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# A CONTINUUM THEORY FOR AN ELASTIC SOLID WITH ELASTIC MICRO-INCLUSIONS

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*Prepared by*
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## Abstract

By analogy with the results of a higher order continuum theory, explicit dispersion relations governing the lowest and next higher modes of propagation of plane, longitudinal waves in an unbounded, elastic particulate composite solid are obtained in terms of the relative properties of the constituent materials. The corresponding ratio of group velocity to phase velocity is likewise evaluated. These and further relationships valid for a special case are exhibited graphically.

## Introduction

On the microscale, all materials exhibit a structure the influence of which is revealed in the course of deformation processes of a comparable scale. This is the case for composite materials also, the primary structure of which, however, comes into play at a relatively larger scale of deformation. For such inhomogeneous materials, when the wavelength of the deformation approaches the characteristic dimension of the heterogeneity, the classical theory of homogeneous continua cannot predict certain observable phenomena, such as dispersive wave propagation and higher modes of motion, that are expected to occur. A continuum model leading to dispersive wave propagation in an unbounded elastic solid is known to require the consideration of higher order terms in the lagrangian density of the solid. Accordingly, one would anticipate the higher order continuum theory of Eringen and Suhubi [1,2] and of Mindlin [3] to supply a sufficiently general framework for the study of the dynamics of composite solids, since it effectively extends the scope of continuum physics to include such wave phenomena. That this is indeed the case is shown in the present work for a particulate composite material. Similar studies of laminated and fiber-reinforced composite materials are due to Herrmann and Achenbach [4].

The results are of practical interest in connection with current attempts at controlling the dynamic response of materials, and at detecting the properties of a heterogeneous material from its response to dynamic loads.

## 1. The Strain Energy

Consider an unbounded, two-phase, particulate composite material in the form of a connected medium surrounding numerous, uniformly distributed, discrete micro-inclusions each having a finite volume. Both phases consist of a homogeneous, linearly elastic and physically isotropic, solid material undergoing small deformations.

Throughout this work, latin indices ranging over 1, 2, 3 denote components relative to rectangular cartesian axes and should be summed over this range when repeated; also, quantities with a superposed prime and bar refer to the inclusion and surrounding materials, respectively.

The centroidal position $x_i$ of the micro-inclusion $\alpha$ at time t is denoted by $x_i^{(\alpha)}$.

The strain $\varepsilon'_{ij}$ at a point in the micro-inclusion $\alpha$ is taken as

$$\varepsilon'_{ij} = e_{ij}^{(\alpha)} + k' e'^{(\alpha)}_{ij} \quad , \tag{1}$$

where

$$\left.\begin{aligned} e'^{(\alpha)}_{ij} &\equiv \varepsilon_{ij}^{(\alpha)} + x'_k \, \gamma_{ijk}^{(\alpha)} \quad , \\ e_{ij}^{(\alpha)} &= e_{ji}^{(\alpha)} \quad , \quad e'^{(\alpha)}_{ij} \neq e'^{(\alpha)}_{ji} \quad , \end{aligned}\right\} \tag{2}$$

$e_{ij}$, $\varepsilon_{ij}$, and $\gamma_{ijk}$ are the strain measures of Suhubi and Eringen [2, eqs. 3.4], $x'_i$ is the position of a point in the micro-inclusion $\alpha$ relative to its $x_i^{(\alpha)}$, and k' is a small, constant parameter.

The strain energy density w for the micro-inclusion $\alpha$ is taken as

$$w^{(\alpha)} = \frac{1}{2} \lambda' \varepsilon'^{(\alpha)}_{ii} \varepsilon'^{(\alpha)}_{jj} + \mu' \varepsilon'^{(\alpha)}_{ij} \varepsilon'^{(\alpha)}_{ij} + \sigma' \varepsilon'^{(\alpha)}_{ij} \varepsilon'^{(\alpha)}_{ji} \quad , \tag{3}$$

where $\lambda'$ and $\mu' + \sigma'$ are the Lamé constants of the inclusion material, the classical shear modulus of which is, therefore, $\mu' + \sigma'$. The total strain energy P for the micro-inclusion $\alpha$ is then

$$P^{(\alpha)} = \int_{v'^{(\alpha)}} w^{(\alpha)} \, dv \tag{4}$$

$$= v'^{(\alpha)} \underline{W}^{(\alpha)} + J'^{(\alpha)}_{ij} W^{(\alpha)}_{ij} , \tag{5}$$

where

$$\left.\begin{aligned} v'^{(\alpha)} &= \int_{v'^{(\alpha)}} dv , & \int_{v'^{(\alpha)}} x'_i \, dv &= 0, \\ J'^{(\alpha)}_{ij} &\equiv \int_{v'^{(\alpha)}} x'_i \, x'_j \, dv , & J'^{(\alpha)}_{ij} &= J'^{(\alpha)}_{ji} , \end{aligned}\right\} \tag{6}$$

$\underline{W}$ and $W_{ij}$ are functions of $e_{ij}$, $\varepsilon_{ij}$, $\gamma_{ijk}$, $\lambda'$, $\mu'$, $\sigma'$, and $k'$, and $\int_{v'^{(\alpha)}} dv$ denotes an integration with respect to $x'_i$ over the volume $v'$ of the micro-inclusion $\alpha$, the second moment $J'_{ij}$ of which, relative to centroidal axes, is a source of structural anisotropy depending on the shape and orientation of the micro-inclusion.

If n denotes the number of micro-inclusions in the composite body B, then the total strain energy $W'$ due to the inclusion material is

$$W' = \sum_{\alpha=1}^{n} P^{(\alpha)} \tag{7}$$

$$\simeq \alpha' \int_B P dv , \tag{8}$$

where

$$\left.\begin{aligned} &P = v' \, \underline{W} + J'_{ij} \, W_{ij} , \\ &\alpha' \equiv \frac{n\eta'}{V'} , \ \eta' \equiv \frac{V'}{V} , \ V' = \sum_{\alpha=1}^{n} v'^{(\alpha)}, \ V = \int_B dv, \end{aligned}\right\} \tag{9}$$

$V'$ and $\eta'$ are respectively the total volume and the phase volume fraction of the inclusion material, $\int_B dv$ denotes an integration with respect to $x_i$ over the volume V of the composite body, and $x_i$ is now the position of a point in the composite body which, following the operation (8), is approximately represented by a continuum model.

