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Report No. 2

DESIGN OF A PITCH ORIENTATIONAL FLIGHT CONTROL SYSTEM

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By

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ABSTRACT

This thesis presents two designs for a pitch orientational flight control system. Both designs employ control techniques based on Liapunov's direct method which are used in conjunction with system state estimation. One design yields a non-linear control law and the other a linear control law, with a linear estimation technique being used in both cases.

The designs are developed using approximate models of the system components, and very satisfactory experimental results are obtained using these approximations. A stability problem arises, however, when the higher order dynamics of the system are considered. This problem is overcome in the linear design by including proper compensation in the controller.

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LIST OF MAJOR SYMBOLS

A_i	$i = 1, 4$ - Coefficients used in control law
A_{ij}	$i = 1, 2 ; j = 1, 2$ - Integral terms in estimation error variance equations
$\overline{A}$	$A_{11} + A_{21}$
$a_1(t)$	$-\omega_a^2$
$a_2(t)$	$-2\zeta_a\omega_a$
a_{01}	$-\omega_0^2$
a_{02}	$-2\zeta_0\omega_0$
a_{ni}	$i = 1, 2$ - Nominal estimator parameters
$b_1(t)$	K_a/T_a
$b_2(t)$	$K_a(1/T_a - 1/T_h)$
B	A 2x2 positive definite, symmetric matrix
C_n	Amplitude of noise spectral density
D (t)	Disturbance Input
$\underline{e}(t)$	System 2x1 error vector
$\hat{\underline{e}}(t)$	Estimated system 2x1 error vector
$\tilde{\underline{e}}(t)$	A 2x1 vector of estimation errors
$\tilde{\underline{e}}_f$	Estimation error caused by f_2
$\tilde{\underline{e}}_n$	Estimation error caused by n
f_2	Forcing term in estimation equation

f_{2ss}	The value of f_2 for steady-state step response
G_i	$i = 1, 2$ - Estimator gains
$\overline{G}_1$	$G_1 - a_{n2}$
$\overline{G}_2$	$G_2 - G_1 a_{n2} - a_{n1}$
$G_h(S)$	Servo actuator transfer function
$G_g(S)$	Gyro transfer function
$G_a(S)$	Aircraft transfer function
$G_c(S)$	Compensator transfer function
$H(S)$	A transfer function in the linear system loop-gain function
h	A scalar variable
$K_a(t)$	Unknown aircraft parameter
$K_d(t)$	Unknown aircraft parameter
$K(t)$	Controlled plant gain
K_{an}	Estimator parameter
$(K_a/T_a)_n$	Estimator parameter
K_{ss}	$(K_a/T_a) / (-a_1)$
$L(S)$	Linear system open loop transfer function
$L(\underline{x}, t)$	A scalar variable
L^*	A scalar constant
$\overline{L}$	A scalar constant
M	Non-linear function
$\overline{M}$	A scalar constant
M^*	$L^* B_{12}$

$n(t)$	An additive noise term
P	A 2x2 estimator gain matrix
Q	A positive definite, symmetric matrix
$\overline{Q}$	Estimator weighting matrix
$r(t)$	System input
S	Laplace transform variable
$T_a(t)$	Unknown aircraft parameter
T_h	Servo actuator time constant
$T(S)$	A 2x2 transfer function matrix
$u(t)$	Signal generated by controller
$v(t)$	Input to servo actuator
V	Liapunov function
W	A 2x2 estimator weighting matrix
$\underline{x}(t)$	A 2x1 system vector
$\underline{x}_d(t)$	A 2x1 model reference vector
$\hat{\underline{x}}(t)$	Estimated 2x1 system vector
$y(t)$	Aircraft angular rate
y_g	Gyro output signal
$\alpha(t)$	An unknown scalar
δ	Elevator angle
Δa_{ci}	$a_i - a_{oi}$, $i = 1, 2$
Δa_{ni}	$a_i - a_{ni}$, $i = 1, 2$

ΔK_a	$K_a - K_{an}$
$\Delta(K_a/T_a)$	$K_a/T_a - (K_a/T_a)_n$
γ	Switching function
$\hat{\gamma}$	Estimated switching function
$\tilde{\gamma}$	$\hat{\gamma} - \gamma$
Φiif	Spectral density of $\tilde{e}_{if}$, $i = 1, 2$
Φff	Spectral density of f_2
σif	Standard deviation of $\tilde{e}_{if}$, $i = 1, 2$
σin	Standard deviation of $\tilde{e}_{in}$, $i = 1, 2$
$\dot{\theta}$	Pitch rate of aircraft
ω_o	Model reference parameter
$\omega_a(t)$	Unknown aircraft parameter
ζ_o	Model reference parameter
$\zeta_a(t)$	Unknown aircraft parameter

INTRODUCTION

One of the well known and major phenomena encountered in the design of aircraft flight control systems is that the transient response of the aircraft changes considerably for different flight conditions. With the advent of the high-performance, variable-geometry type of aircraft, this variation is becoming even more pronounced due to the expanding environment in which the aircraft may operate. In many of the earlier flight control designs, the dynamics of the control system was a function of air data measurements so that satisfactory handling qualities over the entire flight regime could be obtained. This entailed extensive wind tunnel analyses and in-flight calibration to determine optimum parameter settings for various conditions. To overcome these difficulties and to eliminate the need for air-data measurements, adaptive control systems are now in use in many high-performance aircraft. Most of the adaptive techniques are based on the principle of maintaining a constant damping ratio of the closed loop system by varying the system gain. This is done by sensing the system response to either pulse inputs or gusts, determining the damping ratio from this response, and varying the gain accordingly.

In the following report a new flight control system design is presented. The design concentrates only on a pitch orientational flight control using the appropriate longitudinal transfer function for the aircraft. The main advantages of this design over the adaptive techniques mentioned previously is that the system response does not have to be monitored and a variable gain does not have to be implemented. The design is based on a control technique which combines Liapunov's direct method with system state estimation. A model reference is employed in the system, and the object of the control is to force the aircraft to behave as the model through an input initiated by the controller. Two controller designs are presented, one yields a non-linear control law and the other a linear control law. Both designs employ a linear estimator to obtain estimates of quantities required by the controller.

The designs are developed using approximate representations of the servo actuator and the rate gyro. Using these approximations in an analog simulation of the system, the pitch rate of the aircraft follows the output of the model with less than 5% error over the range of parameter variations assumed. It is found, however, that the system is unstable if the higher order dynamics of the actuator and gyro are considered. A compensator is then included in the linear design to overcome this stability problem.

CHAPTER 1

SYSTEM FORMULATION

A block diagram of the control system is shown in Figure 1. The objective of the system is to force the aircraft to behave like the model reference, which in turn is driven by the pitch rate command signal, r. To accomplish this, the pitch rate of the aircraft, $\dot{\theta}$, corrupted by measurement and vibration noise, n, is fed into an estimator along with the elevator position, δ. The purpose of the estimator is to obtain an estimate of $\dot{\theta}$ and its derivative, the estimated values of these quantities being denoted by the vector $\hat{\underline{x}}$. The vector $\hat{\underline{x}}$ is then compared with the output and output derivative of the model reference, which are denoted by $\underline{x}_d$, resulting in the error vector, $\hat{\underline{e}}$. The quantities $\hat{\underline{e}}$, $\hat{\underline{x}}$, δ, and r are then fed into the controller. The control law produces the signal u which drives the system in a manner such that $\hat{\underline{e}}$ is driven toward zero, thus causing the aircraft to behave like the model.

The aircraft is represented by the pitch axis short-period mode transfer function with parameters which vary with time in an unknown manner. This transfer function is

$$Y(S) = \frac{K_a \left(S + \frac{1}{T_a}\right) \delta(S) + K_d SD(S)}{S^2 + 2\zeta_a \omega_a S + \omega_a^2} \tag{1-1}$$

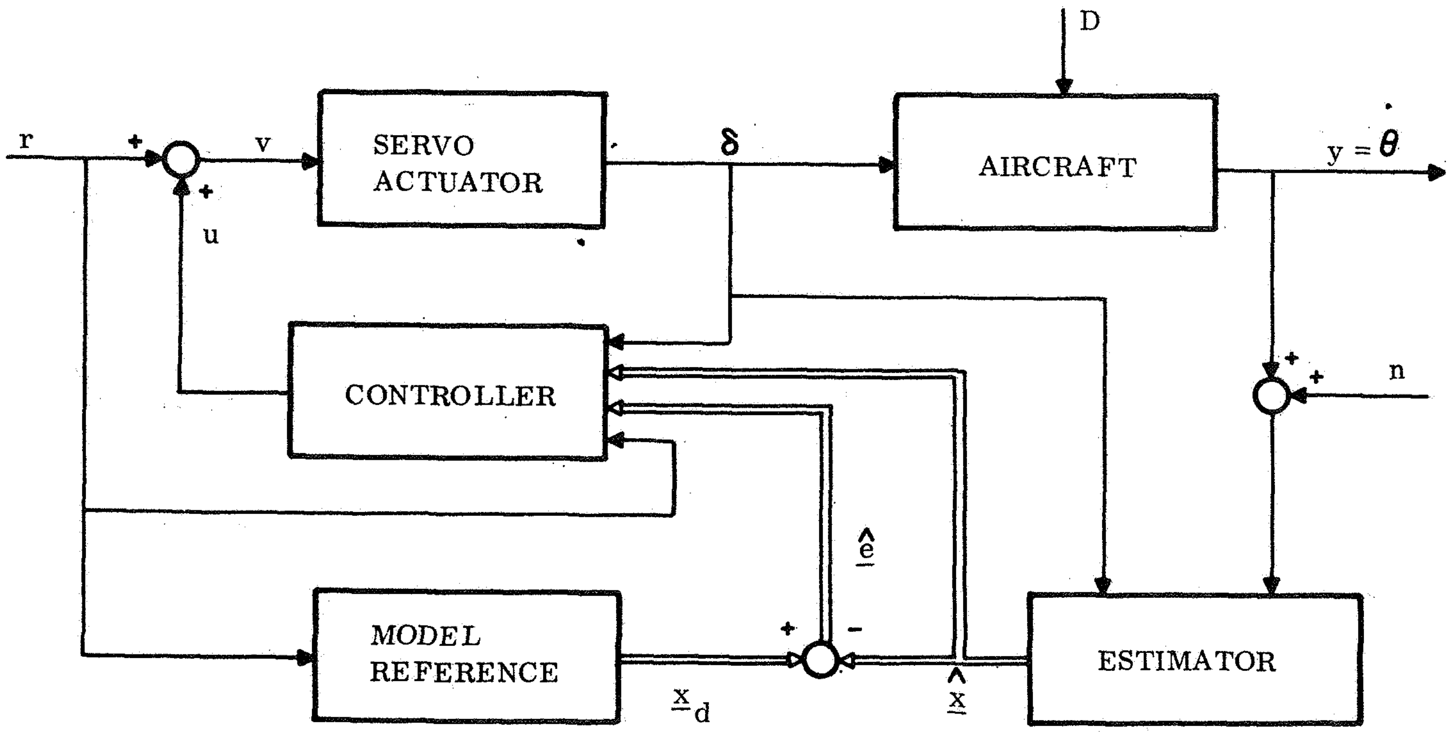


Figure 1 Pitch Orientational Control System

where K_a, K_d, T_a, ζ_a, ω_a are unknown, time-varying parameters of the aircraft and D is a disturbance input. Typical parameter values for a fighter-type aircraft for different flight conditions are given in Table I. It is assumed in the design that this table contains the full range of parameter variations encountered in the aircraft's performance envelope.

To reduce the order of the system for analysis purposes, the transfer function for the servo actuator will be taken as

$$\delta(S) = \frac{1}{T_h S + 1} V(S) \qquad (1-2)$$

where T_h is the actuator time constant. The dynamics of the gyro are presently ignored, with the gyro dynamics and higher-order actuator dynamics being considered in later chapters.

Table I

Aircraft Parameter Variations

CASE	K_a	$\frac{1}{T_a}$	ζ_a	ω_a	K_D
1	2.52	.368	.634	1.10	2.05
2	15.7	1.17	.556	2.82	13.5
3	7.08	.523	.432	1.54	6.23
4	76.2	1.18	.462	7.80	50.9
5	35.9	.452	.155	6.41	32.6
6	13.1	.152	.075	3.90	10.1
7	18.7	.235	.106	4.78	19.5
8	45.2	.846	.532	4.67	30.3
9	11.7	.255	.282	2.47	12.0
10	37.7	2.38	.430	5.10	36.5

CHAPTER II

CONTROLLER DESIGN

A - Non-linear Controller

The non-linear control technique used in the design is outlined in Appendix A. To obtain the system equations in the form required by this technique, (1-1) and (1-2) are converted to the time domain differential equations (2-1) and (2-2).

$$\ddot{y} + 2\zeta_a \omega_a \dot{y} + \omega_a^2 y = K_a \dot{\delta} + \frac{K_a}{T_a} \delta + K_d \dot{D} \qquad (2\text{-}1)$$

$$T_h \dot{\delta} + \delta = v \qquad (2\text{-}2)$$

where

$$v = u + r$$

Solving for $\dot{\delta}$ from (2-2) and substituting into (2-1) yields

$$\ddot{y} + 2\zeta_a \omega_a \dot{y} + \omega_a^2 y = \frac{K_a}{T_h}(u + r) + K_a \left(\frac{1}{T_a} - \frac{1}{T_h}\right) \delta + K_d \dot{D} \qquad (2\text{-}3)$$

Equation (2-3) may be written in the vector form of (A-1). This yields the set of vector differential equations (2-4).

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ b_1 (u + r) + b_2 \delta + K_d \dot{D} \end{bmatrix} \qquad (2\text{-}4)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}$$

where

$$\underline{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix}$$

$$a_1 = -\omega_a^2$$

$$a_2 = -2\zeta_a \omega_a$$

$$b_1 = K_a/T_h$$

$$b_2 = K_a \left(\frac{1}{T_a} - \frac{1}{T_h} \right)$$

The model reference is taken to be a second order system with a natural frequency ω_0 and a damping ratio ζ_0. The model reference equation corresponding to (A-2) is therefore given by

$$\dot{\underline{x}}_d = \begin{bmatrix} 0 & 1 \\ a_{01} & a_{02} \end{bmatrix} \underline{x}_d + \begin{bmatrix} 0 \\ -a_{01} \end{bmatrix} r \tag{2-5}$$

where

$$a_{01} = -\omega_0^2$$

$$a_{02} = -2\zeta_0 \omega_0$$

The matrix Q is defined in (A-6) as

$$-Q = A_0^T B + BA_0$$

Taking Q as a positive definite diagonal matrix, (A-6) is written as

$$\begin{bmatrix} -Q_{11} & 0 \\ 0 & -Q_{22} \end{bmatrix} = \begin{bmatrix} 0 & a_{01} \\ 1 & a_{02} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ a_{01} & a_{02} \end{bmatrix} \tag{2-6}$$

Solving (2-6) for the elements of B gives the following equations

$$B_{12} = -\frac{Q_{11}}{2a_{01}}$$

$$B_{22} = -\frac{2B_{12} + Q_{22}}{2a_{02}} \qquad (2\text{-}7)$$

$$B_{11} = -a_{02} B_{12} - a_{01} B_{22}$$

$$B_{21} = B_{12}$$

If the output of the plant and its derivative were both available, the control signal u would be, in the form (A-10).

$$u = \left[A_1 \left| x_1 \right| + A_2 \left| x_2 \right| + A_3 \left| r \right| + A_4 \left| \delta \right| + \left| \frac{K_d(t)}{b_1(t)} \dot{D} \right|_{max} \right] \text{SIGN}\ \gamma \qquad (2\text{-}8)$$

where

$$\gamma = B_{12} e_1 + B_{22} e_2$$

$$e_1 = x_{d1} - x_1$$

$$e_2 = x_{d2} - x_2$$

$$A_1 = \underset{t}{\text{Max}} \left| \frac{a_1(t) - a_{01}}{b_1(t)} \right|, \quad A_2 = \underset{t}{\text{Max}} \left| \frac{a_2(t) - a_{02}}{b_1(t)} \right|$$

$$A_3 = \underset{t}{\text{Max}} \left| \frac{b_1(t) + a_{01}}{b_1(t)} \right|, \quad A_4 = \underset{t}{\text{Max}} \left| \frac{b_2(t)}{b_1(t)} \right|$$

Since the vector $\underline{x}$ is not available in an uncorrupted form for use in (2-8), its estimate, $\hat{\underline{x}}$, is used instead. This estimate, however, is not perfect and the difference between $\underline{x}$ and $\hat{\underline{x}}$ must be included in (2-8).

