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Contents

Abstract

A quasilinear theory is formulated which includes the microscopic fluctuation fields as well as the coherent waves. The study emphasizes those cases in which the propagating mode of the fluctuation field gives rise to a dominant contribution to the particle correlation. In other words, in these cases the spontaneous Cerenkov emission of plasmons can play a more important role than the two-particle collision process. The instability which is responsible for the turbulence can be of either electrostatic or electromagnetic, or of mixed nature. The effect of an external magnetic field is also considered in the present theory. The final result is valid for an arbitrary ratio of the energy density of the coherent wave to that of the fluctuation field, although the condition of weak turbulence is imposed throughout the analysis.

A Unified Quasilinear Theory of Weakly Turbulent Plasmas

I. Introduction

The existing theories concerning weakly turbulent plasmas may be grouped into two categories; in one, the interaction between the coherent wave and the resonant particles is emphasized, and the other is concerned with the effect of correlation. In the former, which is called the quasilinear theory in the literature (Refs. 1–7),[1] correlation is completely neglected, and in the latter (Refs. 9–16), the macroscopic waves are totally ignored under the assumption of homogeneous plasma. Evidently the two theories are concerned with two extreme situations. For instance, if we define a ratio Γ such that

$$\Gamma \equiv \left(\frac{\text{energy associated with microscopic fluctuation field}}{\text{energy associated with coherent oscillation}}\right)_{\substack{\text{same}\\ \text{mode}}} \tag{1}$$

the quasilinear theory deals with the case $\Gamma \rightarrow 0$ and, on the other hand, the theory emphasizing correlation is concerned with the limit $\Gamma \rightarrow \infty$.

Intuitively one may think that the condition $\Gamma << 1$ describes a more realistic plasma, and thus the quasilinear theory seems to be more interesting. However, we want to point out that the two theories are complementary. For example, in the quasilinear theory, we assume implicitly that the quasilinear stabilization process is characterized by a time scale much shorter than the characteristic time for the establishment of particle correlation. This implies that the validity of the quasilinear theory cannot be fully appreciated unless we can first understand how the plasma turbulence affects the particle correlation. In general, three fundamental processes determine the particle correlation; their relation may be expressed as follows:

$$\text{Correlation of particles}\begin{cases}\text{by virtue of collective processes} & \begin{cases}\text{(A)}\\ \text{(B)}\end{cases}\\ \quad\text{(collective interaction)} & \\ \text{by collisions} \longrightarrow & \text{(C)}\end{cases}$$

[1]For other publications concerning the applications of the quasilinear theory to specific problems, see Ref. 8.

where

(A) = propagating mode of the microscopic fluctuation field originated by both stimulated emission (or absorption) and spontaneous emission

(B) = nonpropagating (heavily damped) mode of the fluctuation field, contributing to the dynamic shielding phenomenon

(C) = direct particle encounters

For turbulent plasmas in many cases, process (A) may prevail over processes (B) and (C). In this memorandum we shall pay special attention to these cases. Since the effective correlation time may be greatly shortened in these cases (this phenomenon is attributed to the enhanced fluctuation of the microscopic density and field due to the presence of instability; see Ref. 17 for publications on the effect of collective processes on various relaxation times in a stable plasma), it is desirable to include the contribution from the fluctuation field associated with the instability.

In the subsequent discussion, we shall restrict our analysis to the weakly turbulent plasmas, in which both mode-coupling and particle-trapping processes (Refs. 8 and 18) are negligible. Moreover, since most instabilities exist in magnetized plasmas, we shall include the effect of an external magnetic field. For the sake of generality, both electrostatic and electromagnetic modes will be considered so that the theory is useful for plasmas with an arbitrary value of β (i.e., the ratio of fluid pressure to magnetic pressure). The present theory is intended to bridge the usual quasilinear theory and the kinetic theory for a homogeneous plasma. Consequently, the result is valid over the complete range of Γ.

From the result of this work, we shall see that the smallness of the ratio defined in Eq. (1) alone is not sufficient to justify the neglect of correlation. An additional criterion must be considered; that is,

$$\begin{pmatrix}\text{stimulated emission}\\ \text{of plasmons with}\\ \text{momentum k}\end{pmatrix} >> \begin{pmatrix}\text{spontaneous emission}\\ \text{of plasmons with the}\\ \text{same momentum}\end{pmatrix} \tag{2}$$

Evidently, in many cases, this condition is violated at a later stage of the relaxation process in the usual quasilinear theory. For instance, (1) in the one-dimensional "bump-on-the-tail" problem (Ref. 18), when the plateau is asymptotically formed and (2) in the case of ion-wave instability, when the spontaneous emission may become significant shortly after an initial stage of evolution because of the high population of resonant electrons. It is true that in the first instance the contribution from the propagating mode of the fluctuation field to the correlation may be expected to be of the same order of magnitude as the collisional process and, in order to be consistent, one should also include the latter effect.

To improve the present work and to extend it to strong turbulence is indeed desirable. Perhaps the methods suggested by Dupree (Ref. 19) and Nishikawa (Ref. 20) can be unified and generalized.

II. Mathematical Formulation

The Klimontovich equations (Refs. 21 and 22) of a multi-species plasma including both electrostatic and electromagnetic interactions can be expressed as follows:

$$\frac{\partial N_s}{\partial t} + \mathbf{v}\cdot\nabla N_s + \frac{e_s}{m_s}\frac{\mathbf{v}}{c}\times\mathbf{B}_0\frac{\partial N_s}{\partial \mathbf{v}} + \frac{e_s}{m_s}\left(\mathbf{E}^m + \frac{\mathbf{v}}{c}\times\mathbf{B}^m\right)\cdot\frac{\partial N_s}{\partial \mathbf{v}} = 0 \tag{3}$$

$$\frac{1}{c}\frac{\partial \mathbf{E}^m}{\partial t} + \frac{4\pi}{c}\sum_s e_s\int d^3v N_s(\mathbf{r},\mathbf{v},t)\,\mathbf{v} - \nabla\times\mathbf{B}^m = 0 \tag{4}$$

$$\frac{1}{c}\frac{\partial \mathbf{B}^m}{\partial t} + \nabla\times\mathbf{E}^m = 0 \tag{5}$$

$$\nabla\cdot\mathbf{B}^m = 0 \tag{6}$$

$$\nabla\cdot\mathbf{E}^m = 4\pi\sum_s e_s\int d^3v N_s(\mathbf{r},\mathbf{v},t) \tag{7}$$

where the subscript s designates the particle species, $\mathbf{B}_0$ denotes a uniform external magnetic field, $\mathbf{E}^m$ and $\mathbf{B}^m$ are the microscopic electromagnetic fields, and $N_s(\mathbf{r},\mathbf{v},t)$ is the random density, which may be defined as

$$N_s(\mathbf{r},\mathbf{v},t) = \sum_s \delta\left[\mathbf{r} - \mathbf{r}_{si}(t)\right]\delta\left[\mathbf{v} - \mathbf{v}_{si}(t)\right] \tag{8}$$

In expression (8), $\mathbf{r}_{si}(t)$ and $\mathbf{v}_{si}(t)$ are the position and velocity vectors at a time t of the ith particles which belong to the s-species. The derivation of these equations may be found in Ref. 23. Now we introduce the Fourier spectral resolution of the microscopic quantities

$$\mathbf{E}^m(\mathbf{r},t) = \sum_{\mathbf{k}} \mathbf{E}_k^m(t)\, e^{i\mathbf{k}\cdot\mathbf{r}} \tag{9}$$

$$\mathbf{B}^m(\mathbf{r},t) = \sum_{\mathbf{k}} \mathbf{B}_k^m(t)\, e^{i\mathbf{k}\cdot\mathbf{r}} \tag{10}$$

$$N_s(\mathbf{r},\mathbf{v},t) = \sum_{\mathbf{k}} N_s^k(\mathbf{v},t)\, e^{i\mathbf{k}\cdot\mathbf{r}} \tag{11}$$

Correspondingly, we obtain

$$\frac{\partial N_s^0}{\partial t} + \frac{e_s}{m_s}\left(\frac{\mathbf{v}}{c}\times\mathbf{B}_0\right)\cdot\frac{\partial N_s^0}{\partial \mathbf{v}} = -\sum_{\mathbf{k}} \frac{e_s}{m_s}\left(\mathbf{E}_{-k}^m + \frac{\mathbf{v}}{c}\times\mathbf{B}_{-k}^m\right)\cdot\frac{\partial N_s^k}{\partial \mathbf{v}} \tag{12}$$

$$\frac{\partial N_s^k}{\partial t} + i\,\mathbf{k}\cdot\mathbf{v}\,N_s^k + \frac{e_s}{m_s}\frac{\mathbf{v}}{c}\times\mathbf{B}_0\frac{\partial N_s^k}{\partial \mathbf{v}} + \frac{e_s}{m_s}\left(\mathbf{E}_k^m + \frac{\mathbf{v}}{c}\times\mathbf{B}_k^m\right)\cdot\frac{\partial N_s^0}{\partial \mathbf{v}} + \frac{e_s}{m_s}\sum_{\mathbf{k}'}\left(\mathbf{E}_{k-k'}^m + \frac{\mathbf{v}}{c}\times\mathbf{B}_{k-k'}^m\right)\cdot\frac{\partial N_s^{k'}}{\partial \mathbf{v}} = 0 \tag{13}$$

$$\frac{1}{c}\frac{\partial \mathbf{E}_k^m}{\partial t} - i\,\mathbf{k}\times\mathbf{B}_k^m + \frac{4\pi}{c}\sum_s e_s\int d^3v\,\mathbf{v}\,N_s^k = 0 \tag{14}$$

$$\frac{1}{c}\frac{\partial \mathbf{B}^m}{\partial t} + i\,\mathbf{k}\times\mathbf{E}_k^m = 0 \tag{15}$$

$$\mathbf{k}\cdot\mathbf{B}_k^m = 0 \tag{16}$$

$$\mathbf{k}\cdot\mathbf{E}_k^m = 4\pi\sum_s e_s\int d^3v\,N_s^k(\mathbf{v},t) \tag{17}$$

Here we consider

$$\mathbf{B}_0^m = \mathbf{E}_0^m = 0$$

In the following, we shall assume that the last term in Eq. (13) may be dropped, in other words, we consider that mode-coupling interaction is negligible. Moreover, we shall take the ensemble-averaged value of Eq. (12) and obtain

$$\frac{\partial F_s}{\partial t} + \frac{e_s}{m_s}\left(\frac{\mathbf{v}}{c}\times\mathbf{B}_0\right)\cdot\frac{\partial F_s}{\partial \mathbf{v}} = -\sum_{\mathbf{k}} \frac{e_s}{m_s n_s}\frac{\partial}{\partial \mathbf{v}}\cdot\left[\langle \mathbf{E}_{-k}^m N_s^k\rangle + \frac{\mathbf{v}}{c}\times\langle \mathbf{B}_{-k}^m N_s^k\rangle\right] \tag{18}$$

where the symbol $\langle\ \rangle$ denotes the ensemble-averaged value and the relation

$$\langle N_s^0(\mathbf{v},t)\rangle = n_s F_s(\mathbf{v},t) \tag{19}$$

has been used. In Eq. (19), F_s is the one-particle distribution function of the s-species particles, which is independent of spatial coordinates; n_s is the average number density of the same species.

