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ON THE DETERMINATION OF MICROMORPHIC MATERIAL CONSTANTS FROM PROPERTIES OF THE CONSTITUENTS

by Richard F. Maye

Prepared by
GENERAL TECHNOLOGY CORPORATION
Lawrenceville, N. J.

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Abstract

Eringen and Suhubi [1], [2] introduced the general theory of micro-elasticity whereby the local deformations of material points are taken into account, the necessity for this consideration being the nonhomogeneity of the material at the local level.

One of the problem areas of micro-elasticity remaining is the determination of the material constants (either analytically or experimentally) which even for the linear isotropic theory number eighteen. In addition, an inertia constant must be determined.

This report is a study of the relationship of a linear micromorphic material to linear classically elastic constituents and investigates the possibility of obtaining the material constants from the properties of the constituents.

1. Introduction

Using relationships given by Eringen and Suhubi [1], [2], it is shown that the strain energy of a linear micromorphic body is equal to the sum of the energies of the constituents when the constituents are assumed to be linear classically elastic. That is,

$$Wdv = \int_{dv} W'dv' \tag{1.1}$$

where W is the macro-strain energy density, dv the macro-differential element, W' the micro-strain energy density, and dv' the micro-differential element. Assuming the constituents to be also isotropic, W' is given by the relation

$$W' = \frac{1}{2}\lambda' u'_{i,i}u'_{j,j} + \frac{1}{2}\mu'(u'_{i,j}u'_{i,j} + u'_{i,j}u'_{j,i}). \tag{1.2}$$

λ' and μ' are the Lamé constants for the constituent (the value of each depends on which constituent is present). Here $u'_{i,j} \equiv \partial u'_i/\partial x'_j$, the x'_j are rectangular coordinates of a point in the body, and u'_i is the displacement vector at a point in the micro-volume.

By substituting (1.2) into (1.1) and using an expression for $u'_{i,j}$ in terms of kinematical macro-quantities, the right-hand side of (1.1) can be expressed in terms of these macro-quantities and properties of the constituents. The left-hand side is given by Eringen and Suhubi in terms of the macro-quantities and micromorphic material moduli. Equating like terms, relationships for the micromorphic material moduli in terms of constituent properties are obtained.

The special case of a two constituent isotropic micromorphic material is considered in detail. For this case, all constituent properties except three are easily evaluated. Determination of the eighteen micromorphic material moduli and the micromorphic inertia constant, ρIo, is therefore reduced to the experimental determination of these three unknowns.

2. Strain Energy Densities

A differential element, dv, of the micromorphic material consists of many micro-differential elements, dv', as shown in figure 1 (where different shaded areas represent different constituents). It is assumed that each constituent is an elastic solid.

Upon deformation, the center of mass of dv moves from position $\underset{\sim}{X}$ to $\underset{\sim}{x}$, and that of dv' goes from position $\underset{\sim}{X}'$ to $\underset{\sim}{x}'$.

According to a first order approximation, the displacement vector, $\underset{\sim}{u}'$, of $\underset{\sim}{X}'$ can be represented in terms of the displacement of the mass center, $\underset{\sim}{u}$, as follows.

$$
\begin{aligned}
u'_k &= u_k + u_{k_{rel}} \\
&= u_k + \phi_{k\ell}\xi_\ell \quad .
\end{aligned}
\tag{2.1}
$$

Here $\phi_{k\ell}$ is a function of $\underset{\sim}{x}$ and time, and ξ_ℓ is the relative position vector of $\underset{\sim}{x}'$ with respect to $\underset{\sim}{x}$ as shown in figure 1.

This motion is identical to that described in [1] after linearization.

Consider the total strain energy of an arbitrary volume, V, with surface S of the micromorphic material obtained by summing the strain energies of the constituents.

$$E = \int_V \int_{dv} W'dv' = \int_V \int_{dv} \frac{1}{2} t'_{ij} e'_{ij} dv' \quad . \tag{2.2}$$

W', t'_{ij}, and e'_{ij} are the constituent strain energy density, stress tensor, and strain tensor respectively at any point $\underset{\sim}{x}'$. Upon carrying the linear strain measure

