General Disclaimer

One or more of the Following Statements may affect this Document

- This document has been reproduced from the best copy furnished by the organizational source. It is being released in the interest of making available as much information as possible.

- This document may contain data, which exceeds the sheet parameters. It was furnished in this condition by the organizational source and is the best copy available.

- This document may contain tone-on-tone or color graphs, charts and/or pictures, which have been reproduced in black and white.

- This document is paginated as submitted by the original source.

- Portions of this document are not fully legible due to the historical nature of some of the material. However, it is the best reproduction available from the original submission.

NASA CR-

RECURRENCE FORMULAE FOR THE HANSEN'S DEVELOPMENTS

by N. X. Vinh

Distribution of this report is provided in the interest of information exchange. Responsibility for the contents resides in the author or organization that prepared it.

Prepared under Contract No. NASr 54(06) by
THE UNIVERSITY OF MICHIGAN
Ann Arbor, Michigan

for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

RECURRENCE FORMULAE FOR THE HANSEN'S DEVELOPMENTS[1]

by N. X. Vinh[2]

ABSTRACT: In this paper we derive some recurrence formulae which can be used to calculate the Fourier expansions of the functions $\left(\frac{r}{a}\right)^n \cos mv$ and $\left(\frac{r}{a}\right)^n \sin mv$ in terms of the eccentric anomaly E or the mean anomaly M. We also establish a recurrence process for computing the series expansions for all n and m when the expansions of two basic series are known. These basic series were given in explicit form in the classical literature. The recurrence formulae are linear in the functions involved and thus make very simple the computation of the series.

[1] This work was supported by NASA contract No. NASr 54(06).

[2] Associate Professor of Aerospace Engineering, Department of Aerospace Engineering, The University of Michigan, Ann Arbor, Michigan 48104.

1. INTRODUCTION

In the development of the disturbing functions in the lunar and planetary theories it is required to find the developments in terms of the mean anomaly M of the functions

$$\Phi^{n,m} = \left(\frac{r}{a}\right)^n \cos mv \quad \text{and} \quad \Psi^{n,m} = \left(\frac{r}{a}\right)^n \sin mv \qquad (1.1)$$

where a is the semi major-axis, r the radial distance and v the true anomaly in elliptic motion. The functions were first considered by Hansen in his Fundamenta (Ref. 1). For each specific pair of values of n and m where n is a positive or negative integer and m is a positive integer, after a series of transformations he arrived to express $\Phi^{n,m}$ and $\Psi^{n,m}$ in terms of the expansions of $\left(\frac{r}{a}\right)^2$ and $\left(\frac{r}{a}\right)^{-2}$ and their derivatives with respect to the eccentricity e.

In general we have

$$\left(\frac{r}{a}\right)^n \cos mv = A_0^{n,m} + A_1^{n,m} \cos M + A_2^{n,m} \cos 2M + \cdots$$
$$\left(\frac{r}{a}\right)^n \sin mv = B_1^{n,m} \sin M + B_2^{n,m} \sin 2M + \cdots \qquad (1.2)$$

The coefficients $A_0^{n,m}$, $A_1^{n,m}$, ..., $B_1^{n,m}$, ..., called the Hansen's coefficients, are functions of e. Later, in another memoir (Ref. 2) Hansen derived very general formulae for the computation of these coefficients. But still the calculation is not easy because it involves Lagrange expansions and multiplication of hypergeometric series so that in 1861 when Cayley established his extensive tables of the Hansen's coefficients for the values of n from -5 to 4, and m from 0 to 5, to the seventh order in e (Ref. 3), he preferred to use similar tables previously computed by Leverrier (Ref. 4) as an easy approach to the problem. Leverrier himself has used the Cauchy's numbers to calculate his tables. Recently Munsen used the eccentric anomaly E in his modified Hansen's theory.

With the current required accuracy in the computation of satellite orbits which can have arbitrarily large values for the eccentricity, it is desirable to have a recurrence process which can be used to generate tables for the expansions of the functions (1.1) in any of the three anomalies and to a higher order in the eccentricity, with a minimum number of computations involved. The establishment of such a process is the purpose of this paper.