Equation (18), in principle, represents the desired kinetic equation, although the right-hand terms are to be determined. The correlation functions $\langle \mathbf{E}_{-k}^m N_s^k\rangle$ and $\langle \mathbf{B}_{-k}^m N_s^k\rangle$ can be derived from Eqs. (13) through (17). However, before going further, some useful definitions and relations should be introduced. First we shall designate

$$\langle N_s^k(\mathbf{v},t)\rangle = n_s f_s(\mathbf{k},\mathbf{v},t) \tag{20}$$

Evidently, if $\mathcal{F}_s(\mathbf{r},\mathbf{v},t)$ is the distribution function of the s-species, $\mathcal{F}_s$ may be expressed as

$$\mathcal{F}_s(\mathbf{r},\mathbf{v},t) = \frac{\langle N_s(\mathbf{r},\mathbf{v},t)\rangle}{n_s} = F_s(\mathbf{v},t) + \sum_{\mathbf{k}} f_s(\mathbf{k},\mathbf{v},t)\, e^{i\mathbf{k}\cdot\mathbf{r}} \tag{21}$$

Moreover, we see that

$$\begin{aligned}
&\langle N_s(\mathbf{r},\mathbf{v},t)\, N_r(\mathbf{r},\mathbf{v},t)\rangle \\
&= \sum_{\mathbf{k}}\sum_{\mathbf{k}'}\langle N_s^k(\mathbf{v},t)\, N_s^{k'-k}(\mathbf{v}',t)\rangle \exp\left[i\mathbf{k}\cdot\mathbf{r} + i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}'\right] \\
&= \langle N_s^0(\mathbf{v},t)\, N_r^0(\mathbf{v}',t)\rangle \\
&\quad + \sum_{\mathbf{k}\neq 0}\langle N_s^k(\mathbf{v},t)\, N_r^{-k}(\mathbf{v}',t)\rangle \exp\left[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')\right] \\
&\quad + \sum_{\mathbf{k}\neq 0}\sum_{\mathbf{k}'}\langle N_s^k(\mathbf{v},t)\, N_r^{k'-k}(\mathbf{v}',t)\rangle \exp\left[i\mathbf{k}\cdot\mathbf{r} + i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}'\right] \\
&= n_s n_r F_s(\mathbf{v},t)\, F_r(\mathbf{v}',t) + n_s n_r G_s(0,\mathbf{v},\mathbf{v}',t) \\
&\quad + n_s\,\delta_{sr}(\mathbf{v}-\mathbf{v}')\, F_s(\mathbf{v},t) \\
&\quad + \sum_{\mathbf{k}\neq 0}\{n_s n_r G_{sr}(\mathbf{k},\mathbf{v},\mathbf{v}',t) + n_s\,\delta(\mathbf{v}-\mathbf{v}')\,\delta_{sr} F_s(\mathbf{v},t) \\
&\quad + n_s n_r f_s(\mathbf{k},\mathbf{v},t)\, f_r(-\mathbf{k},\mathbf{v}',t)\}\exp\left[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')\right] \\
&\quad + \sum_{\mathbf{k}\neq 0}\sum_{\mathbf{k}'}\{n_s n_r g_{sr}(\mathbf{k},\mathbf{k}'-\mathbf{k},\mathbf{v},\mathbf{v}',t) \\
&\quad + n_s\,\delta(\mathbf{v}-\mathbf{v}')\, f_s(\mathbf{k}'-\mathbf{k},\mathbf{v},t) \\
&\quad + n_s n_r f_s(\mathbf{k},\mathbf{v},t)\, f_r(\mathbf{k}'-\mathbf{k},\mathbf{v}',t)\} \\
&\quad \times \exp\left[i\mathbf{k}\cdot\mathbf{r} + i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}'\right]
\end{aligned} \tag{22}$$

where G_{sr} and g_{sr} represent the Fourier components of the pair-correlation function P_{sr} (see Ref. 24 for definition), or we may write

$$P_{sr}(\mathbf{r},\mathbf{r}',\mathbf{v},\mathbf{v}',t) = \sum_{\mathbf{k}} G_{sr}(\mathbf{k},\mathbf{v},\mathbf{v}',t)\, e^{i\mathbf{k}\cdot\mathbf{r}} + \sum_{\mathbf{k}\neq 0}\sum_{\mathbf{k}} g_{sr}(\mathbf{k},\mathbf{k}'-\mathbf{k},\mathbf{v},\mathbf{v}',t) \times \exp[i\mathbf{k}\cdot\mathbf{r} + i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}'] \tag{23}$$

From Eq. (22), we obtain by comparison that

$$\langle N_s^k(\mathbf{v},t)\, N_r^{-k}(\mathbf{v}',t)\rangle = n_s n_r f_s(\mathbf{k},\mathbf{v},t) f_r(-\mathbf{k},\mathbf{v}',t) + n_s n_r G_{sr}(\mathbf{k},\mathbf{v},\mathbf{v}',t) + n_s \delta_{sr}\, \delta(\mathbf{v}-\mathbf{v}')\, F_s(\mathbf{v},t) \tag{24}$$

This relation will be useful in discussions presented in later sections.

III. Initial-Value Problem

Now we intend to solve an initial-value problem according to the following equations:

$$\frac{\partial N_s^k}{\partial t} + i\mathbf{k}\cdot\mathbf{v}\, N_s^k + \frac{e_s}{m_s}\frac{\mathbf{v}}{c}\times\mathbf{B}_0\cdot\frac{\partial N_s^k}{\partial \mathbf{v}} + \frac{e_s n_s}{m_s}\left(\mathbf{E}_k + \frac{\mathbf{v}}{c}\times\mathbf{B}_k\right)\cdot\frac{\partial F_s}{\partial \mathbf{v}} = 0 \tag{25}$$

$$\frac{1}{c}\frac{\partial \mathbf{E}_k}{\partial t} + \frac{4\pi}{c}\sum_s e_s \int d^3v\, \mathbf{v}\, N_s^k(\mathbf{v},t) - i\mathbf{k}\times\mathbf{B}_k = 0 \tag{26}$$

$$\frac{1}{c}\frac{\partial \mathbf{B}_k}{\partial t} - i\mathbf{k}\times\mathbf{E}_k = 0 \tag{27}$$

Two points should be noted: (1) for simplicity, we have dropped the superscript m in the fields $\mathbf{E}^m$ and $\mathbf{B}^m$ and should keep in mind that these fields are microscopic ones, and (2) in Eq. (25) we have replaced N_s^0 by $n_s F_s$ (or, in other words, neglected δN_s^0). Such an approximation does not affect the first-order theory (in the sense discussed by Dupree in Ref. 25) since the error in the evaluation of the function $\langle E_{-k} N_s^k\rangle$ is of second order.

Solving Eq. (25) by characteristic integration along the particle orbit, we find a formal solution of N_s^k; that is,

$$N_s^k(\mathbf{v},t) = \exp[-i\mathbf{k}\cdot\mathbf{r}_s(t)]\, N_s^k(\mathbf{v}_s(t),0) - \frac{n_s e_s}{m_s}\int_0^t d\tau \exp[-i\mathbf{k}\cdot\mathbf{r}_s(\tau)]\left\{\mathbf{E}_k(t-\tau) + \frac{\mathbf{v}_s(\tau)}{c}\times\mathbf{B}_k(t-\tau)\right\}\cdot\frac{\partial F_s(\mathbf{v}_s(\tau))}{\partial \mathbf{v}_s(\tau)} \tag{28}$$

where $\mathbf{r}_s(t)$ and $\mathbf{v}_s(t)$ satisfy the following equations of motion:

$$\frac{d\mathbf{r}_s(t)}{dt} = \mathbf{v}_s(t) \tag{29}$$

$$\frac{d\mathbf{v}_s(t)}{dt} = \frac{e_s}{m_s}\left[\mathbf{E}_k(t) + \frac{\mathbf{v}_s(t)}{c}\times\mathbf{B}_k(t)\right] \tag{30}$$

together with the initial conditions

$$\mathbf{r}_s(0) = 0 \tag{31}$$

and

$$\mathbf{v}_s(0) = \mathbf{v} \tag{32}$$

We notice that Eqs. (25), (26), and (27) resemble the linearized Vlasov equations.[2] This explains why the instability associated with the macroscopic wave may also happen to the microscopic wave. In the following we postulate that each Fourier component $\mathbf{E}_k$ and $\mathbf{B}_k$ may be described by two distinct time variables, say t and ϵt, where ϵ is a small dimensionless parameter; one describes the fast oscillations and the other records the slow amplification. Although the fundamental notion of the multiple time variables is similar to that of Krylov and Bogoliubov (Ref. 26), the only purpose of doing this here is to simplify the Laplace transform method to be used later.[3]

To proceed with this discussion, we shall write

$$\mathbf{E}_k = \mathbf{E}_k(t, \epsilon t) \tag{33}$$

$$\mathbf{B}_k = \mathbf{B}_k(t, \epsilon t) \tag{34}$$

[2]The initial condition of the microscopic density N is not "smooth," as we can see from expression (8). The ensemble-averaged value of the binary product of these initial conditions will contain a "self-correlation" part. It is this part that eventually gives rise to the new contribution which the usual quasilinear theory does not provide.

[3]This technique was used in Section 16 of Ref. 21. Although the discussion there is concerned with the stable case, extension to a weakly unstable plasma can be made without difficulty.

Hence we shall introduce a Laplace transform with respect to the fast time variable t; i.e.,

$$N_s^k(\mathbf{v}, \omega, \epsilon t) = \lim_{\Delta \to 0} \int_0^\infty dt \exp(i\omega t - \Delta t) N_s^k(\mathbf{v}, t, \epsilon t) \qquad (35)$$

$$\mathbf{E}_k(\omega, \epsilon t) = \lim_{\Delta \to 0} \int_0^\infty dt \exp(i\omega t - \Delta t) \mathbf{E}_k(t, \epsilon t) \qquad (36)$$

$$\mathbf{B}_k(\omega, \epsilon t) = \lim_{\Delta \to 0} \int_0^\infty dt \exp(i\omega t - \Delta t) \mathbf{B}_k(t, \epsilon t) \qquad (37)$$

where ω is real. From (28) and (35), we obtain

$$N_s(\mathbf{v}, \omega, \epsilon t) = \lim_{\Delta \to 0} \int_0^\infty dt \exp[i\omega t - \Delta t - i\mathbf{k}\cdot\mathbf{r}_s(t)]$$
$$\times \left\{ N_s^k(\mathbf{v}_s(t), 0) - \frac{n_s e_s i}{m_s \omega} \left(\frac{\mathbf{v}_s(t)}{c} \times \mathbf{B}_k(t = 0) \right) \right.$$
$$\left. \cdot \frac{\partial F_s(\mathbf{v}_s(t))}{\partial \mathbf{v}_s(t)} \right\} - \lim_{\Delta \to 0} \frac{n_s e_s}{m_s} \int_0^\infty dt \exp[i\omega\tau - \Delta\tau - i\mathbf{k}\cdot\mathbf{r}_s(\tau)]$$
$$\times \int_0^\infty d\tau' \exp(i\omega\tau' - \Delta\tau') \left[\mathbf{E}_k(\tau', \epsilon\tau') - \frac{\mathbf{v}_s(t)}{\omega} \right.$$
$$\left. \times (\mathbf{E}_k(\tau', \epsilon\tau') \times \mathbf{k}) \right] \cdot \frac{\partial F_s(\mathbf{v}_s(t))}{\partial \mathbf{v}_s(t)} \qquad (38)$$

where we have made use of the relation

$$\int_0^\infty d\tau\, e^{i\omega\tau} \mathbf{B}_k = \frac{c}{\omega} \int_0^\infty d\tau\, e^{i\omega\tau} (\mathbf{k} \times \mathbf{E}_k) + \frac{i}{\omega} \mathbf{B}_k(t = 0) \qquad (39)$$

which is obtainable from Eq. (27).

If $\mathbf{a}$ denotes a unit polarization vector which is parallel to the field $\mathbf{E}_k$, the last term in Eq. (35) can be written as

$$[N_s^k(\mathbf{v}, \omega, \epsilon t)]_{\text{last}}$$
$$= -\lim_{\Delta \to 0} \frac{n_s e_s}{m_s} \int_0^\infty d\tau \exp[i\omega\tau - \Delta\tau - i\mathbf{k}\cdot\mathbf{r}_s(\tau)]$$
$$\times \left[\mathbf{a} + \frac{(\mathbf{a}\cdot\mathbf{v}_s(\tau))\mathbf{k} - (\mathbf{k}\cdot\mathbf{v}_s(\tau))\mathbf{a}}{\omega} \right]$$
$$\cdot \frac{\partial F_s(v_s(t))}{\partial \mathbf{v}_s(\tau)} \int_0^\infty d\tau' \exp(i\omega\tau' - \Delta\tau') E_k(\tau', \epsilon t)$$
$$= G(\omega) \int_0^\infty d\tau' \exp(i\omega\tau' - \Delta\tau') E_k(\tau', \epsilon\tau')$$
$$\simeq \lim_{\Delta \to 0} G(\omega) \int_0^\infty d\tau' \exp(i\omega\tau' - \Delta\tau')$$
$$\times \left\{ E_k(\tau', \epsilon t) + \epsilon(\tau' - t) \cdot \frac{\partial E_k(\tau', \epsilon t)}{\partial t} + O(\epsilon^2) \right\}$$
$$\simeq G(\omega) E_k(\omega, \epsilon t) + i \frac{\partial G}{\partial \omega} \frac{\partial E_k(\omega, \epsilon t)}{\partial t} + O(\epsilon^2)$$
$$= -\lim_{\Delta \to 0} \frac{n_s e_s}{m_s} \int_0^\infty d\tau \left\{ \left[E_k(\omega, \epsilon t) + \frac{i\,\partial E_k(\omega, \epsilon t)}{\partial t} \frac{\partial}{\partial \omega} \right] \right.$$
$$\times \left[\exp[i\omega\tau - \Delta\tau - i\mathbf{k}\cdot\mathbf{r}_s(\tau)] \right.$$
$$\times \left(\mathbf{a} + \frac{(\mathbf{a}\cdot\mathbf{v}_s(\tau))\mathbf{k} - (\mathbf{k}\cdot\mathbf{v}_s(\tau))\mathbf{a}}{\omega} \right)$$
$$\left. \left. \cdot \frac{\partial F_s(\mathbf{v}_s(\tau))}{\partial \mathbf{v}_s(\tau)} \right] \right\} + O(\epsilon^2) \qquad (40)$$