The error in estimation, $\tilde{\underline{e}}$, and the estimated system error, $\hat{\underline{e}}$, are defined as

$$\tilde{\underline{e}} = \underline{x} - \hat{\underline{x}} \tag{2-9}$$

$$\hat{\underline{e}} = \underline{x}_d - \hat{\underline{x}} \tag{2-10}$$

With (2-9) and (2-10) in (2-8), the result is

$$u = \left[A_1 \left| \hat{x}_1 \right| + A_2 \left| \hat{x}_2 \right| + A_1 \left| \tilde{e}_1 \right| + A_2 \left| \tilde{e}_2 \right| + A_3 \left| r \right| + A_4 \left| \delta \right| \right.$$

$$\left. + \left| \frac{K_d(t)}{b_1(t)} \dot{D} \right|_{max} \right] \text{SIGN } (\hat{\gamma} - \tilde{\gamma}) \tag{2-11}$$

where

$$\hat{\gamma} = B_{12} \hat{e}_1 + B_{22} \hat{e}_2$$

$$\tilde{\gamma} = B_{12} \tilde{e}_1 + B_{22} \tilde{e}_2$$

Since the error in estimation is not a measurable signal, the control signal is taken as

$$u = M \text{ SIGN } (\hat{\gamma}) \tag{2-12}$$

where M corresponds to the bracketed term in (2-11) with the effect of the unknown error $\tilde{\underline{e}}$ taken into account. The term M will be derived in Section IV-B, and the effect of using the switching function $\hat{\gamma}$ insteady of γ will be discussed in Section II-C.

B-Linear Controller

In the system under consideration, the physical nature of the plant will cause the elements of the vector $\underline{x}$ to have some maximum bound. This fact permits the use of a linear control law to generate u. This control law will be derived below.

The time derivative of the Liapunov function V in Appendix A is given by (A-6) as

$$\dot{v} = - e^T Qe + h$$

where h is expressed in (A-9) as

$$h = - 2\gamma \left[\sum_{i=1}^{n} \Delta a_i(t) x_i + a_{01} r + K(t) (u + r) + K(t) \phi (t) \right]$$

This can be rewritten in the form

$$h = - 2\gamma K(t) \left[\frac{L (\underline{x}, t)}{K (t)} + u \right] \tag{2-13}$$

where

$$L(\underline{x}, t) = \sum_{i=1}^{n} \Delta a_i (t) x_i + \left[a_{01} + K(t) \right] r + K(t) \phi (t)$$

The control signal u is taken as

$$u = L^* \gamma \tag{2-14}$$

where

$$L^* = \left| \frac{L (x, t)}{K (t)} \right|_{max}$$

The term h then becomes

$$h = - 2K(t) \left[\frac{L (\underline{x}, t)}{K (t)} \gamma + L^* \gamma^2 \right] \tag{2-15}$$

The most positive h could become is defined as h_{max}, given by

$$h_{max} = -2K(t) \left[-L^* \left| \gamma \right| + L^* \gamma^2 \right] \tag{2-16}$$

$$= -2K(t)\, L^* \left[\gamma^2 - \left| \gamma \right| \right]$$

Therefore if $|\gamma| \geq 1$, $\dot{V}$ will always be negative and V will decrease. For the case where $|\gamma| < 1$, a region in vector space is defined where $\dot{V}$ may be positive, thus a decreasing V is not assured. This region will be investigated for a second order system.

In the second order case, γ is defined as in equation (2-8). The region in the e_1, e_2 error phase plane where $|\gamma| < 1$ is shown in Figure 2. This region can be made arbitrarily narrow by chosing a Q matrix which results in a large value of B_{12} and B_{22}.

For the system being considered, L* as defined in (2-14) is written as

$$L^* = A_1 \left| x_1 \right|_{max} + A_2 \left| x_2 \right|_{max} + A_3 \left| r \right|_{max}$$

$$+ A_4 \left| \delta \right|_{max} + \left| \frac{K_d(t)}{b_1(t)} \dot{D} \right|_{max} \tag{2-17}$$

Placing bounds on x_1, x_2, r, δ, and $\dot{D}$ gives L* as a time-invariant gain acting on γ. Therefore the control signal u in (2-14) in a linear combination of the elements of $\underline{e}$. As explained previously, however, $\underline{e}$ is not directly available, the control signal, therefore, is taken as a function of $\hat{\underline{e}}$ as

$$u = L^* \hat{\gamma} \tag{2-18}$$

where $\hat{\gamma}$ is defined as in (2-11). The effect of using $\hat{\gamma}$ instead of γ is explained in the next section.

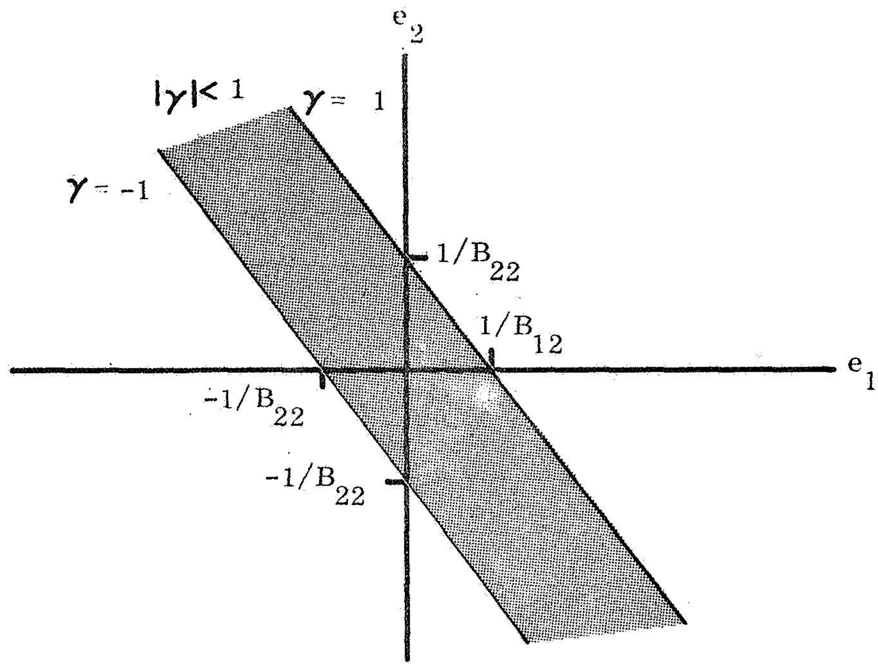


Figure 2 Region Where $|\gamma| < 1$

C - Effect of Estimation Errors

In the two previous sections it was pointed out that the errors in estimation effect the control equations. The main effect of these errors is that the control signal becomes a function of $\hat{\gamma}$ instead of γ . This effect will be analyzed below.

For the non-linear controller, the control would ideally be taken as (2-8). The sign of u is equal to the sign of γ , which is a linear function of $\underline{e}$. The equation for $\gamma = 0$ defines a line in the e_1, e_2 phase plane referred to as the "switching line". This line divides the e_1, e_2 plane into two regions where γ is less than or greater than zero as shown in Figure 3. The control signal, however, is actually taken as a function of $\hat{\gamma}$ in (2-12). The equation for $\hat{\gamma} = 0$ defines another switching line whose location is a function of the estimation

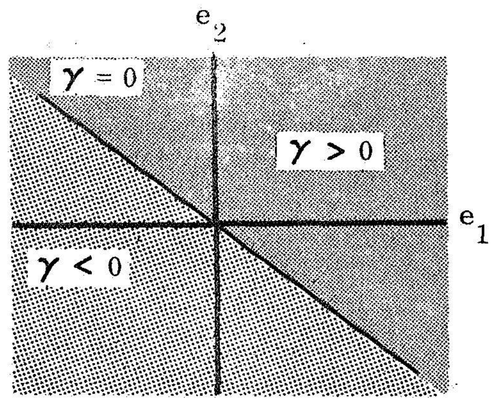


Figure 3 Switching Line

error. This relationship can be seen through (2-9) and (2-10) which yield (2-19).

$$\hat{\underline{e}} = \underline{e} + \tilde{\underline{e}} \tag{2-19}$$

The $\hat{\underline{e}}$ co-ordinates are therefore translated from the $\underline{e}$ co-ordinates by the elements of $\tilde{\underline{e}}$ as shown in Figure 4.

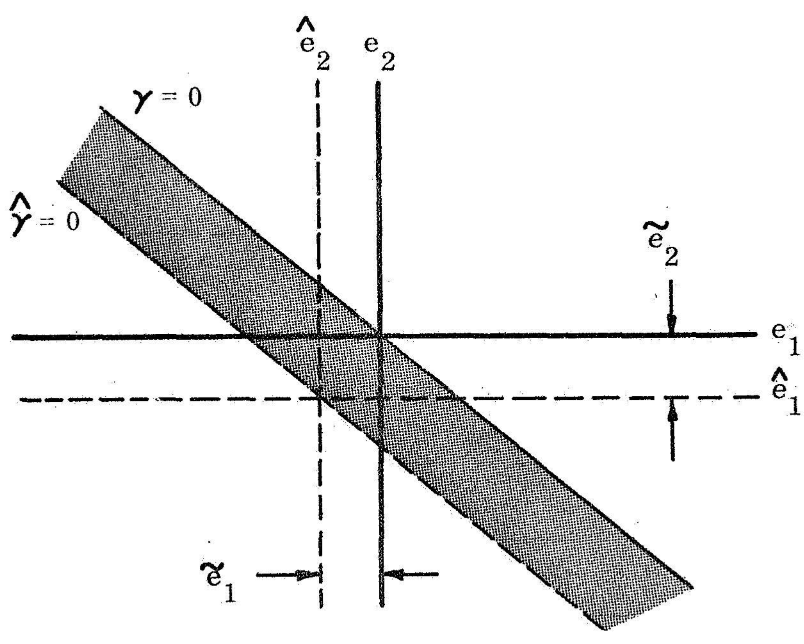
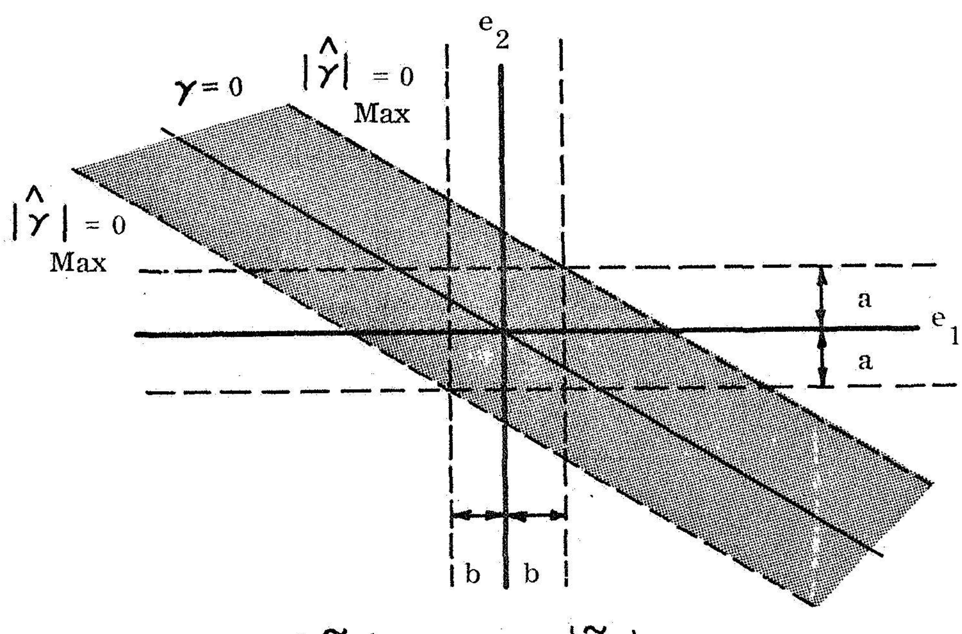


Figure 4 Effect of $\tilde{\underline{e}}$

In the region between the switching lines, the sign of the control signal u will be opposite the sign needed for convergence. For the case where the estimation error $\tilde{\underline{e}}$ is unknown but bounded, a region which contains the switching line $\hat{\gamma} = 0$ is defined as shown in Figure 5. In this region, which will be referred to as the "region of imperfect control", the sign of u may be either positive or negative and convergence of the error vector $\underline{e}$ is not assured. A similar region also occurs for the linear controller as explained below.

For the linear controller, the effect of the estimator error can be found by substituting u as defined in (2-18) into (2-13). Following the previous development, equation (2-16) becomes

$$h_{max} = -\,2K(t)\ \ L^*\left[\gamma\ \hat{\gamma} - |\gamma|\right] \qquad (2\text{-}20)$$



$$a = |\tilde{e}_2|_{Max}, \quad b = |\tilde{e}_1|_{Max}$$

$$|\hat{\gamma}|_{Max} = bB_{12} + aB_{22}$$

Figure 5 Region of Imperfect Control

From (2-20) it is seen that for h_{max} to be negative, $\hat{\gamma}$ has to be greater than one and γ and $\hat{\gamma}$ must have the same sign. The region where this may not be true for the case where $\underline{\tilde{e}}$ is unknown but bounded is shown in Figure 6.

Therefore, the effect of the estimation errors in both controllers is to cause a region around the line defined by $\gamma = 0$ where the sign of $\dot{V}$, the Liapunov function derivative, may be positive. There is a bounded region around the origin of the error plane, however, in which the error vector will ultimately be contained. This region will be investigated in the next section.