$$e'_{ij} = \frac{1}{2} (u'_{i,j} + u'_{j,i}) \tag{2.3}$$

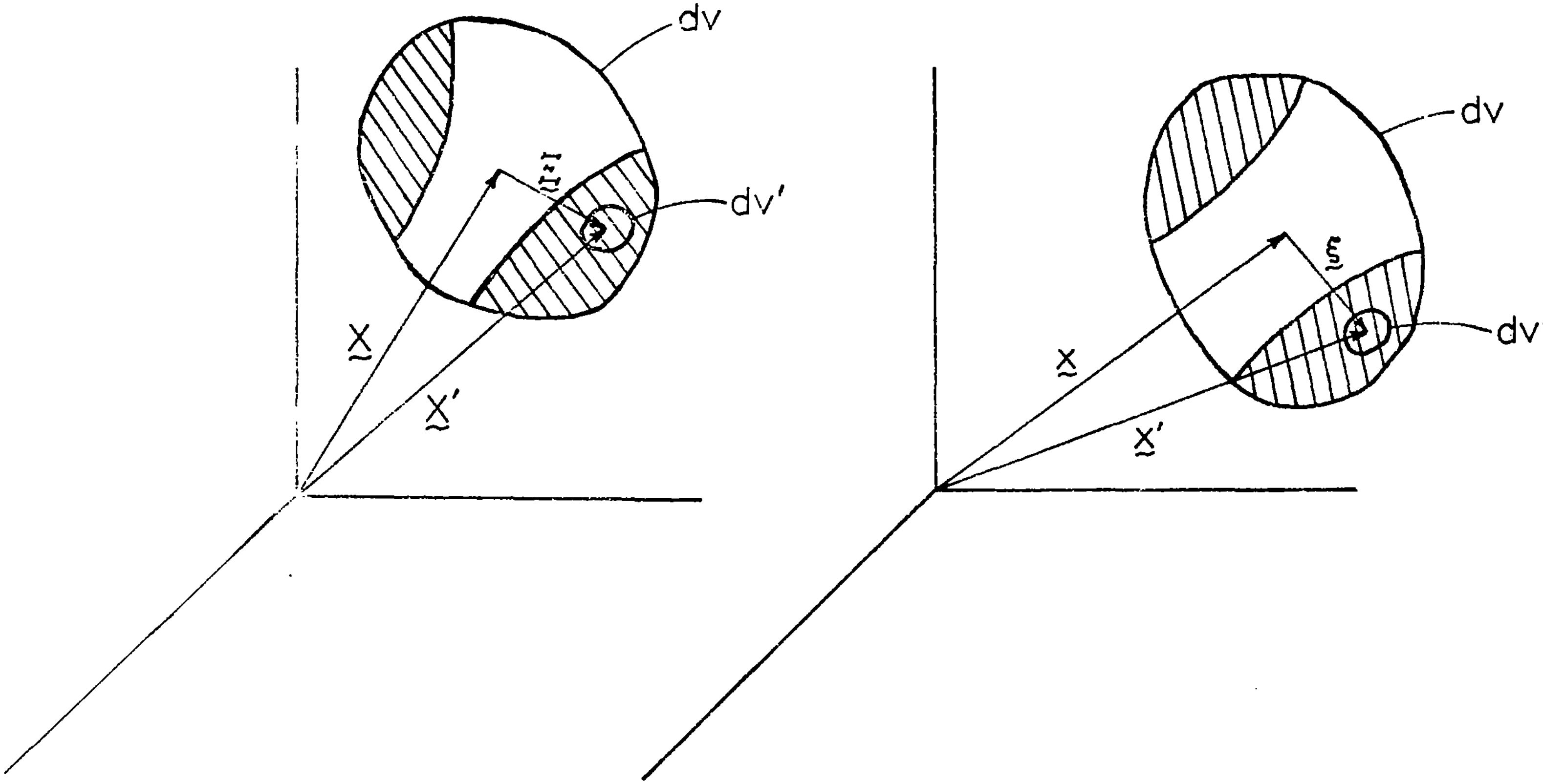

Figure 1. Differential Elements for a Micromorphic Material.

into expression (2.2) we obtain

$$E = \int_V \int_{dv} \frac{1}{2} t'_{ij} u'_{i,j} dv' = \int_V \int_{dv} \frac{1}{2} [(t'_{ij} u'_i)_{,j} - t'_{ij,j} u'_i] dv'. \tag{2.4}$$

Since the surface tractions and u'_i are continuous across the surface of each dv', the first volume integral can be changed to a surface integral.

$$E = \frac{1}{2} \int_S \int_{ds} t'_{ij} u'_i da'_j - \frac{1}{2} \int_V \int_{dv} t'_{ij,j} u'_i dv' . \tag{2.5}$$

ds is a differential surface area of S and $da'_j = n'_j da'$ where n'_j is the outer normal of a micro-differential area da' (also on S).

