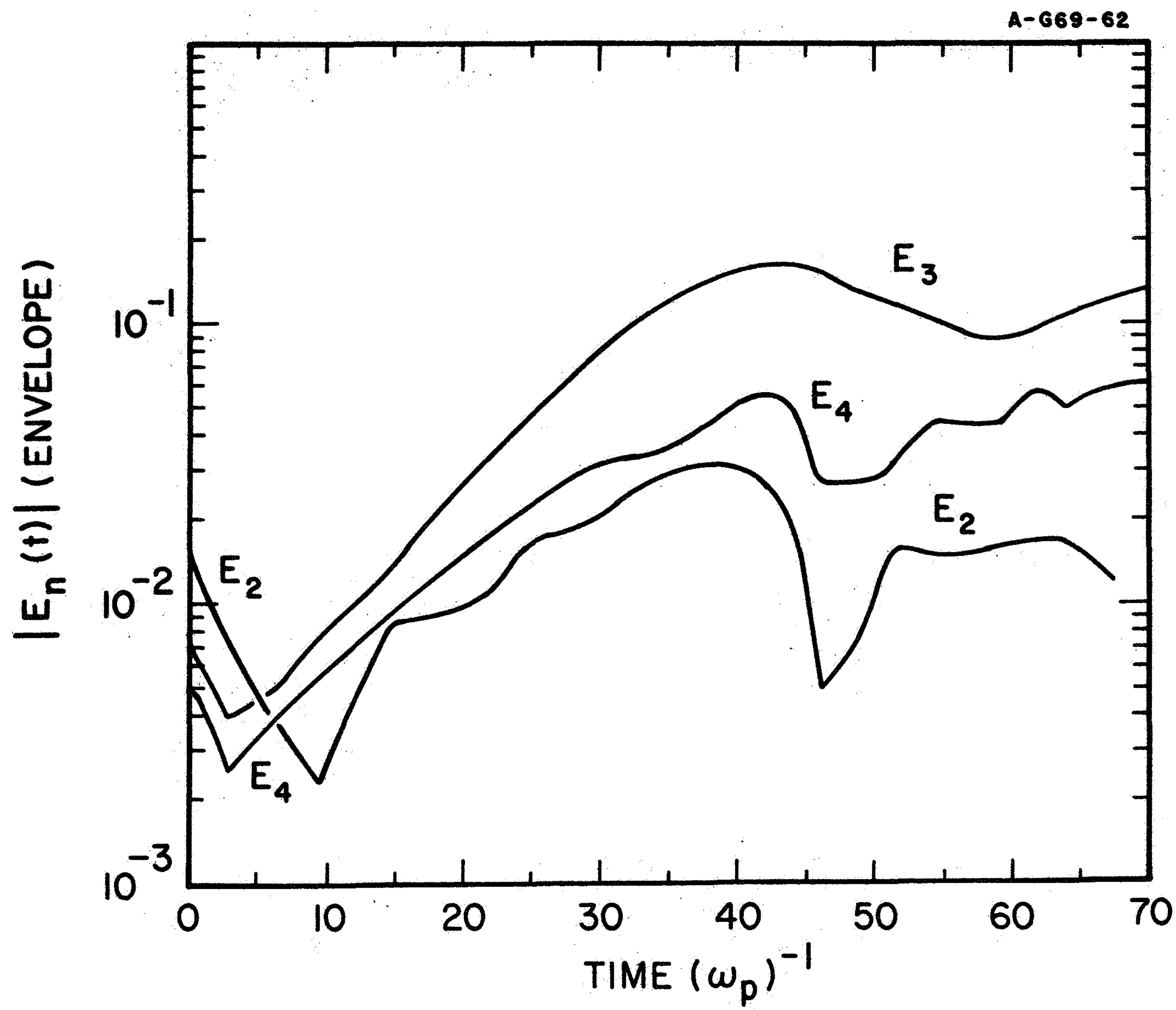
A-G69-62
$|E_n(t)|$ (ENVELOPE)
10^{-1}
10^{-2}
10^{-3}
E_3
E_4
E_2
E_2
E_4
0
10
20
30
40
50
60
70
TIME $(\omega_p)^{-1}$