2. DIFFERENTIAL EQUATIONS

In this section we shall derive a homogeneous linear differential equation of the second order which is satisfied by the functions (1.1).

Consider the vector equation

$$\dot{X} = A(t)X \tag{2.1}$$

where A is the 2×2 matrix

$$A(t) = \begin{bmatrix} f_1(t) & \alpha(f_1 - f_2) \\ \beta(f_1 - f_2) & f_2(t) \end{bmatrix}. \tag{2.2}$$

where $f_1(t)$ and $f_2(t)$ are two arbitrary functions of t of the class C^1, and α and β are two arbitrary constants. It can be shown that A(t) is the most general 2×2 matrix which commutes with its integral. For this paper it suffices to prove the following theorem.

Theorem. The equation (2.1) where A(t) is given by (2.2) can be transformed into a homogeneous linear equation with constant coefficient.

Proof: Let

$$X = \exp\left(\frac{1}{2}\int(f_1 + f_2)dt\right) Z \tag{2.3}$$

$$\dot{X} = \tfrac{1}{2}(f_1 + f_2)\exp(\ )\,Z + \exp(\ )\dot{Z} = \exp(\ )AZ$$

Dividing out by exp ( )

$$\dot{Z} = [A - \tfrac{1}{2}(f_1 + f_2)I]\,Z$$

or

$$\dot{Z} = (f_1 - f_2)BZ \tag{2.4}$$

where B is the constant matrix

$$B = \begin{bmatrix} \frac{1}{2} & \alpha \\ \beta & -\frac{1}{2} \end{bmatrix} \tag{2.5}$$

By using the new independent variable s such that

$$s = \int (f_1 - f_2)dt \tag{2.6}$$

we have the required equation

$$\frac{dZ}{ds} = BZ \tag{2.7}$$

Now, the equivalent second order differential equation of the system (2.1) is

$$\ddot{x} - \left[f_1 + f_2 + \frac{\dot{f}_1 - \dot{f}_2}{f_1 - f_2}\right]\dot{x} - \left[\frac{f_1 \dot{f}_2 - \dot{f}_1 f_2}{f_1 - f_2} + \alpha\beta(f_1 - f_2)^2 - f_1 f_2\right]x = 0 \tag{2.8}$$

where x is any of the two components of X. Since the characteristic equation of the system (2.7) is

$$\lambda^2 = \alpha\beta + \tfrac{1}{4} \tag{2.9}$$

we immediately have, through the changes of variables (2.3) and (2.6), for the general solution of (2.8)

If $\lambda^2 = \alpha\beta + \frac{1}{4} > 0$

$$x(t) = \exp\left(\tfrac{1}{2}\int(f_1 + f_2)dt\right)\left[C_1 \exp(\lambda\int(f_1 - f_2)dt) + C_2 \exp(-\lambda\int(f_1 - f_2)dt)\right]$$

If $\alpha\beta + \frac{1}{4} < 0$

$$x(t) = \exp\left(\tfrac{1}{2}\int(f_1 + f_2)dt\right)\left[C_1 \cos \sqrt{-\lambda^2}\int(f_1 - f_2)dt + C_2 \sin \sqrt{-\lambda^2}\int(f_1 - f_2)dt\right] \tag{2.10}$$

If $\alpha\beta + \frac{1}{4} = 0$

$$x(t) = \exp\left(\tfrac{1}{2}\int(f_1 + f_2)dt\right)\left[C_1 \int(f_1 - f_2)dt + C_2\right]$$

We observe that the functions $\left(\frac{r}{a}\right)^n \cos mv$ and $\left(\frac{r}{a}\right)^n \sin mv$ are special cases of the second solution in (2.10) with t = v.