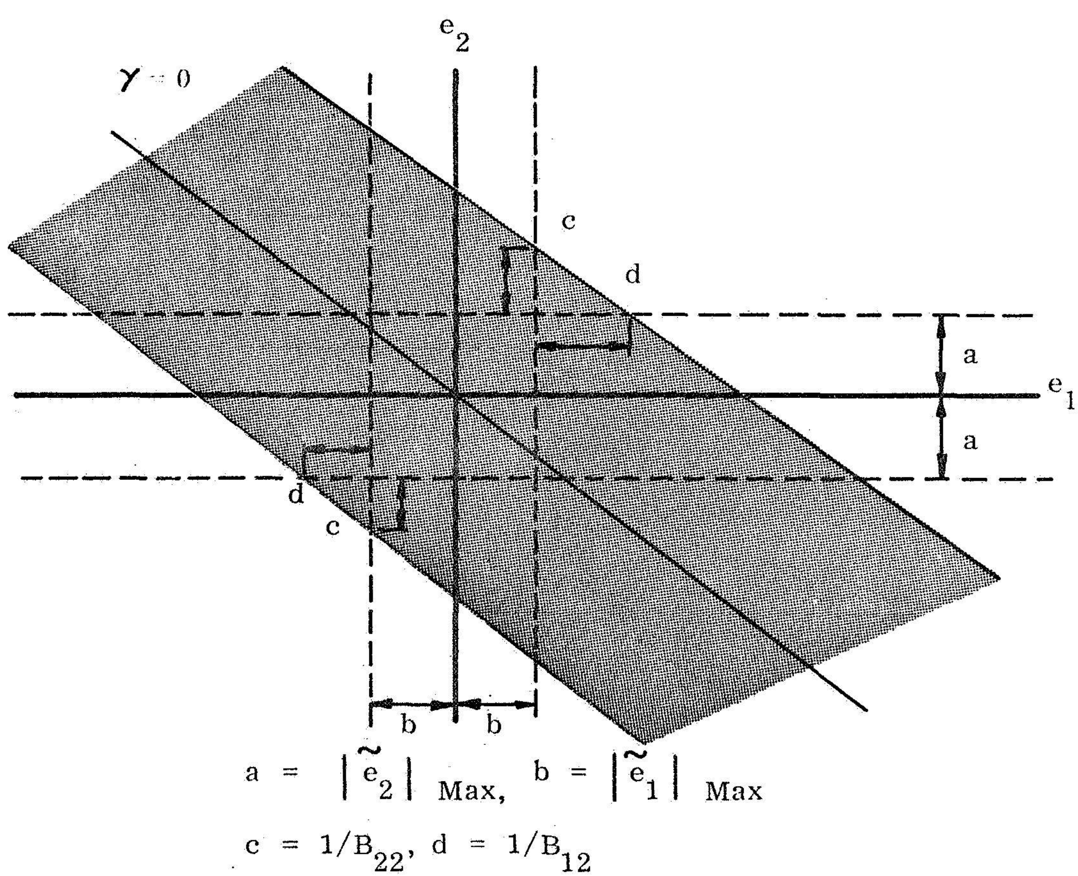


Figure 6 Region of Imperfect Control for Linear Controller

D - System Error Bound

A bound on the system error $\underline{e}$ can be found using a technique developed in Reference 2 This technique will be used below to obtain maximum values of the estimation error $\underline{\tilde{e}}$.

It is seen from Figure 5 that the region of imperfect control for the non-linear controller can be described by the equation

$$\left| \gamma \right| < \bar{L} \tag{2-21}$$

where

$$\bar{L} = B_{12} \left| \tilde{e}_1 \right|_{max} + B_{22} \left| \tilde{e}_2 \right|_{max}$$

If B_{12} and B_{22} are large, (2-21) will also approximate the region of imperfect control for the linear controller as shown in Figure 6. Substituting for the definition of γ, (2-21) can be written as

$$-\bar{L} < B_{12} e_1 + B_{22} e_2 < \bar{L} \tag{2-22}$$

Rewriting (2-22) as a constraint on e_2 yields

$$-\frac{B_{12}}{B_{22}} e_1 - \frac{\bar{L}}{B_{22}} < e_2 < -\frac{B_{12}}{B_{22}} e_1 + \frac{\bar{L}}{B_{22}} \tag{2-23}$$

If e_2 is taken as in (2-24) the inequality contraint (2-23) is satisfied.

$$e_2 = -\frac{B_{12}}{B_{22}} e_1 + \alpha(t) \tag{2-24}$$

where

$$\left| \alpha(t) \right| < \frac{\bar{L}}{B_{22}}$$

Since e_2 is the derivative of e_1, however, (2-24) is a differential equation in e_1 with α (t) acting as an unknown forcing function. The solution of (2-24) can be expressed as a constraint on e_1 as

$$\left| e_1 \right|_{max} \leq \frac{B_{22}}{B_{12}} \left| \alpha (t) \right|_{max}$$

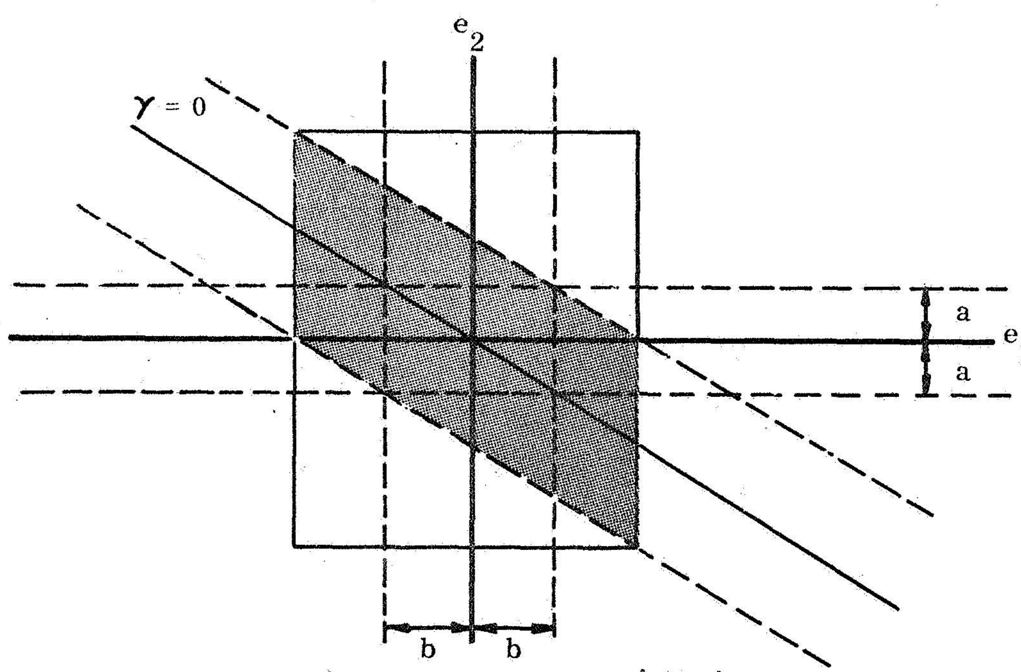
$$< \frac{\bar{L}}{B_{12}}$$

$$< \left| \tilde{e}_1 \right|_{max} + \frac{B_{22}}{B_{12}} \left| \tilde{e}_2 \right|_{max} \tag{2-25}$$

Using (2-25) in conjunction with (2-24) gives the constraint on e_2 as

$$\left| e_2 \right|_{max} < 2 \left[\frac{B_{12}}{B_{22}} \left| \tilde{e}_1 \right|_{max} + \left| \tilde{e}_2 \right|_{max} \right] \tag{2-26}$$

Equations (2-25) and (2-26) define a bounded region in the error phase plane shown in Figure 7 in which the system error vector will ultimately be contained.



$a = \left| \tilde{e}_2 \right|_{Max}$, $b = \left| \tilde{e}_1 \right|_{Max}$

Figure 7 System Error Bound

CHAPTER III

ESTIMATOR DESIGN

A - Estimator Equations

Both controller techniques described in Chapter II require the aircraft's angular rate about the pitch axis and pitch axis angular acceleration. A rate gyro is used to measure the angular rate of the aircraft, but the angular acceleration is not directly measureable. Due to structural vibration and measurement noise, the use of a differentiation circuit to obtain the angular acceleration from the output of the gyro is impractical. An estimate of this signal, however, can be obtained by applying the estimation technique described in Appendix B. This technique yields a linear filter acting on the gyro output. The gyro output signal consists of the actual aircraft rate plus noise. The output of the filter is an estimate of the angular rate and acceleration of the aircraft about the pitch axis. These estimates are used as inputs to the controller as described in Chapter II. The filter equations resulting from this technique will be described below.

The response of the aircraft to elevator inputs and gust disturbances is described by the differential equation (2-1). This is written in vector form as the set of equations (3-1).

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ K_a \dot{\delta} + \frac{K_a}{T_a} \delta + K_d \dot{D} \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x} \qquad (3\text{-}1)$$

where $\underline{x}$, a_1, and a_2 are defined as in (2-4). The first equation is now separated into nominal terms and a term consisting of variations about the nominal as shown in (3-2).

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ a_{n1} & a_{n2} \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ K_{an} \dot{\delta} + \left(\frac{K_a}{T_a}\right)_n \delta \end{bmatrix} + \begin{bmatrix} 0 \\ f_2 \end{bmatrix} \qquad (3\text{-}2)$$

where

$$f_2 = \Delta a_{n1} x_1 + \Delta a_{n2} x_2 + \Delta K_a \dot{\delta} + \Delta \frac{K_a}{T_a} \delta + K_d \dot{D}$$

$$\Delta a_{ni} = a_i - a_{ni} \qquad i = 1, 2$$

$$\Delta K_a = K_a - K_{an}$$

$$\Delta \frac{K_a}{T_a} = \frac{K_a}{T_a} - \left(\frac{K_a}{T_a}\right)_n$$

The first two terms in (3-2) are taken as a nominal vector function whose parameters are time-invariant. The third term is a vector function which accounts for parameter variations about the nominal and also accounts for the unknown disturbance input $\dot{D}$. The gyro output is written as y_g defined by

$$y_g = y + n$$

$$y_g = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x} + n \tag{3-3}$$

where n is an additive noise term.

Equations (3-2) and (3-3) are now in the form of the set of equations (B-1). Application of (B-4) yields the desired estimator equation shown below.

$$\dot{\hat{\underline{x}}} = \begin{bmatrix} 0 & 1 \\ a_{n1} & a_{n2} \end{bmatrix} \hat{\underline{x}} + \begin{bmatrix} 0 \\ K_n \dot{\delta} + \left(\frac{K_a}{T_a}\right)_n \delta \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \left(y_g - \hat{x}_1\right) \tag{3-4}$$

where

$$G_1 = 2\overline{Q} \; P_{11}$$

$$G_2 = 2\overline{Q} \; P_{21}$$

and where P_{11} and P_{21} are elements of the matrix defined by the solution of the matrix Ricatti equation

$$\dot{P} = \begin{bmatrix} 0 & 1 \\ a_{n1} & a_{n2} \end{bmatrix} P + P \begin{bmatrix} 0 & a_{n1} \\ 1 & a_{n2} \end{bmatrix} + 2QP \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P + \frac{1}{2} W^{-1} \tag{3-5}$$

The term f_2 has not been included in (3-4) since nothing is known about it which would improve the estimation.

The matrix equation (3-5) yields a set of non-linear differential equations for the elements of P. These equations have constant coefficients and the initial conditions of the P matrix may be chosen such that $\dot{P} \equiv 0$. Thus, P is constant, and the first column of P is used in the definition for G_1 and G_2, resulting in constant estimator gains. The problem of choosing the weighting matrices Q and W, however, still remains. Since this choice is somewhat arbitrary, the estimator performance will be evaluated directly as a function of the gains G_1 and G_2. These gains will then be chosen on the basis of this evaluation.

B - Frequency Domain Error Analysis

As shown in Section II-C, the errors in estimation directly affect the total system error. Therefore it is desirable to choose the estimator parameters so that these errors will be as small as possible. A measure of the estimator error is also needed for the nonlinear controller in equation (2-11). For these reasons the equations describing the errors in estimation will be derived below.

A vector differential equation for the estimation error $\underline{\tilde{e}}$ as defined in (2-9) can be obtained by subtracting (3-4) from (3-2). This yields

$$\dot{\underline{\tilde{e}}} = \begin{bmatrix} 0 & 1 \\ a_{n1} & a_{n2} \end{bmatrix} \underline{\tilde{e}} + \begin{bmatrix} 0 \\ f_2 \end{bmatrix} - \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} (y_g - \hat{x}_1) \qquad (3\text{-}6)$$

Substituting y_g as defined in (3-3) into (3-6) and combining terms gives

$$\dot{\underline{\tilde{e}}} = \begin{bmatrix} -G_1 & 1 \\ -G_2 + a_{n1} & a_{n2} \end{bmatrix} \underline{\tilde{e}} + \begin{bmatrix} 0 \\ f_2 \end{bmatrix} - \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} n \qquad (3\text{-}7)$$

Equation (3-7) represents a linear system driven by the forcing terms f_2 and n. The eigenvalues of the system are given by

$$\lambda = -\frac{\overline{G_1}}{2} \pm \frac{1}{2} \sqrt{\overline{G_1}^2 - 4\overline{G_2}} \tag{3-8}$$

where

$$\overline{G_1} = G_1 - a_{n2}$$

$$\overline{G_2} = G_2 - G_1 a_{n2} - a_{n1}$$

It is seen that for a stable estimator, both $\overline{G_1}$ and $\overline{G_2}$ must be positive. The frequency domain transition matrix of the system is given by

$$\Phi(S) = \frac{1}{S^2 + \overline{G_1} S + \overline{G_2}} \begin{bmatrix} S - a_{n2} & 1 \\ a_{n1} - G_2 & S + G_1 \end{bmatrix} \tag{3-9}$$

Assuming $\underline{\tilde{e}}(0) = 0$, $\tilde{E}(S)$ is given by

$$\tilde{E}(S) = \Phi(S) \begin{bmatrix} -G_1 N(S) \\ F_2(S) - G_2 N(S) \end{bmatrix} \tag{3-10}$$

Equation (3-10) can be rewritten in the form of (3-11).

$$\tilde{E}(S) = T(S) \begin{bmatrix} F_2(S) \\ N(S) \end{bmatrix} \tag{3-11}$$

where

$$T(S) = \frac{1}{S^2 + \overline{G_1} S + \overline{G_2}} \begin{bmatrix} 1 & -G_1 S + G_1 a_{n2} - G_2 \\ S + G_1 & -G_2 S - G_1 a_{n1} \end{bmatrix}$$

The error in estimation, therefore, is expressed in frequency domain terms by equation (3-11). The transfer function matrix T(S) is a function of the estimator gains G_1 and G_2. The effect of these gains on the estimation error will be investigated in the next section.

C - Determination of Estimator Gains

The estimator gains G_1 and G_2 will be chosen on the basis of their effect on the estimator error. As seen in (3-1), these gains affect the transfer functions relating the forcing terms f_2 and n to $\underline{\tilde{e}}$. Since the measurement noise n is usually kept at a minimal value, the term f_2, which contains the effect of parameter variations, will be treated as the primary source of the estimation error. This term is an unknown quantity and it will be assumed that it may contain frequencies up to and beyond the bandwidth of the terms of the transfer function matrix T(S). The term f_2 will therefore be treated as a white noise input to the system with G_1 and G_2 being chosen to minimize the output error variance.