For the material surrounding the micro-inclusions, the strain energy density W, at a point $x_i$, is taken as

$$W = \frac{1}{2}\bar{\lambda}\, e_{ii}e_{jj} + \bar{\mu}\, e_{ij}e_{ij}, \tag{10}$$

where $\bar{\lambda}$ and $\bar{\mu}$ are the Lamé constants of the material. The total strain energy $\bar{W}$ due to this material is then

$$\bar{W} = \bar{\beta} \int_B W dv, \tag{11}$$

where

$$\bar{\beta} \equiv 1 - \eta' \tag{12}$$

is the phase volume fraction of the material.

Summing the contributions of the inclusion and surrounding materials, the total strain energy of the composite body is written as

$$W' + \bar{W} = \int_B \mathcal{W} dv, \tag{13}$$

where $\mathcal{W}$ is the strain energy density of the composite material determined from

$$\mathcal{W} \equiv \alpha' P + \bar{\beta} W \tag{14}$$

$$= \alpha'(v'\underline{W} + J'_{ij}W_{ij}) + \bar{\beta}W, \tag{15}$$

with

$$\underline{W} \equiv \frac{1}{2} \lambda' e_{ii} e_{jj} + (\mu' + \sigma') e_{ij} e_{ji} + 2k' [\frac{1}{2} \lambda' e_{ii} \varepsilon_{jj}$$

$$+ (\mu' + \sigma') e_{ij} \varepsilon_{ji}] + k'^2 (\frac{1}{2} \lambda' \varepsilon_{ii} \varepsilon_{jj} + \mu' \varepsilon_{ij} \varepsilon_{ij} + \sigma' \varepsilon_{ij} \varepsilon_{ji}), \quad (16)$$

$$W_{ij} \equiv k'^2 (\frac{1}{2} \lambda' \gamma_{kki} \gamma_{\ell\ell j} + \mu' \gamma_{k\ell i} \gamma_{k\ell j} + \sigma' \gamma_{k\ell i} \gamma_{\ell kj}) , \quad (17)$$

and W given by equation (10).

For small deformations, the strain measures appearing in the expressions (10), (16), and (17) are related to the displacements $u_i$ $(x_i,t)$ and the higher order kinematical variables $\phi_{ij}(x_i,t)$ of Suhubi and Eringen [2, eqs. 3.1, 3.2, 3.4] by

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \; \varepsilon_{ij} = \phi_{ij} + u_{j,i}, \; \gamma_{ijk} = -\phi_{ij,k}, \quad (18)$$

where an indicial comma denotes a first partial derivative with respect to the coordinates $x_i$ represented by the latin index following it.

For the case of structural isotropy,

$$J'_{ij} \simeq J' \delta_{ij}, \quad (19)$$

where $\delta_{ij}$ is the Kronecker delta, a condition valid for micro-inclusions nearly spherical in shape, the strain energy density of the particulate composite material given by equation (15) reduces to

$$\mathcal{W} = \frac{1}{2} (\beta' \lambda' + \overline{\beta}\, \overline{\lambda}) e_{ii} e_{jj} + [\beta' (\mu' + \sigma') + \overline{\beta}\, \overline{\mu}] e_{ij} e_{ji}$$

$$+ 2k' \beta' [\frac{1}{2} \lambda' e_{ii} \varepsilon_{jj} + (\mu' + \sigma') e_{ij} \varepsilon_{ji}]$$

$$+ k'^2 \beta' [\frac{1}{2} \lambda' \varepsilon_{ii} \varepsilon_{jj} + \mu' \varepsilon_{ij} \varepsilon_{ij} + \sigma' \varepsilon_{ij} \varepsilon_{ji}$$

$$+ \ell'^2 (\frac{1}{2} \lambda' \gamma_{iij} \gamma_{kkj} + \mu' \gamma_{ijk} \gamma_{ijk} + \sigma' \gamma_{ijk} \gamma_{jik})], \quad (20)$$

where

$$\beta' \equiv \alpha' v' \quad , \qquad \ell'^2 \equiv \frac{J'}{v'} \quad . \tag{21}$$

When each micro-inclusion occupies about the same volume $v'$, $\beta'$ can approximately be replaced by $\eta'$. In view of the condition (19), it is further noted that $\ell'$ is the "radius of gyration" of the volume occupied by one such micro-inclusion about any centroidal axis and, therefore, the only structural parameter which survives; it has the dimension of length.

Comparing our expression (20) with that given by Suhubi and Eringen [2, eq. 4.20], the following identities between the material constants of their linear, isotropic micro-elastic solid and those of an isotropic particulate composite material, respectively, are obtained

$$\left.\begin{aligned}
&\lambda \equiv \beta'\lambda' + \bar{\beta}\,\bar{\lambda}, && \eta \equiv k'(k' + 1)\beta'\lambda',\\
&\mu \equiv \beta'(\mu' + \sigma') + \bar{\beta}\,\bar{\mu}, && \nu \equiv k'\beta'[\mu' + (2k' + 1)\sigma'],\\
&\sigma \equiv k'\beta'(\mu' + \sigma'), && \tau_3 \equiv k'^2\beta'\ell'^2\lambda',\\
&\tau \equiv k'\beta'\lambda', && \tau_7 \equiv 2k'^2\beta'\ell'^2\mu',\\
&\kappa \equiv k'\beta'[(2k' + 1)\mu' + \sigma'], && \tau_{10} \equiv 2k'^2\beta'\ell'^2\sigma',\\
&\tau_1 = \tau_2 = \tau_4 = \tau_5 = \tau_6 = \tau_8 = \tau_9 = \tau_{11} \equiv 0.
\end{aligned}\right\} \tag{22}$$

It is concluded that, the foregoing procedure allows a particular classification of the material constants appearing in the expression for the strain energy density of the theory of Eringen and Suhubi [1,2] according to their order of magnitude in $k'$, while simultaneously relating them to the properties of the constituents of the composite material under consideration. As a result, $\lambda$ and $\mu$, for instance can be interpreted as the effective Lamé constants of an isotropic particulate composite material, to the lowest order in $k'$.