Let

$$-\lambda^2 = -\left(\alpha\beta + \tfrac{1}{4}\right) = m^2$$

$$\exp\left(\tfrac{1}{2}\int(f_1 + f_2)dv\right) = \left(\frac{r}{a}\right)^n = \frac{(1-e^2)^n}{(1+e\cos v)^n}$$

$$\int(f_1 - f_2)dv = v$$

we can deduce

$$f_1(v) = \frac{n e \sin v}{1 + e\cos v} + \tfrac{1}{2}, \quad f_2(v) = \frac{n e \sin v}{1 + e\cos v} - \tfrac{1}{2} \tag{2.11}$$

By substituting into (2.8) we have the differential equation which is satisfied by (1.1)

$$\frac{d^2x}{dv^2} - \frac{2n e \sin v}{1 + e\cos v}\frac{dx}{dv} + \left[m^2 + \frac{n^2 e^2 \sin^2 v}{(1+e\cos v)^2} - \frac{ne(e+\cos v)}{(1+e\cos v)^2}\right]x = 0 \tag{2.12}$$

If we consider $\Phi^{n,m}$ and $\Psi^{n,m}$ as functions of the eccentric anomaly E, then by the change of variables

$$\sin v = \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E}$$

$$\cos v = \frac{\cos E - e}{1 - e\cos E} \tag{2.13}$$

$$\frac{dE}{dv} = \frac{1}{\sqrt{1-e^2}}(1 - e\cos E)$$

we have the differential equation with E as the independent variable

$$(1-e\cos E)^2\frac{d^2x}{dE^2} + (1-2n)e\sin E(1-e\cos E)\frac{dx}{dE}$$
$$+[(1-e^2)(m^2-n^2) + n(2n-1)(1-e\cos E) - n(n-1)(1-e\cos E)^2]x = 0 \tag{2.14}$$

Finally if we consider $\Phi^{n,m}$ and $\Psi^{n,m}$ as functions of the mean anomaly M, then by the transformation

$$M = E - e \sin E$$

$$\frac{dM}{dE} = 1 - e \cos E \qquad (2.15)$$

we have the differential equation which is satisfied by $\Phi^{n,m}$ and $\Psi^{n,m}$, considered as functions of M.

$$(1 - e\cos E)^2 \frac{d^2 x}{dM^2} + 2(1-n)e \sin E \frac{dx}{dM}$$
$$+ \frac{1}{(1 - e\cos E)^2}[(1-e^2)(m^2-n^2) + n(2n-1)(1-e\cos E) - n(n-1)(1-e\cos E)^2]x = 0 \qquad (2.16)$$

In the last equation the coefficients are to be expressed in terms of M using the Kepler's equation (2.15).

The differential equations (2.12), (2.14) and (2.16) with respectively the true anomaly v, the eccentric anomaly E, and the mean anomaly M as independent variable will serve as basic equations in the derivation of the recurrence formulae for the series expansions of $\Phi^{n,m}$ and $\Psi^{n,m}$ in each of the three anomalies. In the following we shall consider the expansions in E and in M.

3. FOURIER EXPANSIONS IN TERMS OF E

Let

$$X^{n,m} = \Phi^{n,m} + i\Psi^{n,m} = \left(\frac{r}{a}\right)^n \exp(imv) \qquad (3.1)$$

We have seen that $X^{n,m}$, considered as function of the eccentric anomaly E, satisfies the differential equation

$$(1 - e\cos E)^2 \frac{d^2 X^{n,m}}{dE^2} + (1-2n)\, e \sin E(1 - e\cos E) \frac{dX^{n,m}}{dE}$$
$$+[(1-e^2)(m^2-n^2) + n(2n-1)(1-e\cos E) - n(n-1)(1-e\cos E)^2]X^{n,m} = 0 \qquad (3.2)$$

From (3.1)

$$\frac{dX^{n,m}}{dE} = n\left(\frac{r}{a}\right)^{n-1} \exp(imv) \frac{d}{dE}\left(\frac{r}{a}\right) + im\left(\frac{r}{a}\right)^n \exp(imv)\frac{dv}{dE}$$

Since

$$\frac{r}{a} = 1 - e\cos E$$

$$\frac{d}{dE}\left(\frac{r}{a}\right) = e \sin E$$

Also

$$\frac{dv}{dE} = \frac{\sqrt{1-e^2}}{1 - e\cos E} = \sqrt{1-e^2}\left(\frac{r}{a}\right)^{-1}$$