The estimation error $\underline{\tilde{e}}$ as given in equation (3-11) can be written as

$$\underline{\tilde{e}} = \underline{\tilde{e}}_f + \underline{\tilde{e}}_n \tag{3-12}$$

where $\underline{\tilde{e}}_f$ is the error caused by the term f_2 and $\underline{\tilde{e}}_n$ is that caused by the noise n. The spectral densities of $\tilde{e}_{1f}$ and $\tilde{e}_{2f}$ can be written as

$$\Phi_{11f} = \left| T_{11} \; (j\omega) \right|^2 \quad \Phi_{ff} \tag{3-13}$$

$$\Phi_{22f} = \left| T_{21} \; (j\omega) \right|^2 \quad \Phi_{ff} \tag{3-14}$$

where Φ_{11f} and Φ_{22f} are the spectral densities of $\tilde{e}_{1f}$ and $\tilde{e}_{2f}$, Φ_{ff} is the spectral density of f_2, and $T_{11}(j\omega)$ and $T_{21}(j\omega)$ are the elements of the first column of T(S). Since f_2 is taken as white noise, its spectral density is constant. The variance of $\tilde{e}_{1f}$ and $\tilde{e}_{2f}$ can be written as

$$\sigma_{1f}^2 = C_f \int_{-\infty}^{\infty} \left| T_{11}(j\omega) \right|^2 d\omega \qquad (3\text{-}15)$$

$$\sigma_{2f}^2 = C_f \int_{-\infty}^{\infty} \left| T_{21}(j\omega) \right|^2 d\omega \qquad (3\text{-}16)$$

where σ_{1f}^2 and σ_{2f}^2 are the variance of $\tilde{e}_{1f}$ and $\tilde{e}_{2f}$ and C_f is the amplitude of the spectral density of f_2.

The gains G_1 and G_2 will now be chosen to minimize the sum of the two integrals in (3-15) and (3-16). Let these integrals be defined by A_{11} and A_{21} where

$$A_{11} = \int_{-\infty}^{\infty} \left| T_{11}(j\omega) \right|^2 d\omega \qquad (3\text{-}17)$$

$$A_{21} = \int_{-\infty}^{\infty} \left| T_{21}(j\omega), \right|^2 d\omega \qquad (3\text{-}18)$$

Equations (3-17) and (3-18) can be evaluated by residues giving A_{11} and A_{21} as

$$A_{11} = \frac{\pi}{\overline{G_1}\ \overline{G_2}} \qquad (3\text{-}19)$$

$$A_{21} = \frac{\pi (G_1^2 + \overline{G_2})}{\overline{G_1}\ \overline{G_2}} \qquad (3\text{-}20)$$

Defining $\bar{A}$ as the sum of A_{11} and A_{21}, then from (3-19) and (3-20) $\bar{A}$ is found as

$$\bar{A} = \frac{\pi(1 + G_1^2 + \bar{G}_2)}{\bar{G}_1\ \bar{G}_2} \qquad (3\text{-}21)$$

$$\bar{A} = \frac{\pi(1 + G_1^2)}{\bar{G}_1\ \bar{G}_2} + \frac{\pi}{\bar{G}_1}$$

As seen from (3-21), $\bar{A}$ will be minimized with respect to G_2 when G_2 approaches infinity. Setting the partial derivitive of $\bar{A}$ with respect to G_1 equal to zero yields

$$\bar{G}_1\ \bar{G}_2\ (2G_1 - a_{n2}) = (1 + G_1^2 + \bar{G}_2)(\bar{G}_2 - a_{n2}\bar{G}_1) \qquad (3\text{-}22)$$

Assuming that the terms a_{n1} and a_{n2} are small compared to G_1 and G_2, equation (3-22) gives an approximate relationship between G_1 and G_2 as

$$G_1 \approx \sqrt{G_2} \qquad (3\text{-}23)$$

Therefore if G_1 and G_2 were chosen on the basis of minimizing the error caused by f_2 alone, G_2 would be chosen as large as possible and G_1 would be chosen as the square-root of G_2. The values of G_1 and G_2, however, also affect the noise transmitted. Assuming that n is white noise, the variance of the estimation error caused by this noise is given by

$$\sigma^2_{1n} = C_n \int_{-\infty}^{\infty} \left| T_{12}(j\omega) \right|^2 d\omega \qquad (3\text{-}24)$$

$$\sigma^2_{2n} = C_n \int_{-\infty}^{\infty} \left| T_{22}(j\omega) \right|^2 d\omega \qquad (3\text{-}25)$$

where σ^2_{1n} and σ^2_{2n} are the variance of $\tilde{e}_{1n}$ and $\tilde{e}_{2n}$ and C_n is the amplitude of the noise spectral density. As before, the integrals in equations (3-24) and (3-25) are defined as A_{12} and A_{22}. Evaluation of these integrals yields

$$A_{12} = \frac{\pi}{\overline{G}_1} \left[G_1^2 + \overline{G}_2 + 2\,a_{n1} + \frac{a_{n1}^2}{\overline{G}_2} \right] \tag{3-26}$$

$$A_{22} = \frac{\pi}{\overline{G}_1} \left[G_2^2 + \frac{G_1^2\, a_{n1}^2}{\overline{G}_2} \right] \tag{3-27}$$

As seen from the above equations, A_{12} and A_{22} increase with increasing G_2. Thus G_2 cannot be arbitrarily large since the noise transmitted may be unreasonable. The estimator gains, therefore, will be chosen by selecting a large value of G_2, taking G_1 according to (3-23), and then evaluating (3-24) and (3-25) to see if the noise transmitted is acceptable.

<u>D - Choice of Nominal Parameter Values</u>

The estimator parameters a_{n1}, a_{n2}, K_{an} and $(K_a/T_a)_n$ still remain to be chosen. The values of these parameters affect the estimation error through the forcing term f_2 defined in (3-2). These parameters should be chosen to minimize this term, although it is not immediately apparent how to do so. The parameters a_{n2} and K_{an} will simply be chosen to minimize the maximum values of Δa_{n2} and ΔK_{an}. The terms a_{n1} and $(K_a/T_a)_n$, however, can be chosen on the basis of reducing the steady state error for a step input to the system as shown below.

In the case where the reference signal r is a step input to the system with no disturbances present, the steady state value of f_2 is denoted by f_{2ss} where

$$f_{2ss} = \Delta a_{n1} x_1 + \Delta \frac{K_a}{T_a} \delta \qquad (3\text{-}28)$$

Also the steady state value of δ is given by

$$\delta = \frac{x_1}{K_{ss}} \qquad (3\text{-}29)$$

where

$$K_{ss} = \frac{(K_a/T_a)}{-a_1}$$

Substituting δ from (3-29) into (3-28) along with the definitions of Δa_{n1} and $\Delta(K_a/T_a)$ yields

$$f_{2ss} = \left[-a_{n1} - \frac{(K_a/T_a)_n}{K_{ss}} \right] x_1 \qquad (3\text{-}30)$$

The terms a_{n1} and $(K_a/T_a)_n$ can then be chosen on the basis of keeping (3-30) small over the range of values of K_{ss}.

CHAPTER IV

FINAL SYSTEM EQUATIONS

A - The Estimator

The form of the estimator equation is given by (3-4) and the method of choosing the estimator parameters is given in Sections III-C and III-D. In this section the final numerical values of the estimator parameters will be given.

The estimator parameters a_{n1}, a_{n2}, K_{an} and $(K_a/T_a)_n$ are chosen as explained in Section III-D. The values of a_1, a_2, K_a/T_a, and K_{ss} for the ten cases in Table I are shown in Table II.

Table II

Case	a_1	a_2	K_a/T_a	K_{ss}
1	- 1.21	-1.40	.927	.766
2	- 7.95	-3.14	18.4	2.31
3	- 2.37	-1.33	3.70	1.56
4	-60.8	-7.21	89.9	1.48
5	-41.1	-1.99	16.2	.395
6	-15.2	- .585	1.99	.131
7	-22.8	-1.01	4.40	.192
8	-21.8	-4.97	38.2	1.75
9	- 6.10	-1.42	2.98	.489
10	-26.0	-4.39	89.7	3.45

The parameters a_{n2} and K_{an} are chosen to minimize the maximum values of Δa_{n2} and ΔK_{an}. The parameter a_2 given in Table II varies approximately from 0 to 7, therefore a_{n2} is taken as 3.5. Similarly from Table I, K_a is seen to vary from 3 to 76. The nominal estimator parameter K_{an} is therefore taken as 40.

The parameters a_{n1} and $(K_a/T_a)_n$ are chosen on the basis of keeping the bracketed term in (3-29) small over the range of K_{ss}. From Table II it is seen that K_{ss} varies from .13 to 3.5. If $(K_a/T_a)_n$ is chosen as .5, then the term $(K_a/T_a)_n/K_{ss}$ varies from .14 to 3.8. The parameter a_{n1} is then taken as -2.0 to give the least variation of $f_{2_{ss}}$.

As explained in Section III-C, the gains G_1 and G_2 should be as large as possible without resulting in an unreasonable amount of noise being transmitted. If G_1 is chosen as 1000, then the value of G_2 is taken as 32 from equation (3-23). These values and the values of a_{n1} and a_{n2} above give the natural frequency and damping ratio of the second order denominator of the elements of the transfer function matrix T(S). These are defined as ω_n and ζ_n and are found as ω_n = 33.5 rad./sec. and $\zeta_n = .53$.

The additive noise term n taken as bandlimited white noise with a standard deviation of .03 deg./sec. and a bandwidth of 750 rad./sec. The bandwidth of the noise is large enough so that the estimation error due to noise can be found from (3-24) and (3-25). The amplitude of the noise spectral density is $C_n = .762(10^{-6})$ and the terms A_{12} and A_{22} are found from (3-26) and (3-27) as $A_{12} = 188$, $A_{22} = 89(10^3)$. This gives the standard deviation of the estimation error as $\sigma_{1n} = .012$ deg./sec. and $\sigma_{2n} = .26$ deg./sec.^{2}. If the 3σ value of the error is taken as the maximum, then

$$\left|\tilde{e}_{1n}\right|_{max} = .036 \text{ deg./sec.}$$
$$\left|\tilde{e}_{2n}\right|_{max} = .78 \text{ deg./sec.}^2 \qquad (4\text{-}1)$$

Since these are acceptable errors due to noise, the estimator parameter values are taken as above.

The final estimator equation is then given by (4-2).

$$\dot{\underline{\hat{x}}} = \begin{bmatrix} 0 & 1 \\ -2 & -3.5 \end{bmatrix} \underline{\hat{x}} + \begin{bmatrix} 0 \\ 40\delta + .5\dot{\delta} \end{bmatrix} + \begin{bmatrix} 32 \\ 1000 \end{bmatrix} (y_g - \hat{x}_1) \qquad (4\text{-}2)$$

B - The Non-Linear Controller

The non-linear control equation was derived in Section II-A and is given by equation (2-12). To obtain the quantity M, bounds on the terms containing $\tilde{e}_1$, $\tilde{e}_2$, and $\dot{D}_{max}$ are needed. The method for obtaining these bounds is given below.

The estimator error caused by f_2 is given by (3-11) as

$$E_{1f}(S) = T_{11}(S)\ F_2(S)$$
$$E_{2f}(S) = T_{21}(S)\ F_2(S) \qquad (4\text{-}3)$$

The natural frequency and damping ratio of the second order denominator of the elements of T(S) were found to be $\omega_n = 33.5$ rad./sec. and $\zeta_n = .53$. The term $T_{11}(S)$ does not have any zeros and the term $T_{21}(S)$ has a first order numerator with a "break" frequency of 32 rad./sec. Examining the gain vs. frequency curves, a bound on the steady-state sinusoidal gain of $T_{11}(S)$ and $T_{21}(S)$ can be taken as 1.5x (transfer function d.c. gain). A constraint equation for $\underline{\tilde{e}}_f$ can therefore be taken as

$$\tilde{e}_{1f}(t) \leq \frac{1.5}{\bar{G}_2} \; f_2(t)$$

$$\tilde{e}_{2f}(t) \leq \frac{1.5G_1}{\bar{G}_2} \; f_2(t) \tag{4-4}$$

The term M defined in (2-12) is now written as

$$M = \left[A_1 \left| \hat{x}_1 \right| + A_2 \left| \hat{x}_2 \right| + A_3 \left| r \right| + A_4 \left| \delta \right| + \bar{M} \right] \tag{4-5}$$

where

$$\bar{M} = \left| \frac{(a_1 - a_{01})}{b_1} \; \tilde{e}_1 \right|_{max} + \left| \frac{(a_2 - a_{02})}{b_1} \; \tilde{e}_2 \right|_{max} + \left| \frac{K_d \dot{D}}{b_1} \right|_{max}$$

Using (3-12) in conjunction with (4-4) and the definition of f_2 given in (3-2), the term $\bar{M}$ can be written as

$$\bar{M} = \left\{ \frac{1.5}{\bar{G}_2 b_1} \; \left[\left| \Delta a_{c1} \right| + G_1 \left| \Delta a_{c2} \right| \right] \left[\left| \Delta a_{n1} \right| \left| x_1 \right|_{max} \right. \right. \tag{4-6}$$

$$\left. + \left| \Delta a_{n2} \right| \left| x_2 \right|_{max} + \left| \Delta Ka \right| \left| \dot{\delta} \right|_{max} + \left| \Delta \frac{Ka}{Ta} \right| \left| \delta \right|_{max} \right]$$

$$+ \left| \frac{\Delta a_{c1}}{b_1} \right| \left| \tilde{e}_{1n} \right|_{max} + \left| \frac{\Delta a_{c2}}{b_1} \right| \left| \tilde{e}_{2n} \right|_{max} + \left| \frac{Kd}{b_1} \left(\frac{1.5}{\bar{G}_2} \right. \right.$$

$$\left. \left. + 1 \right) \right| \left| \dot{D} \right|_{max} \Bigg\}$$

where

$$\Delta a_{ci} = a_i - a_{oi} \qquad i = 1, 2$$

To evaluate M, the following bounds are assumed.

$$\left| \dot{x}_1 \right|_{max} = 50 \text{ deg./sec.}, \quad \left| \dot{x}_2 \right|_{max} = 100 \text{ deg./sec.}^2$$

$$\left| \delta \right|_{max} = 50 \text{ deg.}, \quad \left| \dot{\delta} \right|_{max} = 50 \text{ deg./sec.}$$

The disturbance term D is a pertubation of the angle of attack of the aircraft due to wind gusts. The rate of change of D will be assumed to have a maximum value of $\left| \dot{D} \right|_{max} = 100$ deg./sec. The maximum values of $\tilde{e}_{1n}$ and $\tilde{e}_{2n}$ are given in (4-1) as .036 deg./sec. and .78 deg./sec.^{2}.

With the above bounds substituted into (4-6), and with the servo time constant taken as $T_h = .05$ and the model reference parameters taken as $\zeta_o = .8$ and $\omega_o = 3$, the values of $\overline{M}$ can be computed for the ten cases in Table I. Computing these values, the maximum value of $\overline{M}$ is found to be 10.6 degrees. Similarly, the values for A_1, A_2, A_3, and A_4 as defined in (2-8) were computed for the ten cases as .2, .1, 1.0 and 1.0 respectively. The final form of M can now be found from (4-5).