Figure 3

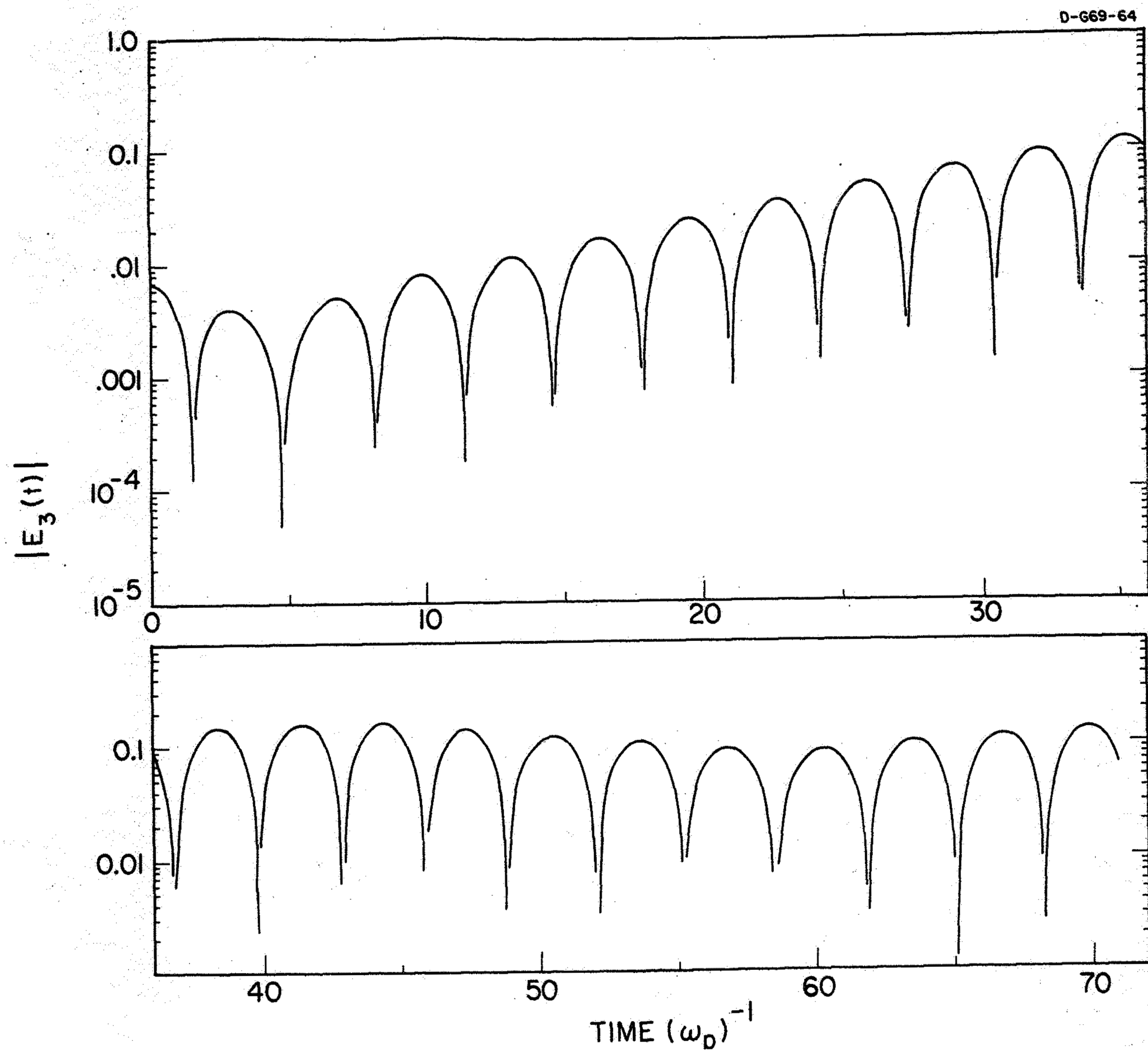
D-G69-64
1.0
0.1
.01
.001
10^{-4}
10^{-5}
$|E_3(t)|$
0
10
20
30
0.1
0.01
40
50
60
70
TIME $(\omega_D)^{-1}$