In what follows, a similar procedure is adopted in the evaluation of the kinetic energy density.

## 2. The Kinetic Energy

For the composite material under consideration, the velocity $v_i$ at a point in the micro-inclusion $\alpha$ is taken as

$$v_i^{(\alpha)} = k'\dot{u}_i^{(\alpha)} + a'\dot{u}'_i{}^{(\alpha)} , \tag{23}$$

where

$$\dot{u}'_i{}^{(\alpha)} \equiv x'_j \dot{\phi}_{ij}^{(\alpha)} , \tag{24}$$

$a'$ is a small, constant parameter, and a superposed dot denotes a first partial derivative with respect to the time.

The kinetic energy density k for the micro-inclusion $\alpha$ is taken as

$$k^{(\alpha)} = \frac{1}{2} \rho' v_i^{(\alpha)} v_i^{(\alpha)}, \tag{25}$$

where $\rho'$ denotes the mass density of the inclusion material. The total kinetic energy T for the micro-inclusion $\alpha$ is then

$$T^{(\alpha)} = \int_{V'(\alpha)} k^{(\alpha)} dv \tag{26}$$

$$= V'^{(\alpha)} \underline{K}^{(\alpha)} + J'_{ij}{}^{(\alpha)} K_{ij}{}^{(\alpha)} , \tag{27}$$

where $\underline{K}$ and $K_{ij}$ are functions of $\dot{u}_i$, $\dot{\phi}_{ij}$, $\rho'$, $k'$, and $a'$, and the reappearance of $J'_{ij}$ is noted.

The total kinetic energy $T'$ due to the inclusion material is

$$T' = \sum_{\alpha=1}^{n} T^{(\alpha)} \tag{28}$$

$$\simeq \alpha' \int_B \mathrm{T} dv, \tag{29}$$

where

$$T = v'\underline{K} + J'_{ij}K_{ij}. \tag{30}$$

For the material surrounding the micro-inclusions, the kinetic energy density K is taken as

$$K = \frac{1}{2}\bar{\rho}\,\dot{u}_i\dot{u}_i , \tag{31}$$

where $\bar{\rho}$ denotes the mass density of the material. The total kinetic energy $\bar{T}$ due to this material is then

$$\bar{T} = \bar{\beta}\int_B K dv . \tag{32}$$

Summing the contributions of the inclusion and surrounding materials, the total kinetic energy of the composite body is written as

$$T' + \bar{T} = \int_B \mathcal{T}\, dv , \tag{33}$$

where $\mathcal{T}$ is the kinetic energy density of the composite material determined from

$$\mathcal{T} \equiv \alpha' T + \bar{\beta}K \tag{34}$$

$$\simeq \alpha'(v'\underline{K} + J'_{ij}K_{ij}) + \bar{\beta}K , \tag{35}$$

with

$$\underline{K} \equiv \frac{1}{2}k'^2\rho'\dot{u}_i\dot{u}_i , \qquad K_{ij} \equiv \frac{1}{2}a'^2\rho'\dot{\phi}_{ki}\dot{\phi}_{kj} , \tag{36}$$

and K given by equation (31).

For the case when the condition (19) holds, the kinetic energy density of the particulate composite material given by equation (35) reduces to

$$\mathcal{T} = \frac{1}{2}\,(k'^2\beta'\rho' + \bar{\beta}\,\bar{\rho})\dot{u}_i\dot{u}_i + \frac{1}{2}\,a'^2\beta'\ell'^2\rho'\dot{\phi}_{ij}\dot{\phi}_{ij} \;, \tag{37}$$

where the reappearance of $\ell'$ is noted.

Comparing our expression (37) with that underlying Suhubi and Eringen's equations of motion [2, eqs. 5.11, 5.12], the following identities between the remaining material constants of their linear, isotropic micro-elastic solid and those of an isotropic particulate composite material, respectively, are obtained.

$$\rho_o \equiv k'^2\beta'\rho' + \bar{\beta}\,\bar{\rho} \;, \qquad I_o \equiv \frac{a'^2\beta'\ell'^2\rho'}{k'^2\beta'\rho' + \bar{\beta}\,\bar{\rho}} \;. \tag{38}$$

The remainder of our results are for an isotropic particulate composite material.

## 3. The Propagation of Longitudinal Waves

The lagrangian density of the composite material under consideration, from which equations of motion could be obtained, can be evaluated from the expressions (20) and (37) for the strain and kinetic energy densities, respectively. The equations of motion in terms of the kinematical variables $u_i$, $\phi_{ij}$, and of the appropriate material constants, have, however, already been given by Suhubi and Eringen [2, eqs. 5.11, 5.12]. In view of the identities (22) and (38) these equations can now be employed to study the propagation of waves in a particulate composite material.

Considering the propagation of plane, harmonic waves, the kinematical variables can be taken to vary as exp $[i(\xi x_1 - \omega t)]$, in which $\xi$ is the wave number and $\omega$ is the natural angular frequency. Then, by setting

$$a'^2 \equiv k'^2 \;, \qquad k' \to 0 \tag{39}$$

in the corresponding equations governing the propagation of longitudinal waves in a particulate composite material, the following relation

$$\left(\frac{\ell'^2}{c'^2_1}\right)\omega_1^4 - \left[1 + \left(1 + \frac{\beta'\rho'}{\bar{\beta}\bar{\rho}} + \frac{1}{c_1^2}\right)\ell'^2\xi_1^2\right]\omega_1^2$$

$$+ \bar{c}_1^2\left[1 + \left(1 + \frac{\beta'\rho'}{\bar{\beta}\bar{\rho}}c_1^2\right)\ell'^2\xi_1^2\right]\xi_1^2 = 0 \qquad (40)$$

is obtained, where

$$c_1'^2 \equiv \frac{\lambda' + 2(\mu' + \sigma')}{\rho'}, \quad \bar{c}_1^2 \equiv \frac{\bar{\lambda} + 2\bar{\mu}}{\bar{\rho}}, \quad c_1^2 \equiv \frac{c'^2_1}{\bar{c}_1^2}, \qquad (41)$$

$\xi_1$ and $\omega_1$ are respectively the wave number and the natural angular frequency of the longitudinal waves, and $c'_1$ and $\bar{c}_1$ are the velocities at which longitudinal waves propagate in the inclusion and surrounding materials, respectively, the ratio of which is a dimensionless physical parameter $c_1$. It is of interest to note that the constant $\sigma'$ introduced in equation (3) does not appear by itself in the final result, and that, in the absence of either the inclusion ($\ell'$, $\beta' \to 0$) or the surrounding material ($\bar{\beta} \to 0$), equation (40) yields a classical dispersion relation between $\omega_1^2$ and $\xi_1^2$ for the remaining material.