Hereafter we shall neglect the second-order terms. Returning to Eqs. (26) and (27), and making use of Eqs. (38) and (40), we find

$$[\omega^2 \epsilon_{ij}^\dagger(\mathbf{k}, \omega) - c^2 k^2 \delta_{ij} + c^2 k_i k_j] E_k^j(\omega, \epsilon t)$$
$$+ i\omega \frac{\partial}{\partial \omega} \left[\frac{c^2 k_i k_j - c^2 k^2 \delta_{ij}}{\omega} + \omega \epsilon_{ij}^\dagger(\mathbf{k}, \omega) \right] \frac{\partial E_k^j(\omega, \epsilon t)}{\partial t}$$
$$= H_i^\dagger(\mathbf{k}, \omega) + K_i^\dagger(\mathbf{k}, \omega) \qquad (41)$$

where

$$\mathbf{H}^+ = -\lim_{\Delta \to 0} 4\pi i \sum_s e_s \int d^3 v\, \mathbf{v}$$
$$\times \int_0^\infty dt \exp[i\omega t - \Delta t - i\mathbf{k}\cdot\mathbf{r}_s(t)]\, N_s^k(\mathbf{v}_s(t), 0) \qquad (42)$$

$$\mathbf{K}^+ = -ic[\mathbf{k} \times \mathbf{B}_k(t = 0)] + i\omega \mathbf{E}_k(t = 0)$$
$$- 4\pi \lim_{\Delta \to 0} \sum_s e_s \int d^3 v\, \mathbf{v}$$
$$\times \int_0^\infty dt \exp[i\omega t - \Delta t - i\mathbf{k}\cdot\mathbf{r}_s(t)]$$
$$\times \frac{n_s e_s}{m_s} \left(\frac{\mathbf{v}_s(t) \times \mathbf{B}_k(t = 0)}{c} \right) \cdot \frac{\partial F_s(\mathbf{v}_s(t))}{\partial \mathbf{v}_s(t)} \qquad (43)$$

$$\epsilon^{+}_{ij}(\mathbf{k},\omega) = \delta_{ij} + \frac{4\pi i}{\omega} \lim_{\Delta \to 0} \sum_{s} \frac{n_s e_s^2}{m_s} \int d^3 v\, v_i$$

$$\times \int_0^\infty dt \exp[i\omega t - \Delta t - i\mathbf{k}\cdot\mathbf{r}_s(t)]$$

$$\times \left\{ \left[1 - \frac{\mathbf{k}\cdot\mathbf{v}_s(t)}{\omega}\right] \frac{\partial F_s(\mathbf{v}_s(t))}{\partial v_{sj}(t)} + \frac{v_{sj}(t)}{\omega}\left(\mathbf{k}\cdot\frac{\partial F_s(\mathbf{v}_s(t))}{\partial \mathbf{v}_s(t)}\right)\right\} \tag{44}$$

In $\epsilon^{+}_{ij}(\mathbf{k},\omega)$, which is the usual dielectric tensor, the subscript + indicates that the real frequency ω should be considered as the limit of a complex frequency with a vanishing positive imaginary part. Let us write

$$\epsilon^{+}_{ij}(\mathbf{k},\omega) = Re\,\epsilon^{+}_{ij}(\mathbf{k},\omega) + i\, Im\,\epsilon^{+}_{ij}(\mathbf{k},\omega) \tag{45}$$

Hereafter we shall consider that $Im\,\epsilon^{+}_{ij}$ is of $O(\varepsilon)$ in the transparent region. Neglecting the high-order term in Eq. (41), we find, by collecting the real and imaginary parts, that

$$a_i[\omega^2 Re\,\epsilon^{+}_{ij} - c^2k^2\delta_{ij} + c^2k_ik_j]\,a_j E_k^2 = Re\,\{[H_i^{+}(\mathbf{k},\omega) + K_i^{+}(\mathbf{k},\omega)]\,E_k^{*} a_i\} \tag{46}$$

$$2\omega^2 a_i\, Im\,\epsilon^{+}_{ij}\, a_j E_k^2 + \omega\frac{\partial}{\partial\omega}\left[a_i\left(\frac{-c^2k^2\delta_{ij} + c^2k_ik_j}{\omega} + \omega\, Re\,\epsilon^{+}_{ij}\right)a_j\right]\frac{\partial E_k^2}{\partial t} = 2IM\,[(H_i^{+}(\mathbf{k},\omega) + K_i^{+}(\mathbf{k},\omega))\,a_i E_k^{*}] \tag{47}$$

where $E_k^2 = E_k(\omega,t)\,E_k^{*}(\omega,t)$, and E_k^{*} denotes the complex conjugate of E_k. Obviously, from Eq. (46), we obtain the dispersion equation

$$|\omega^2 Re\,\epsilon^{+}_{ij}(\mathbf{k},\omega) - c^2k^2\delta_{ij} + c^2k_ik_j| = 0 \tag{48}$$

Thus, in the transparent region, Eq. (47) may be rewritten as

$$\frac{\partial}{\partial\omega}(\omega^2 a_i\, Re\,\epsilon^{+}_{ij}\, a_j)\frac{\partial E_k^2}{\partial t} = -2\omega^2 a_i\, Im\,\epsilon^{+}_{ij}\, a_j E_k^2 + 2Im\,[(H_i^{+} + K_i^{+})\,a_i E_k^{*}] \tag{49}$$

Equation (49) is useful in the subsequent discussion.

IV. The Equation of Wave Amplitude

If we consistently keep all the terms in Eq. (49) to the lowest order in ε, then we need to insert only the zeroth-order expression for E_k^{*} into the right-hand side. According to Eq. (41), we have

$$E_k^{*}(\omega,t) = \frac{[H_l^{-}(\mathbf{k},\omega) + K_l^{-}(\mathbf{k},\omega)]\,a_l}{a_\mu \alpha^{-}_{\mu\nu} a_\nu} \tag{50}$$

where H^{-} and K^{-} represent the complex conjugate of H^{+} and K^{+}, respectively. Moreover, for simplicity, we have defined

$$\alpha^{\pm}_{\mu\nu} \equiv \omega^2\epsilon^{\pm}_{\mu\nu} - c^2k^2\delta_{\mu\nu} + c^2k_\mu k_\nu \tag{51}$$

Hence, after taking the ensemble-averaged value, we can write Eq. (47) as

$$\frac{\partial\langle E_k^2(\omega,\varepsilon t)\rangle}{\partial t} = 2\gamma\langle E_k^2(\omega,\varepsilon t)\rangle + 2Im\left\{\frac{\langle[H_l^{+} + K_l^{+}]\,a_l\,[H_m^{-} + K_m^{-}]\,a_m\rangle}{(a_\mu\alpha^{-}_{\mu\nu}a_\nu)\left[\omega\frac{\partial}{\partial\omega}a_i\left(\frac{c^2k_ik_j - c^2k^2\delta_{ij}}{\omega} + \omega\, Re\,\epsilon^{+}_{ij}(\mathbf{k},\omega)\right)a_j\right]}\right\} \tag{52}$$

where

$$\gamma = -\frac{\omega^2 a_i\, Im\,\epsilon^{+}_{ij}(\mathbf{k},\omega)\, a_j}{\omega\frac{\partial}{\partial\omega}a_\mu\left(\frac{c^2k_\mu k_\nu - c^2k^2\delta_{\mu\nu}}{\omega} + \omega\, Re\,\epsilon^{+}_{\mu\nu}\right)a_\nu} \tag{53}$$

Let us restrict our discussion to the field associated with the unstable mode. For that case, we may write

$$E_k(t,\varepsilon t) = E_{0k}(\varepsilon t)\exp(-i\omega_q t)$$

Thus,

$$\langle E_k^2(\omega,\varepsilon t)\rangle = \lim_{\Delta\to 0}\int_0^\infty d\tau \exp[i(\omega-\omega_q)\tau - \Delta\tau]\int_0^\infty d\tau' \times \exp[-i(\omega-\omega_q)\tau' - \Delta\tau'] \times \langle E_{0k}^{*}(\varepsilon\tau)\,E_{0k}(\varepsilon\tau')\rangle$$

$$= \lim_{\Delta\to 0}\int_0^\infty\int_0^\infty d\tau\, d\tau' \exp[i(\omega-\omega_q)(\tau-\tau') - \Delta(\tau+\tau')] \times \{\langle E_{0k}^{*}(\varepsilon t)\,E_{0k}(\varepsilon t')\rangle + O(\varepsilon)\}$$

$$= \lim_{\Delta\to 0}\frac{\pi}{\Delta}\,\delta(\omega-\omega_q)\,\langle E_k^2(\varepsilon t)\rangle + O(\varepsilon) \tag{54}$$

When we substitute expression (54) into Eq. (52), we may ignore the first-order terms in (54) since both $\partial \langle E_k^2 \rangle / \partial t$ and $2\gamma \langle E_k^2 \rangle$ are already spontaneously first order in ε. Furthermore, we know that when $\omega = \omega_q(k)$, we may write

$$\frac{1}{a_\mu \alpha_{\mu\nu}^-(\mathbf{k}, \omega_q)\, a_\nu} = \frac{i\pi\delta(\omega - \omega_q)}{\dfrac{\partial}{\partial\omega_q}[a_\mu Re\, \alpha_{\mu\nu}^-(\mathbf{k}, \omega_q)\, a_\nu]}$$

$$= \frac{i\pi\delta(\omega - \omega_q)}{\dfrac{\partial}{\partial\omega_q}[\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^+ a_\nu]} \qquad (55)$$

and

$$\frac{1}{\omega_q \dfrac{\partial}{\partial\omega_q}\left[a_i \left(\dfrac{c^2 k_i k_j - c^2 k^2 \delta_{ij}}{\omega_q} + \omega_q Re\, \epsilon_{ij}^+ \right) a_j \right]}$$

$$= \frac{1}{\dfrac{\partial}{\partial\omega_q}[\omega_q^2 a_i Re\, \epsilon_{ij}^+(\mathbf{k}, \omega_q)\, a_j]} \qquad (56)$$

From Eq. (52) and relations (54), (55), and (56), we find, after integrating over ω throughout Eq. (52), that

$$\frac{\partial \langle E_k^2(\varepsilon t)\rangle}{\partial t} = 2\gamma(\mathbf{k}, \omega_q)\langle E_k^2(\varepsilon t)\rangle + \lim_{\Delta\to 0} \frac{2\Delta \langle H_l^+(\mathbf{k}, \omega_q)\, a_l\, H_m^-(\mathbf{k}, \omega_q)\, a_m\rangle}{\left| \dfrac{\partial}{\partial\omega_q}[\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^+(\mathbf{k}, \omega_q)\, a_\nu] \right|^2} \qquad (57)$$

Notice that in the source term of Eq. (57) all terms proportional to $K^{\pm}$ vanish. The reason $\Delta \langle H_l^+ a_l H_m^- a_m \rangle$ survives is that it contains a part which behaves like Δ^{-1} (see Appendix A). The final form of Eq. (57) is

$$\frac{\partial}{\partial t}\langle E_k^2(\varepsilon t)\rangle = 2\gamma(\mathbf{k}, \omega_q)\langle E_k^2(\varepsilon t)\rangle + 64\pi^4 \omega_q^2 \sum_s \sum_{n=-\infty}^{+\infty} n_s e_s^2 \int_0^\infty dv_\perp\, v_\perp \int_{-\infty}^{+\infty} dv_z \times \frac{\delta(\omega_q - n\Omega_s - k_z v_z)\,|Q_n^+|^2 F_s(v_z, v_\perp^2)}{\left| \dfrac{\partial}{\partial\omega_q}[\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^+(\mathbf{k}, \omega_q)\, a_\nu] \right|^2} \qquad (58)$$

where

$$|Q_n^+|^2 = \left[v_z a_z + \frac{n\Omega_s a_\perp}{k_\perp}\cos\psi \right]^2 J_n^2\left(\frac{k_\perp v_\perp}{\Omega_s}\right) + a_\perp^2 v_\perp^2 \sin^2\psi\, J_n'^2\left(\frac{k_\perp v_\perp}{\Omega_s}\right) \qquad (59)$$

In Eq. (58) and expression (59), the subscripts z and $\perp$ denote the components of vectors parallel and perpendicular to the magnetic field, ψ is the angle between the vectors $\mathbf{a}_\perp$ and $\mathbf{k}_\perp$, $\Omega_s = e_s B_0 / m_s c$, and J_n and J_n' are the Bessel function of nth order and its derivative, respectively. The first term on the right-hand side may be conceived to be proportional to the stimulated emission of plasmons associated with momentum $\mathbf{k}$ and the second term proportional to the spontaneous emission.