Therefore

$$e \sin E(1 - e\cos E)\frac{dX^{n,m}}{dE} = n e^2 \sin^2 E X^{n,m} + i m \sqrt{1-e^2}\, e \sin E X^{n,m}$$

Using the relations

$$e^2 \sin^2 E = e^2 - e^2 \cos^2 E = -(1-e^2) + 2\left(\frac{r}{a}\right) - \left(\frac{r}{a}\right)^2$$

$$\sqrt{1-e^2} \sin E = \left(\frac{r}{a}\right) \sin v$$

$$i \sin v = \exp(i v) - \cos v = \exp(i v) + \frac{1}{e} - \frac{(1-e^2)}{e}\left(\frac{r}{a}\right)^{-1}$$

we have

$$e \sin E(1 - e\cos E)\frac{dX^{n,m}}{dE} = m e X^{n+1,m+1} - n X^{n+2,m} + (m+2n)X^{n+1,m}$$
$$- (m+n)(1-e^2)X^{n,m}$$

By substituting into Eq. (3.2) we have the recurrence formula

$$m(2n-1)e X^{n-1,m+1} = \frac{d^2X^{n,m}}{dE^2} + n^2X^{n,m} - (m+n)(2n-1)X^{n-1,m}$$
$$+ (m+n)(m+n-1)(1-e^2)X^{n-2,m} \qquad (3.3)$$

where X can be Φ or Ψ. This formula can be used to go from cos m v (or sin m v) to cos (m+1)v (or sin (m+1)v). Changing m into -m and noticing that $X^{n,-m} = \overline{X}^{n,m}$ where $\overline{X}^{n,m}$ is the complex conjugate of $X^{n,m}$ we have

$$m(1-2n)e \overline{X}^{n-1,m-1} = \frac{d^2\overline{X}^{n,m}}{dE^2} + n^2\overline{X}^{n,m} + (n-m)(1-2n)\overline{X}^{n-1,m}$$
$$+ (n-m)(n-m-1)(1-e^2)\overline{X}^{n-2,m} \qquad (3.4)$$

where $\overline{X}$ can be Φ or Ψ. This formula can be used to go from cos (m+1)v (or sin (m+1)v) to cos m v (or sin m v).

Combining the Eqs (3.3) and (3.4) we easily obtain

$$e[X^{n,m+1} + X^{n,m-1}] = 2(1-e^2)X^{n-1,m} - 2X^{n,m} \tag{3.5}$$

where X can be Φ or Ψ. This last relation can be derived directly from the polar equation of elliptic orbit.

The process for constructing tables of the expansions of $\Phi^{n,m}$ and $\Psi^{n,m}$ is as follows.

Expansions of $\left(\frac{r}{a}\right)^n \cos m v$

First step

Let m = 0 in (3.3) and we have

$$\frac{d^2}{dE^2}\left(\frac{r}{a}\right)^n + n^2\left(\frac{r}{a}\right)^n - n(2n-1)\left(\frac{r}{a}\right)^{n-1} + n(n-1)(1-e^2)\left(\frac{r}{a}\right)^{n-2} = 0 \tag{3.6}$$

This recurrence formula can be used to calculate the series for $\left(\frac{r}{a}\right)^n$ for all values of n when those for n = -1, and n = -2 have been obtained.

Let

$$\left(\frac{r}{a}\right)^n = \sum A_p^n \cos pE \tag{3.7}$$

$$p = 0, 1, 2\ldots$$

Then we have the recurrence formula for the coefficients A_p^n

$$(n^2 - p^2)A_p^n - n(2n-1)A_p^{n-1} + n(n-1)(1-e^2)A_p^{n-2} = 0 \tag{3.8}$$

When n is negative the series is infinite. When $n \geq 0$ the series terminates at the term cos n E. In this case the last coefficient cannot be calculated by formula (3.8) but by setting E = 0 in Eq. (3.7) it can readily be seen that

$$A_n^n = (1-e)^n - \sum_{p=0}^{n-1} A_p^n \tag{3.9}$$

The starting series for n = -1 and n = -2 can be easily calculated by the classical methods. We have (Ref. 5)