Letting

$$\omega_1^{**2} \equiv \frac{\ell'^2\omega_1^2}{\bar{c}_1^2}, \qquad \xi_1^{**2} \equiv \ell'^2\xi_1^2, \qquad \beta^2 \equiv \frac{\beta'\rho'}{\bar{\beta}\bar{\rho}}, \qquad (42)$$

where $\beta^2$ is clearly another dimensionless physical parameter, equation (40) can be rewritten as

$$\omega_1^{**4} - [c_1^2 + (1 + c_1^2 + \beta^2 c_1^2)\xi_1^{**2}]\omega_1^{**2}$$

$$+ c_1^2[1 + (1 + \beta^2 c_1^2)\xi_1^{**2}]\xi_1^{**2} = 0, \qquad (43)$$

the solution of which,

$$\omega_{11,\ 12}^{**2} = \frac{1}{2} [c_1^{\ 2} + (1 + c_1^{\ 2} + \beta^2 c_1^{\ 2})\xi_1^{\ **2}] \{1 \pm [1$$

$$- \frac{4c_1^{\ 2}[1 + (1 + \beta^2 c_1^{\ 2})\xi_1^{\ **2}]\xi_1^{\ **2}}{[c_1^{\ 2} + (1 + c_1^{\ 2} + \beta^2 c_1^{\ 2})\xi_1^{\ **2}]^2} ]^{1/2} \} , \tag{44}$$

an explicit dispersion relation between the dimensionless variables $\omega_1^{\ **2}$ and $\xi_1^{\ **2}$, is shown in figure 1 for two representative pairs of values of $c_1^{\ 2}$ and $\beta^2$. The latter are the only parameters explicitly appearing in equation (43) after the structural parameter $\ell'$ is absorbed into $\omega_1^{\ **2}$ and $\xi_1^{\ **2}$. From the definitions (41) and (42), it is seen that $\beta^2 c_1$ represents the product of the ratio of the phase volume fractions $\beta'/\bar{\beta}$, with $\beta' \simeq \eta'$, and that of the characteristic mechanical impedances $\rho' c_1'/\bar{\rho}\ \bar{c}_1$, for longitudinal waves, of the inclusion and surrounding materials. In terms of this quantity, effective reflection and transmission coefficients can be defined, when necessary, for the longitudinal motion of particulate composite materials.

In figure 1, the curves labelled 1 and 3 represent the lowest, fundamental mode $\omega_{11}^{\ **2}$ for $c_1^{\ 2} = 0.4$, $\beta^2 = 0.4$, and $c_1^{\ 2} = 2$, $\beta^2 = 0.5$, respectively. The curves labelled 2 and 4 represent the next higher mode $\omega_{12}^{\ **2}$ for the same pairs of values of $c_1^{\ 2}$, $\beta^2$, respectively. For $\xi_1^{\ **2} = 0$, the latter mode yields

$$\omega_{12}^{\ **2} = c_1^{\ 2}, \tag{45}$$

independently from $\beta^2$; the ratio $c_1'/\ell'$ is the corresponding angular cutoff frequency. The lines

$$\omega_1^{\ **2} = \xi_1^{\ **2} , \qquad \omega_1^{\ **2} = c_1 (1 + \xi_1^{\ **2}) , \tag{46}$$

the first of which is shown in figure 1, are the asymptotes of the dispersion curves for $\beta^2 \to 0$. These lines intersect in the first quadrant of the $\omega_1^{**2}$, $\xi_1^{**2}$ plane for $c_1^2 < 1$. For $\beta^2 \to \infty$, the curve representing the fundamental mode tends to the line

$$\omega_1^{**2} = c_1^2 \xi_1^{**2} \quad . \tag{47}$$

The first of the lines given by equations (46) is also generated by this curve for $c_1^2 = 1$, independently from $\beta^2$.

From the first of equations (46) and equation (47), it is concluded that the dispersion relation expressed by equation (44) exhibits the type of limiting behavior that is appropriate, namely, an absence of dispersion, for the particular values of the parameters $c_1^2$ and $\beta^2$ considered. This observation also holds for the remainder of our results.

Since

$$\omega_1 = \frac{2\pi V_1}{\lambda_1} , \quad \xi_1^{**} = \frac{2\pi \ell'}{\lambda_1} , \quad U_1 = V_1 - \lambda_1 \frac{dV_1}{d\lambda_1} , \tag{48}$$

where $V_1$ is the phase velocity, $U_1$ is the group velocity, and $\lambda_1$ is the wavelength of the longitudinal waves propagating through the particulate composite material, it follows from equation (44) that

$$(U_1/V_1)_{1,2} = 1 - \left(\frac{1}{f_1}\right) \left[1 \pm \frac{(f_2 - \frac{1}{2}) f_3}{(f_1 f_4)^{1/2} \pm f_4}\right] , \tag{49}$$

where

$$\left.\begin{aligned} f_1 &\equiv 1 + \left(1 + \beta^2 + \frac{1}{c_1^2}\right)\xi_1^{**2} , & f_2 &\equiv \left(\frac{1}{f_1}\right)[1 + (1 + \beta^2 c_1^2)\xi_1^{**2}], \\ f_3 &\equiv \frac{4}{c_1^2}\xi_1^{**2} , & f_4 &\equiv f_1 - f_2 f_3 . \end{aligned}\right\} \tag{50}$$

The dimensionless ratio of group velocity to phase velocity expressed by equation (49) as a function of $\xi_1{**}^2$ is shown in figure 2 for the same pairs of values of the parameters $c_1^2$ and $\beta^2$ as in figure 1. The curves labelled 1, 3 and 2, 4 correspond to the fundamental mode $(U_1/V_1)_1$ and to the next higher mode $(U_1/V_1)_2$, respectively. The line

$$U_1/V_1 \doteq 1 \ , \tag{51}$$

shown in figure 2, is generated by the curve corresponding to the fundamental mode for $c_1^2 = 1$.

