A Differential Calculus for Multifunctions

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1. Introduction.

Let E and F be nonempty sets. A multifunction $\Omega: E \to F$ is a subset of $E \times F$ with domain equal to E; equivalently Ω is a mapping (or function) from E into the collection of nonempty subsets of F. Multifunctions have many diverse and interesting applications in control problems and the theory of contingent equations (for example, see [B-4-6, C-1-3, F-1, H-1-4, K-1-3, L-1-2, and O-2]), in mathematical economics [A-1, D-1], and in various branches of analysis (for example in the study of subdifferentials of convex functions [M-3]). By now the theory of integration of multifunctions has been rather well developed and the applications of this theory to control problems and mathematical economics have been discussed [A-1, C-1, D-1, H-1-2, H-4, J-1-2, O-1-2]. It is our purpose in this paper to develop a differential calculus for a reasonably generous class of multifunctions, and to point out some of the applications. The calculus is developed by taking advantage of some ideas used in [D-1], especially Rådström's embedding theorem [R-1], to give our definition of the derivative of a multifunction. By means of Rådström's embedding principle we are able to convert the discussion into one concerning differentials of ordinary functions $f: E \to F$ where E and F are normed linear spaces [D-2]. At least two steps have been taken toward developing a differential calculus for multifunctions, one by Bridgland [B-4] and another by Hukuhara [H-4]. Our theory subsumes that of Hukuhara and Bridgland. A discussion of their results and a comparison with those of this paper are given in Section 4, and some examples are included to illustrate differences. In Section 2 we give the notation and terminology to be employed throughout the paper. Also in this section is a description of those aspects of Rådström's embedding operation which we shall need later on. A number of examples of differentiable multifunctions are presented in Section 3, and finally in Section 5 we give some applications.

2. Preliminaries.

Let F be a real normed linear space. The symbol $\mathcal{B}(F)$ will be used to denote the collection of all nonempty, closed, bounded, and convex subsets of F. Whenever the normed linear space F is understood we shall just suppress F and write $\mathcal{B}$ for $\mathcal{B}(F)$. If A and B are subsets of F, there is defined $A+B \equiv \{a+b \mid a \in A, b \in B\}$ and $\lambda A \equiv \{\lambda a \mid a \in A\}$ where $\lambda \in R$ and R denotes the field of real numbers. The symbol $co(A)$ denotes the convex hull of A, for $A \subset F$. If F is reflexive[1], then $\mathcal{B}(F)$ with the addition defined above is a commutative semigroup which satisfies the cancellation law [R-1]. Moreover, if α,β are real scalars, $A,B \in \mathcal{B}(F)$, then

$$\alpha(A+B) = \alpha A+\alpha B, \quad \alpha(\beta A) = (\alpha\,\beta)A, \quad 1A = A,$$

and if $\alpha,\beta \geq 0$, then $(\alpha+\beta)A = \alpha A+\beta A$. Note that the assumption that F is reflexive is used to show that $A,B \in \mathcal{B}(F)$ implies $A+B \in \mathcal{B}(F)$, and the convexity of the elements of $\mathcal{B}(F)$ is used both in the proof of the cancellation law and in the proof of $(\alpha+\beta)A = \alpha A+\beta A$, $\alpha,\beta \geq 0$. Moreover, the proof of the cancellation law also uses the fact that elements of $\mathcal{B}(F)$ are closed and bounded subsets of F.

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[1] In the results that follow the requirement that F be reflexive can be replaced by the assumption that F is a B-space if we agree to deal only with the subcollection $\mathcal{K}(F)$ consisting of those elements of $\mathcal{B}(F)$ which are compact. Also the completeness of F intervenes only when we want $\mathcal{B}(F)$ to be complete.

If X and Y are sets, if $H \subset X \times Y$, and if $A \subset X$, then $H[A]$ denotes the set $\{y \in Y \mid \exists\, x \in A\colon (x,y) \in H\}$. Let (X,ρ) be a metric space, and define $J_\epsilon = \{(x_1,x_2) \mid \rho(x_1,x_2) \leq \epsilon\}$. Thus if $A \subset X$, then $J_\epsilon[A]$ is an "ϵ-neighborhood of A". If F is a normed linear space with metric ρ determined by the norm, and if A,B are bounded subsets of F, then the Hausdorff distance [B-1] between A and B is denoted by $d_H(A,B)$ which is defined by the relation

$$(2.1) \qquad d_H(A,B) \equiv \inf\{\epsilon > 0 \mid J_\epsilon[A] \supset B \text{ and } J_\epsilon[B] \supset A\}.$$

We observe that if F is complete, then $(\mathcal{B}(F), d_H)$ is complete. The proof of this assertion is essentially the same as the proof of (5.6) in [D-1, pg. 362]. One quickly establishes that a Cauchy sequence of nonempty closed and bounded sets in F must converge to a closed and bounded set in F (see [K-4, pg. 314] or [M-1, Prop. 4.1, pg. 161]). Price's inequality [P-1, (2.9), pg. 4],

$$(2.2) \qquad d_H(\mathrm{co}(A),\ \mathrm{co}(B)) \leq d_H(A,B),$$

where A and B are closed, bounded, nonempty subsets of F, then implies that a Cauchy sequence in $\mathcal{B}(F)$ must converge to an element of $\mathcal{B}(F)$.

Rådström's embedding theorem [R-1, Theorem 2] tells us that in case F is reflexive, there is a real normed linear space $\mathfrak{B}(F)$ (or simply $\mathfrak{B}$ when F is understood) and an isometric mapping $\pi\colon \mathcal{B} \to \mathfrak{B}$, where $\mathcal{B}$ is metrized by d_H, such that $\pi(\mathcal{B})$ is a convex cone in $\mathfrak{B}$.

Furthermore addition in $\mathfrak{B}$ induces addition in $\mathscr{B}$ and multiplication by non-negative scalars in $\mathfrak{B}$ induces the corresponding operation in $\mathscr{B}$. $\mathfrak{B}$ can be chosen minimal in the sense that if $\mathfrak{B}_1$ is any other real normed linear space into which $\mathscr{B}$ has been embedded in the above fashion, then $\mathfrak{B}_1$ contains a subspace containing $\mathscr{B}$ which is isomorphic to $\mathfrak{B}$. It is appropriate to describe in some detail the space $\mathfrak{B}$, since we must take advantage of some of its peculiar properties in the sequel. An equivalence relation $\sim$ is defined on $\mathscr{B}^2 \equiv \mathscr{B} \times \mathscr{B}$ by stating that $(A,B) \sim (C,D)$ if $A+D = B+C$. The equivalence class containing (A,B) is denoted by $< A,B >$. The space $\mathfrak{B}$ is taken to be the quotient space $\mathscr{B}^2/\sim$, where addition in $\mathfrak{B}$ is defined by $< A,B > + < C,D > \equiv < A+C, B+D >$, and if $\alpha \geq 0$, then $\alpha < A,B > \equiv < \alpha A, \alpha B >$ while if $\alpha < 0$, then $\alpha < A,B > \equiv < |\alpha| B, |\alpha| A >$. With addition and scalar multiplication so defined $\mathfrak{B}$ becomes a real linear space. The embedding $\pi: \mathscr{B} \to \mathfrak{B}$ is given by $\pi(A) \equiv < A,0 >$, $A \in \mathscr{B}$, i.e., $< A,0 >$ is the equivalence class $\{(A+D,D) | D \in \mathscr{B}\}$. We shall adopt the convention of denoting $\pi(A)$ by $\hat{A}$ when A is an element of $\mathscr{B}$, and hence the convex cone $\pi(\mathscr{B}) = \hat{\mathscr{B}}$. A metric $\mathfrak{d}_H$ on $\mathfrak{B}$ is defined by

$$\mathfrak{d}_H(< A,B >, < C,D >) \equiv d_H(A+D, B+C).$$

The zero element of $\mathfrak{B}$ is the equivalence class $\{(D,D) | D \in \mathscr{B}\}$ which will be denoted by $< 0,0 >$. Since d_H is translation invariant and positively homogeneous, the relation $\| < A,B > \| = \mathfrak{d}_H(< A,B >, < 0,0 >)$

actually defines a norm on $\mathfrak{B}$ such that $\delta_H(\langle A,B\rangle, \langle C,D\rangle) = \| \langle A,B\rangle - \langle C,D\rangle \|$.