$$\left(\frac{r}{a}\right)^{-1} = \frac{1}{\sqrt{1-e^2}}(1 + 2\beta\cos E + 2\beta^2\cos 2E + \cdots) \tag{3.10}$$

where β is given by

$$\beta = \frac{1-\sqrt{1-e^2}}{e} \tag{3.11}$$

and in general

$$\beta^p = \left(\frac{e}{2}\right)^p + p\left(\frac{e}{2}\right)^{p+2} + \frac{p}{2!}(p+3)\left(\frac{e}{2}\right)^{p+4} + \frac{p}{3!}(p+4)(p+5)\left(\frac{e}{2}\right)^{p+6}$$
$$+ \frac{p}{4!}(p+5)(p+6)(p+7)\left(\frac{e}{2}\right)^{p+8} + \cdots \tag{3.12}$$

Also

$$\left(\frac{r}{a}\right)^{-2} = (1-e^2)^{-\frac{3}{2}} + (1-e^2)^{-\frac{3}{2}} \sum_{p=1}^{\infty} 2\beta^p (1 + p\sqrt{1-e^2}) \cos pE \tag{3.13}$$

Second step

Calculate all $\left(\frac{r}{a}\right)^n \cos v$ by taking $m = 0$ in the recurrence formula (3.5). Explicitly we have

$$\left(\frac{r}{a}\right)^n \cos v = \frac{(1-e^2)}{e}\left(\frac{r}{a}\right)^{n-1} - \frac{1}{e}\left(\frac{r}{a}\right)^n \tag{3.14}$$

Third step

Use (3.5) again with $m = 1$ to calculate all $\left(\frac{r}{a}\right)^n \cos 2v$

$$\left(\frac{r}{a}\right)^n \cos 2v = \frac{2(1-e^2)}{e}\left(\frac{r}{a}\right)^{n-1} \cos v - \frac{2}{e}\left(\frac{r}{a}\right)^n \cos v - \left(\frac{r}{a}\right)^n \tag{3.15}$$

The process continues by successive applications of (3.5) to calculate $\left(\frac{r}{a}\right)^n \cos mv$.

Since the recurrence formulae are linear in the functions involved, the calculation process is extremely simple. But while this process is very simple, it suffers from two unavoidable defects:

The Hansen's coefficients behave like the Bessel's coefficients. Each time we apply formula (3.5) the order in e of the coefficients is decreased by one unit. Therefore to compute tables up to $\cos m v$ to the order of e^p we need to compute the basic series for $\left(\frac{r}{a}\right)^{-1}$, and $\left(\frac{r}{a}\right)^{-2}$ to the order of e^{p+m}. Since these series are given in explicit forms the defect does not create any real handicap.

When n is a negative integer, to compute $\left(\frac{r}{a}\right)^n \cos(m+1)v$, it involves the expansion of $\left(\frac{r}{a}\right)^{n-1} \cos m v$. Therefore to compute tables down to $\left(\frac{r}{a}\right)^n$ $\cos m v$, n being a negative integer, in the first step mentioned above we should compute down to the expansion of $\left(\frac{r}{a}\right)^{n-m}$. This defect again does not create any serious problem since by the recurrence formula (3.8) we can easily calculate $\left(\frac{r}{a}\right)^n$ for any negative n.

Expansions of $\left(\frac{r}{a}\right)^n \sin m v$

We first compute all $\left(\frac{r}{a}\right)^n \sin v$ by the relation

$$\left(\frac{r}{a}\right)^n \sin v = \frac{\sqrt{1-e^2}}{n e} \frac{d}{dE} \left(\frac{r}{a}\right)^n \tag{3.16}$$

When n = 0 we can calculate the expansion of $\sin v$ by

$$\frac{d}{dE}(\sin v) = \sqrt{1-e^2} \left(\frac{r}{a}\right)^{-1} \cos v \tag{3.17}$$

$$= \frac{(1-e^2)^{\frac{3}{2}}}{e} \left(\frac{r}{a}\right)^{-2} - \frac{(1-e^2)^{\frac{1}{2}}}{e} \left(\frac{r}{a}\right)^{-1}$$

or directly by (Ref. 5)

$$\sin v = (1-\beta^2) \sum_{p=1}^{\infty} \beta^{p-1} \sin p E \tag{3.18}$$

Next we can successively apply the recurrence formula (3.5) to calculate all $\left(\frac{r}{a}\right)^n \sin 2v$, and so on.