In general, the ratio of group velocity U to phase velocity V is a characteristic property of a dispersive dynamical system. It indicates the relative ease with which energy can be transported through the system by traveling waves. For instance, if $dV/d\lambda$, in which $\lambda$ is the wavelength, is negative, then $U/V > 1$ and the system exhibits the type of dispersion which renders the transport of energy more difficult, with the longer waves traveling slower than the shorter waves. The reverse holds when $dV/d\lambda$ is positive, in which case, $U/V < 1$. For a nondispersive system, V is independent of $\lambda$ and $U/V = 1$. Therefore, this ratio can be adopted as a measure of dispersion.

In our particular case, $(U_1/V_1)_1 > 1$ for $c_1^2 > 1$, $(U_1/V_1)_1 < 1$ for $c_1^2 < 1$, and $(U_1/V_1)_1 = 1$ for $c_1^2 = 1$, except for the limiting values of $\xi_1{**}^2 = 0$ and $\xi_1{**}^2 \to \infty$ for which $(U_1/V_1)_1 = 1$ independently from $c_1^2$. This suggests a simple scheme, based solely on the magnitude of the parameter $c_1^2$, according to which the fundamental modes of longitudinal motion of particulate composite materials can be classified.

## 4. A Special Case

Mindlin [3, eq. 9.31] has derived a low frequency, very long wavelength approximation to his higher order continuum theory. The corresponding version of this approximation for a particulate composite material, obtained by employing the correlation between the theory of Eringen and Suhubi [1,2] and that of Mindlin [3] established by Eringen [5, sec. 9.c)], together with the operation (39), leads to the following explicit dispersion relation for plane, harmonic, longitudinal waves

$$\omega_1{}^{*2} = \frac{(1 + c_1{}^2\xi_1{}^{*2})\xi_1{}^{*2}}{1 + \xi_1{}^{*2}} , \tag{52}$$

where

$$\omega_1{}^{*2} \equiv \frac{\ell^2\omega_1{}^2}{\bar{c}_1{}^2} , \quad \xi_1{}^{*2} \equiv \ell^2\xi_1{}^2 , \quad \ell^2 \equiv \frac{\ell'^2\beta'\rho'}{\bar{\beta}\,\bar{\rho}} = \ell'^2\beta^2 . \tag{53}$$

The absence of a higher mode is to be expected. It is noted that $c_1$ is now the only dimensionless physical parameter explicitly appearing in equation (52) after both parameters $\ell'$ and $\beta^2$ are absorbed into the new dimensionless variables $\omega_1{}^{*2}$ and $\xi_1{}^{*2}$.

For this case,

$$U_1/V_1 = 1 - \frac{(1 - c_1{}^2)\xi_1{}^{*2}}{(1 + c_1{}^2\xi_1{}^{*2})(1 + \xi_1{}^{*2})} , \tag{54}$$

and by setting

$$\frac{\partial}{\partial\xi_1{}^{*2}} (U_1/V_1) = 0 , \tag{55}$$

it follows that the extremum of $U_1/V_1$ is given by

$$(U_1/V_1)_e = \frac{2c_1}{1 + c_1} , \qquad \xi_1{}^{*2} = \frac{1}{c_1} . \tag{56}$$

The dispersion relation between $\omega_1{}^{*2}$ and $\xi_1{}^{*2}$ expressed by equation (52) is shown in figure 3 for a range of values of the parameter $c_1{}^2$.

The dimensionless ratio of group velocity to phase velocity expressed by equation (54) as a function of $\xi_1{}^{*2}$ is shown in figure 4 for the same values of the parameter $c_1{}^2$ as in figure 3, and for $c_1{}^2 \to \infty$.

Finally, the influence of the parameter $c_1$ on the extremum of the ratio of group velocity to phase velocity expressed by equations (56) is shown in figure 5.

These results lend further support to the concluding remarks of the preceding section.

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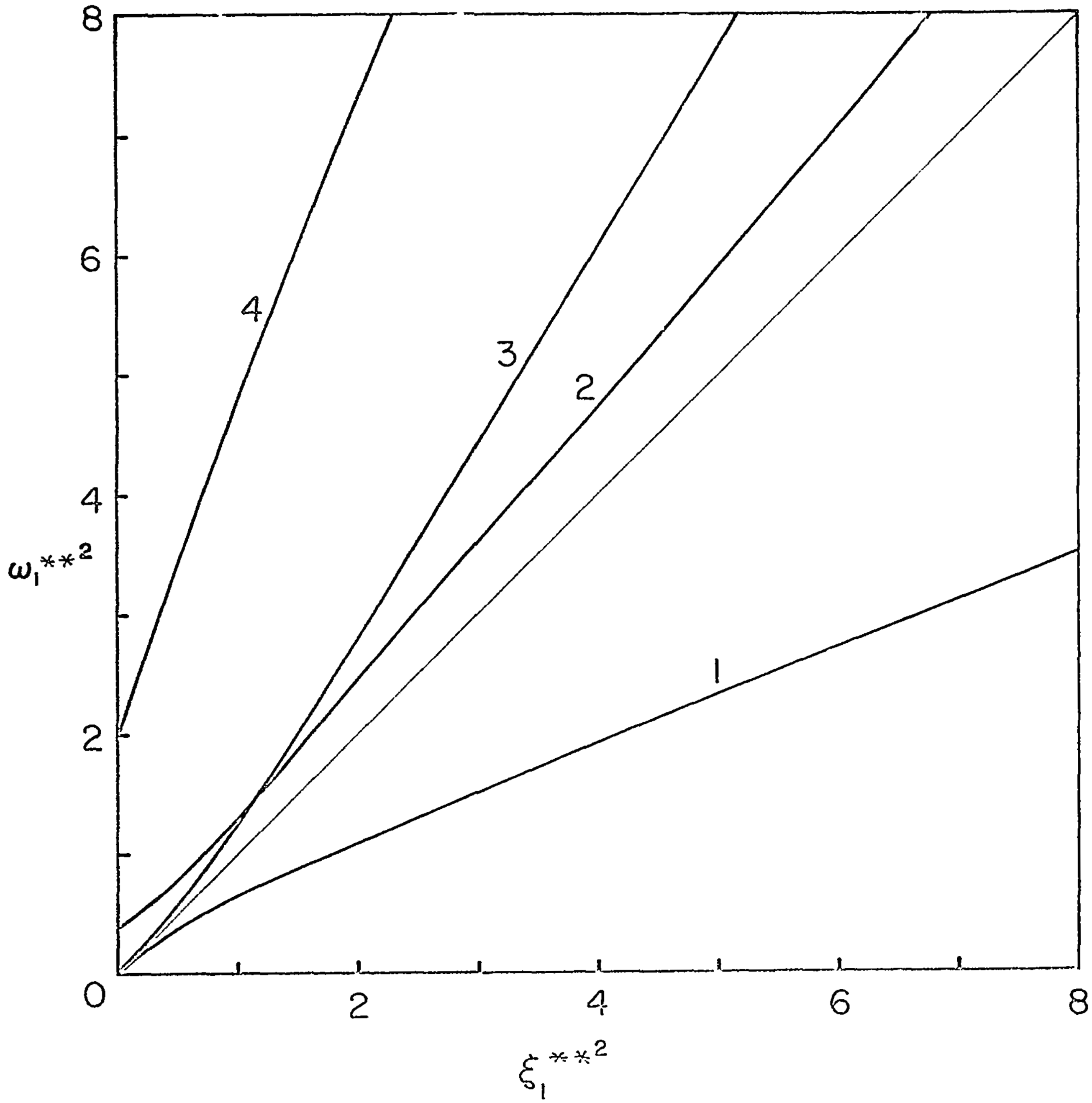