In deriving Eq. (58), we have assumed that

$$F_s(\mathbf{v}) = F_s(v_z, v_\perp^2) \qquad (60)$$

The validity of this postulated condition may be questioned if the unstable mode is propagated in a direction neither parallel nor perpendicular to the external magnetic field $\mathbf{B}_0$. One important point is that if the plasma is initially cylindrically symmetrical, there should be no preferred direction of propagation in the plane of symmetry. This is to say that if the aforementioned unstable mode exists, then other modes must exist which have the same propagation characteristics but are cylindrically symmetrical with respect to the magnetic field. Consequently the symmetry condition is not violated.

Finally, since E_k is the microscopic field, the quantity $\langle E_k^2 \rangle$ may be expressed as the sum of two parts, i.e.,

$$\langle E_k^2 \rangle = \widetilde{E}_k^2 + \langle \delta E_k^2 \rangle$$

where $\widetilde{E}_k$ is the ensemble-averaged field or the macroscopic field and δE_k the fluctuation field. In the usual quasilinear theory, it is assumed that

$$\widetilde{E}_k^2 >> \langle \delta E_k^2 \rangle$$

and, on the other hand, the theory with correlation considers

$$\langle \delta E_k^2 \rangle >> \widetilde{E}_k^2 \to 0$$

V. The Kinetic Equation

Formally the kinetic equation may be expressed as

$$\frac{\partial F_s}{\partial t} = -\sum_{\mathbf{k}} \frac{e_s}{m_s n_s} \frac{\partial}{\partial \mathbf{v}} \int_{-\infty}^{+\infty} d\omega \lim_{\Delta \to 0} \frac{\Delta}{\pi} \times \left[\mathbf{a} + \frac{(\mathbf{v} \cdot \mathbf{a})\mathbf{k} - (\mathbf{v} \cdot \mathbf{k})\mathbf{a}}{\omega} \right] \langle E_k^* (\omega, \varepsilon t) N_s^k (\mathbf{v}, \omega, \varepsilon t) \rangle \tag{61}$$

Since F_s is considered to be symmetrical in the $v_\perp$-plane and $\langle E_{-k}^* N_s^{-k} \rangle = \langle E_k^* N_s^k \rangle^*$, we can average the right-hand term of Eq. (61) over the azimuthal angle and retain only the real part of it. Thus,

$$\frac{\partial F_s}{\partial t} = -\sum_{\mathbf{k}} \frac{e_s}{m_s n_s} \lim_{\Delta \to 0} \frac{\Delta}{2\pi^2} \int_0^{2\pi} d\phi \int_{-\infty}^{+\infty} d\omega \frac{\partial}{\partial \mathbf{v}} \cdot Re \left\{ \left[\mathbf{a} + \frac{(\mathbf{v} \cdot \mathbf{a})\mathbf{k} - (\mathbf{v} \cdot \mathbf{k})\mathbf{a}}{\omega} \right] \langle E_k^* (\omega, \varepsilon t) N_s^k (\mathbf{v}, \omega, \varepsilon t) \rangle \right\} \tag{62}$$

The evaluation of the right-hand term is shown in Appendix B. Here we shall simply state the result:

$$\frac{\partial F_s}{\partial t} = \sum_{\mathbf{k}} \frac{\pi e_s^2}{m_s} \sum_{n=-\infty}^{+\infty} \left(k_z \frac{\partial}{\partial v_z} + \frac{n\Omega_s}{v_\perp} \frac{\partial}{\partial v_\perp} \right) \times \left\{ |Q_n^+|^2 \delta (\omega_q - n\Omega_s - k_z v_z) \times \left[\frac{4\pi F_s (v_z, v_\perp^2)}{\frac{\partial}{\partial \omega_q} (\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^+ (\mathbf{k}, \omega_q) a_\nu)} + \frac{\langle E_k^2 (\varepsilon t) \rangle}{\omega_q^2 m_s} \left(k_z \frac{\partial F_s}{\partial v_z} + \frac{n\Omega_s}{v_\perp} \frac{\partial F_s}{\partial v_\perp} \right) \right] \right\} \tag{63}$$

The quantity $|Q_n^+|^2$ is defined by Eq. (59).

Equation (63) is obtained in the quasilinear approximation in which the resonant diffusion process is considered to be most important and the nonresonant contribution and the ordinary collision integral are neglected. The expression for the nonresonant contribution is somewhat lengthy and may be found in Appendix B. The term proportional to $\langle E_k^2 \rangle$ is obtainable from the conventional quasilinear theory. The extra term, or what we call the friction term, is attributed to correlation.

VI. Conservation of Energy

It is important to examine the law of conservation of energy from the equation just derived. However, since the resonant diffusion process is important only in a restricted region in the velocity space, it is conceivable that in the discussion of conservation of energy and momentum one must consider both resonant and nonresonant contributions to the kinetic equation.

Let us study the resonant contribution by making use of Eq. (63). We obtain

$$\left(\frac{\partial U}{\partial t} \right)_{\text{resonant}} = \sum_s \frac{\partial}{\partial t} \left[\frac{n_s m_s}{2} \int d^3 v\, v^2 F_s \right]_{\text{resonant}}$$

$$= -\sum_{\mathbf{k}} \sum_s \frac{2\pi^2 n_s e_s^2}{m_s} \int_0^{\infty} dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_z \times \sum_{n=-\infty}^{+\infty} (k_z v_z + n\Omega_s) |Q_n^+|^2 \delta (\omega_q - n\Omega_s - k_z v_z) \left[\frac{4\pi m_s F_s}{\frac{\partial}{\partial \omega_q} (\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^+ a_\nu)} + \frac{\langle E_k^2 (\varepsilon t) \rangle}{\omega_q^2} \left(k_z \frac{\partial F_s}{\partial v_z} + \frac{n\Omega_s}{v_\perp} \frac{\partial F_s}{\partial v_\perp} \right) \right] \tag{64}$$

However, from Eq. (58) and definition (53), we find

$$\frac{\partial}{\partial \omega_q} (\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^+ (\mathbf{k}, \omega_q) a_\nu) \frac{\partial}{\partial t} \frac{\langle E_k^2 \rangle}{8\pi} = \sum_s \frac{2\pi^2 n_s e_s^2}{m_s} \times \int_0^{\infty} dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_z \sum_{n=-\infty}^{+\infty} |Q_n^+|^2 \delta (\omega_q - n\Omega_s - k_z v_z) \times \left[\frac{4\pi m_s F_s \omega_q^2}{\frac{\partial}{\partial \omega_q} (\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^+ a_\nu)} + \langle E_k^2 (\varepsilon t) \rangle \left(k_z \frac{\partial F_s}{\partial v_z} + \frac{n\Omega_s}{v_\perp} \frac{\partial F_s}{\partial v_\perp} \right) \right] \tag{65}$$

In Eq. (65), we have made use of expression for

$$a_i\, Im \epsilon_{ij}^+ (\mathbf{k}, \omega_q) a_j$$

which is shown in Appendix C.

Comparing Eqs. (64) and (65), we find

$$\left(\frac{\partial U}{\partial t}\right)_{\text{resonant}} = -\sum_{\mathbf{k}} \frac{1}{\omega_q}\frac{\partial}{\partial \omega_q}\left[\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^{+}(\mathbf{k}, \omega_q) a_\nu\right]\frac{\partial}{\partial t}\frac{\langle E_k^2\rangle}{8\pi} \tag{66}$$

To calculate the nonresonant contribution, we shall employ the result given by Eq. (B-19) in Appendix B. We see that

$$\left(\frac{\partial U}{\partial t}\right)_{\text{nonresonant}} = \sum_{s} \frac{n_s m_s}{2}\int d^3 v\, v^2 (I_s^{\text{II}})_{\text{nonresonant}}$$

$$= -\sum_{\mathbf{k}}\sum_{s}\frac{4\pi n_s e_s^2}{4 m_s \omega_q^2}\sum_{n=-\infty}^{+\infty}\int_0^{\infty} dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_z$$

$$\times\left\{a_z^2 v_z J_n^2(v_z)\frac{\partial F_s}{\partial v_z} + a_\perp^2 \cos^2\psi\left(\frac{n\Omega_s}{k_\perp}\right)^2 J_n^2 \frac{1}{v_\perp}\frac{\partial F_s}{\partial v_\perp}\right.$$

$$+ a_\perp^2 v_\perp^2 \sin^2\psi J_n'^2 \frac{1}{v_\perp}\frac{\partial F_s}{\partial v_\perp}$$

$$+\left[-\omega_q \frac{\partial}{\partial \omega_q} P\frac{1}{\omega_q - n\Omega_s - v_z k_z} + P\frac{1}{\omega_q - k_z v_z - n\Omega_s}\right]$$

$$\left.\times |Q_n^{+}|^2\left(k_z\frac{\partial F_s}{\partial v_z} + \frac{n\Omega_s}{v_\perp}\frac{\partial F_s}{\partial v_\perp}\right)\right\} \tag{67}$$

However, from Appendix C, we know that

$$a_\mu Re\, \epsilon_{\mu\nu}^{+}(\mathbf{k}, \omega_q) a_\nu$$

$$= 1 + 2\pi\sum_{s}\frac{\omega_s^2}{\omega_q^2}\int_0^{\infty} dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_z$$

$$\times\sum_{n=-\infty}^{+\infty}\left\{P\frac{|Q_n^{+}|^2}{\omega_q - n\Omega_s - k_z v_z}\left(k\frac{\partial F_s}{\partial v_z} + \frac{n\Omega_s}{v_\perp}\frac{\partial F_s}{\partial v_\perp}\right)\right.$$

$$+ a_z^2 v_z J_n^2\frac{\partial F_s}{\partial v_z} + \frac{a_\perp^2 n^2\Omega_s^2}{k_\perp^2}\cos^2\psi J_n^2\frac{1}{v_\perp}\frac{\partial F_s}{\partial v_\perp}$$

$$\left.+ a_\perp^2 v_\perp^2 \sin^2\psi J_n'^2\frac{1}{v_\perp}\frac{\partial F_s}{\partial v_\perp}\right\} \tag{68}$$

From (67) and (68), we obtain

$$\left(\frac{\partial U}{\partial t}\right)_{\text{nonresonant}} = -\sum_{\mathbf{k}}\left\{\left[a_\mu Re\, \epsilon_{\mu\nu}^{+}(\mathbf{k}, \omega_q) a_\nu\right]\frac{\partial}{\partial t}\frac{\langle E_k^2\rangle}{8\pi}\right.$$

$$+ \frac{1}{\omega_q}\frac{\partial}{\partial \omega_q}\left(\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^{+} a_\nu\right)$$

$$\left.\times\frac{\partial}{\partial t}\frac{\langle E_k^2\rangle}{8\pi} - \frac{\partial}{\partial t}\frac{\langle E_k^2\rangle}{8\pi}\right\} \tag{69}$$

However, we know that

$$\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^{+}(\mathbf{k}, \omega_q) = c^2 k^2 - c^2 a_\mu k_\mu k_\nu a_\nu$$

Thus,

$$\left(\frac{\partial U}{\partial t}\right)_{\text{nonresonant}}$$

$$= -\sum_{\mathbf{k}}\left\{\left[\frac{c^2 k^2 - c^2 a_\mu k_\mu k_\nu a_\nu}{\omega_q^2} + 1\right]\frac{\partial}{\partial t}\frac{\langle E_k^2\rangle}{8\pi}\right\} \tag{70}$$

However, it is easy to verify that

$$\left[\frac{c^2 k^2 - c^2 a_\mu k_\mu k_\nu a_\nu}{\omega_q^2}\right]\langle E_k^2\rangle = \frac{c^2}{\omega_q^2}(\mathbf{k}\times\mathbf{a})^2\langle E_k^2\rangle = \langle B_k^2\rangle \tag{71}$$

From Eqs. (66), (69), and (70), we conclude that

$$\left(\frac{\partial U}{\partial t}\right)_{\text{total}} + \frac{\partial}{\partial t}\sum_{\mathbf{k}}\left(\frac{\langle E_k^2\rangle}{8\pi} + \frac{\langle B_k^2\rangle}{8\pi}\right) = 0 \tag{72}$$

This proves that the sum of the particle energy and energy associated with the wave fields is constant in time.