4. FOURIER EXPANSIONS IN TERMS OF M

First we notice that these developments can be deduced from those in in terms of E by using the classical series expansions (Ref. 6)

$$\sin mE = m \sum_{p=1}^{\infty} \frac{\sin pM}{p} [J_{p-m}(pe) + J_{p+m}(pe)]$$

$$\cos mE = A_0 + m \sum_{p=1}^{\infty} \frac{\cos pM}{p} [J_{p-m}(pe) - J_{p+m}(pe)] \qquad (4.1)$$

$$\begin{aligned} A_0 &= 1 \text{ if } m = 0 \\ &= -\tfrac{1}{2}e \text{ if } m = 1 \\ &= 0 \text{ if } m > 1 \end{aligned}$$

where $J_k(pe)$ is the Bessel's coefficient of order k and argument pe.

If direct computation is desired, we can start with the differential equation (2.16) and, by using the same type of derivation as in the preceding section, we obtain the recurrence formula

$$\begin{aligned} 2m(n-1)eX^{n-3,m+1} = {} & \frac{d^2X^{n,m}}{dM^2} + n(n-1)X^{n-2,m} \\ & -(2n^2+2mn-3n-2m)X^{n-3,m} \\ & +(m+n)(m+n-2)(1-e^2)X^{n-4,m} \end{aligned} \qquad (4.2)$$

where X can be Φ or Ψ.

By taking n = 1 we have

$$\frac{d^2}{dM^2}\left(\frac{r}{a}\right)\begin{matrix}\cos mv\\ \sin mv\end{matrix} + \left(\frac{r}{a}\right)^{-2}\begin{matrix}\cos mv\\ \sin mv\end{matrix} + (m^2-1)(1-e^2)\left(\frac{r}{a}\right)^{-3}\begin{matrix}\cos mv\\ \sin mv\end{matrix} = 0 \qquad (4.3)$$

By further taking m = 1 we have the classical formula (Ref. 6)

$$\frac{d^2}{dM^2}\left(\frac{r}{a}\right)\begin{matrix}\cos v\\ \sin v\end{matrix} + \left(\frac{r}{a}\right)^{-2}\begin{matrix}\cos v\\ \sin v\end{matrix} = 0 \qquad (4.4)$$

Putting m = 0 in (4.2) we have Hansen's recurrence formula (Ref. 1, p 167)

$$\frac{d^2}{dM^2}\left(\frac{r}{a}\right)^{n} + n(n-1)\left(\frac{r}{a}\right)^{n-2} - n(2n-3)\left(\frac{r}{a}\right)^{n-3} + n(n-2)(1-e^2)\left(\frac{r}{a}\right)^{n-4} = 0 \qquad (4.5)$$

We also have as before

$$e[X^{n,m+1} + X^{n,m-1}] = 2(1-e^2)X^{n-1,m} - 2X^{n,m} \qquad (4.6)$$

where X can be Φ or Ψ.

The process for computing tables of the expansions of $\Phi^{n,m}$ and $\Psi^{n,m}$ is as follows.

Expansions of $\left(\frac{r}{a}\right)^n \cos mv$

<u>First step</u>

The recurrence formula (4.5) is used to calculate the series for $\left(\frac{r}{a}\right)^n$ for all values of n when those for certain values have been obtained.