Upon substituting (2.1) into (2.5) and remembering that u_i and ϕ_{ik} are independent of integration over ds or dv, we obtain

$$2E = \int_S [u_i \int_{ds} t'_{ji} da'_j + \phi_{ik} \int_{ds} t'_{ji} \xi_k da'_j]$$

$$- \int_V [u_i \int_{dv} t'_{ji,j} dv' + \phi_{ik} \int_{dv} t'_{ji,j} \xi_k dv'] . \tag{2.6}$$

The following terms (averaged quantities) defined by Eringen and Suhubi [1] are now employed.

$$\int_{ds} t'_{ji} da'_j \equiv t_{ji} da_j ,$$

$$\int_{ds} t'_{ji} \xi_k da'_j \equiv \lambda_{jik} da_j , \tag{2.7}$$

$$\int_{dv} t'_{ij} dv' \equiv s_{ij} dv .$$

$da_j = n_j ds$ where n_j is the outer normal of ds. t_{ij}, s_{ij}, and λ_{jik} are referred to as stress quantities.

It can also be shown (see appendix A) that

$$\int_{dv} t'_{ji,j} dv' = t_{ji,j} dv ,$$

$$\int_{dv} t'_{ji,j} \xi_k dv' = (t_{ki} - s_{ki} + \lambda_{jik,j}) dv . \qquad (2.8)$$

For an unprimed quantity, an index following a comma represents partial differentiation with respect to x_j, e.g. $t_{ji,j} \equiv \frac{\partial t_{ji}}{\partial x_j}$.

Using (2.7) and (2.8), expression (2.6) becomes

$$2E = \int_S [u_i t_{ji} + \phi_{ik} \lambda_{jik}] da_j$$

$$- \int_V [u_i t_{ji,j} + \phi_{ik} (t_{ki} - s_{ki} + \lambda_{jik,j})] dv . \qquad (2.9)$$

Changing the surface integral to a volume integral, (2.9) becomes

$$2E = \int_V [u_{i,j} t_{ji} + \phi_{ik,j} \lambda_{jik} - \phi_{ik}(t_{ki} - s_{ki})] dv. \qquad (2.10)$$

Again we employ linear strain and microstrain measures

$$e_{ij} \equiv \frac{1}{2} (u_{i,j} + u_{j,i}) ,$$

$$\varepsilon_{ij} \equiv \phi_{ij} + u_{j,i} ,$$

$$\gamma_{ijk} \equiv - \phi_{ij,k} \qquad (2.11)$$

introduced in [1]. With these, (2.10) now becomes

$$2E = \int_V [t_{ji}(2e_{ij} - \varepsilon_{ij}) - \lambda_{jik}\gamma_{ikj} + s_{ij}(\varepsilon_{ij} - e_{ij})]dv \; . \tag{2.12}$$

Therefore, the strain energy density, W, for the micromorphic material obtained by summing the strain energies of the constituents is

$$\begin{aligned} W &= \frac{1}{2} t_{ji}(2e_{ij} - \varepsilon_{ij}) - \frac{1}{2} \lambda_{jik}\gamma_{ikj} + \frac{1}{2}s_{ij}(\varepsilon_{ij} - e_{ij}) \\ &= (t_{ji} - \frac{1}{2} s_{ij})e_{ij} + (\frac{1}{2} s_{ij} - \frac{1}{2} t_{ji})\varepsilon_{ij} - \frac{1}{2} \lambda_{jik}\gamma_{ikj} \; . \end{aligned} \tag{2.13}$$

This expression can be shown to be equivalent to that given for the strain energy density, Σ, in [2] (see appendix B).

Therefore, the total strain energy of the micromorphic body can be obtained by summing the strain energies of the constituents.

3. Extension of Clapeyron's Theorem to Micromorphic Theory

Proceeding in a manner similar to section 2, Clapeyron's Theorem can be extended to micromorphic theory.

According to Sokolnikoff [3] we have: Clapeyron's Theorem - If a body is in equilibrium under a given system of body forces F_i and surface forces $T_i^{\nu} \equiv t_{\nu i} n_{\nu}$, then the strain energy of deformation is equal to one-half the work that would be done by the external forces *(of the equilibrium state) acting through the displacements* u_i *from* the unstressed to the stressed state, i.e.,

$$2 \int_V W dv = \int_V F_i u_i dv + \int_S t_{ji} u_i da_j \quad . \tag{3.1}$$

Applying the theorem to dv of the micromorphic body, and then summing over the complete body, we have (ρ' and f'_i are the constituent mass density and body force per mass density respectively).

$$2 \int_V W dv = \int_V \int_{dv} \rho' f'_i u'_i dv' + \int_S \int_{ds} t'_{ji} u'_i da'_j \quad . \tag{3.2}$$

Note again that the tractions and u'_i are continuous across the surface of each dv'. Using equation (2.1) we obtain

$$\begin{aligned} 2 \int_V W dv = & \int_V [u_i \int_{dv} \rho' f'_i dv' + \phi_{ik} \int_{dv} \rho' f'_i \xi_k dv'] \\ & + \int_S [u_i \int_{ds} t'_{ji} da'_j + \phi_{ik} \int_{ds} t'_{ji} \xi_k da'_j] \quad . \end{aligned} \tag{3.3}$$