VII. Stationary Solution

In this section, we shall discuss the form of the time-independent solution of F_s. From the amplitude equation, Eq. (59), we find that

$$\int_0^{\infty} dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_z \sum_{n=-\infty}^{+\infty}\delta(\omega_q - n\Omega_s - k_z v_z)|Q_n^{+}|^2$$

$$\times\left\{4\pi\left(k_z\frac{\partial F_s}{\partial v_z} + \frac{n\Omega_s}{v_\perp}\frac{\partial F_s}{\partial v_\perp}\right)\langle E_k^2\rangle + m_s(16\pi^2\omega_q^2)\right.$$

$$\left.\times\frac{F_s(v_z, v_\perp^2)}{\frac{\partial}{\partial \omega_q}\left[\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^{+} a_\nu\right]}\right\} = 0 \tag{73}$$

However, from the kinetic equation we also conclude that any moment T_s, i.e.,

$$T_s = 2\pi \int_0^\infty dv_\perp v_\perp \int dv_z \Phi_s(v_z, v_\perp) F_s(v_z, v_\perp^2)$$

must be independent of time. Thus,

$$\int_0^\infty dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_z \sum_{n=-\infty}^{+\infty} \delta(\omega_q - n\Omega_s - k_z v_z) |Q_n^+|^2$$

$$\times \left(\frac{\partial \Phi}{\partial v_z} + \frac{n\Omega_s}{v_\perp}\frac{\partial \Phi}{\partial v_\perp}\right)\left\{4\pi\left(k_z \frac{\partial F_s}{\partial v_z} + \frac{n\Omega_s}{v_\perp}\frac{\partial F_s}{\partial v_\perp}\right)\langle E_k^2\rangle\right.$$

$$\left. + m_s (16\pi^2 \omega_q^2) \frac{F_s(v_z, v_\perp^2)}{\frac{\partial}{\partial \omega_q}(\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^+ a_\nu)}\right\} = 0 \qquad (74)$$

From Eqs. (73) and (74), we see that because Φ is arbitrary, it must be true that

$$\left\{\left(k_z \frac{\partial F_s}{\partial v_z} + \frac{n\Omega_s}{v_\perp}\frac{\partial F_s}{\partial v_\perp}\right)\langle E_k^2\rangle\right.$$

$$\left. + \frac{4\pi m_s \omega_q^2 F_s(v_z, v_\perp^2)}{\frac{\partial}{\partial \omega_q}(\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^+ a_\nu)}\right\}_{k_z v_z + n\Omega_s = \omega_q} = 0 \qquad (75)$$

Let us assume that $F_s(v_z, v_\perp^2) = g_1(v_z) g_2(v_\perp^2)$. Hence,

$$\{[k_z g_1' g_2 + 2n\Omega_s g_1 g_2'] \alpha + g_1 g_2\}_{k_z v_z + n\Omega = \omega_q} = 0$$

or

$$\left[k_z \frac{g_1'}{g_1} + 2n\Omega_s \frac{g_2'}{g_2}\right]_{k_z v_z + n\Omega_s = \omega_q} \alpha = -1 \qquad (76)$$

where

$$\alpha = \frac{\langle E_k^2\rangle}{4\pi} \frac{1}{m_s \omega_q^2} \frac{\partial}{\partial \omega_q}(\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^+ a_\nu) \qquad (77)$$

We now define

$$\frac{g_1'}{g_1} = f_1(v_z) \qquad \text{and} \qquad \frac{g_2'}{g_2} = f_2(v_\perp^2) \qquad (78)$$

Since α is independent of v_z, $v_\perp^2$ and n, the only choice of f_1 and f_2 is

$$f_1 = -2v_z A_s \qquad (79)$$

$$f_2 = -A_s \qquad (80)$$

where A_s is a constant. Thus, from Eq. (76), we find the condition

$$2\omega_q \alpha A_s = 1 \qquad (81)$$

From Eqs. (78), (79), and (80) we can easily show that

$$F_s = g_1(v_z) g_2(v_\perp^2) = B_s \exp[-A_s(v_z^2 + v_\perp^2)] \qquad (82)$$

In Eq. (82), B_s is a normalization coefficient. Since we require

$$2\pi \int_0^\infty dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_z F_s(v_z, v_\perp^2) = 1 \qquad (83)$$

we find that

$$B_s = \left(\frac{A_s}{\pi}\right)^{3/2} \qquad (84)$$

and A_s may be identified with the thermal energy, such that

$$A_s = \frac{m_s}{2\chi T}$$

Therefore, Eq. (82) shows that the time-independent solution of F_s is the Maxwellian distribution. When this state is reached, we see that, from Eq. (77),

$$\langle E_k^2\rangle = \langle \delta E_k^2\rangle = \frac{4\pi\chi T}{\frac{1}{\omega_q}\frac{\partial}{\partial \omega_q}(\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^+ a_\nu)} \qquad (85)$$

for the mode with wave vector k and frequency ω_q. However, since in this case $-\omega_q(\mathrm{k})$ is also a root of the dispersion equation, the total value of $\langle \delta E_k^2\rangle$ should be

$$\langle \delta E_k^2\rangle_t = \frac{8\pi\chi T \omega_q}{\frac{\partial}{\partial \omega_q}(\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^+ a_\nu)} \qquad (86)$$

Equation (86) represents the energy spectrum of the fluctuation field. When there is no external field and $\delta \mathbf{E}$ is longitudinal, (86) reduces to

$$\langle \delta E_k^2 \rangle_t \simeq 4\pi \chi T \tag{87}$$

which is well known.

VIII. Special Cases

In this section, we shall consider a number of special cases for which the equations yield simpler forms. Discussion of these cases may facilitate the future applications of the theory.

A. Electrostatic Instabilities Without an External Magnetic Field

The quasilinear equations for this case reduce to the following form:

$$\frac{\partial F_s}{\partial t} = \sum_{\mathbf{k}} \frac{\pi e_s^2}{m_s k^2} \left(\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{v}}\right) \times \left\{ \delta(\omega \mathbf{k} \cdot \mathbf{v}) \left[\frac{4\pi F_s(\mathbf{v})}{\frac{\partial}{\partial \omega_q}(Re\, \epsilon^+(\mathbf{k}, \omega_q))} + \frac{\langle E_k^2 \rangle}{m_s} \left(\mathbf{k} \cdot \frac{\partial F_s}{\partial \mathbf{v}}\right) \right] \right\} \tag{88}$$

$$\frac{\partial \langle E_k^2 \rangle}{\partial t} = 2\gamma \langle E_k^2 \rangle + 8\pi^2 \sum_s \frac{\omega_s^2 m_s}{k^2} \times \int d^3 v \, \frac{\delta(\omega_q - \mathbf{k} \cdot \mathbf{v}) F_s(\mathbf{v})}{\left| \frac{\partial}{\partial \omega_q}(Re\, \epsilon^+(\mathbf{k}, \omega_q)) \right|^2} \tag{89}$$

where

$$Re\, \epsilon^+(\mathbf{k}, \omega_q) = 1 + \sum_s \frac{\omega_s^2}{k^2} P \int d^3 v \left(\mathbf{k} \cdot \frac{\partial F_s}{\partial \mathbf{v}}\right) \frac{1}{(\omega - \mathbf{k} \cdot \mathbf{v})} \tag{90}$$

$$\gamma = \frac{\pi}{\frac{\partial}{\partial \omega_q}(Re\, \epsilon^+(\mathbf{k}, \omega_q))} \sum_s \frac{\omega_s^2}{k^2} \int d^3 v \, \delta(\omega - \mathbf{k} \cdot \mathbf{v}) \left(\mathbf{k} \cdot \frac{\partial F_s}{\partial \mathbf{v}}\right) \tag{91}$$

If we consider electron oscillations and approximate

$$\frac{\partial}{\partial \omega_q}(Re\, \epsilon^+(\mathbf{k}, \omega_q)) \simeq \frac{2}{\omega_q} \tag{92}$$

then it is found that the form of Eqs. (88) and (89) reduces to that obtained previously by Harris,[4] who formulated the problem from a quantum mechanical approach, and other authors (Refs. 12 and 13). However, two points should be noted. First, in our theory the field E_k is the total wave field (macroscopic field plus fluctuation field). In other words, we have

$$\langle E_k^2 \rangle = \widetilde{E}_k^2 + \langle \delta E_k^2 \rangle$$

where $\widetilde{E}_k$ is the usual macroscopic field. If the plasma satisfies the conditions

$$\widetilde{E}^2 >> \langle \delta E_k^2 \rangle \tag{94}$$

and

$$\widetilde{E}^2 >> \left[\frac{4\pi m_s F_s(\mathbf{v})}{\left(\mathbf{k} \cdot \frac{\partial F_s}{\partial \mathbf{v}}\right) \frac{\partial}{\partial \omega_q}(Re\, \epsilon^+(\mathbf{k}, \omega_q))} \right]_{v = \omega_q/k} \tag{95}$$

then Eqs. (88) and (89) reduce to the conventional quasilinear equations. From this point of view, we can also see that condition (94) alone is not sufficient to validate the usual quasilinear equations. Condition (95) indicates that when $(\mathbf{k} \cdot \partial F_s/\partial \mathbf{v})_{v=\omega/k}$ becomes very small—for instance, when the "plateau" is asymptotically formed in the one-dimensional "bump-on-the-tail" problem (Refs. 1 and 2), the efficiency of stimulated emission of plasmons becomes very low, and the spontaneous emission may become significant.

Second, in deriving the equations, we have assumed that the unstable mode has a frequency $\omega_q(\mathbf{k})$. In some cases, when F_s is symmetrical in velocity space, $-\omega_q(\mathbf{k})$ can also be a root of the dispersion equation. Then, if we define each propagating mode as consisting of both plus and minus frequencies, the friction term in Eq. (88) and the source term in Eq. (89) should be modified by a factor of 2 and the energy density is defined as twice the present value.

Finally, it is important to note that in the study of a turbulent plasma in which ion-wave instability is playing a major role, the spontaneous emission can be more significant than that in the case of a growing Langmuir wave, since the phase velocity of the ion wave is low compared with the electron thermal velocity.

[4]Notice that the contribution from the nonresonant particles vanishes automatically since it is proportional to $\partial \langle E_k^2 \rangle / \partial t$.

B. Electrostatic Instabilities With an External Magnetic Field

The unified quasilinear equations in this case may be obtained by setting $\psi = 0$ (longitudinal waves):

$$\frac{\partial F_s}{\partial t} = \sum_{\mathbf{k}} \sum_{n=-\infty}^{+\infty} \frac{\pi e_s^2}{m_s k^2} \left(k_z \frac{\partial}{\partial v_z} + \frac{n\Omega_s}{v_\perp} \frac{\partial}{\partial v_\perp} \right) \times \left\{ \delta(\omega_q - n\Omega_s - k_z v_z) J_n^2 \left(\frac{k_\perp v_\perp}{\Omega_s} \right) \times \left[\frac{4\pi F_s(v_z, v_\perp^2)}{\frac{\partial}{\partial \omega_q} [Re\, \epsilon^+(\mathbf{k}, \omega_q)]} + \frac{\langle E_k^2 \rangle}{m_s} \left(k_z \frac{\partial F_s}{\partial v_z} + \frac{n\Omega_s}{v_\perp} \frac{\partial F_s}{\partial v_\perp} \right) \right] \right\} \tag{96}$$

$$\frac{\partial \langle E_k^2 \rangle}{\partial t} = 2\gamma \langle E_k^2 \rangle + 16\pi^3 \sum_s \frac{m_s \omega_s^2}{k^2} \int_0^\infty dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_z \times \sum_{n=-\infty}^{+\infty} \frac{\delta(\omega_q - n\Omega_s - k_z v_z) J_n^2 F_s(v_z, v_\perp^2)}{\left| \frac{\partial}{\partial \omega_q} [Re\, \epsilon^+(\mathbf{k}, \omega_q)] \right|^2} \tag{97}$$

where the growth rate is

$$\gamma = 2\pi \sum_s \frac{\omega_s^2}{k^2} \int_0^\infty dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_z \times \sum_{n=-\infty}^{+\infty} \delta(\omega_q - n\Omega_s - k_z v_z) J_n^2 \left(\frac{k_\perp v_\perp}{\Omega_s} \right) \times \left(k_z \frac{\partial F_s}{\partial v_z} + \frac{n\Omega_s}{v_\perp} \frac{\partial F_s}{\partial v_\perp} \right) \frac{1}{\frac{\partial}{\partial \omega_q} [Re\, \epsilon^+(\mathbf{k}, \omega_q)]} \tag{98}$$

We can see easily that when $(\omega_q/k_z) < v_s \equiv$ thermal velocity of the s-species particles, the friction term in Eq. (96) and the source term in Eq. (97) can be very important. Yet, these terms are not included in the usual quasilinear theory.

C. Cyclotron or Alfvén Waves in a Magnetized Plasma

In this case, $a_z = k_\perp = 0$ and $\psi = \pi/2$. Thus, we find that $|Q_n^+|^2 = 0$ if $n \neq \pm 1$, and $|Q_n^+|^2 = v_\perp^2/4$ if $n = \pm 1$.