Figure 2

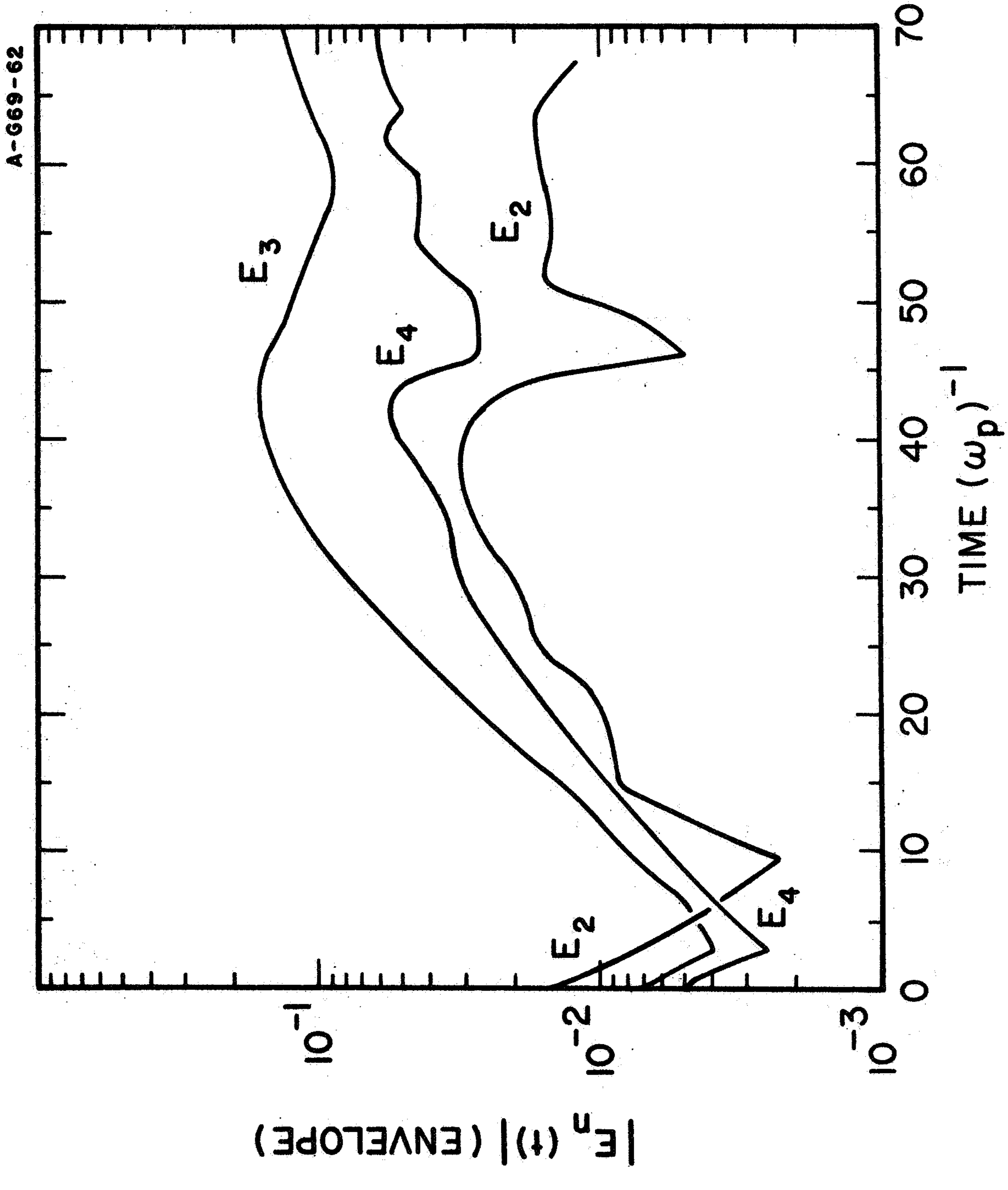
A-G69-62
E_3
E_4
E_2
E_2
E_4
$|E_n(t)|$ (ENVELOPE)
10^{-1}
10^{-2}
10^{-3}
0
10
20
30
40
50
60
70
TIME $(\omega_p)^{-1}$