The definitions (2.7) and the following (also given in [1]) are applied.

$$\begin{aligned} & \int_{dv} \rho' f'_i dv' \equiv \rho f_i dv, \\ & \int_{dv} \rho' f'_i \xi_k dv' \equiv \rho \ell_{ik} dv \quad . \end{aligned} \tag{3.4}$$

ρ, f_i, and ℓ_{ik} are the mass density, body force per mass density, and first body moment per mass density respectively of the micromorphic material.

We have, therefore, the result

$$2 \int_V W dv = \int_V [u_i \rho f_i + \phi_{ik} \rho \ell_{ik}] \, dv + \int_S [u_i t_{ji} + \phi_{ik} \lambda_{jik}] da_j \quad . \tag{3.5}$$

4. Micromorphic Strain Energy Density in Terms of Constituent Properties

It has been shown in section 2 that the strain energy of a micromorphic body is equal to the sum of the energies of the constituents. That is, for the differential element, dv, shown in Figure 1,

$$W dv = \int_{dv} W' dv' \quad . \tag{4.1}$$

W' is obtained from the classical relation

$$W' = \frac{1}{2} \lambda' e'_{ii} e'_{jj} + \mu' e'_{ij} e'_{ij} \; . \tag{4.2}$$

λ' and μ' are the Lamé constants for the constituent (the value of each depends on which constituent is present).

Carrying (2.3) into (4.2) we have

$$W' = \frac{1}{2} \lambda' u'_{i,i} u'_{j,j} + \frac{1}{2} \mu' (u'_{i,j} u'_{i,j} + u'_{i,j} u'_{j,i}) \; . \tag{4.3}$$

It is necessary to find an expression for $u'_{i,j}$ in terms of macro-quantities as is the case for u'_i introduced in [1], i.e.

$$u'_i = u'_i(x'_j, t) = u_i(x_j, t) + \phi_{ij}(x_j, t)\xi_j \quad . \tag{4.4}$$

If we consider u'_i expressible as a function of (x'_j, t) then clearly for (4.4) to be valid, we must have

$$x_i = x_i(x'_j) \quad , \qquad \xi_i = \xi_i(x'_j) \quad . \tag{4.5}$$

Differentiating u'_i by the chain rule gives

$$u'_{i,j} = \frac{\partial u'_i}{\partial x_k} \frac{\partial x_k}{\partial x'_j} + \frac{\partial u'_i}{\partial \xi_k} \frac{\partial \xi_k}{\partial x'_j}$$

$$= \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial \phi_{i\ell}}{\partial x_k} \xi_\ell\right) \frac{\partial x_k}{\partial x'_j} + \phi_{ik} \frac{\partial \xi_k}{\partial x'_j}$$

$$= (u_{i,k} + \phi_{i\ell,k}\xi_\ell)A_{kj} + \phi_{ik}H_{kj}, \tag{4.6}$$

where

$$A_{kj} \equiv \frac{\partial x_k}{\partial x'_j} \quad , \qquad H_{kj} \equiv \frac{\partial \xi_k}{\partial x'_j} \quad . \tag{4.7}$$

The following relationship between x'_i, x_i, and ξ_i is now used, [1].

$$x'_i = x_i + \xi_i \quad . \tag{4.8}$$

Differentiating with respect to x'_j we have

$$\delta_{ij} = A_{ij} + H_{ij} \tag{4.9}$$

or

$$H_{ij} = \delta_{ij} - A_{ij} \quad . \tag{4.10}$$

Expression (4.6) for $u'_{i,j}$ now becomes

$$u'_{i,j} = (u_{i,k} + \phi_{i\ell,k}\xi_\ell - \phi_{ik})A_{kj} + \phi_{ij} \quad . \tag{4.11}$$

In general, A_{kj} is a function of x'_j. However, an examination of the analysis which follows shows use of a non-constant value would give material moduli for the micromorphic material which are functions of position. That is, the material would not be homogeneous. We exclude this case and consider A_{kj} constant.