So far we have considered that the wave is linearly polarized. In order to discuss a cyclotron wave, which is circularly polarized, we may consider it as the superposition of two linearly polarized modes with a phase difference of $\pi/2$. This can be readily studied, and the result is given as follows:

$$\frac{\partial F_s}{\partial t} = \sum_{\mathbf{k}} \frac{\pi e_s^2}{m_s} \left(k_z \frac{\partial}{\partial v_z} \pm \frac{\Omega_s}{v_\perp} \frac{\partial}{\partial v_\perp} \right) \times \left\{ \delta(\omega_q - (\pm\Omega_s) - k_z v_z) v_\perp^2 \times \left[\frac{2\pi F_s(v_z, v_\perp^2)}{\frac{\partial}{\partial \omega_q} (\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^+ a_\nu)} + \frac{\langle E_k^2 \rangle}{2\omega_q^2 m_s} \left(k_z \frac{\partial F_s}{\partial v_z} \pm \frac{\Omega_s}{v_\perp} \frac{\partial F_s}{\partial v_\perp} \right) \right] \right\} \tag{99}$$

$$\frac{\partial \langle E_k^2 \rangle}{\partial t} = 2\gamma \langle E_k^2 \rangle + 4\pi^3 \omega_q^2 \sum_s \omega_s^2 m_s \int_0^\infty dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_z \times \delta(\omega_q - (\pm\Omega_s) - k_z v_z) \frac{v_\perp^2 F_s(v_z, v_\perp^2)}{\left| \frac{\partial}{\partial \omega_q} (\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^+(k_z, \omega_q) a_\nu) \right|^2} \tag{100}$$

where

$$\gamma = \frac{\pi^2}{2} \sum_s \omega_s^2 \int_0^\infty dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_z\, \delta[\omega_q - (\pm\Omega_s) - k_z v_z] \times v_\perp^2 \frac{\left(k_z \frac{\partial F_s}{\partial v_z} \pm \frac{\Omega_s}{v_\perp} \frac{\partial F_s}{\partial v_\perp} \right)}{\frac{\partial}{\partial \omega_q} (\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^+ a_\nu)} \tag{101}$$

It is understood that, in the expression

$$\frac{\partial}{\partial \omega_q} (\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^+ a_\nu)$$

we should set $a_z = k_\perp = 0$. Furthermore, the proper sign to be chosen depends upon whether the wave is right- or left-circularly polarized. It is useful to remember that Ω_s is negative for the electrons and positive for the ions. Thus from the argument of the delta function in Eq. (99) or (100), we see that for the right-circularly polarized wave (electron cyclotron wave) we should choose the minus sign, otherwise the plus sign.

IX. Summary and Discussion

One flaw in the usual quasilinear theory is that the theory breaks down when the growth rate diminishes. As this condition is approached, the process of resonant diffusion becomes more and more inefficient. For the one-dimensional "bump-on-the-tail" problem, the "dying" diffusion process eventually leads to an ambiguous result, namely, the formation of a plateau at a certain part of the distribution function; and then the "quasilinear" interaction comes to a complete stop. It is true that this difficulty may be resolved by considering the three-dimensional problem. However, the basic problem is still there, although it appears less severe.

In this memorandum, we present a unified theory which includes both the *macroscopic* coherent field and the *microscopic* fluctuation field. We are especially interested in the case in which the contribution to the pair correlation from the propagating mode of the fluctuation field is considerably more important than that due to the nonpropagating modes and direct particle encounters. For the sake of generality, we have considered both electrostatic and electromagnetic interactions, and also a magnetized plasma. The result may be summarized as follows:

$$\frac{\partial F_s}{\partial t} = \sum_{k} \frac{\pi e_s^2}{m_s} \sum_{n=-\infty}^{+\infty} \left(k_z \frac{\partial}{\partial v_z} + \frac{n\Omega_s}{v_\perp} \frac{\partial}{\partial v_\perp} \right) \times \left\{ |Q_n^+|^2 \, \delta(\omega_q - n\Omega_s - k_z v_z) \times \left[\frac{4\pi F_s(v_z, v_\perp^2)}{\frac{\partial}{\partial \omega_q}(\omega_q^2 a_\mu \, Re \, \epsilon_{\mu\nu}^+(\mathbf{k}, \omega_q) a_\nu)} + \frac{\langle E_k^2(\varepsilon t)\rangle}{\omega_q^2 m_s} \left(k_z \frac{\partial F_s}{\partial v_z} + \frac{n\Omega_s}{v_\perp} \frac{\partial F_s}{\partial v_\perp} \right) \right] \right\} \tag{102}$$

$$\frac{\partial}{\partial t} \langle E_k^2(\varepsilon t)\rangle = 2\gamma(\mathbf{k}, \omega_q) \langle E_k^2(\varepsilon t)\rangle + 64\pi^4 \omega_q^2 \sum_s \sum_{n=-\infty}^{+\infty} n_s e_s^2 \int_0^\infty dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_z \times \frac{\delta(\omega_q - n\Omega_s - k_z v_z) |Q_n^+|^2 F_s(v_z, v_\perp^2)}{\left| \frac{\partial}{\partial \omega_q} [\omega_q^2 a_\mu \, Re \, \epsilon_{\mu\nu}^+(\mathbf{k}, \omega_q) a_\nu \right|^2} \tag{103}$$

where

$$|Q_n^+|^2 = \left[v_z a_z + \frac{n\Omega_s a_\perp}{k_\perp} \cos\psi \right]^2 J_n^2\left(\frac{k_\perp v_\perp}{\Omega_s}\right) + a_\perp^2 v_\perp^2 \sin^2\psi J_n'^2\left(\frac{k_\perp v_\perp}{\Omega_s}\right)$$

Equations (102) and (103) show that the time-independent solution of F_s is the Maxwellian distribution. The conservation of particle and wave energy is also proved by taking into account both the resonant and nonresonant contributions.

The "friction" term in Eq. (102) and the source term in Eq. (103) represent the spontaneous Cerenkov emission of the unstable mode. The usual "collision integral"[6] is considered to be negligible in this case.

For the high-frequency electron wave, the spontaneous emission term in Eq. (102) is admittedly small and of the same order of magnitude as the collision term (Ref. 29). However, for the unstable ion wave, the spontaneous emission can be much more important than the collisional contribution since the population of the emitting electrons is high.

In deriving the amplitude equation, we have made use of the adiabatic approximation; that is, we have treated the distribution function as a time-independent quantity. Strictly speaking, such an approximation may be inconsistent with the theory and, in principle, we should include the effect of slow variation of F_s in the derivation. However, such correction turns out to be rather insignificant in the quasilinear theory, as pointed out by several authors (Refs. 6 and 30). Intuitively, this consequence is conceivable from the following point of view. It is expected that the correction terms to the Landau growth rate are proportional to the moments of $\partial F_s/\partial t$ since γ is independent of particle velocity. However, according to the usual quasilinear theory, we know that only a narrow region of the velocity distribution function evolves significantly due to the resonant diffusion process, and the change of the moment of the entire distribution function with respect to time cannot be very large under ordinary

[6]The general collision integral has not been derived in this memorandum. However, when electromagnetic interaction is not important, the collision integral has been discussed by Rostoker (Ref. 28).

circumstances. Thus, the correction to the growth rate cannot be important.

The theory established in the present work is fairly general except for the assumption of weak turbulence. However, it is possible to improve the theory so that it can describe strong turbulence. It also appears desirable to generalize the present theory to study instabilities originated by plasma inhomogeneity such as, for instance, the drift-wave instability.

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Appendix A
Evaluation of the Source Term

The source term may be represented as

$$S = \lim_{\Delta \to 0} 2\Delta \int_{-\infty}^{+\infty} d\omega \frac{\delta\,[a_m \, Re\, \alpha^{-}_{ml}\, a_l]}{\frac{\partial}{\partial \omega}[a_i \, Re\, \alpha^{+}_{ij}(k,\omega)\, a_j]} \langle H^{+}_{\mu} a_{\mu} H^{-}_{\nu} a_{\nu} \rangle$$

$$= \lim_{\Delta \to 0} 2\Delta \int_{-\infty}^{+\infty} d\omega \frac{\delta(\omega - \omega_q)\, 16\pi^2 \omega^2}{\frac{\partial}{\partial \omega} \left| [\omega^2 a_i \, Re\, \epsilon^{+}_{ij}(\mathbf{k},\omega)\, a_j] \right|^2}$$

$$\times \sum_s e_s \sum_r e_r \int d^3 v\, (\mathbf{v} \cdot \mathbf{a}) \int d^3 v'\, (\mathbf{v}' \cdot \mathbf{a})$$

$$\times \int_0^{\infty} dt \exp[i\omega t - \Delta t - i\mathbf{k} \cdot \mathbf{r}_s(t)]$$

$$\times \int_0^{\infty} dt' \exp[-i\omega t' - \Delta t' + i\mathbf{k} \cdot \mathbf{r}'_r(t')]$$

$$\times n_s \delta_{sr} \delta\, [\mathbf{v}_s(t) - \mathbf{v}'_r(t')]\, F_s(\mathbf{v}_s(t)) \qquad \text{(A-1)}$$

where we have made use of Eq. (24). (Notice: other terms do not contribute.)

If we "translate" the velocities such that $\mathbf{v}_s(t) \to \mathbf{v}$ and $\mathbf{v}_s(t') \to \mathbf{v}'$, then correspondingly, we should translate $\mathbf{v}$ to $\mathbf{v}_s(-t)$ and $\mathbf{v}'$ to $\mathbf{v}_r(-t')$. Hence, we can write

$$S = \lim_{\Delta \to 0} \Delta \frac{32\pi^2 \omega_q^2}{\left| \frac{\partial}{\partial \omega_q} [\omega_q^2 a_i \, Re\, \epsilon^{+}_{ij}(\mathbf{k},\omega_q)\, a_j] \right|^2} \sum_s n_s e_s^2 \int d^3 v'$$

$$\times \int_0^{\infty} dt \exp[i\omega t - \Delta t + i\mathbf{k} \cdot \mathbf{r}_s(-t)]$$

$$\times \int_0^{\infty} dt' \exp[-i\omega t' - \Delta t' - i\mathbf{k} \cdot \mathbf{r}_s(-t')]$$

$$\times [\mathbf{v}_s(-t) \cdot \mathbf{a}]\, [\mathbf{v}'_s(-t') \cdot \mathbf{a}]$$

$$\times \delta(\mathbf{v} - \mathbf{v}')\, F_s(\mathbf{v}) \qquad \text{(A-2)}$$

In Eq. (A-2), we have replaced $\mathbf{r}_s(t)$ by $-\mathbf{r}_s(-t)$, and $\mathbf{r}'_s(t)$ by $-\mathbf{r}'_s(-t')$ because of the velocity translation. Integrating over $\mathbf{v}'$ and utilizing the relations

$$\mathbf{v}_s(-t) \cdot \mathbf{a} = v_z a_z + v_\perp a_\perp [\cos(\phi + \Omega_s t) \cos\psi - \sin(\phi + \Omega_s t) \sin\psi] \qquad \text{(A-3)}$$

$$i\mathbf{k} \cdot [\mathbf{r}_s(-t) - \mathbf{r}'_s(-t')] = -ik_z v_z (t - t') - i\frac{k_\perp v_\perp}{\Omega_s} [\sin(\phi + \Omega_s t) - \sin(\phi + \Omega_s t')] \qquad \text{(A-4)}$$

where ϕ is the angle between the vectors $\mathbf{k}_\perp$ and $\mathbf{v}_\perp$ and other quantities are defined after Eq. (59), we find that

$$S = \lim_{\Delta \to 0} \frac{32\pi^2 \omega_q^2 \Delta \sum_s n_s e_s^2}{\left| \frac{\partial}{\partial \omega_q} [\omega_q^2 a_i \, Re\, \epsilon^{+}_{ij}(\mathbf{k},\omega_q)\, a_j] \right|^2} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \int_0^{2\pi} d\phi$$

$$\times \int_0^{\infty} dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_z \int_0^{\infty} dt \int_0^{\infty} dt'$$

$$\times \exp[i\omega_q (t - t') - \Delta(t + t') - in\Omega_s t + im\Omega_s t' - im\phi + in\phi] \left\{ \left[a_z v_z + a_\perp \cos\psi \left(\frac{n\Omega_s}{k_\perp} \right) \right] \times J_n(\alpha) - ia_\perp v_\perp \sin\psi J'_n(\alpha) \right\}$$

$$\times \left\{ \left[a_z v_z + a_\perp \cos\psi \left(\frac{m\Omega_s}{k_\perp} \right) \right] \times J_m(\alpha) + ia_\perp v_\perp \sin\psi J'_m(\alpha) \right\} F_s(\mathbf{v}) \qquad \text{(A-5)}$$

where we have used the identity

$$e^{i\alpha \sin\phi} = \sum_{n=-\infty}^{+\infty} J_n(\alpha)\, e^{in\phi}$$

and ϕ is defined as

$$\alpha = \frac{k_\perp v_\perp}{\Omega_s}$$

Since we consider that

$$F_s(\mathbf{v}) = F_s(v_z, v_\perp^2)$$

the ϕ-integration is non-zero only when $m = n$. We finally conclude that

$$S = \frac{6\pi^4 \omega_q^2 \sum_s n_s e_s^2}{\left| \frac{\partial}{\partial \omega_q} [\omega_q^2 a_i \, Re \, \epsilon_{ij}^{+} (\mathbf{k}, \omega_q) a_j] \right|^2} \times \sum_{n=-\infty}^{+\infty} \int_0^{\infty} dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_z |Q_n^{+}|^2 \times \delta (\omega_q - n\Omega_s - k_z v_z) F_s (v_z, v_\perp^2) \qquad \text{(A-6)}$$

where

$$Q_n^{\pm} = \left[v_z a_z + a_\perp \cos\psi \left(\frac{n\Omega_s}{k_\perp} \right) \right] J_n(\alpha) \pm i a_\perp v_\perp \sin\psi J_n'(\alpha) \qquad \text{(A-7)}$$

or

$$|Q_n^{+}|^2 = \left[v_z a_z + a_\perp \cos\psi \left(\frac{n\Omega_s}{k_\perp} \right) \right]^2 J_n^2(\alpha) + a_\perp^2 v_\perp^2 \sin^2\psi J_n'^2(\alpha) \qquad \text{(A-8)}$$