A function $f: E \to F$ where E and F are arbitrary normed linear spaces is said to be equal to $o(\|h\|)$ if $\|f(x)\|/\|h\| \to 0$ as $\|h\| \to 0$.

Let F be a reflexive Banach space, and let E be a normed linear space. A multifunction, $\Omega: G \to \mathscr{B}(F)$, where G is an open subset of E, is defined to be <u>π-differentiable at $x_o \in G$</u> if the function, $\hat{\Omega}: G \to \mathfrak{B}(F)$, $x \to \hat{\Omega}(x)$, $x \in G$ is differentiable at $x_o \in G$. As usual Ω is <u>π-differentiable on G</u> if it is π-differentiable at every point of G. Thus Ω is π-differentiable at $x_o \in G$ means that there is a continuous linear mapping $\hat{\Omega}'(x_o): E \to \mathfrak{B}$ such that

$$\hat{\Omega}(x) - \hat{\Omega}(x_o) - \hat{\Omega}'(x_o)(x-x_o) = o(\|x-x_o\|). \tag{2.3}$$

If we write $\hat{\Omega}'(x_o)(\Delta x) = \langle A_{\Delta x}, B_{\Delta x}\rangle$, $\Delta x \in E$, and $A_{\Delta x}, B_{\Delta x} \in \mathscr{B}$, then in terms of the Hausdorff metric (2.3) means

$$d_H(\Omega(x) + B_{x-x_o}, \Omega(x_o) + A_{x-x_o}) = o(\|x-x_o\|). \tag{2.3'}$$

If E is finite dimensional with basis $\xi_1, \xi_2, \ldots, \xi_n$, then $\Delta x = \sum \Delta x^i \xi_i$, $\Delta x \in E$. If $\hat{\Omega}'(x_o)(\xi_i) = \langle A_{\xi_i}, 0\rangle$, $i = 1,2,\ldots,n$, then we say that Ω is <u>conically differentiable at $x_o \in G$</u> and we have

$$\hat{\Omega}'(x_o)(\Delta x) = \sum \Delta x^i < A_{\xi_i}, 0 >.$$

We should mention that $\mathfrak{B}(F)$ need not be complete when F is complete [D-1, pg. 363], but nonetheless if F is complete, then so is $\mathscr{B}(F)$ and hence $\hat{\mathscr{B}}(F)$. Even though $\mathfrak{B}(F)$ is not complete most of the basic rules of the differential calculus [D-2, Chapt. VIII] can still be applied to the mapping $x \to \hat{\Omega}(x)$, $x \in G$, and we shall feel free to do so in the subsequent sections of this paper.

Remark. It would be interesting and useful to have these embedding results for certain collections of closed, convex, and nonempty subsets of F where F is finite dimensional with $\xi_1, \ldots, \xi_n$ as basis. For example the set $\mathscr{L}_{\xi+}$ of all nonempty, closed, and convex sets $A \subset F$ such that $a \in A$, $a = \sum a^i \xi_i$ imply $a^i \geq 0$, $i = 1, 2, \ldots, n$, is interesting. If addition and scalar multiplication (with nonnegative scalars) in $\mathscr{L}_{\xi+}$ are defined as before, then all the data needed to extend Rådström's embedding result to $\mathscr{L}_{\xi+}$ (with the uniform topology determined by the norm on F [M-1, pg. 153]) are fulfilled except the crucial cancellation law. For example, take $F = R^2$, $\xi_1 = (1,0)$, $\xi_2 = (0,1)$. Define sets $A \equiv \{(x,y) | \; y = 2x, \; x \geq 0\}$, $B \equiv \{(x,y) | y = x, \; x \geq 0\}$, and $C \equiv A+B$. Then we have $A+C = B+C$ and yet $A \neq B$.

3. Examples of Differentiable Multifunctions.

We shall next exhibit some examples of π-differentiable multifunctions. Although very simple, these examples illustrate the notion of π-differentiability and are useful in the applications discussed in Section 5.

Example 3.1. Let E be a normed linear space, F a reflexive Banach space. Let A be a fixed element of $\mathscr{B}(F)$ and r a differentiable mapping, $r: G \to F$, where G is an open subset of E. Consider the multifunction $\Omega: G \to \mathscr{B}(F)$ defined by $\Omega(x) = \{r(x)\}+A$, $x \in G$. Thus Ω is a fixed set moving along a differentiable curve r in the space F. It is easy to see that Ω is π-differentiable with $\hat{\Omega}'(x_o)(\Delta x) = < \{r'(x_o)(\Delta x)\},0 >$, $x_o \in G$, $\Delta x \in E$, since

$$
\begin{aligned}
\|\hat{\Omega}(x)-\hat{\Omega}(x_o)-<\{r'(x_o)(x-x_o)\},0>\| &= \|<\{r(x)\}+A,0> - <\{r(x_o)\}+A,0>-< \{r'(x_o)(x-x_o)\},0 >\| \\
&= d_H(\{r(x)\}+A,\ \{r(x_o)+r'(x_o)(x-x_o)\}+A) \\
&= d_H(\{r(x)\},\ \{r(x_o)+r'(x_o)(x-x_o)\}) \\
&= \|r(x)-r(x_o)-r'(x_o)(x-x_o)\| \\
&= o(\|x-x_o\|).
\end{aligned}
$$

It remains to note that $\Delta x \to < \{r'(x_o)(\Delta x)\},0 >$ is a bounded linear operator. That the operator is additive in Δx is clear, and the homogeneity follows from the fact that $< \alpha A,0 > = \alpha < A,0 >$ for every $\alpha \in R$ whenever $A = \{a\}$ is a singleton set in F. Finally we have

$\| < \{r'(x_o)(\Delta x)\}, 0 > \| = \|r'(x_o)(\Delta x)\| \leq \|r'(x_o)\|\|\Delta x\|$, $\Delta x \in E$.

If in this example E is finite dimensional, then Ω is conically differentiable. In fact if $\xi_1, \ldots, \xi_n$ is a basis for E, and $\Delta x = \sum \Delta x^i \xi_i$, then

$$\hat{\Omega}'(x_o)(\Delta x) = \sum \Delta x^i < \{r'(x_o)(\xi_i)\}, 0 >.$$

Example 3.2. Let E, F, G, A be as in Example 3.1. Let g be a differentiable mapping from G into R such that g does not change sign on G, i.e., $g(x) \geq 0$, $x \in G$ or $g(x) \leq 0$, $x \in G$. Let $\Omega: G \to \mathcal{B}(F)$ be the multifunction defined by $\Omega(x) = g(x)A$, $x \in G$. We consider first the case where $g(x) \geq 0$, $x \in G$. In this case we have $\pi(g(x)A) = < g(x)A, 0 > = g(x) < A, 0 >$, which implies that $\hat{\Omega}'(x_o)(\Delta x) = g'(x_o)(\Delta x) < A, 0 >$, $x_o \in G$, $\Delta x \in E$. On the other hand for $\alpha \leq 0$, we have $< \alpha A, 0 > = < -\alpha(-A), 0 > = \alpha < 0, -A >$. Consequently, if $g(x) \leq 0$, $x \in G$, then we have $\pi(g(x)A) = < g(x)A, 0 > = g(x) < 0, -A >$ from which we deduce that $\hat{\Omega}'(x_o)(\Delta x) = g'(x_o)(\Delta x) < 0, -A >$. If E is finite dimensional with basis $\xi_1, \xi_2, \ldots, \xi_n$, then Ω is conically differentiable at $x_o \in G$ whenever $g \geq 0$ and $g'(x_o)(\xi_i) \geq 0$, $i = 1, 2, \ldots, n$ (respectively, $g \leq 0$ and $g'(x_o)(\xi_i) \leq 0$, $i = 1, 2, \ldots, n$), and in fact in either case we have the relation $\hat{\Omega}'(x_o)(\Delta x) = \sum \Delta x^i < g'(x_o)(\xi_i)A, 0 >$ for $\Delta x = \sum \Delta x^i \xi_i \in E$.