Using (4.11) and (4.3) in (4.1) we obtain

$$
\begin{aligned}
Wdv = \int_{dv} \{ \tfrac{1}{2} \lambda' [& A_{ki}A_{mj}[u_{i,k}u_{j,m} + 2u_{i,k}\phi_{jp,m}\xi_p - 2u_{i,k}\phi_{jm} \\
& + \phi_{i\ell,k}\phi_{jp,m}\xi_\ell\xi_p - 2\phi_{ik}\phi_{jp,m}\xi_p + \phi_{ik}\phi_{jm}] \\
& + 2\, A_{ki}[u_{i,k}\phi_{jj} + \phi_{jj}\phi_{ip,k}\xi_p - \phi_{jj}\phi_{ik}] + \phi_{ii}\phi_{jj}] \\
& + \tfrac{1}{2}\mu'[A_{kj}A_{mj}\ [u_{i,k}u_{i,m} + 2u_{i,k}\phi_{ip,m}\xi_p - 2u_{i,k}\phi_{im} \\
& + \phi_{i\ell,k}\phi_{ip,m}\xi_\ell\xi_p - 2\phi_{im}\phi_{ip,k}\xi_p + \phi_{ik}\phi_{im}] \\
& + 2\, A_{kj}[u_{i,k}\phi_{ij} + \phi_{ij}\phi_{ip,k}\xi_p - \phi_{ik}\phi_{ij}] + \phi_{ij}\phi_{ij} \\
& + A_{kj}A_{mi}[u_{i,k}u_{j,m} + 2u_{i,k}\phi_{jp,m}\xi_p - 2u_{i,k}\phi_{jm} \\
& + \phi_{i\ell,k}\phi_{jp,m}\xi_\ell\xi_p - 2\phi_{jm}\phi_{ip,k}\xi_p + \phi_{ik}\phi_{jm}] \\
& + 2A_{kj}[u_{i,k}\phi_{ji} + \phi_{ji}\phi_{ip,k}\xi_p - \phi_{ik}\phi_{ji}] + \phi_{ij}\phi_{ji}]\}dv' \ .
\end{aligned}
\tag{4.12}
$$

We note that u_i, ϕ_{ij}, and their derivatives and A_{kj} are independent of the integration variables. The following integrals are introduced for convenience. Some of these will be evaluated explicitly later.

$$\int_{dv} \lambda' dv' \equiv 2B dv \quad , \quad \int_{dv} \mu' dv' \equiv 2C dv \; ,$$

$$\int_{dv} \lambda' \xi_m \xi_k dv' \equiv 2D_{mk} dv \quad , \quad \int_{dv} \mu' \xi_m \xi_k dv' \equiv 2E_{mk} dv \; , \tag{4.13}$$

$$\int_{dv} \lambda' \xi_m dv' \equiv 2F_m dv \quad , \quad \int_{dv} \mu' \xi_m dv' \equiv 2G_m dv \; .$$

The micromorphic strain energy density now takes the following form.

$$\begin{aligned}
W = {} & B\{A_{ki}A_{mj}[u_{i,k}u_{j,m} - 2u_{i,k}\phi_{jm} + \phi_{ik}\phi_{jm}] + 2A_{ki}[u_{i,k}\phi_{jj} - \phi_{jj}\phi_{ik}] + \phi_{ii}\phi_{jj}\} \\
& + C\{A_{kj}A_{mj}[u_{i,k}u_{i,m} - 2u_{i,k}\phi_{im} + \phi_{ik}\phi_{im}] + 2A_{kj}[u_{i,k}\phi_{ij} - \phi_{ik}\phi_{ij}] + \phi_{ij}\phi_{ij} \\
& + A_{kj}A_{mi}[u_{i,k}u_{j,m} - 2u_{i,k}\phi_{jm} + \phi_{ik}\phi_{jm}] + 2A_{kj}[u_{i,k}\phi_{ji} - \phi_{ik}\phi_{ji}] + \phi_{ij}\phi_{ji}\} \\
& + D_{\ell p}\{A_{ki}A_{mj}[\phi_{i\ell,k}\phi_{jp,m}]\} \\
& + E_{\ell p}\{A_{kj}A_{mj}[\phi_{i\ell,k}\phi_{ip,m}] + A_{kj}A_{mi}[\phi_{i\ell,k}\phi_{jp,m}]\} \\
& + F_p\{A_{ki}A_{mj}[2u_{i,k}\phi_{jp,m} - 2\phi_{ik}\phi_{jp,m}] + 2A_{ki}[\phi_{jj}\phi_{ip,k}]\} \\
& + G_p\{A_{kj}A_{mj}[2u_{i,k}\phi_{ip,m} - 2\phi_{im}\phi_{ip,k}] + 2A_{kj}[\phi_{ij}\phi_{ip,k}] \\
& + A_{kj}A_{mi}[2u_{i,k}\phi_{jp,m} - 2\phi_{jm}\phi_{ip,k}] + 2A_{kj}[\phi_{ji}\phi_{ip,k}]\} \; .
\end{aligned} \tag{4.14}$$