Figure 3

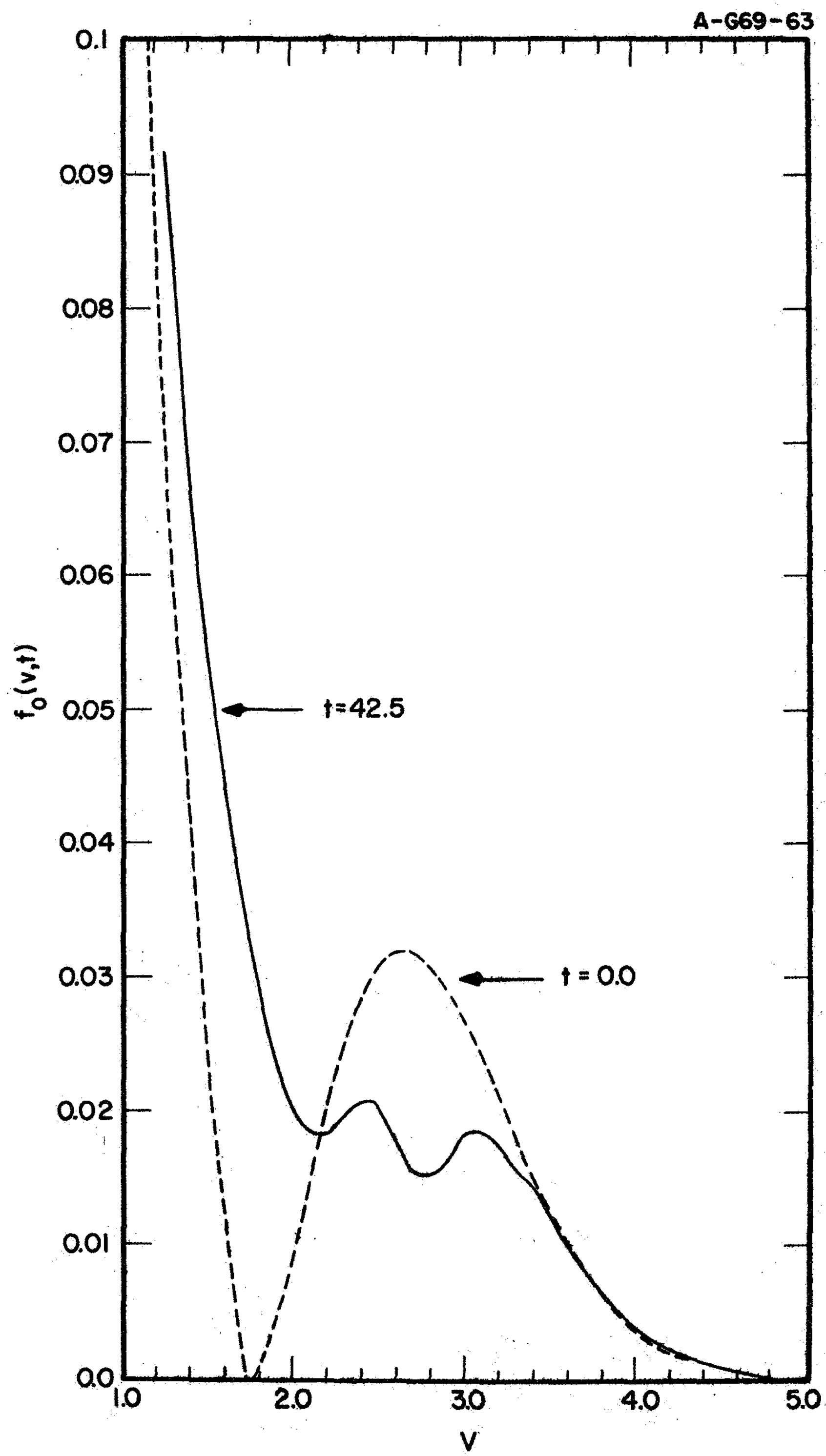
A-G69-63
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.0
$f_0(v,t)$
t=42.5
t = 0.0
1.0
2.0
3.0
4.0
5.0
V

Figure 4