From the foregoing we infer that if g is differentiable on G, then Ω is π-differentiable on $G^+ \equiv g^{-1}(0, \infty)$ and on $G^- \equiv$

$g^{-1}(-\infty,0)$. Ω will also be differentiable at points $x_o \in G$ where $g(x_o) = 0$ if there is a neighborhood U of x_o such that $g(x) \geq 0$ for each $x \in U$ or $g(x) \leq 0$ for each $x \in U$. In addition if x_o is a point of G such that every neighborhood of x_o contains points where g is positive and also points where g is negative, then Ω is not π-differentiable at x_o unless A is a singleton point set.

The requirement that g not change sign on G if Ω is to be π-differentiable may appear somewhat strange. However, the reason the condition is necessary for π-differentiability can perhaps best be seen by considering a special case of Example 3.2. Let $E = G = R$, let $F = R^2$, and define $\Omega(t) = t\sigma_2$, where σ_2 is the unit disc in R^2. One might very well expect this multifunction to be differentiable. However, in view of the preceding paragraph this multifunction is not differentiable at $t = 0$. In fact we have that $\hat{\Omega}'(t_o)(\Delta t) = \Delta t < \sigma_2, 0 >$ for $t_o > 0$, and $\hat{\Omega}'(t_o)(\Delta t) = \Delta t < 0, -\sigma_2 >$, for $t_o < 0$, but $\hat{\Omega}$ is not differentiable at $t_o = 0$ even though the right and left hand derivatives both exist at $t_o = 0$. It is interesting to interpret this geometrically. The graph of the multifunction Ω in $R \times R^2$ consists of a cone whose vertex is at the origin and whose axis is the t-axis (i.e., $R \times \{(0,0)\}$). The cross sections of the cone through $(t,0,0)$ parallel to the plane $\{0\} \times R^2$ are the discs $t\sigma_2$. We see that the behavior of this multifunction is similar to that of the real valued function $t \to |t|$ which has a corner at $t_o = 0$. In fact note that the graphs of Ω, $t \to t\sigma_2$, $t \in R$ and $\tilde{\Omega}$, $t \to |t|\sigma_2$, $t \in R$ are precisely the same. An additional comment about a multifunction of this type will be made in Example 3.3.

The next example supports the correctness of our formulation of the definition of the derivative of a multifunction inasmuch as it seems reasonable that Theorems 3.1, 3.2, and Corollary 3.1 of Example 3.3 ought to be true for any satisfactory theory of differentiation for multifunctions.

Example 3.3. This final example is somewhat more complicated notationally than the previous ones. In essence we shall consider multifunctions which are cross-products of compact intervals of real numbers where the endpoints of these intervals are differentiable real valued functions. In order to investigate the π-differentiability of such multifunctions some auxiliary results and notation are needed.

If $[a^j, b^j]$, $j = 1, \ldots, n$ are compact intervals in R, then the product $\prod_{j=1}^{n} [a^j, b^j]$ is in $\mathcal{B}(R^n)$. Addition and scalar multiplication on such sets is evidently described by the following relations:

$$\lambda\left(\prod_{j=1}^{n} [a^j, b^j]\right) = \begin{cases} \prod_{j=1}^{n} [\lambda a^j, \lambda b^j], & \lambda \geq 0 \\ \prod_{j=1}^{n} [\lambda b^j, \lambda a^j], & \lambda < 0, \end{cases}$$

and

$$\sum_{i=1}^{k} \prod_{j=1}^{n} [a_i^j, b_i^j] = \prod_{j=1}^{n} \left[\sum_{i=1}^{k} a_i^j, \sum_{i=1}^{k} b_i^j\right].$$

It will be useful to have some estimates on the Hausdorff distance between two sets which are n-fold products of compact intervals. In the computations we use the Euclidean norm on R^n.

Lemma 3.1.

(a) $d_H(\prod_{j=1}^{n}[a^j,b^j], \prod_{j=1}^{n}[\tilde{a}^j,\tilde{b}^j]) \geq \max_{j=1,2,\dots,n} (|a^j-\tilde{a}^j|, |b^j-\tilde{b}^j|)$;

(b) $d_H(\prod_{j=1}^{n}[a^j,b^j], \prod_{j=1}^{n}[\tilde{a}^j,\tilde{b}^j]) \leq \sum_{j=1}^{n}(|a^j-\tilde{a}^j| + |b^j-\tilde{b}^j|)$.

Proof of (a). Denote the sets $\prod_{j=1}^{n}[a^j,b^j]$ and $\prod_{j=1}^{n}[\tilde{a}^j,\tilde{b}^j]$ in $\mathcal{B}(R^n)$ by A and $\tilde{A}$ respectively. Choose an arbitrary $\epsilon > d_H(A,\tilde{A})$; then by (2.1) we have $J_\epsilon[A] \supset \tilde{A}$ and $J_\epsilon[\tilde{A}] \supset A$. If $a^j < \tilde{a}^j$, we get from $J_\epsilon[\tilde{A}] \supset A$ the relation $\epsilon \geq \tilde{a}^j - a^j = |\tilde{a}^j - a^j|$. On the other hand, if $a^j > \tilde{a}^j$, then $J_\epsilon[A] \supset \tilde{A}$ implies that $\epsilon \geq a^j - \tilde{a}^j = |\tilde{a}^j - a^j|$. Thus for $j = 1,2,\dots,n$, we have $\epsilon \geq |a^j - \tilde{a}^j|$. Similar arguments show that $\epsilon \geq |b^j - \tilde{b}^j|$, $j = 1,2,\dots,n$. Consequently, the right hand side of (a) does not exceed ϵ. Since $\epsilon > d_H(A,\tilde{A})$ was arbitrary this proves (a).

Proof of (b). Define for $j = 1,2,\dots,n$

$$x_1^j = \tilde{a}^j - (|a^j-\tilde{a}^j| + |b^j-\tilde{b}^j|)$$

$$x_2^j = \tilde{b}^j + (|a^j-\tilde{a}^j| + |b^j-\tilde{b}^j|)$$

and put $X = \prod_{j=1}^{n}[x_1^j,x_2^j]$. Evidently $X \supset \tilde{A}$ and since $x_1^j \leq a^j$, $x_2^j \geq b^j$, $j = 1,2,\dots,n$, X also contains A. Define $\epsilon = \sum_{j=1}^{n}(|a^j-\tilde{a}^j| + |b^j-\tilde{b}^j|)$; then since

$$[\sum_{j=1}^{n}(|a^j-\tilde{a}^j| + |b^j-\tilde{b}^j|)^2]^{1/2} \leq \sum_{j=1}^{n}(|a^j-\tilde{a}^j| + |b^j-\tilde{b}^j|),$$

we have $A \subset X \subset J_\epsilon[\tilde{A}]$. In a similar manner one can show that $J_\epsilon[A] \supset \tilde{A}$. Hence $\epsilon \geq d_H(A,\tilde{A})$, which proves (b).