The only remaining parameters to be chosen in (2-12) are the terms B_{12} and B_{22} appearing in the definition of $\hat{\gamma}$. The effect of these terms on the system error can be seen from equations (2-25) and (2-26). For a given estimation error, as the ratio of B_{22} to B_{12} increases, $\left| e_1 \right|_{max}$ increases and $\left| e_2 \right|_{max}$ decreases. Since the purpose of the control technique is to control x_1 or the rate of the aircraft, e_1 is of primary interest and therefore the ratio of B_{22} to B_{12} should be taken as small as possible. Substituting the values of

a_{01} and a_{02} into the first two equations in (2-7), the ratio B_{22}/B_{12} can be found as given below.

$$\frac{B_{22}}{B_{12}} = .208 + 1.88 \frac{Q_{22}}{Q_{11}} \tag{4-7}$$

If the ratio Q_{22}/Q_{11} is kept very small, then B_{22}/B_{12} will be close to its minimum value of .208. Chossing $Q_{11} = 9.6$ and $Q_{22} = .22$ gives the values of B_{12} and B_{22} as $B_{12} = 1.0$, $B_{22} = .25$.

Substituting the expression for M and $\hat{\gamma}$ in (2-12) using the above parameters, the final form of the control signal for the non-linear controller is as follows.

$$u = \left[.2 \left| \hat{x}_1 \right| + .1 \left| \hat{x}_2 \right| + \left| r \right| + \left| \delta \right| + 10.6 \right] \text{SIGN} (\hat{e}_1 + .25 \hat{e}_2) \tag{4-8}$$

C - The Linear Controller

The linear control equation was derived in Section II-C and is given by (2-18). Substituting for $\hat{\gamma}$, this equation is written as

$$u = L^* (B_{12} \hat{e}_1 + B_{22} \hat{e}_2) \tag{4-9}$$

If, as in the previous section, $B_{22}/B_{12} = .25$, (4-9) becomes

$$u = M^* (\hat{e}_1 + .25 \hat{e}_2) \tag{4-10}$$

where

$$M^* = L^* B_{12}$$

The term L^* is defined in (2-17). The bounds on x_1, x_2, $\dot{\delta}$, and D are the same as in the previous section and the bound on $\dot{r}$ is taken as 50 deg./sec.

The term L* is then found by taking its maximum value for the ten cases in Table I. The resultant value is found to be 108 degrees. If the gain M* is taken as 1000, B_{12} is then equal to 9.3. Therefore, as shown in Figure 2, the region of imperfect control due to the controller is narrow even for this extreme case. Hence, the system error depends mainly on the estimation error as shown in Figure 7.

The final control equation for the linear controller is then taken as

$$u = 1000\ (\hat{e}_1 + .25\ \hat{e}_2) \tag{4-11}$$

CHAPTER V

EXPERIMENTAL RESULTS

The system was simulated on an analog computer using the estimator described by equation (4-2) and using both the non-linear and linear controllers given in equations (4-8) and (4-11). The ten cases of parameter variations given in Table I were run for both step and sinusoidal reference inputs. The disturbance response was simulated using an impulsive gust input and also a turbulence input consisting of bandlimited random noise. In all cases the results of the non-linear and linear controllers was almost identical, and therefore only one set of results is shown.

The uncontrolled plant step responses are shown in Figures 8 and 9 for the ten cases to illustrate the wide variation caused by the plant's changing parameters. Figures 10 and 11 show the controlled plant response to a step input and Figures 12 and 13 show the response to a sinusoidal input. It is seen that in both cases the plant output follows the model reference output very closely. The disturbance response is shown in Figures 14 and 15. Figure 14 shows the response to an approximate impulse disturbance modeled by the disturbance input

$$D = \alpha_o (1 - e^{-t}) \qquad (5-1)$$

where α_o is the initial angle of attack of the aircraft which was taken as 5 degrees. Figure 15 shows the response to an air turbulence type disturbance