Expression (4.14) must now be compared with the linearized form of the theory developed by Eringen and Suhubi [1], [2]. In that development, the strain measures defined by (2.11) are utilized. The most general expression for W in the linear theory of micromorphic elastic solids must be a quadratic expression in these strain measures. Therefore,

$$W = \frac{1}{2} c_{ijk\ell} e_{ij} e_{k\ell} + \frac{1}{2} b_{ijk\ell} \varepsilon_{ij} \varepsilon_{k\ell} + \frac{1}{2} a_{ijk\ell mn} \gamma_{ijk} \gamma_{\ell mn}$$

$$+ d_{ijk\ell m} \varepsilon_{ij} \gamma_{k\ell m} + f_{ijk\ell m} \gamma_{ijk} e_{\ell m} + g_{ijk\ell} \varepsilon_{ij} e_{k\ell} \tag{4.15}$$

where the following symmetries must hold.

$$c_{ijk\ell} = c_{k\ell ij} = c_{jik\ell},$$

$$b_{ijk\ell} = b_{k\ell ij},$$

$$a_{ijk\ell mn} = a_{\ell mnijk}, \tag{4.16}$$

$$f_{ijk\ell m} = f_{ijkm\ell},$$

$$g_{ijk\ell} = g_{ij\ell k}.$$

Substituting (2.11) into (4.15) and comparing with (4.14), it is found that the symmetry conditions, (4.16), are satisfied if A_{ij} is of a special form, namely

$$A_{ij} = A\delta_{ij} . \tag{4.17}$$

This is a sufficient condition and has not been shown to be a necessary condition. Further investigation into the generality of A_{ij} is needed.

With (4.17) imposed, the following expressions are found for the constants in (4.15).

$$
\begin{aligned}
a_{ijk\ell mn} &= 2A^2[D_{jm}\delta_{ki}\delta_{n\ell} + E_{jm}(\delta_{nk}\delta_{i\ell} + \delta_{k\ell}\delta_{ni})] , \\
d_{ijk\ell m} &= -2A(1-A)[F_\ell\delta_{mk}\delta_{ji} + G_\ell(\delta_{mj}\delta_{ik} + \delta_{mi}\delta_{jk})] , \\
b_{ijk\ell} &= 2(1-A)^2[B\delta_{ji}\delta_{\ell k} + C(\delta_{j\ell}\delta_{ki} + \delta_{jk}\delta_{\ell i})] , \\
f_{k\ell mij} &= -2A(2A-1)[F_\ell\delta_{ij}\delta_{mk} + G_\ell(\delta_{im}\delta_{jk} + \delta_{ik}\delta_{mj})] , \\
g_{k\ell ij} &= 2(2A-1)(1-A)[B\delta_{ij}\delta_{\ell k} + C(\delta_{i\ell}\delta_{kj} + \delta_{ik}\delta_{\ell j})] , \\
c_{ijk\ell} &= 2(2A-1)^2[B\delta_{ij}\delta_{k\ell} + C(\delta_{i\ell}\delta_{jk} + \delta_{ik}\delta_{j\ell})] .
\end{aligned}
\tag{4.18}
$$

For the case where the micromorphic material is isotropic, the following conditions must be imposed.

$$
\begin{aligned}
F_\ell &= G_\ell = 0 , \\
D_{ij} &= D\,\delta_{ij} , \\
E_{ij} &= E\,\delta_{ij} .
\end{aligned}
\tag{4.19}
$$

Using (4.17) and (4.19), expression (4.14) can be compared with the expression of energy for a linear isotropic micromorphic material [2, pp. 396, 397]. The material moduli of Eringen and Suhubi, λ , μ , η , τ, σ, ν, κ, τ_1, τ_2, ..., τ_{11} in terms of the constituent properties are found to be

$$
\begin{aligned}
\lambda &= 2(2A-1)^2B , & \mu &= 2(2A-1)^2C , \\
\tau &= 2(2A-1)(1-A)B , & \sigma &= 2(2A-1)(1-A)C , \\
\eta &= 2A(1-A)B , & \nu &= 2A(1-A)C , \\
\kappa &= 2A(1-A)C , & \tau_6 &= 2A^2D , \\
\tau_7 &= 2A^2E , & \tau_{11} &= 2A^2E , \\
\tau_i &= 0 \quad (i = 1, 2, 3, 4, 5, 8, 9, 10) .
\end{aligned}
\tag{4.20}
$$

Expressions for B, C, D, E and the inertia term, ρI_o, which appears in [1] and [2] are now considered for a material made of two constituents. The following operations may be performed on $(4.13)_1$.

$$\int_{dv} \lambda' dv' = \sum_{p=1}^{2} \int_{dv_p} \lambda^p dv' = \lambda^1 \int_{dv_1} dv' + \lambda^2 \int_{dv_2} dv'$$

$$= \lambda^1 dv_1 + \lambda^2 dv_2 \tag{4.21}$$

where λ^1 and λ^2 are the first Lamé constant of materials 1 and 2 respectively.