Now let a^j, b^j be real-valued mappings defined on an open set $G \subset R^m$ such that $a^j(x) \leq b^j(x)$, $x \in G$, $j = 1,2,\ldots,n$. Define the multifunction $\Omega\colon G \to \mathscr{B}(R^n)$ by the relation

$$\Omega(x) = \prod_{j=1}^{n} [a^j(x), b^j(x)].$$

Suppose Ω is π-differentiable at $x_o \in G$. Then since R^m is finite dimensional there exist $A_i(x_o)$, $B_i(x_o) \in \mathscr{B}(R^n)$ such that

$$\hat{\Omega}'(x_o)(\Delta x) = \sum_{i=1}^{m} \Delta x^i < A_i(x_o),\ B_i(x_o) > \tag{3.1}$$

for all $\Delta x = (\Delta x^1, \Delta x^2, \ldots, \Delta x^m) \in R^m$. Now let us assume further that

$$A_i(x_o) = \prod_{j=1}^{n} [a_i^j(x_o), b_i^j(x_o)],\quad B_i(x_o) = \prod_{j=1}^{n} [\tilde{a}_i^j(x_o), \tilde{b}_i^j(x_o)], \tag{3.2}$$

for $i = 1,2,\ldots,m$ (This is always true if $n = 1$, since $\mathscr{B}(R)$ consists only of compact intervals). By direct calculation we obtain the relation

$$\|\hat{\Omega}(x_o+\Delta x) - \hat{\Omega}(x_o) - \sum_{i=1}^{m} \Delta x^i < A_i(x_o),\ B_i(x_o) > \| = d_H\Big(\prod_{j=1}^{n} [\omega_o^j, \omega_1^j],\ \prod_{j=1}^{n} [\omega_2^j, \omega_3^j]\Big) \tag{3.3}$$

where

$$\omega_0^j = a^j(x_o+\Delta x) + \sum_{\Delta x^i \geq 0} \Delta x^i \tilde{a}_i^j(x_o) - \sum_{\Delta x^i < 0} \Delta x^i a_i^j(x_o),$$

$$\omega_1^j = b^j(x_o+\Delta x) + \sum_{\Delta x^i \geq 0} \Delta x^i \tilde{b}_i^j(x_o) - \sum_{\Delta x^i < 0} \Delta x^i b_i^j(x_o),$$

$$\omega_2^j = a^j(x_o) + \sum_{\Delta x^i \geq 0} \Delta x^i a_i^j(x_o) - \sum_{\Delta x^i < 0} \Delta x^i \tilde{a}_i^j(x_o),$$

$$\omega_3^j = b^j(x_o) + \sum_{\Delta x^i \geq 0} \Delta x^i b_i^j(x_o) - \sum_{\Delta x^i < 0} \Delta x^i \tilde{b}_i^j(x_o),$$

for $j = 1,2,\ldots,n$. From Lemma 3.1(a) we determine that the right hand side of (3.3) is greater than or equal to $\max_{j=1,2,\ldots,n} \{|\omega_0^j-\omega_2^j|, |\omega_1^j-\omega_3^j|\}$ and consequently this expression is $o(\|\Delta x\|)$ since Ω is π-differentiable at x_o. Now if we observe that

$$|\omega_0^j-\omega_2^j| = |a^j(x_o+\Delta x)-a^j(x_o) - \sum_{i=1}^{m} \Delta x^i(a_i^j(x_o)-\tilde{a}_i^j(x_o))|$$

$$|\omega_1^j-\omega_3^j| = |b^j(x_o+\Delta x)-b^j(x_o) - \sum_{i=1}^{m} \Delta x^i(b_i^j(x_o)-\tilde{b}_i^j(x_o))|,$$

$j = 1,2,\ldots,n$, then the following theorem will thereby be proved:

<u>Theorem 3.1.</u> Let G be an open subset of R^m. If the multifunction $x \to \Omega(x) = \prod_{j=1}^{n}[a^j(x),b^j(x)]$, $x \in G$, is π-differentiable at $x_o \in G$ and if $\hat{\Omega}'(x_o)(\Delta x)$ satisfies (3.1) and (3.2), then the mappings a^j, b^j: $G \to R$ are differentiable at x_o, $j = 1,2,\ldots,n$, and their differentials are given by

$$a^{j\prime}(x_o)(\Delta x) = \sum_{i=1}^{m} \Delta x^i(a_i^j(x_o) - \tilde{a}_i^j(x_o))$$

$$b^{j\prime}(x_o)(\Delta x) = \sum_{i=1}^{m} \Delta x^i (b_i^j(x_o) - \tilde{b}_i^j(x_o))$$

for $j = 1,2,\ldots,n$ and $\Delta x = (\Delta x^1, \Delta x^2, \ldots, \Delta x^m)$.

Happily, there is a converse to Theorem 3.1, the proof of which uses the estimate in Lemma 3.1(b).

<u>Theorem 3.2</u>. Let G be an open subset of R^m, and let mappings $a^j, b^j\colon G \to R$ be given with $a^j \leq b^j$, $j = 1,2,\ldots,n$. If the mappings a^j, b^j, $j = 1,2,\ldots,n$ are differentiable at $x_o \in G$, then the multi-function $x \to \Omega(x) = \prod_{j=1}^{n} [a^j(x), b^j(x)]$ is π-differentiable at $x_o \in G$, and

$$(1^o)\quad \hat{\Omega}'(x_o)(\Delta x) = \sum_{i=1}^{m} \Delta x^i < \prod_{j=1}^{n} [\alpha_i^j(x_o) + \frac{\partial a^j}{\partial x^i}(x_o), \beta_i^j(x_o) + \frac{\partial b^j}{\partial x^i}(x_o)],$$
$$\prod_{j=1}^{n} [\alpha_i^j(x_o), \beta_i^j(x_o)] >$$

where the $\alpha_i^j(x_o)$, $\beta_i^j(x_o)$ can be arbitrary real numbers satisfying

$$(2^o)\quad \begin{aligned} &\alpha_i^j(x_o) \leq \beta_i^j(x_o) \\ &\alpha_i^j(x_o) + \frac{\partial a^j}{\partial x^i}(x_o) \leq \beta_i^j(x_o) + \frac{\partial b^j}{\partial x^i}(x_o), \end{aligned}$$

$i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$.

<u>Proof</u>. Let $\alpha_i^j(x_o)$, $\beta_i^j(x_o)$ be chosen subject to (2^o), and let $F(x_o, \Delta x)$ denote the right hand side of (1^o). Then $\|\hat{\Omega}(x_o+\Delta x) - \hat{\Omega}(x_o) - F(x_o,\Delta x)\|$ is exactly the quantity on the left hand side of (3.3) if in (3.2) we take

$$a_i^j(x_o) = \alpha_i^j(x_o) + \frac{\partial a^j}{\partial x^i}(x_o), \quad b_i^j(x_o) = \beta_i^j(x_o) + \frac{\partial b^j}{\partial x^i}(x_o)$$

$$\tilde{a}_i^j(x_o) = \alpha_i^j(x_o), \quad \tilde{b}_i^j(x_o) = \beta_i^j(x_o),$$

$i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$. Hence using (3.3), Lemma 3.1(b), and the assumption that the functions a^j, b^j are differentiable at x_o we obtain

$$\|\hat{\Omega}(x_o+\Delta x)-\hat{\Omega}(x_o)-F(x_o,\Delta x)\| \le \sum_{j=1}^{n} \Big(\Big| a^j(x_o+\Delta x)-a^j(x_o)-\sum_{i=1}^{m} \Delta x^i \frac{\partial a^j}{\partial x^i}(x_o)\Big|$$

$$+ \Big| b^j(x_o+\Delta x)-b^j(x_o) - \sum_{i=1}^{m} \Delta x^i \frac{\partial b^j}{\partial x^i}(x_o)\Big|\Big)$$

$$= o(\|\Delta x\|),$$

where $\Delta x = (\Delta x^1, \Delta x^2, \ldots, \Delta x^m) \in R^m$.