Appendix B
Evaluation of the Interaction Term

The interaction term is denoted by I_s, i.e.,

$$I_s \equiv -\frac{e_s}{m_s n_s}\int_0^{2\pi} d\phi \frac{\partial}{\partial \mathbf{v}} \cdot \int_{-\infty}^{+\infty} d\omega \lim_{\Delta \to 0}\left(\frac{\Delta}{2\pi^2}\right)$$
$$\times \sum_{\mathbf{k}} \left[\mathbf{a} + \frac{(\mathbf{v}\cdot\mathbf{a})\,\mathbf{k} - (\mathbf{v}\cdot\mathbf{k})\,\mathbf{a}}{\omega}\right]$$
$$\times \langle E_k^*(\omega, \varepsilon t)\, N_s^k(\mathbf{v}, \omega, \varepsilon t)\rangle \qquad \text{(B-1)}$$

where

$$N_s^k(v, \omega, \varepsilon t) = \int_0^\infty dt \exp\left[i\omega t - \Delta t - i\,\mathbf{k}\cdot\mathbf{r}_s(t)\right]$$
$$\times N_s^k(\mathbf{v}_s(t), 0) - \frac{n_s e_s}{m_s}$$
$$\times \int_0^\infty dt \left[E_k(\omega, \varepsilon t) + i\frac{\partial E_k(\omega, \varepsilon t)}{\partial t}\frac{\partial}{\partial \omega}\right]$$
$$\times \left\{\exp\left[i\omega\tau - \Delta\tau - i\,\mathbf{k}\cdot r_s(\tau)\right]\right.$$
$$\times \left[\mathbf{a} + \frac{(\mathbf{v}_s(\tau)\cdot\mathbf{a})\,\mathbf{k} - (\mathbf{v}_s(\tau)\cdot\mathbf{k})}{\omega}\right]$$
$$\left.\cdot \frac{\partial F_s(\mathbf{v}_s(\tau))}{\partial \mathbf{v}_s(\tau)}\right\} + R_s^k \qquad \text{(B-2)}$$

$$E_k^*(\omega, \varepsilon t) = \frac{4\omega i\pi}{a_\mu \alpha_{\mu\nu}^-(\mathbf{k}, \omega)\, a_\nu} \sum_r e_r \int d^3 v' (\mathbf{a}\cdot\mathbf{v}')$$
$$\times \int_0^\infty d\tau \exp\left[i\omega\tau - \Delta\tau + i\,\mathbf{k}\cdot\mathbf{r}_r(\tau)\right]$$
$$\times N_r^{-k}(\mathbf{v}_r(\tau), 0) + \mathcal{E}_k \qquad \text{(B-3)}$$

In Eqs. (B-2) and (B-3), R_s^k and $\mathcal{E}_k$ are two remainder terms. Since, in the limit $\Delta \to 0$, R_s^k and $\mathcal{E}_k$ contribute only to terms of order $O(\Delta)$, we may ignore both of them.

Hence,

$$\lim_{\Delta\to 0}\frac{\Delta}{2\pi^2}\langle E_k^*(\omega, \varepsilon t)\, N_s^k(\mathbf{v}, \omega, \varepsilon t)\rangle$$
$$= \lim_{\Delta\to 0}\frac{\Delta}{2\pi^2}\left\{-\frac{4\pi^2\omega\delta(\omega - \omega_q)}{\frac{\partial}{\partial\omega}\left[\omega^2 a_\mu Re\, \epsilon_{\mu\nu}^-(\mathbf{k}, \omega)\, a_\nu\right]}\right.$$
$$\times \sum_r e_r \int d^3 v' (\mathbf{v}'\cdot\mathbf{a}) \int_0^\infty d\tau \int_0^\infty d\tau'$$
$$\times \exp\left[i\omega(\tau - \tau') - \Delta(\tau + \tau') - i\,\mathbf{k}\cdot\mathbf{r}_s(\tau)\right.$$
$$\left. + i\,\mathbf{k}\cdot\mathbf{r}_r(\tau')\right]\langle N_s^k(\mathbf{v}_s(\tau), 0)\, N_r^{-k}(\mathbf{v}_r'(\tau'), 0)\rangle$$
$$- \frac{n_s e_s}{m_s}\int_0^\infty d\tau \exp\left[i\omega\tau - \Delta\tau - i\,\mathbf{k}\cdot\mathbf{r}_s(\tau)\right]$$
$$\times \langle E_k^*(\omega, \varepsilon t)\, E_k(\omega, \varepsilon t)\rangle$$
$$\times \left[\frac{(\mathbf{v}_s(\tau)\cdot\mathbf{a})\,\mathbf{k} - (\mathbf{v}_s(\tau)\cdot\mathbf{k})\,\mathbf{a}}{\omega} + \mathbf{a}\right]\cdot \frac{\partial F_s(\mathbf{v}_s(\tau))}{\partial \mathbf{v}_s(\tau)}$$
$$- i\frac{n_s e_s}{m_s}\int_0^\infty d\tau \frac{1}{2}\frac{\partial\langle E_k^*(\omega, \varepsilon t)\, E_k(\omega, \varepsilon t)\rangle}{\partial t}$$
$$\times \frac{\partial}{\partial\omega}\left[\exp\left(i\omega\tau - \Delta\tau - i\,\mathbf{k}\cdot\mathbf{r}_s(\tau)\right)\right.$$
$$\left.\times \left(\frac{(\mathbf{v}_s(\tau)\cdot\mathbf{a})\,\mathbf{k} - (\mathbf{v}_s(\tau)\cdot\mathbf{k})\,\mathbf{a}}{\omega} + \mathbf{a}\right)\right]$$
$$\left.\cdot \frac{\partial F_s(\mathbf{v}_s(\tau))}{\partial \mathbf{v}_s(\tau)}\right\} \qquad \text{(B-4)}$$

As before, we see that in $\langle N_s^k N_r^{-k}\rangle$ only the term which consists of the delta function $\delta[\mathbf{v}_s(\tau) - \mathbf{v}_r'(\tau')]$ would survive in the limit $\Delta \to 0$. Thus, we may simply write

$$\langle N_s^k(\mathbf{v}_s(\tau), 0)\, N_r^{-k}(\mathbf{v}_r'(\tau'), 0)\rangle$$
$$= n_s\,\delta_{sr}\,\delta\left[\mathbf{v}_s(\tau) - \mathbf{v}_r'(\tau')\right] F_s(\mathbf{v}_s(\tau)) \qquad \text{(B-5)}$$

Again, since we are particularly interested in the unstable mode $\omega = \omega_q(\mathbf{k})$, we may use expression (54) and retain the lowest-order contribution.

Hereafter, we write $I_s = I_s^{\mathrm{I}} + I_s^{\mathrm{II}}$. From Eqs. (B-5) and (B-6), we have

$$I_s^{\mathrm{I}} = \sum_{\mathbf{k}} \frac{e_s^2}{m_s}\int_0^{2\pi} d\phi \frac{\partial}{\partial \mathbf{v}} \cdot \int_{-\infty}^{+\infty} d\omega \lim_{\Delta\to 0}$$
$$\times \frac{2\Delta\omega\delta(\omega - \omega_q)}{\frac{\partial}{\partial\omega}\left[\omega^2 a_\mu Re\, \epsilon_{\mu\nu}^-(\mathbf{k}, \omega)\, a_\nu\right]}\left[\mathbf{a} + \frac{(\mathbf{v}\cdot\mathbf{a})\,\mathbf{k} - (\mathbf{v}\cdot\mathbf{k})\,\mathbf{a}}{\omega}\right]$$
$$\times \int d^3 v' (\mathbf{v}'\cdot\mathbf{a}) \int_0^\infty dt \int_0^\infty dt'$$
$$\times \exp\left[i\omega(t - t') - \Delta(t + t') - i\,\mathbf{k}\cdot\mathbf{r}_s(t) + i\,\mathbf{k}\cdot\mathbf{r}_s'(t')\right]$$
$$\times \delta\left[\mathbf{v}_s(t) - \mathbf{v}_s'(t')\right] F_s(\mathbf{v}_s(t)) \qquad \text{(B-6)}$$

$$I_s^{II} = \sum_{\mathbf{k}} \frac{e_s^2}{m_s} \int_0^{2\pi} d\phi \frac{\partial}{\partial \mathbf{v}} \cdot \int_{-\infty}^{+\infty} d\omega \frac{\delta(\omega - \omega_q)}{2\pi}$$

$$\times \left[\mathbf{a} + \frac{(\mathbf{v}\cdot\mathbf{a})\mathbf{k} - (\mathbf{v}\cdot\mathbf{k})\mathbf{a}}{\omega} \right]$$

$$\times \left[\langle E_k^2(t) \rangle + \frac{i}{2} \frac{\partial \langle E_k^2(t) \rangle}{\partial t} \frac{\partial}{\partial \omega} \right]$$

$$\times \left\{ \int_0^\infty d\tau \exp\left[i\omega\tau - 0_+\tau - i\mathbf{k}\cdot\mathbf{r}_s(\tau)\right] \right.$$

$$\left. \times \left[\mathbf{a} + \frac{(\mathbf{v}_s(\tau)\cdot\mathbf{a})\mathbf{k} - (\mathbf{v}_s(\tau)\cdot\mathbf{k})\mathbf{a}}{\omega} \right] \frac{\partial F_s(\mathbf{v}_s(\tau))}{\partial \mathbf{v}_s(\tau)} \right\} \quad \text{(B-7)}$$

In the following, I_s^I and I_s^{II} will be discussed separately. First, in Eq. (B-6) we notice $F_s(\mathbf{v}_s(t)) = F_s(\mathbf{v})$ because we consider $F_s(\mathbf{v}) = F_s(v_z, v_\perp^2)$. Integrating over $\mathbf{v}'$ and ω, we find that

$$I_s^I = \sum_{\mathbf{k}} \frac{2e_s^2 \omega_q}{m_s} \int_0^{2\pi} d\phi \frac{\partial}{\partial \mathbf{v}} \cdot \left[\mathbf{a} + \frac{(\mathbf{v}\cdot\mathbf{a})\mathbf{k} - (\mathbf{v}\cdot\mathbf{k})\mathbf{a}}{\omega} \right]$$

$$\times \lim_{\Delta\to 0} \frac{\Delta}{\frac{\partial}{\partial \omega_q}\left[\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^+(\mathbf{k}, \omega_q) a_\nu\right]} \int_0^\infty dt$$

$$\times \int_0^\infty dt' \exp\left[i\omega_q(t - t') - \Delta(t + t')\right.$$

$$\left. - i\mathbf{k}\cdot\mathbf{r}_s(t - t')\right] \left[\mathbf{a}\cdot\mathbf{v}_s(t - t')\right] F_s(\mathbf{v}) \quad \text{(B-8)}$$

In obtaining expression (B-8), we have introduced a "velocity translation" by equating $\mathbf{v}_s'(t')$ to $\mathbf{v}_s(t)$. Correspondingly, we find that

$$\mathbf{r}_s'(t') = \mathbf{r}_s'(t') - \mathbf{r}_s'(0) \to \mathbf{r}_s(t) - \mathbf{r}_s(t - t') \quad \text{(B-9)}$$

$$\mathbf{v}' \to \mathbf{v}_s(t - t') \quad \text{(B-10)}$$

We shall designate $\tau = t - t'$ and employ the relations

$$\mathbf{a}\cdot\mathbf{v}_s(\tau) = a_z v_z + v_\perp a_\perp$$

$$\times \left[\cos\psi \cos(\phi - \Omega_s\tau) - \sin\psi \sin(\phi - \Omega_s\tau)\right] \quad \text{(B-11)}$$

$$\mathbf{k}\cdot\mathbf{r}_s(\tau) = k_z v_z \tau - \frac{k_\perp v_\perp}{\Omega_s} \sin(\phi - \Omega_s\tau) \quad \text{(B-12)}$$

After some algebra, we obtain

$$I_s^I = \sum_{\mathbf{k}} \frac{4\pi^2 e_s^2}{m_s} \sum_{n=-\infty}^{+\infty} \left(k_z \frac{\partial}{\partial v_z} + \frac{n\Omega_s}{v_\perp} \frac{\partial}{\partial v_\perp} \right)$$

$$\times \left\{ |Q_n^k|^2 \delta(\omega_q - n\Omega_s - k_z v_z) \frac{F_s(\mathbf{v})}{\frac{\partial}{\partial \omega_q}\left[\omega_q^2 a_\mu Re\, \epsilon_{\mu\nu}^- a_\nu\right]} \right\} \quad \text{(B-13)}$$

where all notations have been defined previously (see Appendix A).