Let

$$\left(\frac{r}{a}\right)^n = \sum_{p=0}^{\infty} \text{[illegible]} \cos pM \tag{4.7}$$

Then we have the recurrence formula for the coefficients A_p^n

$$p^2 A_p^n = n(n-1)A_p^{n-2} - n(2n-3)A_p^{n-3} + n(n-2)(1-e^2)A_p^{n-4} \tag{4.8}$$

In particular the constant term is given by

$$(n+1)(n+2)A_0^n - (n+2)(2n+1)A_0^{n-1} + n(n+2)(1-e^2)A_0^{n-2} = 0 \tag{4.9}$$

Examination of the formulae reveals that the expansions for all values of n can be evaluated in terms of the expansions for n = 1, 2, -2 and -4. But the expansions for n = 1, and n = -4 can be evaluated in terms of the expansions for n = 2, -2 and -3 through the relations

$$\begin{aligned} \left(\frac{r}{a}\right) &= (1-e^2) + \frac{e}{2}\frac{d}{de}\left(\frac{r}{a}\right)^2 \\ (1-e^2)\left(\frac{r}{a}\right)^{-4} &= \left(\frac{r}{a}\right)^{-3} + \frac{e}{2}\frac{d}{de}\left(\frac{r}{a}\right)^{-2} \end{aligned} \tag{4.10}$$

Hence we only need to compute the two basic starting series $\left(\frac{r}{a}\right)^2$ and $\left(\frac{r}{a}\right)^{-2}$ by the classical methods. These are given explicitly in the classical literature. We have

$$\left(\frac{r}{a}\right)^2 = 1 + \frac{3}{2}e^2 - \sum_{p=1}^{\infty} \frac{4}{p^2} J_p(pe) \cos pM \tag{4.11}$$

Series expansion of $\left(\frac{r}{a}\right)^{-2}$ is more involved. Probably the simplest way is to use the series (3.13) and the transformation (4.1) with the knowledge that the constant term in the expansion is $(1-e^2)^{-\frac{1}{2}}$. Another simple way to have the series expansion of $\left(\frac{r}{a}\right)^{-2}$ is to use the relation

$$\left(\frac{r}{a}\right)^{-2} = \frac{1}{\sqrt{1-e^2}} \frac{dv}{dM} \tag{4.12}$$

The expansion of v in terms of M has been calculated by Schubert (Ref.7) as far as e^{20}.

Second step

Once the expansions of $\left(\frac{r}{a}\right)^n$ for all values of n have been obtained we can successively use the recurrence formula (4.6) with $m = 0, 1, \ldots$ to calculate the expansions of $\left(\frac{r}{a}\right)^n \cos v$, $\left(\frac{r}{a}\right)^n \cos 2v, \ldots$ as described in the preceding section.

Expansions of $\left(\frac{r}{a}\right)^n \sin m v$

We first compute all $\left(\frac{r}{a}\right)^n \sin v$ by the relation

$$\left(\frac{r}{a}\right)^n \sin v = \frac{\sqrt{1-e^2}}{(n+1)e} \frac{d}{dM} \left(\frac{r}{a}\right)^{n+1} \tag{4.13}$$

When $n = -1$ we can use the relation

$$\frac{d}{dM}\left(\frac{r}{a}\right)^{-1} \sin v = \frac{\sqrt{1-e^2}}{e}\left(\frac{r}{a}\right)^{-2} - 3\frac{\sqrt{1-e^2}}{e}\left(\frac{r}{a}\right)^{-3} + 2\frac{(1-e^2)^{\frac{3}{2}}}{e}\left(\frac{r}{a}\right)^{-4} \tag{4.14}$$

to calculate $\left(\frac{r}{a}\right)^{-1} \sin v$.

Next we can use the recurrence formula (4.6) to calculate all $\left(\frac{r}{a}\right)^n \sin 2v$, and so on.

As discussed before to compute tables for the expansions of $\Phi^{n,m}$ and $\Psi^{n,m}$ up to the value m and from a negative $-n$ $(n > 0)$ to a positive n up to the order of e^p we should compute the basic series for $\left(\frac{r}{a}\right)^2$ and $\left(\frac{r}{a}\right)^{-2}$ to the order of e^{p+m}, and first calculate the series from $\left(\frac{r}{a}\right)^{-n-m}$ to $\left(\frac{r}{a}\right)^{n+1}$.