Let us note that in Theorem 3.2 one may choose $\alpha_i^j(x_o) = \beta_i^j(x_o) = 0$ if $\frac{\partial a^j}{\partial x^i}(x_o) \le \frac{\partial b^j}{\partial x^i}(x_o)$, and if $\frac{\partial a^j}{\partial x^i}(x_o) > \frac{\partial b^j}{\partial x^i}(x_o)$ one may take $\alpha_i^j(x_o) = -\frac{\partial a^j}{\partial x^i}(x_o)$, $\beta_i^j(x_o) = -\frac{\partial b^j}{\partial x^i}(x_o)$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$. In particular, then, for $n = 1$ we have

$$\hat{\Omega}'(x_o)(\Delta x) = \Sigma^{(+)} \Delta x^i < [\frac{\partial a}{\partial x^i}(x_o), \frac{\partial b}{\partial x^i}(x_o)], 0 >$$

$$+ \Sigma^{(-)} \Delta x^i < 0, [-\frac{\partial a}{\partial x^i}(x_o), -\frac{\partial b}{\partial x^i}(x_o)] >,$$

the sum $\Sigma^{(+)}$ being taken over those i for which $\partial(b-a)(x_o)/\partial x^i \ge 0$, and the sum $\Sigma^{(-)}$ being taken over the complementary set of indices i.

Thus, if $\frac{\partial a}{\partial x^i}(x_o) \le \frac{\partial b}{\partial x^i}(x_o)$ for $i = 1,2,\ldots,m$, then we have that $\Omega(x) = [a(x),b(x)]$ is conically differentiable at x_o.

As an immediate corollary to Theorems 3.1, 3.2 we have:

<u>Corollary 3.1</u>. Let G be an open subset of R^m, and let mappings $a,b: G \to R$ be given satisfying $a(x) \le b(x)$, $x \in G$. Then the multifunction $x \to \Omega(x) \equiv [a(x),b(x)]$ is π-differentiable at $x_o \in G$ if and only if a and b are differentiable at $x_o \in G$.

Let us return to the discussion in Example 3.2, and consider the multifunction $t \to \Omega(t) = t\sigma_1$ where $\sigma_1 = [-1,1]$. We have shown that Ω is π-differentiable except at $t = 0$, and $\hat{\Omega}'(t_o)(\Delta t) = \Delta t < \sigma_1, 0 >$ for $t_o > 0$, $\hat{\Omega}'(t_o)(\Delta t) = \Delta t < 0, -\sigma_1 >$ for $t_o < 0$. Note that $\Omega(t) = [a(t),b(t)]$, where $a(t) = -|t|$, $b(t) = |t|$. Hence Corollary 3.1 gives a complete analysis of the π-differentiability properties of this multifunction.

4. Other Definitions for Derivatives of Multifunctions.

We now compare π-differentiability of multifunctions with two other types of differentiability discussed by Hukuhara [H-4] and by Bridgland [B-4]. If F is a reflexive Banach space, and $A,B \in \mathscr{B}(F)$, then the Hukuhara difference $A \overset{h}{-} B$ (if it exists) is defined to be the set $C \in \mathscr{B}(F)$ such that $C+B = A$ [cf. H-4, pg. 210]. It is to be noted that in general $A \overset{h}{-} B \neq A+(-1)B = A-B$, and that $A \overset{h}{-} B$ exists only if some translate of B is contained in A. Actually our remarks in this section are for $F = R^n$ since that is the context in which Hukuhara's and Bridgland's results were developed. However, the notion of the difference $A \overset{h}{-} B$ is useful in Section 5 so we gave above a more general definition than we need at the present time. In the remainder of this section we take $F = R^n$. Hukuhara [H-4] gave the following definition.

Definition. Let I be an interval of real numbers. Let a multifunction $\Omega: I \to \mathscr{B}(R^n)$ be given. Ω is Hukuhara differentiable at $t_o \in I$ if there exists $D_h\Omega(t_o) \in \mathscr{B}(R^n)$ such that the limits

$$(4.1) \qquad \lim_{\Delta t \to 0^+} \frac{\Omega(t_o+\Delta t) \overset{h}{-} \Omega(t_o)}{\Delta t}$$

and

$$(4.2) \qquad \lim_{\Delta t \to 0^+} \frac{\Omega(t_o) \overset{h}{-} \Omega(t_o-\Delta t)}{\Delta t}$$

both exist and are equal to $D_h\Omega(t_o)$.

Of course, implicit in the definition of $D_h\Omega(t_o)$ is the existence of the differences $\Omega(t_o+\Delta t) \overset{h}{-} \Omega(t_o)$ and $\Omega(t_o) \overset{h}{-} \Omega(t_o-\Delta t)$ for all $\Delta t > 0$ sufficiently small. Using the difference quotient in (4.2) is not equivalent to using the difference quotient in

$$(4.2') \qquad \lim_{\Delta t \to 0^-} \frac{\Omega(t_o+\Delta t) \overset{h}{-} \Omega(t_o)}{\Delta t},$$

contrary to the situation for ordinary functions from I into a topological vector space. In general the existence of $A \overset{h}{-} B$, $A,B \in \mathscr{B}(R^n)$ implies nothing about the existence of $B \overset{h}{-} A$. Thus we raise the question of which of the limits (4.2) or (4.2') is preferable for defining the "left hand derivative of Ω at t_o"? In [H-4] Hukuhara defines the integral of a continuous multifunction $F: [a,b] \to \mathscr{B}(R^n)$ and shows that $D_h\int_a^t F(s)ds = F(t)$. One must use (4.2) rather than (4.2'), if this type of result is to be true as the following simple example shows: Let $A \in \mathscr{B}(R^n)$ and define $F(t) = A$, $t \in R$; then for any $t \geq 0$ we have $\int_o^t F(s)ds = tA$. Taking $\Omega(t) = tA$, $t \geq 0$ we see that the difference quotient in (4.2') does not exist.

Differentiability in the sense of Hukuhura implies conical (π-) differentiability. Before proving this let us observe that if $A \overset{h}{-} B$ exists for some $A,B \in \mathscr{B}(R^n)$, then

$$(4.3) \qquad \langle A,0 \rangle - \langle B,0 \rangle = \langle A \overset{h}{-} B,0 \rangle.$$

<u>Lemma 4.1</u>. If a multifunction $\Omega: I \to \mathscr{B}(R^n)$ is Hukuhara differenti-

able at $t_o \in I$ with derivative $D_h\Omega(t_o)$, then Ω is π-differentiable with $\hat{\Omega}'(t_o)(\Delta t) = \Delta t < D_h\Omega(t_o), 0 >$, $\Delta t \in R$.

__Proof.__ Using (4.3) one obtains for $\Delta t > 0$,

$$\left\| \frac{\hat{\Omega}(t_o+\Delta t) - \hat{\Omega}(t)}{\Delta t} - < D_h\Omega(t_o), 0 > \right\| =$$
$$d_H\left(\frac{\Omega(t_o+\Delta t) \overset{h}{-} \Omega(t_o)}{\Delta t}, D_h\Omega(t_o)\right),$$

and consequently both sides of the equality converge to 0 as $\Delta t \to 0^+$.

Similarly for $\Delta t < 0$ we have (with $k = -\Delta t$),

$$\left\| \frac{\hat{\Omega}(t_o+\Delta t) - \hat{\Omega}(t_o)}{\Delta t} - < D_h\Omega(t_o), 0 > \right\| =$$
$$\left\| \frac{\hat{\Omega}(t_o) - \hat{\Omega}(t_o-k)}{k} - < D_h\Omega(t_o), 0 > \right\| =$$
$$d_H\left(\frac{\Omega(t_o) \overset{h}{-} \Omega(t_o-k)}{k}, D_h\Omega(t_o)\right),$$

and the last term converges to 0 as $k \to 0^+$. Hence the lemma is proved.

The following result is interesting and can be used to show that the converse of Lemma 4.1 is false.

__Proposition 4.1.__ If the multifunction $\Omega: I \to \mathcal{B}(R^n)$ is Hukuhara differentiable on I, then the real valued function $t \to \text{diam}(\Omega(t))$, $t \in I$ is nondecreasing on I.