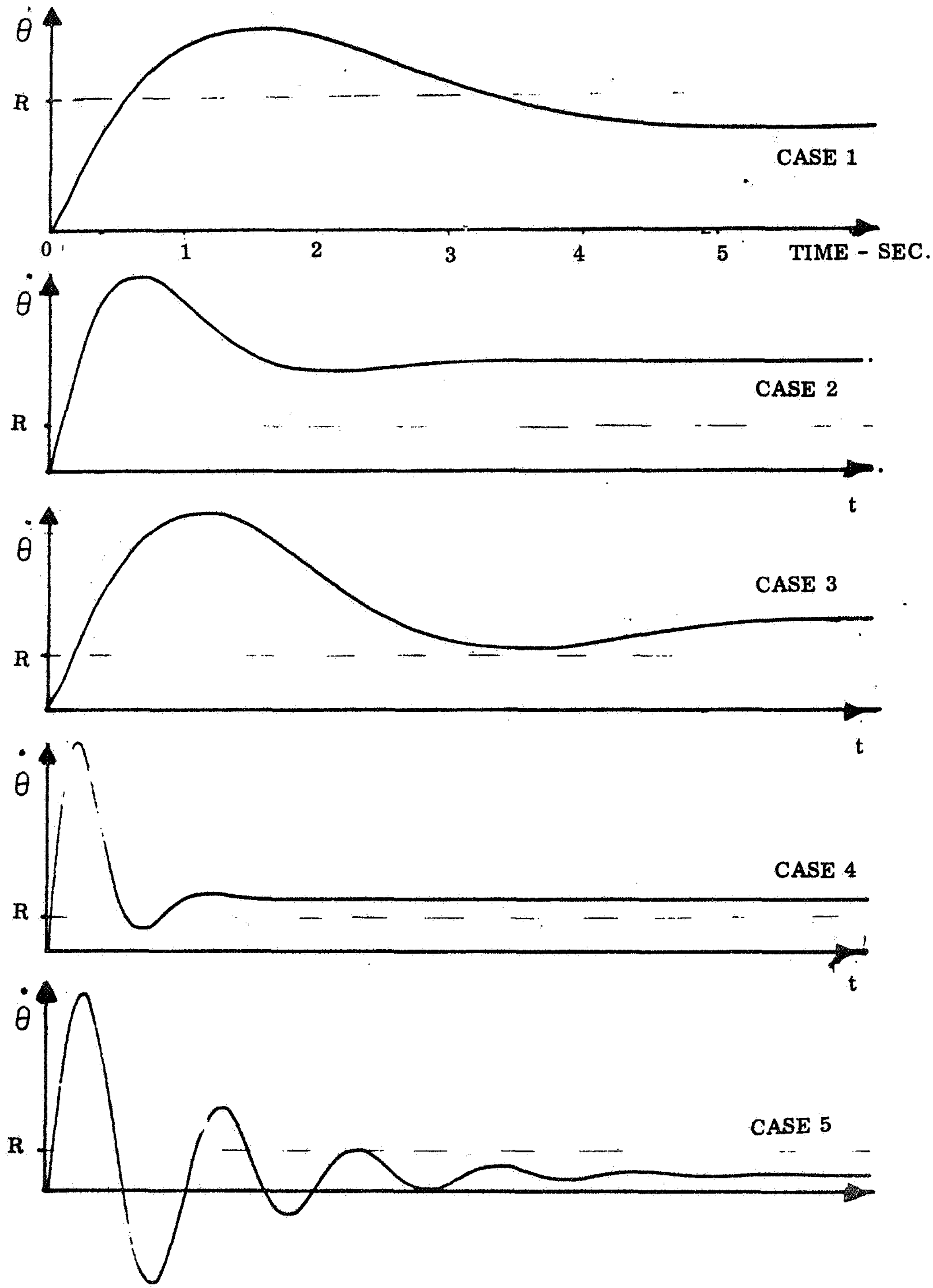


Figure 8 Uncontrolled Plant Response

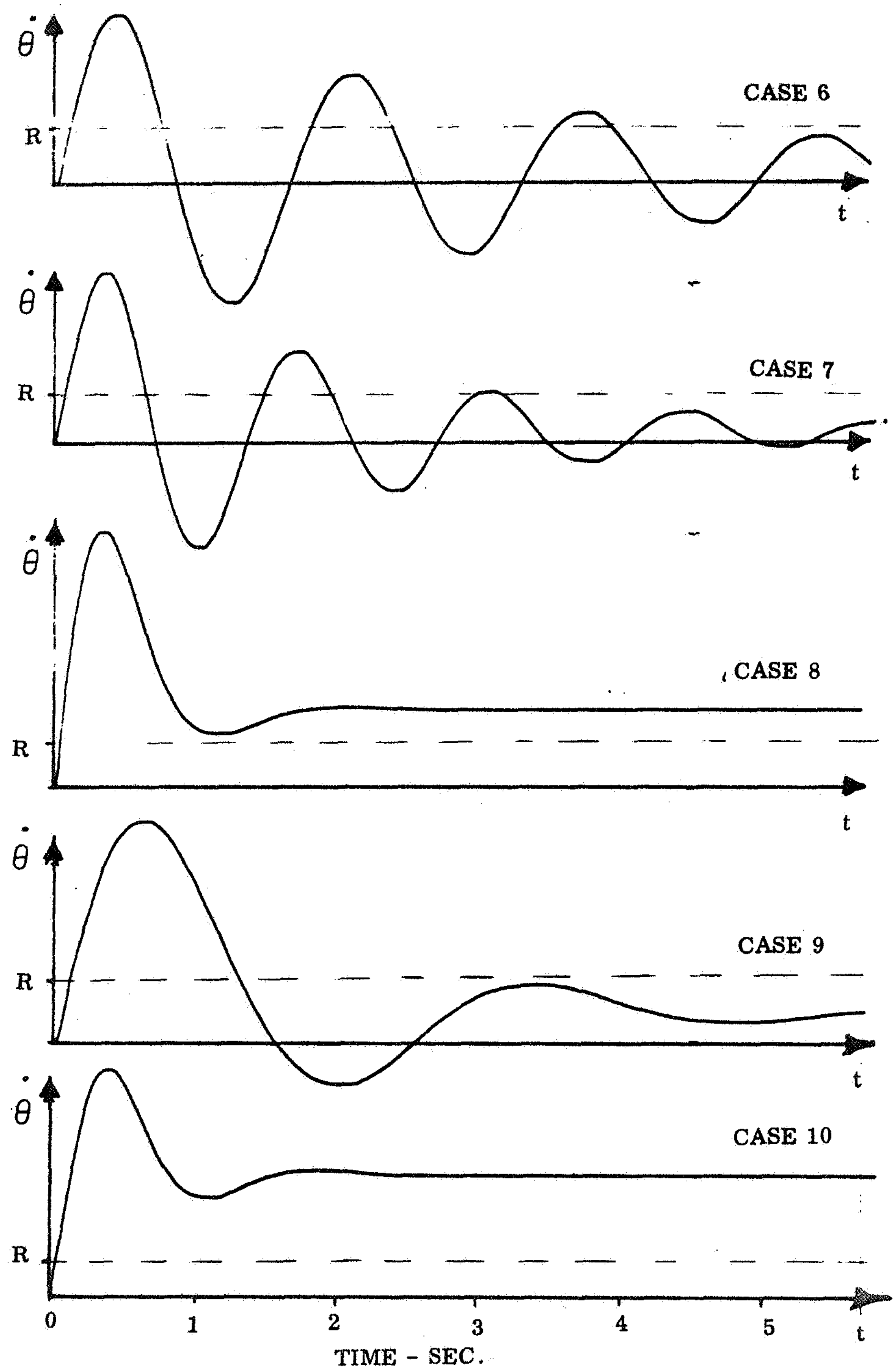


Figure 9 Uncontrolled Plant Response

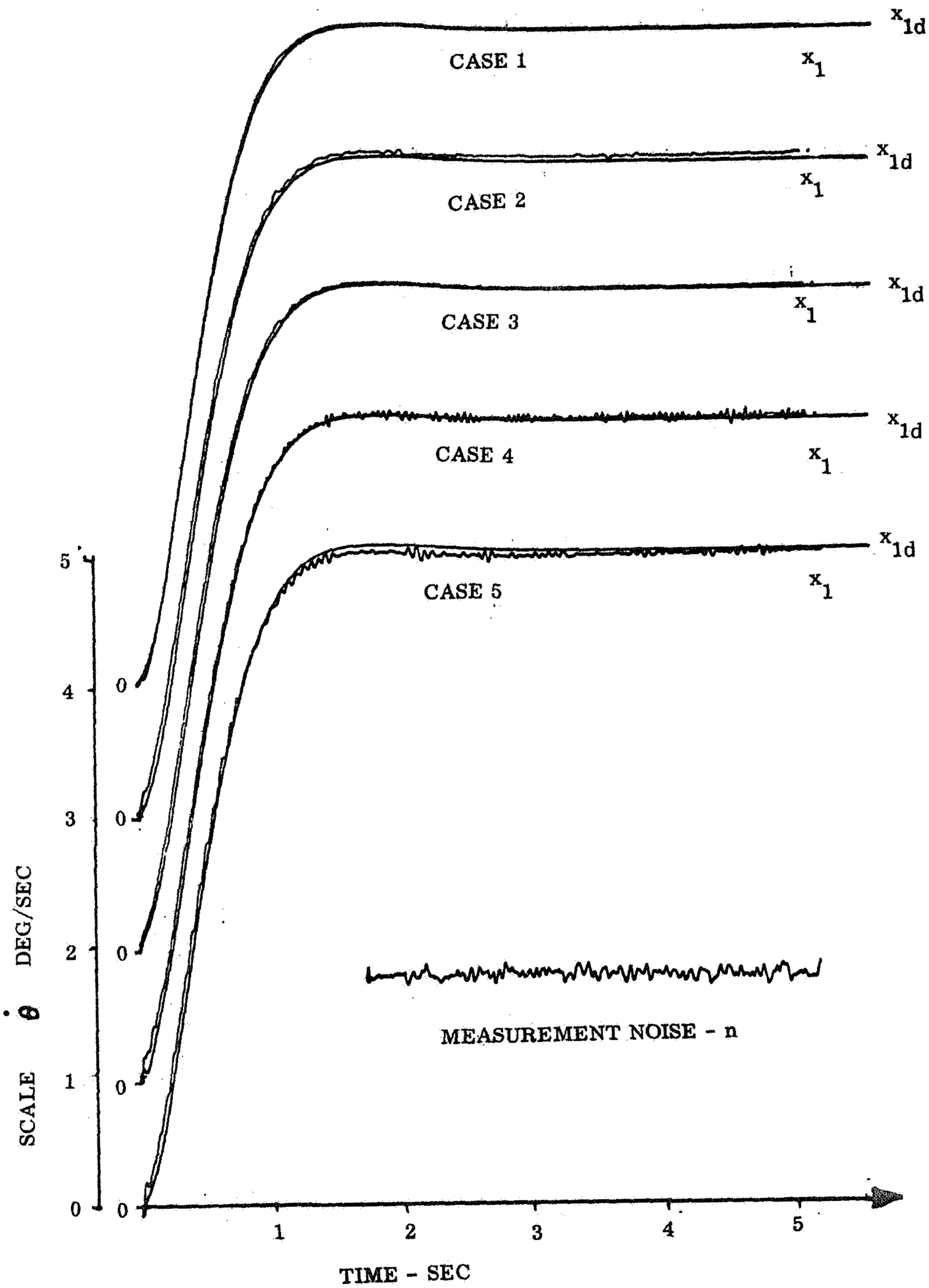


Figure 10 Controlled Plant Response - Step Input

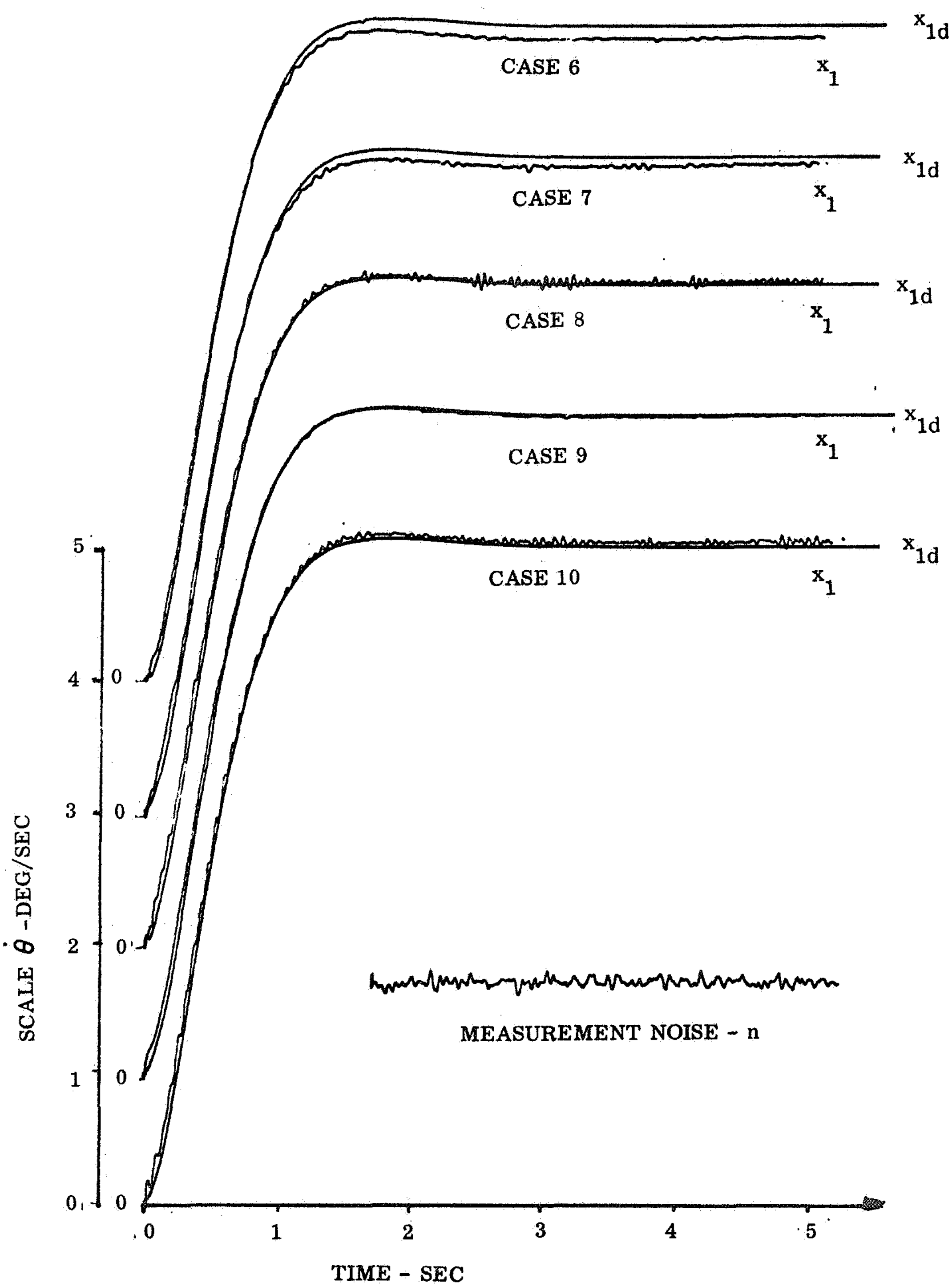


Figure 11 Controlled Plant Response - Step Input

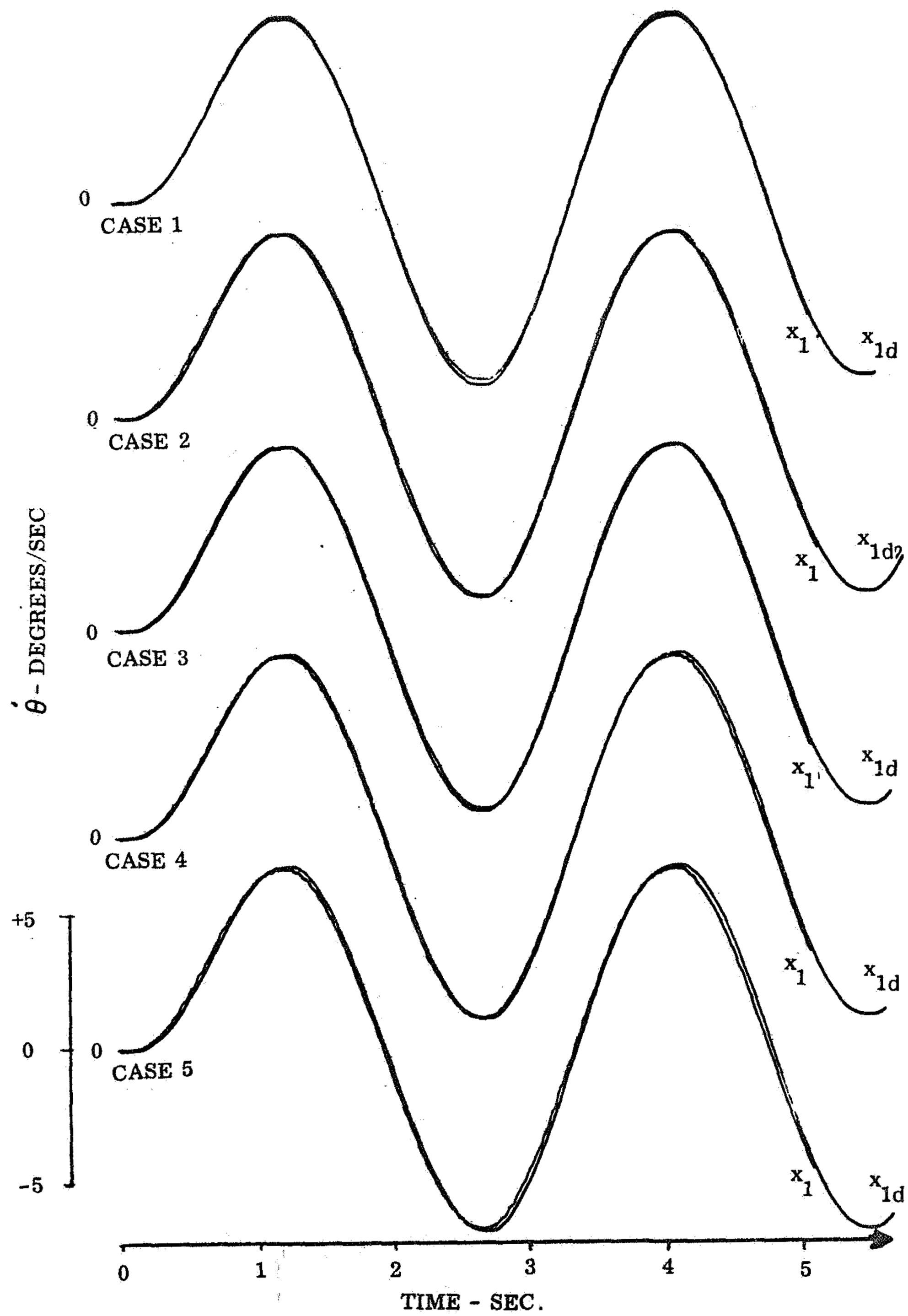


Figure 12 Controlled Plant Response - Sinusoidal Input

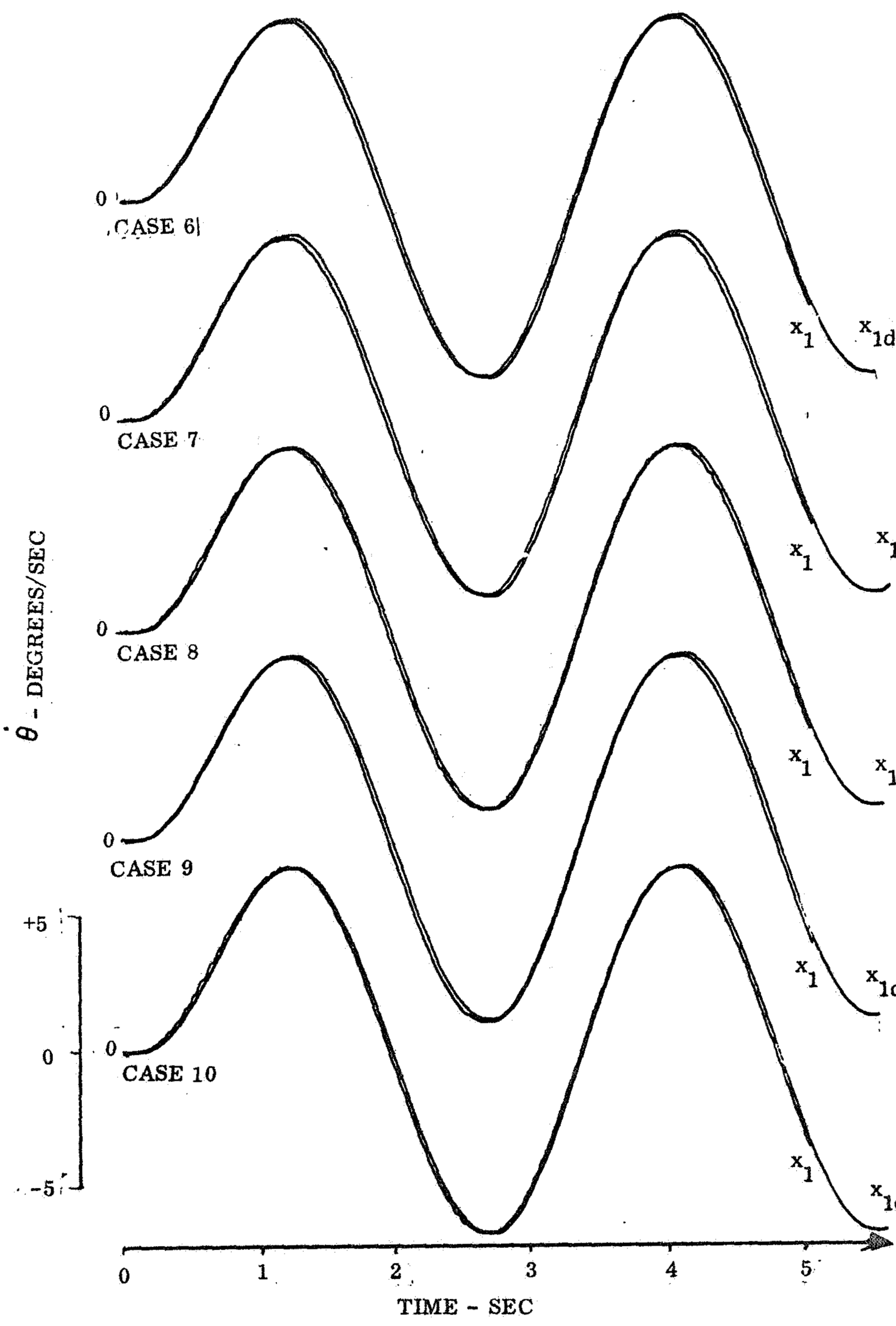


Figure 13 Controlled Plant Response - Sinusoidal Input

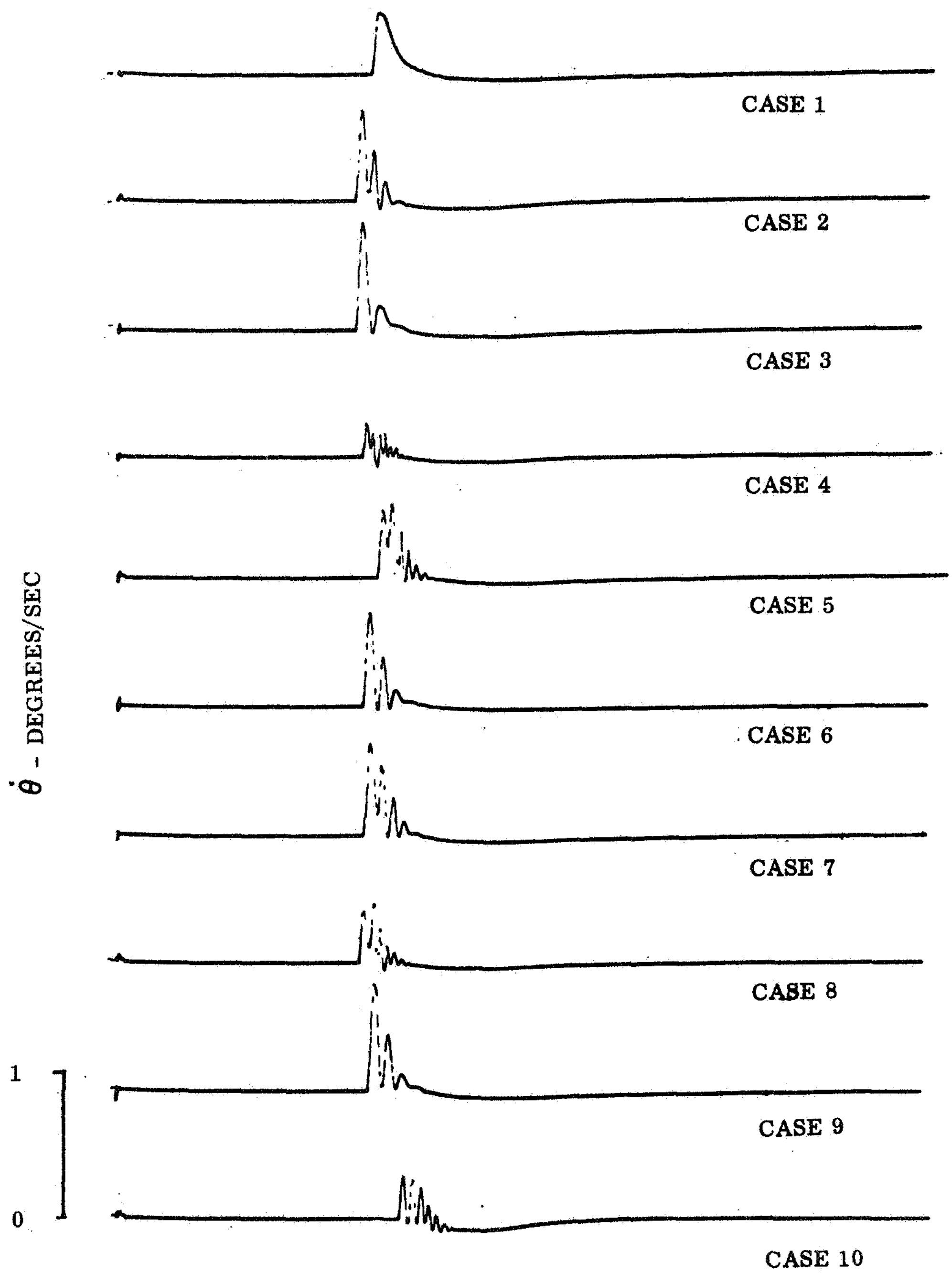


Figure 14 Impulsive Disturbance Response

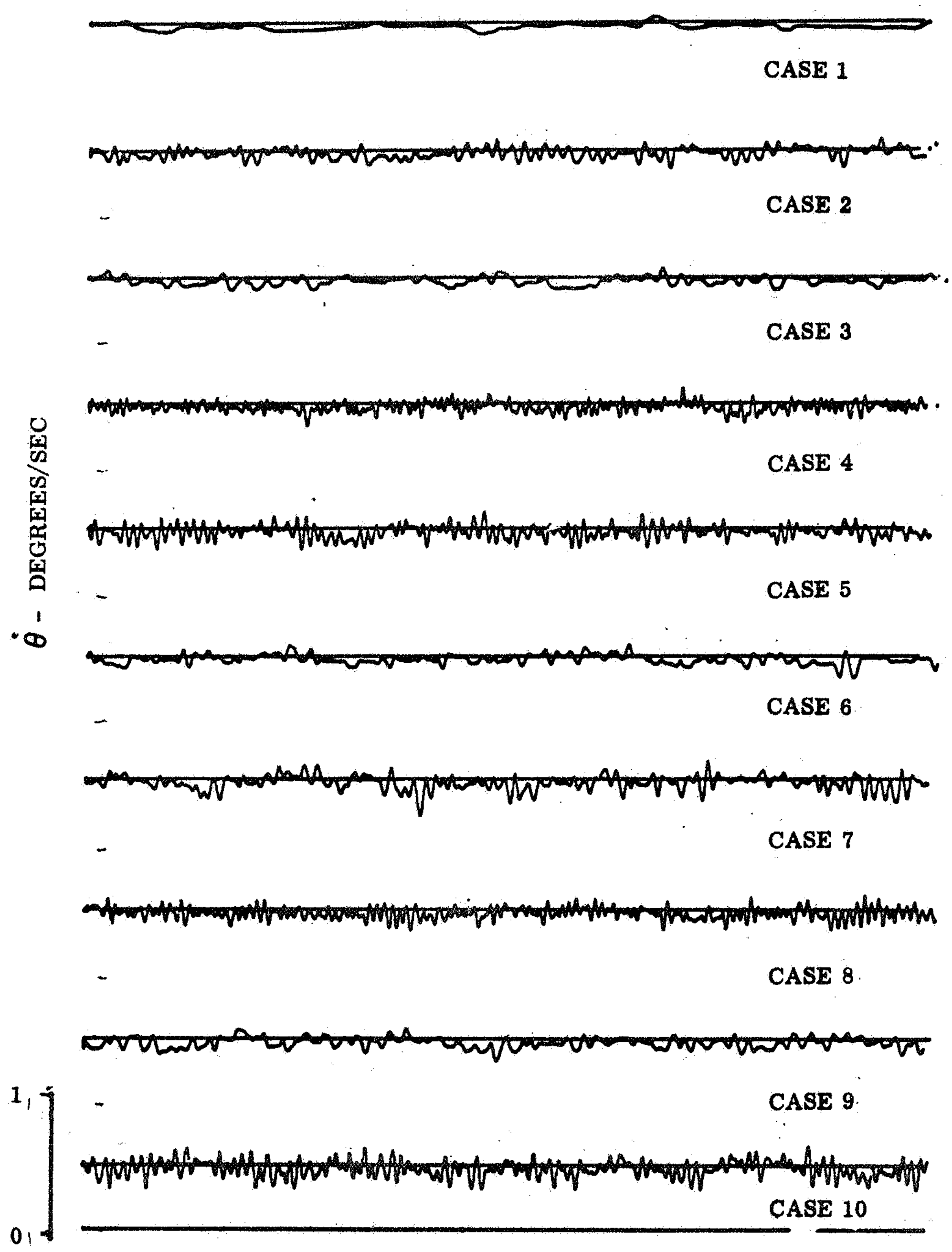


Figure 15 Turbulence Response

which was modeled as band-limited white noise with a standard deviation of 1 degree and a bandwidth of 1 rad./sec. The results show that in all cases the disturbances are quickly damped and there is no excessive response due to turbulence.

CHAPTER VI

EFFECT OF HIGHER ORDER DYNAMICS

In the system as formulated in Chapter I, the servo actuator was represented as a first-order system and the dynamics of the rate gyro were ignored. These approximations are valid for low gain systems where any higher order dynamics will not affect system stability. However, as seen by the linear control law given by (4-11), the linear system as designed incorporates a high gain feedback. This results in stability problems if the higher order effects are included in the system model and it requires that additional compensation be included in the system. To determine this compensation, the system using the linear control law will be analyzed in the frequency domain using higher order models of the actuator and gyro.

The system using the linear control law can be represented in block diagram form as shown in Figure 16, where $G_h(S)$, $G_g(S)$, and $G_a(S)$ are the servo actuator, gyro, and aircraft transfer function respectfully. To determine the estimator transfer function, the time domain estimator equation is transformed into the frequency domain. The matrix equation (4-2) yields

$$\dot{\hat{x}}_1 = \hat{x}_2 + 32\ (y_g - \hat{x}_1) \tag{6-1}$$

$$\dot{\hat{x}}_2 = -2\hat{x}_1 - 3.5\hat{x}_2 + 40\ \dot{\delta} + .5\delta + 1000\ (y_g - \hat{x}_1) \tag{6-2}$$

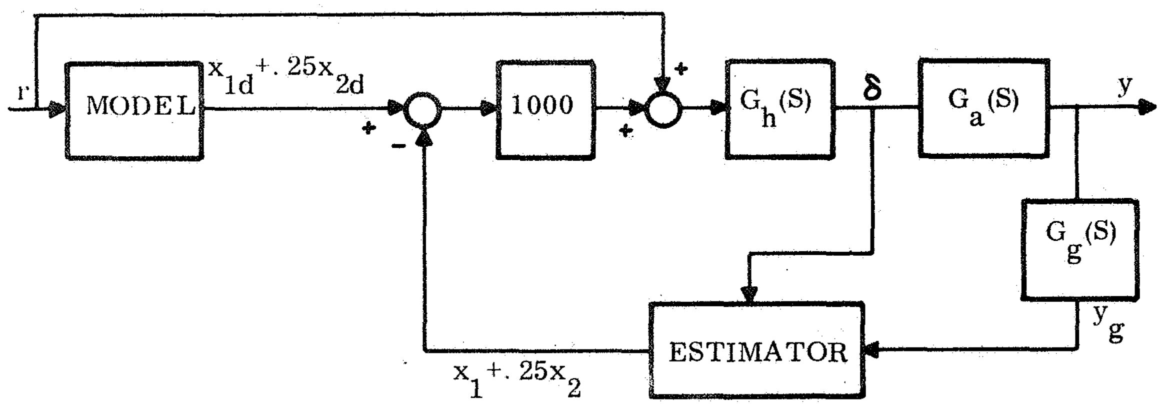


Figure 16 Linear System

Conversion of (6-1) and (6-2) from the time domain into the frequency domain gives the transfer functions relating $\hat{\underline{x}}$ to y_g and δ as

$$\hat{x}_1(S) = \frac{(32S + 1112)\, Y_g(S) + (40S + .5)\, \delta(S)}{S^2 + 35.5S + 1114} \tag{6-3}$$

$$\hat{x}_2(S) = \frac{(1000S - 64)\, Y_g(S) + (40S^2 + 1280S + 16)\, \delta(S)}{S^2 + 35.5S + 1114} \tag{6-4}$$