5. CONCLUSION

In this paper we have derived recurrence formulae to calculate the series expansions of $\left(\frac{r}{a}\right)^n \cos m v$ and $\left(\frac{r}{a}\right)^n \sin m v$ in terms of the eccentric anomaly E or the mean anomaly M. We also have established a recurrence process which can be used to compute the series expansions for all n and m when the expansions of two basic series are known. The expansions in terms of the true anomaly v are similar to those in terms of the eccentric anomaly E. By observing that

$$\left(\frac{r}{a}\right)^n = (1 - e\cos E)^n = \frac{(1-e^2)^n}{(1+e\cos v)^n} \tag{5.1}$$

for the expansions in v we only need to change n into -n, e into -e, a into $a(1-e^2)$, and E into v in Eq. (3.6) and next change the sign of all the exponents to have

$$(1-e^2)\frac{d^2}{dv^2}\left(\frac{r}{a}\right)^n + n^2(1-e^2)\left(\frac{r}{a}\right)^n - n(2n+1)\left(\frac{r}{a}\right)^{n+1} + n(n+1)\left(\frac{r}{a}\right)^{n+2} = 0 \tag{5.2}$$

In applying the recurrence formulae, each time we go to a next higher multiple anomaly the order in e in the Hansen's coefficients is decreased by one. This is caused by a property of the Hansen's coefficients, called the D'Alembert characteristic by E. W. Brown (Ref. 8); namely, the lowest order in e in the coefficient of cos pM (or sin pM) in the expansion of $\left(\frac{r}{a}\right)^n \cos m v$ (or $\left(\frac{r}{a}\right)^n \sin m v$) is $|p - m|$. This property is also true for the expansions in E and in v.

In his tables Cayley also gave the series expansion of $\log\left(\frac{r}{a}\right)$ in terms of M. Explicitly we have (Ref. 8)

$$\log\left(\frac{r}{a}\right) = -\log(1+\beta^2) - 2\sum_{p=1}^{\infty}\frac{\beta^p}{p}\cos pE \tag{5.3}$$

and

$$\log\left(\frac{r}{a}\right) = -\log(1+\beta^2) + e\beta - 2\sum_{s=1}^{\infty}\frac{1}{s}\sum_{p=1}^{\infty}\beta^p\left[J_{s-p}(se) - J_{s+p}(se)\right]\cos sM \tag{5.4}$$

In our process we can have those expansions by integrating term by term the following relations

$$\frac{d}{dE} \log \frac{r}{a} = \frac{e}{\sqrt{1 - e^2}} \sin v \qquad (5.5)$$

with the constant term in the integration being $\log \frac{(1 + \sqrt{1 - e^2})}{2}$ and

$$\frac{d}{dM} \log \frac{r}{a} = \frac{e}{\sqrt{1 - e^2}} \left(\frac{r}{a}\right)^{-1} \sin v \qquad (5.6)$$

with the constant term being $\log \frac{(1 + \sqrt{1 - e^2})}{2} + 1 - \sqrt{1 - e^2}$.

REFERENCES

1. Hansen, P.A., "Fundamenta Nova Investigationis Orbitae vera quam Luna perlustrat," Gotha, 1838.

2. Hansen, P.A., "Entwicklung der negativen und ungraden Potenzen...," Memoirs of the Royal Society of Sciences, Sachs, Vol IV, pp 283-376, 1854.

3. Cayley, A., "Tables of the Developments of Functions in the Theory of Elliptic Motion," Memoirs of the Royal Astronomical Society, Vol XXIX, pp 191-306, 1861.

4. Leverrier, U. J. J., Tables in the "Annales de l'Observatoire de Paris," Vol I, pp 346-348, 1855.

5. Brown, E. W., and Shook, C. A., "Planetary Theory," Cambridge: The University Press, 1933.

6. Plummer, H. C., "An Introductory Treatise on Dynamical Astronomy," Cambridge: The University Press, 1918.

7. Schubert, Th., "Astronomie Théorique," St. Petersburg, 1822.

8. Brouwer D., and Clemence G.M., "Methods of Celestial Mechanics," Academic Press, N.Y., 1961.