Proof. If Ω is Hukuhara differentiable at a point $t_o \in I$, then there is a $\delta(t_o) > 0$ such that $\Omega(t_o+\Delta t) \overset{h}{-} \Omega(t_o)$ and $\Omega(t_o) \overset{h}{-} \Omega(t_o-\Delta t)$ are defined for $0 < \Delta t < \delta(t_o)$. Since $A \overset{h}{-} B$, $A, B \in \mathcal{B}(R^n)$ is defined only if some translate of B is contained in A, then $A \overset{h}{-} B$ exists only if $\operatorname{diam}(A) \geq \operatorname{diam}(B)$. Let $t_1, t_2 \in I$ be fixed with $t_1 < t_2$. Then for each $\tau \in [t_1, t_2]$ there is a $\delta(\tau) > 0$ such that $\operatorname{diam}(\Omega(s)) \leq \operatorname{diam}(\Omega(\tau))$ for $s \in [\tau-\delta(\tau), \tau]$ and $\operatorname{diam}(\Omega(s)) \geq \operatorname{diam}(\Omega(\tau))$ for $s \in [\tau, \tau+\delta(\tau)]$. The collection

$$\{I_\tau \mid \tau \in [t_1, t_2],\ I_\tau = (\tau-\delta(\tau), \tau+\delta(\tau))\}$$

forms an open covering of $[t_1, t_2]$. Choose a finite subcover $I_{\tau_1}, \ldots, I_{\tau_N}$ with $\tau_i < \tau_{i+1}$; then there results $\operatorname{diam}(\Omega(t_1)) \leq \operatorname{diam}(\Omega(\tau_1))$ and $\operatorname{diam}(\Omega(\tau_N)) \leq \operatorname{diam}(\Omega(t_2))$. There will be no loss in generality to assume $I_{\tau_i} \cap I_{\tau_{i+1}} \neq \Phi$ $i = 1, \ldots, N-1$. Thus for each $i = 1, \ldots, N-1$ there exists an $s_i \in I_{\tau_i} \cap I_{\tau_{i+1}}$ with $\tau_i < s_i < \tau_{i+1}$, and hence $\operatorname{diam}(\Omega(\tau_i)) \leq \operatorname{diam}(\Omega(s_i)) \leq \operatorname{diam}(\Omega(\tau_{i+1}))$. Therefore we have $\operatorname{diam}(\Omega(t_1)) \leq \operatorname{diam}(\Omega(t_2))$, which proves the proposition.

Remark. Note that the existence of the limits in (4.1) and (4.2) was not used in the proof of Proposition 4.1. In fact instead of the hypothesis that Ω is Hukuhara differentiable on I one could substitute the assumption that for each $t \in I$ the differences $\Omega(t+\Delta t) \overset{h}{-} \Omega(t)$ and $\Omega(t) \overset{h}{-} \Omega(t-\Delta t)$ both exist for all sufficiently small $\Delta t > 0$.

It should also be pointed out that the fact that a multifunction $\Omega: I \to \mathcal{B}(R^n)$ is Hukuhara differentiable on I and $\text{diam}(\Omega(t)) > 0$ for $t \in I$ need not imply Ω is monotone with respect to set inclusion. For example, if $\Omega(t) = [t,2t]$, $0 < t < 1$, then $D_h\Omega(t) = [1,2]$, $0 < t < 1$ and yet $\Omega(t_1) \not\subset \Omega(t_2)$ and $\Omega(t_2) \not\subset \Omega(t_1)$ for any t_1, t_2, $0 < t_1 < t_2 < 1$.

One can show that if $\Omega: I \to \mathcal{B}(R^n)$ is conically differentiable on I, and if the Hukuhara differences $\Omega(t+\Delta t) \overset{h}{-} \Omega(t)$ and $\Omega(t) \overset{h}{-} \Omega(t-\Delta t)$ exist for each $t \in I$ provided $\Delta t > 0$ is sufficiently small, then Ω is Hukuhara differentiable. Moreover, if $\hat{\Omega}'(t)(\Delta t) = \Delta t < \Omega'(t), 0 >$, $t \in I$, $\Delta t \in R$, then $D_h\Omega(t) = \Omega'(t)$. However, in general $\Omega: I \to \mathcal{B}(R^n)$ π-differentiable on I does not imply Ω is Hukuhara differentiable on I as the following example shows: Let σ_n be the closed unit ball in R^n, and consider the multifunction $t \to \Omega(t)$, $t \in (0,2\pi)$ where,

$$\Omega(t) = (2 + \sin t)\sigma_n.$$

This function is π-differentiable on $(0,2\pi)$ and $\hat{\Omega}'(t)(\Delta t) = (\Delta t)(\cos t) < \sigma_n, 0 >$, $t \in (0,2\pi)$, $\Delta t \in R$. In view of Proposition 4.1 Ω is not Hukuhara differentiable on $(0,2\pi)$ since $\text{diam}(\Omega(t)) = 2(2 + \sin t)$ is not non-decreasing on $(0,2\pi)$.

If $S \subset B^A$, where A and B are sets and B^A denotes the set of all functions from A into B, then we use $S[a]$ to designate the set $\{\varphi(a) | \varphi \in S\}$. If $I = [0,T]$ is a compact interval of real numbers, then $C(I,R^n)$, or simply $C(I)$, denotes the Banach space of all continuous functions from I into R^n with the norm of uniform convergence on I. In [B-4] Bridgland gave the following definition.

Definition. Let S be a nonempty compact subset of $C(I)$. Then S is said to be Huygens differentiable at $t_o \in I$ if there exists $(DS)(t_o) \in \mathscr{B}(R^n)$ such that

$$(4.4) \qquad \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} d_H(S[t_o+\Delta t],\ S[t_o] + \Delta t(DS)(t_o)) = 0$$

and

$$(4.5) \qquad \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} d_H(S[t_o],\ S[t_o-\Delta t] + \Delta t(DS)(t_o)) = 0.$$

$(DS)(t_o)$, if it exists, is called the Huygens derivative of S at t_o.

Huygens differentiability of compact sets in $C(I)$ is also related to the conical differentiability of multifunctions in a manner shown by the following remarks.

Lemma 4.2. If S is a nonempty compact subset of $C(I)$ which is Huygens differentiable on I with Huygens derivative $(DS)(t)$ for $t \in I$, and if Ω is the multifunction defined by $\Omega(t) = co(S[t])$, $t \in I$, then $\Omega: I \to \mathscr{B}(R^n)$ is conically differentiable on I and $\hat{\Omega}'(t)(\Delta t) = \Delta t < (DS)(t), 0 >$, $t \in I$, $\Delta t \in R$.

Proof. Since S is a compact subset of $C(I)$, it follows that $S[t]$ is a compact subset of R^n for each $t \in I$. Hence $co(S[t]) \in \mathscr{B}(R^n)$, $t \in I$. Now $(DS)(t) \in \mathscr{B}(R^n)$, $t \in I$, and so $< (DS)(t), 0 > \in \hat{\mathscr{B}}(R^n)$, $t \in I$. Thus for $\Delta t > 0$ we have by (2.2),

$$\left\| \frac{\hat{\Omega}(t+\Delta t) - \hat{\Omega}(t)}{\Delta t} - < (DS)(t), 0 > \right\| =$$

$$\frac{1}{\Delta t} d_H(co(S[t+\Delta t]),\ co(S[t]) + \Delta t(DS)(t)) =$$

$$\frac{1}{\Delta t} d_H(co(S[t+\Delta t]),\ co(S[t] + \Delta t(DS)(t))) \leq$$

$$\frac{1}{\Delta t} d_H(S[t+\Delta t],\ S[t] + \Delta t(DS)(t)) \to 0 \quad \text{as} \quad \Delta t \to 0^+.$$

Similarly, for $\Delta t < 0$ and $k = -\Delta t > 0$ we have

$$\left\| \frac{\hat{\Omega}(t+\Delta t) - \hat{\Omega}(t)}{\Delta t} - < (DS)(t), 0 > \right\| \leq$$

$$\frac{1}{k} d_H(S[t],\ S[t-k] + k(DS)(t)) \to 0 \quad \text{as} \quad k \to 0^+.$$

Actually Bridgland's definition of differentiability is essentially equivalent to conical differentiability. Lemma 4.2 gives the sense in which we mean that Huygens differentiability implies conical differentiability. Conversely if $\Omega: I \to \mathscr{B}(R^n)$ is conically differentiable on I, then Theorem 5.1 in Section 5 says that there is a compact, convex $S \subset C(I)$ such that $S[t] = \Omega(t)$, $t \in I$. Thus if $\hat{\Omega}'(t)(\Delta t) = \Delta t < \Omega'(t), 0 >$, $t \in I$, $\Delta t \in R$, then $(DS)(t) = \Omega'(t)$, $t \in I$.