Let ζ be the volume density of material 1. Then

$$\zeta = \frac{\text{Volume 1}}{\text{Total Volume}} = \frac{dv_1}{dv} , \quad 1-\zeta = \frac{\text{Volume 2}}{\text{Total Volume}} = \frac{dv_2}{dv} . \tag{4.22}$$

Therefore,

$$2Bdv = [\lambda^1 \zeta + \lambda^2 (1-\zeta)]dv , \tag{4.23}$$

or

$$2B = \lambda^1 \zeta + \lambda^2 (1-\zeta) . \tag{4.24}$$

In a similar manner, one obtains (μ^1 and μ^2 are the second Lamé constant for materials 1 and 2 respectively)

$$2C = \mu^1 \zeta + \mu^2 (1-\zeta) . \tag{4.25}$$

It is thus seen that 2B and 2C are determined by the "Rule of Mixtures".

Consider now the following operations on $(4.13)_3$ for the isotropic microelastic material.

$$2D\delta_{mk} dv = \int_{dv} \lambda' \xi_m \xi_k dv'$$

$$= \sum_{p=1}^{2} \int_{dv_p} \lambda^p \xi_m \xi_k dv'$$

$$= \lambda^1 \int_{dv_1} \xi_m \xi_k \, dv' + \lambda^2 \int_{dv_2} \xi_m \xi_k dv' \tag{4.26}$$

$$= 2[\lambda^1 Q + \lambda^2 R]\delta_{mk} dv$$

where

$$2Q\delta_{mk} dv \equiv \int_{dv_1} \xi_m \xi_k dv' \quad , \qquad 2R\delta_{mk} dv \equiv \int_{dv_2} \xi_m \xi_k dv' \quad . \tag{4.27}$$

Therefore,

$$D = \lambda^1 Q + \lambda^2 R \quad . \tag{4.28}$$

In a similar manner one can obtain

$$E = \mu^1 Q + \mu^2 R \quad . \tag{4.29}$$

Noting that the inertia term, ρI_o, is given by [1]

$$\rho I_o \delta_{mk} dv \equiv \int_{dv} \rho' \xi_m \xi_k dv' \quad , \tag{4.30}$$

one obtains (ρ^1 and ρ^2 are the mass densities for materials 1 and 2 respectively)

$$\rho I_o = 2[\rho^1 Q + \rho^2 R] \quad . \tag{4.31}$$

This completes the determination of the micromorphic material constants and ρI_o for a two constituent material. All quantities except A, Q, and R can be found once ζ, ρ^1, ρ^2, λ^1, λ^2, μ^1, and μ^2 are given. A, Q, and R are left for experimental investigation.

A physical interpretation of A can be made. Conditions (4.17) and (4.10) used in (4.7) give

$$\frac{\partial x_k}{\partial x'_j} = A\delta_{kj} \quad , \qquad \frac{\partial \xi_k}{\partial x'_j} = (1-A)\delta_{kj} \quad . \tag{4.32}$$

Integration, setting the arbitrary integration constants equal to zero gives

$$x_k = Ax'_k , \qquad \xi_k = (1-A)x'_k . \tag{4.33}$$

When A goes to unity, ξ_k goes to zero and x_k becomes x'_k. Then, from (4.27), Q and R become zero, and therefore, from (4.28) and (4.29), D and E become zero. Looking at (4.20), it is seen that all constants except λ and μ go to zero. λ and μ, however, are calculated by the "Rule of Mixtures" and are actually just the Lamé constants for the material which now behaves as a classically elastic material.

It appears then that A is a measure of whether a material behaves as classically elastic or exhibits other phenomena. Also, since A relates x_k to x'_k, it appears to be a measure of how large dv (which is actually considered finite) must be taken such that averaging is possible--however, there must be a limit to how large dv can be taken. With the above considerations in mind, it is conjectured that

$$1 \leq A \leq M \quad , \tag{4.34}$$

where M is some finite value.

A is probably very close to unity since most tensile tests on composite materials give close agreement with the "Rule of Mixtures" values.