<u>Figure 1</u>. The dispersion relation based on equation (44). The curves labelled 1, 2 and 3, 4 are for values of $c_1^2 = 0.4$, $\beta^2 = 0.4$, and $c_1^2 = 2$, $\beta^2 = 0.5$, respectively.

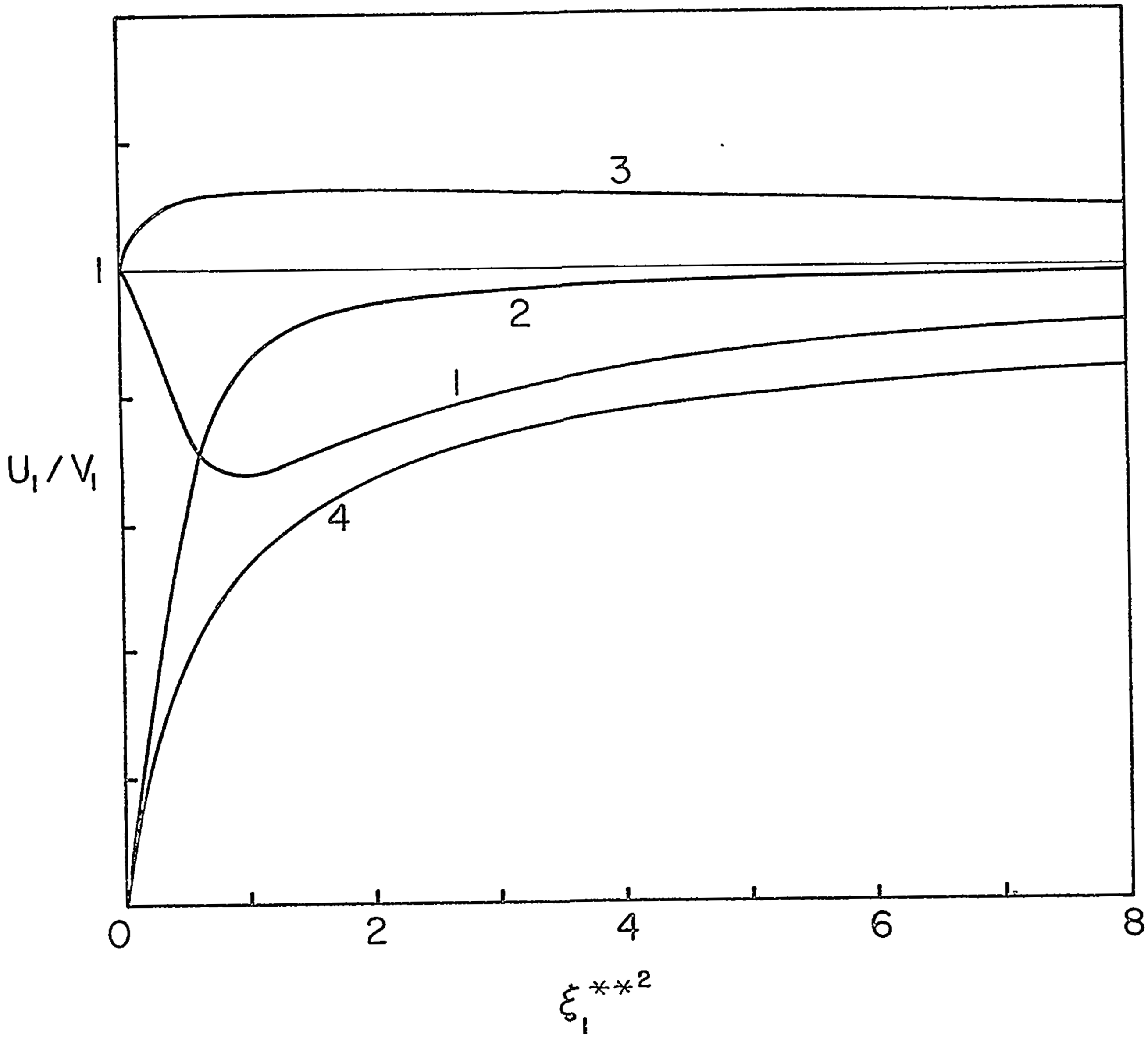


Figure 2. The ratio of group velocity to phase velocity based on equation (49). The curves labelled 1, 2 and 3, 4 are for values of $c_1^2 = 0.4$, $\beta^2 = 0.4$, and $c_1^2 = 2$, $\beta^2 = 0.5$, respectively.