Bridgland uses the Huygens derivative mainly to prove a theorem concerning the Huygens derivative of the indefinite integral of a multifunction, i.e., if $F: I \to \mathcal{B}(R^n)$ is measurable and integrably bounded, and if one defines $S_I(F) = \{\varphi: I \to R^n | \varphi(0) = 0, \dot{\varphi}$ integrable on I, and $\dot{\varphi}(t) \in F(t)$ a.e. on $I\}$, then $S_I(F)$ is a compact, convex subset of $C(I)$ and $(DS_I(F))(t) = F(t)$ a.e. on I. Thus in this situation the multifunction $t \to \Omega(t) \equiv (S_I(F))[t]$ is conically differentiable almost everywhere on I, and $\hat{\Omega}'(t)(\Delta t) = \Delta t < F(t), 0 >$ for almost every $t \in I$ and every $\Delta t \in R$. Noting that $S_I(F)[t] = \int^{\#}_{[0,t]} F(s)ds$ where the integral on the right hand side of the equality is Aumann's integral [A-1], one sees the connections between the results given in Section 5 (specifically Lemma 5.6 and Theorem 5.3) and those given by Bridgland [B-4], Hermes [H-2], and Hukuhara [H-4].

5. Applications.

In this final section we give some miscellaneous results and applications of the differential calculus for multifunctions.

Lemma 5.1. Let F and G be reflexive Banach spaces. Let $U: F \to G$ be a continuous linear mapping which maps elements of $\mathscr{B}(F)$ into closed sets in G. Then the induced mapping $\hat{U}: \mathfrak{B}(F) \to \mathfrak{B}(G)$ defined by $\hat{U}: < C,D > \to < U(C),U(D) >$ is a continuous linear mapping.

Proof. It is easy to verify that $\hat{U}$ is linear. Let $< C_n,D_n >$ be a null sequence in $\mathfrak{B}(F)$. Then given $\epsilon > 0$ there is a positive integer n_o such that $n \geq n_o$ implies $J_\lambda[C_n] \supset D_n$ and $J_\lambda[D_n] \supset C_n$ where $\lambda = \epsilon/(1+\|U\|)$. The linearity of U now reveals that $J_\epsilon[U(C_n)] \supset U(D_n)$ and $J_\epsilon[U(D_n)] \supset U(C_n)$ whenever $n \geq n_o$. This shows that $\hat{U}(< C_n,D_n >)$ is a null sequence, thereby proving $\hat{U}$ is continuous.

Corollary 5.1. Let F and G be reflexive Banach spaces. Let $U: F \to G$ be a continuous linear mapping which maps elements of $\mathscr{B}(F)$ into closed sets in G. Let E be a normed linear space, let W be an open subset of E and let $\Omega: W \to \mathscr{B}(F)$ be π-differentiable. Then the composite function $\Phi: x \to U(\Omega(x))$, $x \in W$ is also π-differentiable.

Proof. It suffices to show that the mapping $\hat{\Phi}: x \to \hat{\Phi}(x)$, $x \in W$, is differentiable. Observe that $\hat{\Phi} = \hat{U} \circ \hat{\Omega}$ where $\hat{U}$ is the induced linear mapping defined in Lemma 5.1. Whence $\hat{\Phi}: W \to \mathfrak{B}(G)$ is differentiable and $\hat{\Phi}' = \hat{U} \circ \hat{\Omega}'$.

Remark. If E if finite dimensional with basis $\xi_1, \xi_2, \ldots, \xi_n$, then Corollary 5.1 has the following interpretation. Given $x \in W$, $\hat{\Omega}'(x)(\xi_i) = \langle C_{ix}, D_{ix} \rangle$, $i = 1,2,\ldots,n$ and $\Delta x = \sum \Delta x^i \xi_i$ the corollary implies

$$\hat{\Phi}'(x)(\Delta x) = \sum \Delta x^i \langle U(C_{ix}), U(D_{ix}) \rangle.$$

Lemma[1] 5.2. Let F be a uniformly convex Banach space [W-1, pg. 108]. Let T be a first countable topological space. Let $\Omega\colon T \to \mathscr{B}(F)$ be a continuous multifunction. Then there is a unique function $f\colon T \to F$ satisfying

$$(5.1) \qquad \|f(t)\| = \inf\{\|u\| \mid u \in \Omega(t)\}, \; f(t) \in \Omega(t), \; t \in T,$$

and f is continuous.

Proof. The Milman-Pettis theorem [W-1, pg. 109] says F is reflexive. Thus $\Omega(t)$ is weakly compact for each $t \in T$. The existence and uniqueness of the function $f\colon T \to F$ satisfying (5.1) follows from Theorem 2 of [W-1, pg. 110]. Thus we need only verify that f is continuous. Let t_n be a sequence in T converging to t_o. Then by the continuity of Ω the sequence $\Omega(t_n)$ converges to $\Omega(t_o)$. Given $\epsilon > 0$ pick an integer n_o such that $n \geq n_o$ implies

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[1] This lemma is still true if the $\Omega(t)$, $t \in T$ are only assumed to be nonempty, closed, convex subsets of F, if it is understood that the collection of nonempty, closed subsets of F has the uniform topology determined by the norm on F [M-1, Def. 1.6, pg. 153].

(5.2) $$J_\epsilon[\Omega(t_o)] \supset \Omega(t_n) \text{ and } J_\epsilon[\Omega(t_n)] \supset \Omega(t_o).$$

Thus for $n \geq n_o$ there exist $g(t_n) \in \Omega(t_o)$ and $g^*(t_n) \in \Omega(t_n)$ such that $\|f(t_n)-g(t_n)\| \leq \epsilon$ and $\|f(t_o)-g^*(t_n)\| \leq \epsilon$. Consequently we have that $\|f(t_o)\| \leq \|g(t_n)\| \leq \epsilon+\|f(t_n)\|$ and $\|f(t_n)\| \leq \|g^*(t_n)\| \leq \epsilon+\|f(t_o)\|$ for $n \geq n_o$. Combining these inequalities there results $|\|f(t_n)\| - \|f(t_o)\|| \leq \epsilon$, for $n \geq n_o$, and whence $\lim\|f(t_n)\| = \|f(t_o)\|$. If $f(t_{n_k})$ is any subsequence of $f(t_n)$ which converges weakly to $u_o \in F$, then a sequence , σ_k, of convex linear combinations of the $f(t_{n_k})$ converge strongly to u_o [D-3, Cor. 14, pg. 422]. Observe that for any $\epsilon > 0$, $J_\epsilon[\Omega(t_o)]$ is closed and convex. Given $\epsilon > 0$ the relation (5.2) is true for all sufficiently large n. Thus $\sigma_k \in J_\epsilon[\Omega(t_o)]$ for all sufficiently large k, and hence $u_o \in J_\epsilon[\Omega(t_o)]$. Since $\epsilon > 0$ was arbitrary, it follows that $u_o \in \Omega(t_o)$, and this together with the weak lower semicontinuity of the norm [W-1, pg. 212] imply that $\liminf\|f(t_{n_k})\| = \|f(t_o)\| \geq \|u_o\| \geq \|f(t_o)\|$. Moreover, in view of the fact that the element in $\Omega(t_o)$ of minimal norm is unique we have $f(t_o) = u_o$. Thus we have shown that if any subsequence of $f(t_n)$ converges weakly to u_o, then $u_o = f(t_o)$. The sequence $f(t_n)$ is bounded because $\|f(t_n)\| \to \|f(t_o)\|$, and therefore relatively weakly compact; this together with the preceding sentence is enough to show $\|f(t_n)-f(t_o)\| \to 0$ in view of [D-3, Ex. 28, pg. 74].