We next study I_s^{II}. If we define

$$\Lambda = \mathbf{a} + \frac{(\mathbf{v}_s(t)\cdot\mathbf{a})\mathbf{k} - (\mathbf{v}_s(t)\cdot\mathbf{k})\mathbf{a}}{\omega_q} \quad \text{(B-14)}$$

then we see that

$$\Lambda \cdot \frac{\partial F_s(\mathbf{v}_s(t))}{\partial \mathbf{v}_s(t)} = \left[(\hat{b}\cdot\Lambda)\hat{b} + (\hat{b}\times\Lambda)\times\hat{b}\cos\Omega_s t\right.$$

$$\left. - (\Lambda\times\hat{b})\sin\Omega_s t\right] \cdot \frac{\partial F_s(\mathbf{v})}{\partial \mathbf{v}} \quad \text{(B-15)}$$

where $F_s(\mathbf{v}) = F_s(v_z, v_\perp^2)$ and $\hat{b}$ is a unit vector parallel to $\mathbf{B}_0$. From Eqs. (B-7), (B-14), and (B-15), we can show that

$$I_s^{II} = \sum_{\mathbf{k}} \frac{e_s^2}{2m_s^2 \pi} \int_0^{2\pi} d\phi \left\{ \frac{\partial}{\partial v_z} \right.$$

$$\times \left[\left(a_z + \frac{v_\perp a_\perp k_z \cos(\phi + \psi) - k_\perp v_\perp a_z \cos\phi}{\omega_q} \right) \right.$$

$$\left. \times \mathbb{P}(k_z, k_\perp, v_z, v_\perp, \omega_q) \right] + \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} \left[v_\perp \left(a_\perp \cos(\phi + \psi) \right. \right.$$

$$\left. + \frac{v_z a_z k_\perp \cos\phi - v_z a_\perp k_z \cos(\phi + \psi)}{\omega_q} \right)$$

$$\left. \left. \times \mathbb{P}(k_z, k_\perp, v_z, v_\perp, \omega_q) \right] \right\} \quad \text{(B-16)}$$

where

$$\mathbb{P} = \left[\langle E_k^2(t)\rangle + \frac{i}{2}\frac{\partial\langle E_k^2\rangle}{\partial t}\frac{\partial}{\partial\omega_q}\right]\left\{\int_0^\infty dt\right.$$

$$\times \exp\left[i\omega_q t - O_+ t - ik_z v_z t\right]$$

$$\times \exp\left[i\frac{k_\perp v_\perp}{\Omega_s}\left[\sin(\phi - \Omega_s t) - \sin\phi\right]\right]\left[\left(a_z\right.\right.$$

$$\left. + \frac{v_\perp a_\perp k_z \cos(\phi + \psi - \Omega_s t) - k_\perp v_\perp a_z \cos(\phi - \Omega_s t)}{\omega_q}\right)$$

$$\times \frac{\partial F_s}{\partial v_z} + \left(a_\perp \cos(\phi + \psi - \Omega_s t)\right.$$

$$\left.\left.\left. + \frac{v_z a_z k_\perp \cos(\phi - \Omega_s t) - v_z k_z a_\perp \cos(\phi + \psi - \Omega_s t)}{\omega_q}\right)\frac{\partial F_s}{\partial v_\perp}\right]\right\} \tag{B-17}$$

After some algebra, we find that I_s^{II} may be expressed as the sum of two parts, say $(I_s^{II})_{\text{resonant}}$ and $(I_s^{II})_{\text{nonresonant}}$, and they take the following form:

$$(I_s^{II})_{\text{resonant}} = \sum_{\mathbf{k}} \frac{\pi e_s^2}{m_s^2 \omega_q^2} \sum_{n=-\infty}^{+\infty} \left(k_z \frac{\partial}{\partial v_z} + \frac{n\Omega_s}{v_\perp}\frac{\partial}{\partial v_\perp}\right),$$

$$\times \left\{\delta(\omega_q - n\Omega_s - k_z v_z)\,|Q_n^+|^2\right.$$

$$\left.\times \left(k_z \frac{\partial F_s}{\partial v_z} + \frac{n\Omega_s}{v_\perp}\frac{\partial F_s}{\partial v_\perp}\right)\right\}\langle E_k^2\rangle \tag{B-18}$$

$$(I_s^{II})_{\text{nonresonant}} = \sum_{\mathbf{k}} \frac{e_s^2}{2m_s^2\omega_q^2} \sum_{n=-\infty}^{+\infty} \left\{\frac{\partial}{\partial v_z}\left[\left(a_z(\omega_q - n\Omega_s)J_n(\alpha)\right.\right.\right.$$

$$+ a_\perp \frac{k_z}{k_\perp} n\Omega_s \cos\psi J_n(\alpha)$$

$$\left.\left. - ia_\perp v_\perp k_z \sin\psi J_n'(\alpha)\right)\mathbb{R}(v_z, v_\perp, k_z, k_\perp, \omega_q)\right]$$

$$+ \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left[\left(a_\perp(\omega_q - v_z k_z)\cos\psi J_n(\alpha)\frac{n\Omega_s}{k_\perp}\right.\right.$$

$$+ a_z v_z n\Omega_s J_n(\alpha)$$

$$\left. - ia_\perp v_\perp(\omega_q - k_z v_z)\sin\psi J_n'(\alpha)\right)$$

$$\left.\left.\times \mathbb{R}(v_z, v_\perp, k_z, k_\perp, \omega_q)\right]\right\} \tag{B-19}$$

where

$$\mathbb{R} = \left\{a_z J_n(\alpha)\frac{\partial F_s}{\partial v_z} + a_\perp \frac{\cos\psi}{k_\perp} J_n(\alpha)\frac{n\Omega_s}{v_\perp}\frac{\partial F_s}{\partial v_\perp}\right.$$

$$+ ia_\perp \sin\psi J_n'(\alpha)\frac{\partial F_s}{\partial v_\perp} + \left[P\frac{1}{\omega_q - n\Omega_s - k_z v_z}\right.$$

$$\left. - \omega_q \frac{\partial}{\partial \omega_q} P \frac{1}{\omega_q - n\Omega_s - k_z v_z}\right]$$

$$\left.\times Q_n^+\left(k_z \frac{\partial F_s}{\partial v_z} + \frac{n\Omega_s}{v_\perp}\frac{\partial F_s}{\partial v_\perp}\right)\right\}\frac{\partial\langle E_k^2\rangle}{\partial t} \tag{B-20}$$

In the quasilinear theory, $(I_s^{II})_{\text{resonant}}$ is supposed to be much more important than $(I_s^{II})_{\text{nonresonant}}$, which represents the contribution from the nonresonant part, and thus the latter is often neglected.

Appendix C

Evaluation of $a_\mu \epsilon^{+}_{\mu\nu}(\mathbf{k}, \omega)\, a_\nu$

By definition, we have

$$
\begin{aligned}
a_\mu \epsilon^{+}_{\mu\nu}(\mathbf{k}, \omega)\, a_\nu = {} & a_\mu a_\mu - \frac{4\pi i}{\omega} \lim_{\Delta \to 0} \sum_s \frac{n_s e_s^2}{m_s} \int d^3 v\, (\mathbf{v} \cdot \mathbf{a}) \\
& \times \int_0^\infty dt \exp\left[i\omega t - \Delta t - i\,\mathbf{k} \cdot \mathbf{r}_s(t)\right] \\
& \times \left[\mathbf{a} + \frac{\mathbf{k}\,(\mathbf{a} \cdot \mathbf{v}_s(t)) - (\mathbf{v}_s(t) \cdot \mathbf{k})\,\mathbf{a}}{\omega}\right] \\
& \cdot \frac{\partial F_s(\mathbf{v}_s(t))}{\partial \mathbf{v}_s(t)} \\
= {} & 1 - \frac{i}{\omega} \sum_s \omega_s^2 \lim_{\Delta \to 0} \int d^3 v \int_0^\infty dt\, (\mathbf{v}_s(-t) \cdot \mathbf{a}) \\
& \times \exp\left[i\omega t - \Delta t + i\,\mathbf{k} \cdot \mathbf{r}_s(-t)\right] \\
& \times \left[\mathbf{a} + \frac{\mathbf{k}\,(\mathbf{v} \cdot \mathbf{a}) - (\mathbf{v} \cdot \mathbf{k})\,\mathbf{a}}{\omega}\right] \frac{\partial F_s(\mathbf{v})}{\partial \mathbf{v}} \\
= {} & 1 - \frac{i}{\omega} \sum_s \omega_s^2 \lim_{\Delta \to 0} \int d^3 v \int_0^\infty dt \\
& \times \left[v_z a_z + v_\perp a_\perp \left(\cos\psi \cos(\phi + \Omega_s t)\right.\right. \\
& \left.\left. - \sin\psi \sin(\phi + \Omega_s t)\right)\right] \exp\left[i\omega t - \Delta t - i k_z v_z t\right. \\
& \left. - i\frac{k_\perp v_\perp}{\Omega_s}\left(\sin(\phi + \Omega_s t) - \sin\phi\right)\right] \left\{\left[a_z\right.\right. \\
& + \frac{v_\perp a_\perp k_z (\cos\psi \cos\phi - \sin\psi \sin\phi) - k_\perp v_\perp a_z \cos\phi}{\omega}\Big] \\
& \times \frac{\partial F_s}{\partial v_z} + \Big[a_\perp (\cos\psi \cos\phi - \sin\psi \sin\phi) \\
& + \frac{v_z a_z k_\perp \cos\phi - v_z k_z a_\perp \cos\psi \cos\phi}{\omega} \\
& \left.\left. + \frac{v_z k_z a_\perp \sin\psi \sin\phi}{\omega}\right] \frac{\partial F_s}{\partial v_\perp}\right\}
\end{aligned}
\tag{C-1}
$$

where $\omega_s^2 = 4\pi n_s e_s^2 / m_s$ and the other notations are similar to those used before. After some manipulations, we obtain

$$
\begin{aligned}
a_\mu \epsilon^{+}_{\mu\nu}(\mathbf{k}, \omega)\, a_\nu = {} & 1 + 2\pi \sum_s \frac{\omega_s^2}{\omega^2} \int_0^\infty dv_\perp v_\perp \int_0^\infty dv_z \left\{a_z^2 v_z \frac{\partial F_s}{\partial v_z}\right. \\
& + \sum_{n=-\infty}^{+\infty} \left[\frac{a_\perp^2 n^2 \Omega_s^2}{k_\perp^2} \cos^2\psi\, J_n^2\left(\frac{k_\perp v_\perp}{\Omega_s}\right)\right. \\
& \left. + a_\perp^2 v_\perp^2 \sin^2\psi\, J_n'^2\left(\frac{k_\perp v_\perp}{\Omega_s}\right)\right] \frac{1}{v_\perp} \\
& \left. \times \frac{\partial F_s}{\partial v_\perp}\right\} + 2\pi \sum_s \frac{\omega_s^2}{\omega^2} \int_0^\infty dv_\perp v_\perp \int_0^\infty dv_z \\
& \times \sum_{n=-\infty}^{+\infty} \frac{|Q_n^{+}|^2}{\omega - k_z v_z - n\Omega_s + iO_+} \\
& \times \left\{k_z \frac{\partial F_s}{\partial v_z} + \frac{n\Omega_s}{v_\perp} \frac{\partial F_s}{\partial v_\perp}\right\}
\end{aligned}
\tag{C-2}
$$