The relation between $Y_g(S)$ and $\delta(S)$ is

$$Y_g(S) = \left[G_a(S)\, G_g(S)\right] \delta(S) \tag{6-5}$$

Substituting (6-5) into (6-3) and (6-4), the output of the estimator as shown in Figure 16 can be written as

$$\hat{x}_1(S) + .25\hat{x}_2(S) = H(S)\, \delta(S) \tag{6-6}$$

where

$$H(S) = \frac{(282S + 1096)\ G_g(S)\ G_a(S) + 10S^2 + 360S + 4.5}{S^2 + 35.5S + 1114}$$

The block diagram in Figure 16 is now rewritten as shown in Figure 17.

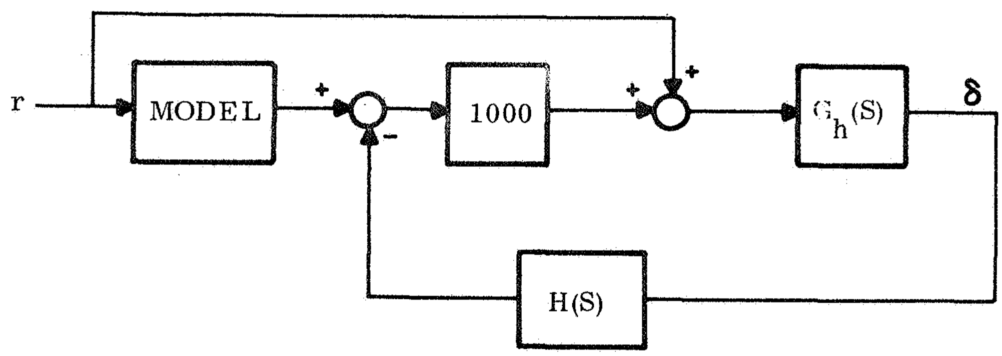


Figure 17 Linear System

The stability of the system shown in Figure 17 is determined by the loop gain function $1000G_h(S)\ H(S)$. The transfer functions for the actuator and gyro are taken as

$$G_h(S) = \frac{4900}{(.05S + 1)\ (S^2 + 70S + 4900)} \qquad (6\text{-}7)$$

$$G_g(S) = \frac{22500}{S^2 + 150S + 22500} \qquad (6\text{-}8)$$

The second order term of the actuator transfer function and the gyro transfer function were both obtained from a linear design technique presented in Reference [6]. Using (6-7), (6-8) and H(S) as defined in (6-6), the loop gain function is given by (6-9).

$$L(S) = 1000G_h(S)H(S)$$

$$L(S) = \frac{\left\{(1000)(4900)\left[(10S^2 + 360S + 4.5)(S^2 + 150S + 22500)(S^2 + 2\zeta_a\omega_a S + \omega_a^2) + (22500)K_a(S + 1/T_a)(282S + 1096)\right]\right\}}{\left\{(.05S + 1)(S^2 + 70S + 4900)(S^2 + 150S + 22500)(S^2 + 35.5S + 1114)(S^2 + 2\zeta_a\omega_a S + \omega_a^2)\right\}} \quad (6\text{-}9)$$

where K_a, T_a, ω_a, ζ_a are defined as in (1-1).

The magnitude and phase plots of L(S) for Cases 1 and 4 are shown in Figure 19. For frequencies above 10 rad. / sec., the phase plots for the other eight cases fall within the phase plots of the two cases shown. From this figure it is obvious that the system as it now stands is unstable. To remedy this situation, a compensator is included in the system as shown in Figure 18.

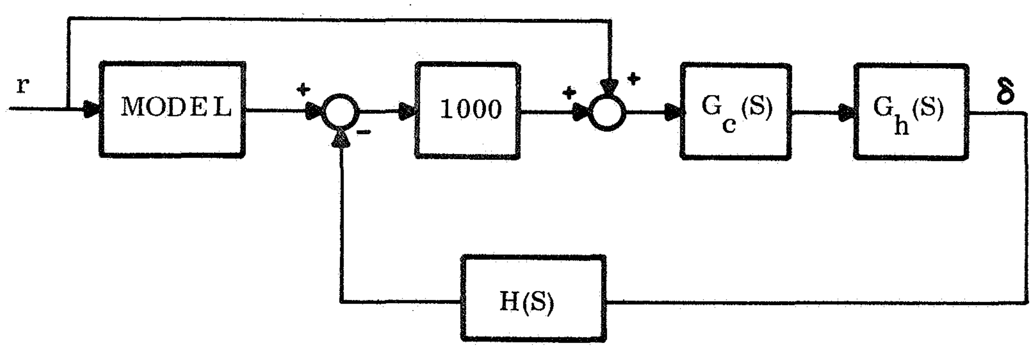


Figure 18 Compensated System

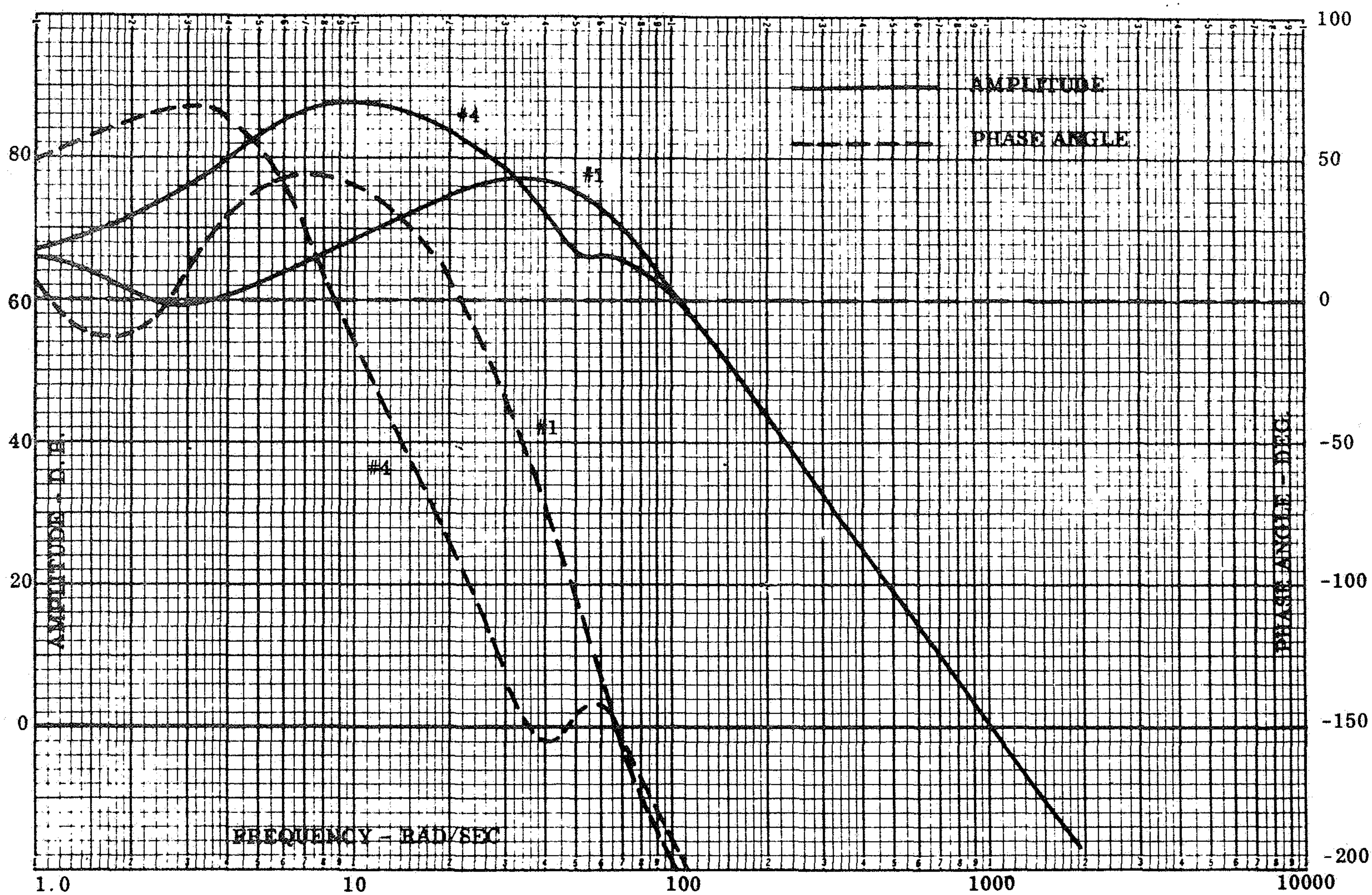


Figure 19. Magnitude and Phase For Uncompensated System

The compensator, $G_c(S)$, is taken as

$$G_c(S) = \frac{\left(\frac{S}{150} + 1\right)^2 \left(\frac{S}{5000} + 1\right)}{\left(\frac{S}{1200} + 1\right)\left[\frac{S^2}{(30,000)^2} + \frac{2(.5)}{30,000} S + 1\right]} \qquad (6\text{-}10)$$

The gain and phase plots of the open loop compensated system are shown in Figure 20. It is seen that the system is now stable with adequate phase margin. Also, since the gain and phase angle are hardly affected below 50 rad., this additional compensation should not appreciably affect the response of the system.