As a final step, using (4.20), the linear isotropic micromorphic displacement equations of motion [2] may be written as

$$\begin{aligned} & 2A^2(B+C)u_{k,k\ell} + 2A^2Cu_{\ell,kk} + 2A(1-A)B\phi_{kk,\ell} \\ & \quad + 2A(1-A)C(\phi_{\ell k} + \phi_{k\ell})_{,k} + \rho f_\ell = \rho\ddot{u}_\ell \\ & 2A^2(D+E)\phi_{km,k\ell} + 2A^2E\phi_{\ell m,kk} - 2(A-1)^2B\phi_{kk}\delta_{m\ell} \\ & \quad -2(A-1)^2C(\phi_{m\ell} + \phi_{\ell m}) - 2A(1-A)Bu_{k,k}\delta_{m\ell} \\ & \quad -2A(1-A)C(u_{m,\ell} + u_{\ell,m}) + \rho\ell_{\ell m} = \rho I_o\ddot{\phi}_{\ell m} . \end{aligned} \tag{4.35}$$

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Appendix A

I) $$\int_V \int_{dv} t'_{ji,j}\, dv' = \int_S \int_{ds} t'_{ji}\, da'_j$$

$$= \int_S t_{ji}\, da_j = \int_V t_{ji,j}\, dv \, .$$

Therefore,

$$\int_{dv} t'_{ji,j}\, dv' = t_{ji,j}\, dv \, .$$

II) $$\int_V \int_{dv} t'_{ji,j}\xi_k\, dv' = \int_V \int_{dv} t'_{ji,j}(x'_k - x_k)\, dv'$$

$$= - \int_V x_k \int_{dv} t'_{ji,j}\, dv' + \int_V \int_{dv} t'_{ji,j}\, x'_k\, dv'$$

$$= - \int_V x_k\, t_{ji,j}\, dv + \int_V \int_{dv} [(t'_{ji}\, x'_k)_{,j} - t'_{ki}]dv'$$

$$= - \int_V x_k t_{ji,j} dv + \int_S \int_{ds} t'_{ji} x'_k da'_j - \int_V s_{ki} dv$$

$$= - \int_V x_k t_{ji,j} dv + \int_S x_k \int_{ds} t'_{ji}\, da'_j + \int_S \int_{ds} t'_{ji}\xi_k da'_j$$

$$- \int_V s_{ki} dv$$

$$= - \int_V x_k t_{ji,j} dv + \int_S x_k t_{ji} da_j + \int_S \lambda_{jik} da_j - \int_V s_{ij} dv$$

$$= - \int_V x_k t_{ji,j} dv + \int_V x_k t_{ji,j} dv + \int_V t_{ki} dv$$

$$+ \int_V \lambda_{jik,j} dv - \int_V s_{ki} dv$$

$$= \int_V (t_{ki} + \lambda_{jik,j} - s_{ki}) dv \quad .$$

Therefore,

$$\int_{dv} t'_{ji,j}\xi_k dv' = (t_{ki} + \lambda_{jik,j} - s_{ki})dv \ .$$

Appendix B

A comparison of expressions can be made in the following way.

In the linear theory of [1] and [2], the stress components are given in terms of the derivatives of the strain energy density.

$$t_{k\ell} = \frac{\partial \Sigma}{\partial e_{k\ell}} + \frac{\partial \Sigma}{\partial \varepsilon_{k\ell}} , \qquad s_{k\ell} = \frac{\partial \Sigma}{\partial e_{k\ell}} + \frac{\partial \Sigma}{\partial \varepsilon_{k\ell}} + \frac{\partial \Sigma}{\partial \varepsilon_{\ell k}}$$

$$\lambda_{k\ell m} = - \frac{\partial \Sigma}{\partial \gamma_{\ell m k}} .$$

Therefore, we obtain

$$t_{k\ell} - \frac{1}{2} s_{k\ell} = \frac{1}{2} \frac{\partial \Sigma}{\partial e_{k\ell}} + \frac{1}{2} \left(\frac{\partial \Sigma}{\partial \varepsilon_{k\ell}} - \frac{\partial \Sigma}{\partial \varepsilon_{\ell k}} \right)$$

$$s_{k\ell} - t_{k\ell} = \frac{\partial \Sigma}{\partial \varepsilon_{\ell k}} .$$

Then,

$$(t_{ji} - \frac{1}{2} s_{ij}) e_{ij} + \frac{1}{2} (s_{ij} - t_{ji}) \varepsilon_{ij} - \frac{1}{2} \lambda_{jik} \gamma_{ikj}$$

$$= \frac{1}{2} \frac{\partial \Sigma}{\partial e_{ij}} e_{ij} + \frac{1}{2} \frac{\partial \Sigma}{\partial \varepsilon_{ij}} \varepsilon_{ij} + \frac{1}{2} \frac{\partial \Sigma}{\partial \gamma_{ikj}} \gamma_{ikj} = \Sigma$$

The last equality is obtained from Euler's relation for homogeneous functions [4] since Σ is homogeneous quadratic in its arguments.

NASA-Langley, 1969 —— 32 CR-1320