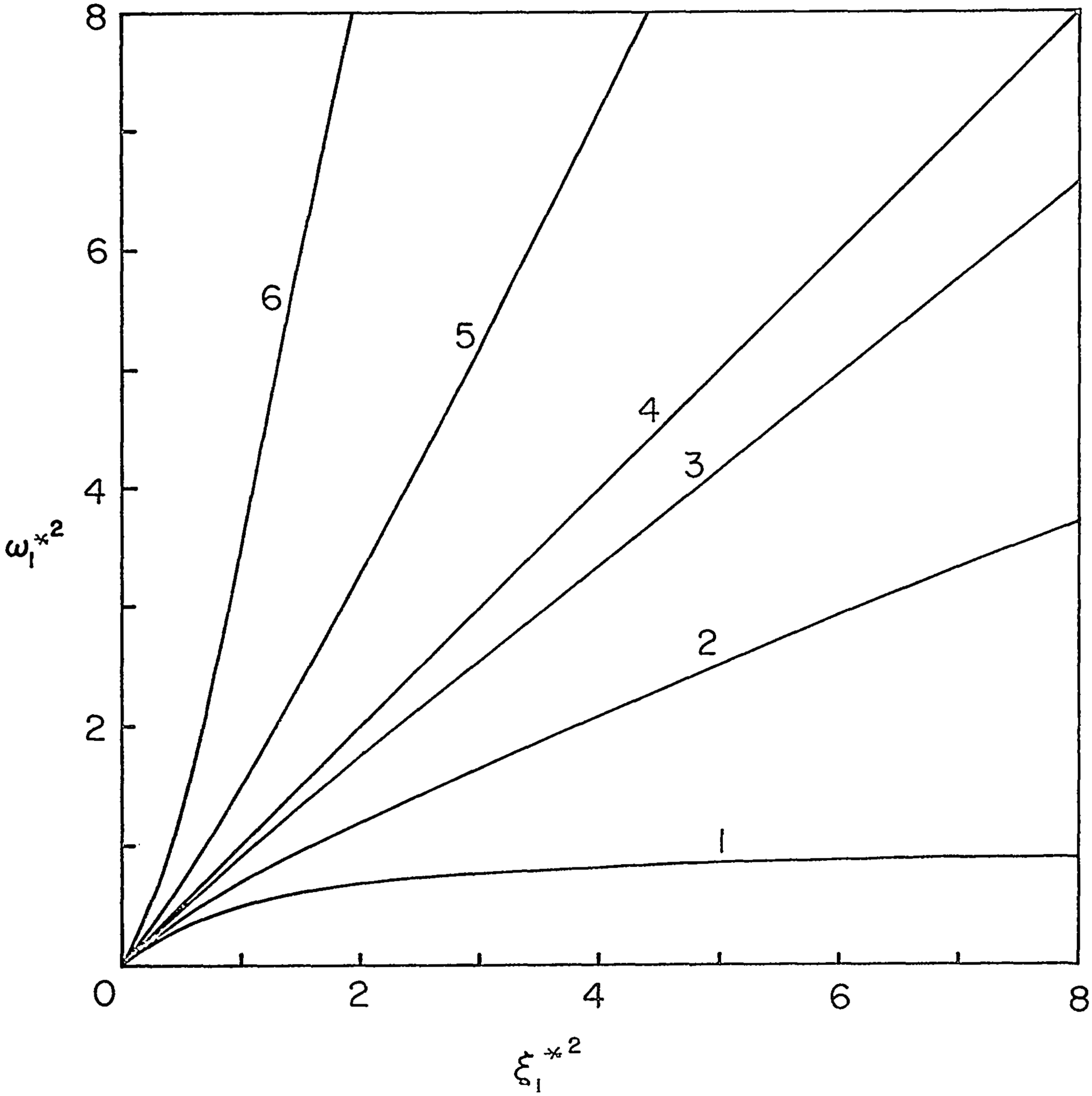


Figure 3. The dispersion relation based on equation (52). The curves labelled 1 to 6 are for values of $c_1^2$ = 0, 0.4, 0.8, 1, 2, and 6, respectively.