As shown above, when higher order dynamics are considered, the system using the linear control law can be made stable by addition of proper compensation. In the case of the non-linear controller, however, the required compensation cannot be found through linear analysis. To determine the stability characteristics of the non-linear system, the system would probably have to be simulated with the higher order dynamics included in the simulation. Proper compensation would then be sought through experimental means.

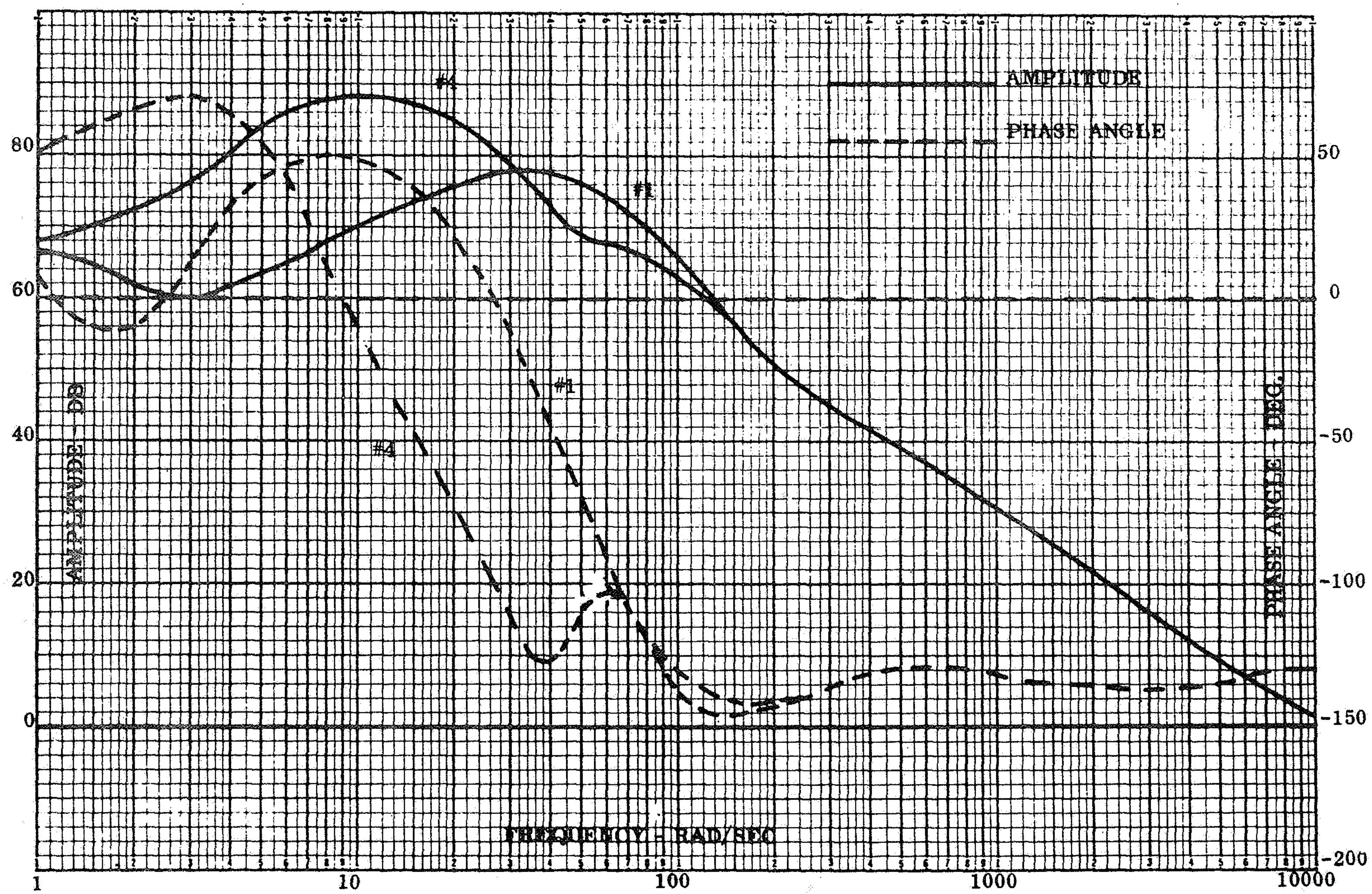


Figure 20. Magnitude and Phase for Compensated System

CHAPTER VII

CONCLUSIONS

In this report two designs for a flight control system were presented; one using a linear control law and the other using a non-linear control law. Both designs employ a linear estimator to obtain an estimate of pitch rate and its derivative for use in the control laws. Using approximate models for the actuator and gyro, both designs yielded excellent results. It was found, however, that stability problems existed if the higher order dynamics of the actuator and gyro were considered. In the case of the linear system, a compensator was included in the system to obtain stability. No stability analysis was performed on the non-linear system, however, and the system would have to be simulated to determine its stability characteristics. This stability problem necessitates that topics for further investigation include other higher order effects, such as aircraft bending modes, which could affect the stability of the system.

APPENDIX A

CONTROL TECHNIQUE EMPLOYING LIAPUNOV'S DIRECT METHOD

The control technique given in Reference [1] as applied to linear systems is outlined below.

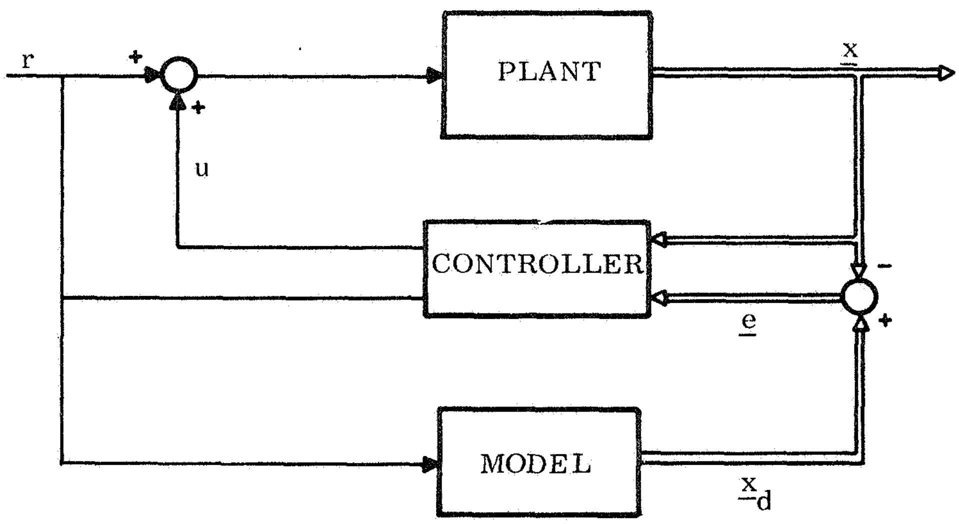


Figure A-1 Block Diagram of Control Technique

The plant in Figure A-1 is described by the vector differential equation

$$\dot{\underline{x}} = A(t)\underline{x} + \underline{b}\ f \tag{A-1}$$

where $\underline{x}$ is an n-vector, f is a scalar function containing the control signal u and the reference signal r, $\underline{b}^T = [0, \ 0, \ \cdot\ \cdot, \ 1]$, and A(t) is an nxn matrix of the form

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1(t) & a_2(t) & a_3(t) & \cdot & \cdot & a_n(t) \end{bmatrix}$$

The control objective is to force the plant to behave like the model, which is in turn described by the n-vector differential equation

$$\dot{\underline{x}}_d = A_o \underline{x}_d + \underline{b}_o r \qquad \text{(A-2)}$$

where A_0 is a time-invariant nxn stability matrix of the same form as A(t) with the last row consisting of elements $a_{01}, a_{02}, \cdots, a_{0n}$ and where $\underline{b}_0^T = \left[0, 0, \cdots, -a_{01}\right]$.

An error vector is defined as

$$\underline{e} = \underline{x}_d - \underline{x} \qquad \text{(A-3)}$$

Equation (A-3) along with (A-1) and (A-2) yields the error vector differential equation

$$\dot{\underline{e}} = A_o \underline{e} - \Delta A(t)\underline{x} + \underline{b}_o r - \underline{b} f \qquad \text{(A-4)}$$

where

$$\Delta A(t) = A(t) - A_o$$

The Liapunov function

$$V = \underline{e}^T B \underline{e} \qquad \text{(A-5)}$$

is associated with (A-4). The time derivative of V is found to be

$$\dot{V} = -e^T Qe + h \qquad \text{(A-6)}$$

where

$$-Q = A_o^T B + BA_o$$

$$h = 2\underline{e}^T B \left[-\Delta A(t)\, \underline{x} + \underline{b}_o r + \underline{b} f \right]$$

If A_0 is a stable matrix, and if Q is chosen as a positive matrix, then B is also positive definite. This makes V a positive definite function; thus if h can be maintained nonpositive by the choice of u, $\dot{V}$ will be negative definite and $\underline{e}$ will approach zero. The term h can be expressed as

$$h = -2\gamma \left[\sum_{i=1}^{n} \Delta a_i(t) x_i + a_{01} r + f \right] \qquad \text{(A-7)}$$

where

$$\Delta a_i(t) = a_i(t) - a_{oi}$$

$$\gamma = \sum_{i=1}^{n} B_{in} e_i$$

Also, for plants with linear gains, the function f may be written in the form

$$f = K(t) \left[(u + r) + \phi(t) \right] \qquad \text{(A-8)}$$

where K(t) is a time varying gain and ϕ (t) is a generalized function which includes the remaining terms in f. Substituting for f in (A-7)

$$h = -2\gamma \left[\sum_{i=1}^{n} \Delta a_i(t) x_i + a_{01} r + K(t)(u + r) + K(t)\phi(t) \right] \qquad \text{(A-9)}$$

The term h can be maintained non-positive if u is taken as

$$u = \left[X_1 + X_2 + X_3 \right] \text{ SIGN } \gamma$$

where

$$X_1 = \sum_{i=1}^{n} \left| \frac{\Delta a_i(t)}{K(t)} \right|_{Max} \left| x_i \right|$$

$$X_2 = \left| \frac{K(t) + a}{K(t)} \right|_{Max} \left| r \right|$$

$$X_3 = \left| \phi(t) \right|_{Max}$$

APPENDIX B

SEQUENTIAL LEAST SQUARES ESTIMATION TECHNIQUE

The estimation technique reported in Ref. [3] and extended in Ref. [4] and [5] is outlined below.

Given a system described by the vector differential equations

$$\dot{x} = g_0 (t, x) + \Delta g(t, x) + K(t, x)\, u(t) \qquad \text{(B-1)}$$

$$y(t) = h(t, x) + v(t)$$

where x is an n-vector

$g_0 (t, x)$ is a nominal vector function

$\Delta g(t, x)$ is a vector function whose variation with time is unknown

u(t) is a p-vector unknown input

K(t, x) is an nxp vector function whose variation with time is unknown

y(t) is an m-vector output

h(t, x) is an m-vector function

v(t) is an m-vector of measurement errors

The problem is to obtain an optimum estimate of the n-vector x. The estimator is chosen to satisfy the equation

$$\dot{\hat{x}} = g_0 (t, \hat{x}) + w(t, \hat{x}, y) \qquad \text{(B-2)}$$

where $\hat{x}$ is an optimum estimate of x and w is the estimator input to be determined. The vector $\hat{x}$ is optimum in the sense that it minimizes the cost functional

$$J = \int_{0}^{t_f} (e_1^T \bar{Q} e_1 + e_2^T W e_2) \; dt \qquad \text{(B-3)}$$

where

$$e_1(t) = y - h(t, \hat{x})$$

$$e_2(t) = f(t, \hat{x}) - w(t, \hat{x}, y)$$

$$f(t, \hat{x}) = \Delta g(t, \hat{x}) + K(t, \hat{x}) \; u(t)$$

and where $\bar{Q}$ and W are weighting matrices. The problem now is to minimize J with respect to w, subject to the estimator constraint equation. This variational problem is solved using Pontryagin's maximum principle where a Hamiltonian is maximized with respect to w. This results in a two-point boundary value problem which is solved using invariant imbedding. This technique yields the solution

$$\dot{\hat{x}}(t) = g_0(t, \hat{x}) + 2P(t) \, H(t, \hat{x}) \, Q \left[y(t) - h(t, \hat{x}) \right] + f(t, \hat{x}) \qquad \text{(B-4)}$$

where

$$H(t, \hat{x}) = \left[\frac{\partial h(t, \hat{x})}{\partial \hat{x}} \right]^T$$

and where P (t) is an nxn matrix defined by the matrix Riccati equation

$$\dot{P} = g_{0, \hat{x}}(t, \hat{x}) \, P + P g_{0, \hat{x}}^T(t, \hat{x}) +$$

$$2P \frac{\partial}{\partial x} \left\{ H(t, \hat{x}) \, Q \left[y(t) - h(t, \hat{x}) \right] \right\} P + \frac{1}{2} W^{-1}$$

where

$$g_{0,\hat{x}}(t, \hat{x}) = \frac{\partial g_0(t, \hat{x})}{\partial \hat{x}}$$

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