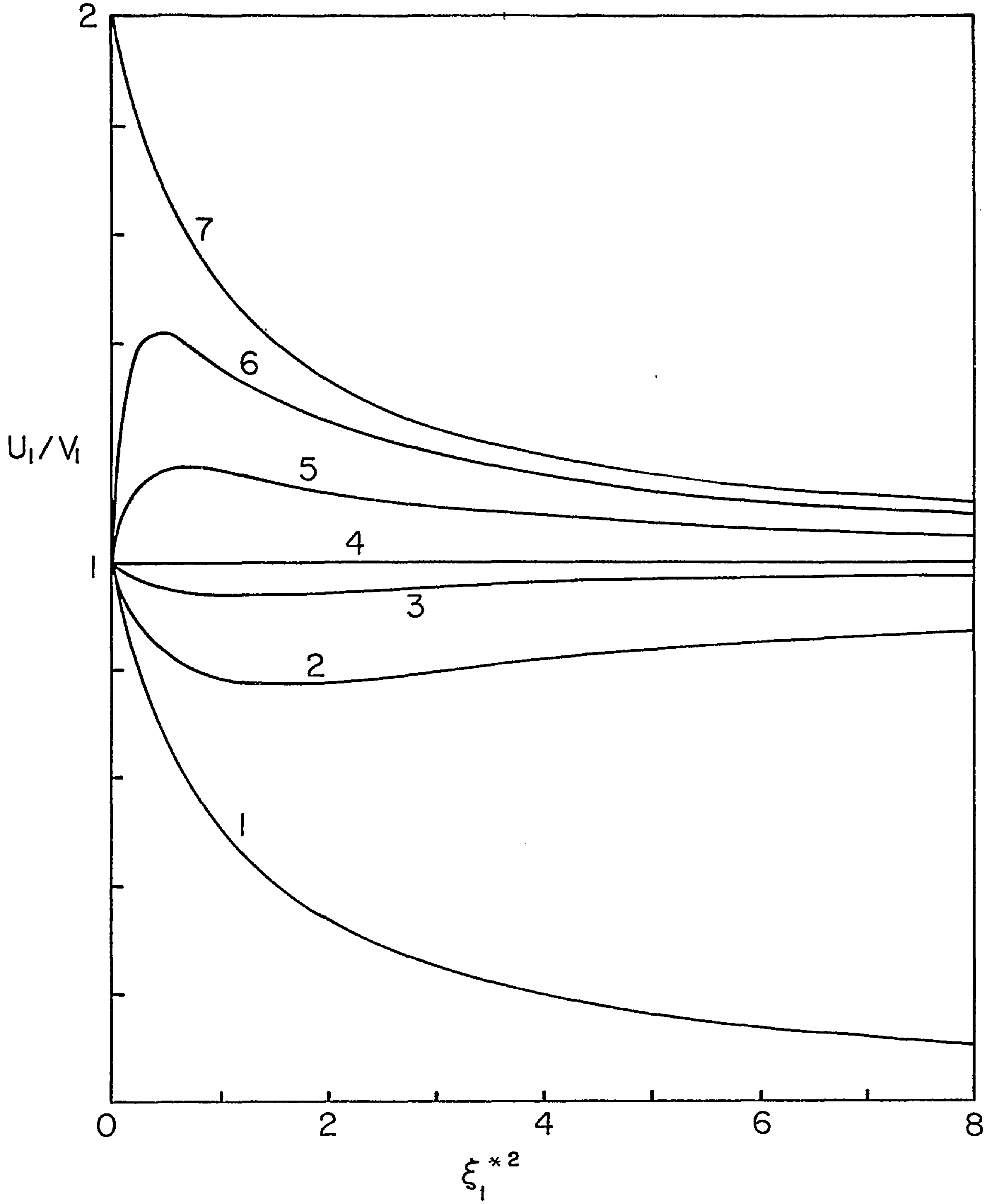


Figure 4. The ratio of group velocity to phase velocity based on equation (54). The curves labelled 1 to 7 are for values of $c_1^2$ = 0, 0.4, 0.8, 1, 2, 6, and $c_1^2 \to \infty$, respectively.

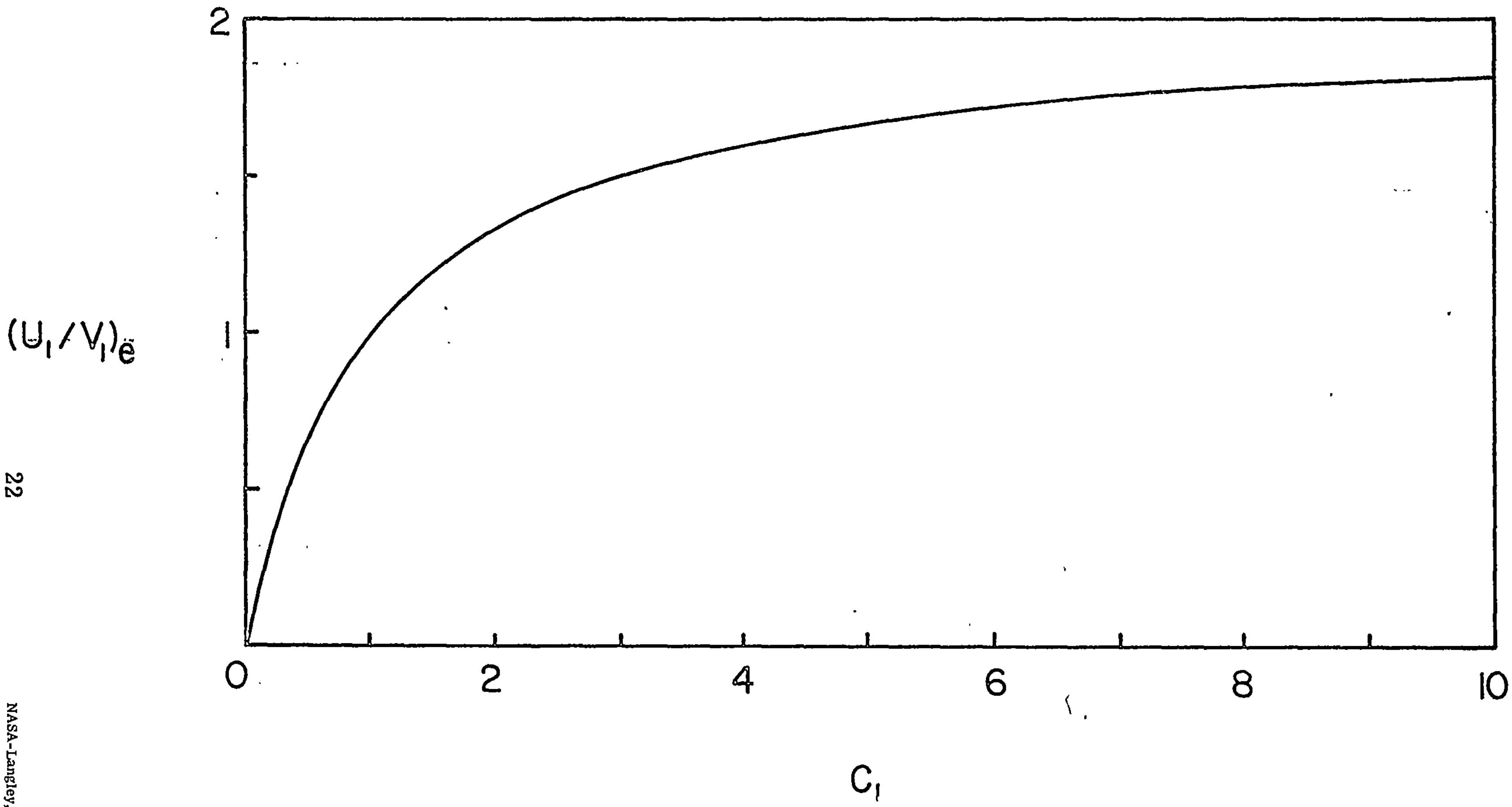


**Figure 5.** The influence of $c_1$ on the extremum of the ratio of group velocity to phase velocity based on equations (56).



NASA-Langley, 1969 — 32 CR-1321