The next lemma is similar to a result obtained by Filippov in the finite dimensional case [F-1, pg. 614].

Lemma 5.3. Let F be a Hilbert space. Let A, B be given elements of $\mathcal{B}(F)$. Let D denote the Hausdorff distance $d_H(A,B)$. Let a and b be the unique elements in A and B respectively satisfying

$$\|a\| = \min\{\|x\| \mid x \in A\}, \quad \|b\| = \min\{\|x\| \mid x \in B\},$$

and let $\beta = \max(\|a\|, \|b\|)$. Then there results

$$(5.3) \qquad \big| \|a\| - \|b\| \big| \le D,$$

and

$$(5.4) \qquad \|a-b\| \le \sqrt{D^2 + 4\beta D} \; .$$

Proof. From the definition of $d_H(A,B) = D$ we have that for every positive integer n

$$J_{D+1/n}[A] \supset B \quad \text{and} \quad J_{D+1/n}[B] \supset A.$$

Consequently there is an $a_n \in A$ such that $\|b-a_n\| \le D + 1/n$ and there is a $b_n \in B$ such that $\|a-b_n\| \le D + 1/n$. One then obtains $\|b\| \le \|b_n\| \le D + 1/n + \|a\|$ and $\|a\| \le \|a_n\| \le D + 1/n + \|b\|$, which together yield $\big|\|a\|-\|b\|\big| \le D + 1/n$. Since n is an arbitrary positive integer we have proved (5.3). In order to prove (5.4) we consider two cases: (i) $2\beta < D$ and (ii) $2\beta \ge D$. In case (i), we have $\|a-b\| \le \|a\|+\|b\| \le 2\beta < D \le \sqrt{D^2+4\beta D}$. Thus we turn our attention to case (ii). Using the same a_n and b_n as above we have that $\|\frac{a+b}{2} - \frac{a_n+a}{2}\| = \frac{\|b-a_n\|}{2} \leq D/2 + 1/2n$, and this implies that $\|a\| \leq \|\frac{a+a_n}{2}\| \leq D/2 + 1/2n + \|\frac{a+b}{2}\|$. Hence we have that $\|a+b\| \geq 2\|a\| - D - 1/n$ for every positive integer n, and we conclude that $\|a+b\| \geq 2\|a\|-D$. In a similar manner one can establish that $\|a+b\| \geq 2\|b\|-D$, and hence $\|a+b\| \geq 2\beta-D \geq 0$. Therefore we have $\|a+b\|^2 \geq 4\beta^2-4\beta D+D^2$. From the parallelogram law we find that

$$\begin{aligned}\|a-b\|^2 = 2\|a\|^2 + 2\|b\|^2 - \|a+b\|^2 &\leq 2\|a\|^2 + 2\|b\|^2 - 4\beta^2 + 4\beta D - D^2\\ &\leq 2\|a\|^2 + 2\|b\|^2 - 4\|a\|\|b\| + 4\beta D - D^2\\ &= 2(\|a\|-\|b\|)^2 + 4\beta D - D^2\\ &\leq D^2+4\beta D\end{aligned}$$

and this completes the proof of the lemma.

<u>Theorem 5.1.</u> Let T be a locally compact Hausdorff space, and let F be a Hilbert space. Let $C_W(T,F)$ denote the family of all weakly continuous functions from T into F, and let $C_W(T,F)$ have the topology of uniform convergence on the compact subsets of T [B-2, Pt. 2, pg. 278], where F is given the weak topology (i.e., the $\sigma(F,F^*)$-topology). If $\Omega: T \to \mathscr{B}(F)$ is a continuous multifunction, where $\mathscr{B}(F)$ is (as usual) metrized with the Hausdorff metric determined by the norm on F, then there is a compact, convex set $S \subset C_W(T,F)$ such that $S[t] = \Omega(t)$ for every $t \in T$. Moreover, the set S can be chosen so that it is strongly equicontinuous.

Proof. The space $C_W(T,F)$ is actually a topological vector space if addition and scalar multiplication are defined in the usual way. For each $x \in F$ there is a uniquely defined function $f_x: T \to F$ determined by the relation

$$\|x-f_x(t)\| = \min\{\|x-u\| \mid u \in \Omega(t)\}, \ f_x(t) \in \Omega(t), \ t \in T.$$

The family $\{f_x \mid x \in F\}$ is strongly equicontinuous at each $t_o \in T$. Let $U(t_o)$ be a compact neighborhood of t_o. Then due to the continuity of Ω there is a $\mu > 0$ such that $\|x\| \leq \mu$, $x \in \Omega(t)$, $t \in U(t_o)$. Thus $\|f_x(t)\| \leq \mu$ for $x \in F$, $t \in U(t_o)$. From Lemma 5.3 we infer that

$$(5.5) \qquad \|f_x(t)-f_x(t_o)\| \leq \sqrt{D_{tt_o}^2 + 4\mu D_{tt_o}} \ , \ t \in U(t_o)$$

where $d_H(\Omega(t),\Omega(t_o)) = d_H(\Omega(t)-x, \Omega(t_o)-x)$ is denoted by D_{tt_o}. Consequently the relation (5.5) implies that the set $S' \equiv \{f_x \mid x \in F\}$ is strongly equicontinuous at each $t_o \in T$. It is clear that $S'[t] = \Omega(t)$ for each $t \in T$. We define S to be the closure in $C_W(T,F)$ of the convex hull of S'. Certainly $co(S')$ is strongly equicontinuous at each $t_o \in T$, since S' has this property. Hence given $\epsilon > 0$ and $t_o \in T$ there is a compact neighborhood $V(t_o)$ of t_o such that $t \in V(t_o)$ implies $\|f(t)-f(t_o)\| \leq \epsilon$, for all $f \in co(S')$. On the other hand if $\{f_\alpha, \alpha \in A\}$ is a net in $co(S')$ converging to g in $C_W(T,F)$, then the weak lower semicontinuity of the norm on F, $\|\cdot\|$, [W-1, pg. 212] and the inequalities, $\|f_\alpha(t)-f_\alpha(t_o)\| \leq \epsilon$, $\alpha \in A$, $t \in V(t_o)$, imply that $\|g(t)-g(t_o)\| \leq \epsilon$ for $t \in V(t_o)$. Hence S is strongly equicontinuous at each $t_o \in T$, and this is more than enough to ensure that S is equicontinuous at each $t_o \in T$, when S is considered as a subset of $C_W(T,F)$. Thus Ascoli's theorem [B-2, Pt. 2, pg. 290] can now be applied to give that S is a compact, convex subset of $C_W(T,F)$. Evidently $S[t] = \Omega(t)$ for each $t \in T$.

Remark. If in the above theorem one takes $F = R^n$ and $T = I$, I an interval in R, then $C_W(T,F)$ is just $C(I,R^n)$ with the usual topology of uniform convergence on compact subsets of I. For $\Omega: I \to \mathscr{B}(R^n)$ continuous, the theorem then guarantees the existence of a compact convex $S \subset C(I,R^n)$ such that $S[t] = \Omega(t)$ for each $t \in I$.

Given a π-differentiable multifunction $x \to \Omega(x)$, $x \in E$, it is of interest to know if one can determine a differentiable function (selection) $x \to f(x)$, $x \in E$, such that $f(x) \in \Omega(x)$, for each $x \in E$. We have not achieved a really satisfactory answer for this at this time. In certain finite dimensional situations one can show that the centroid of $\Omega(x)$, $x \in E$ is such a selection. It is only natural to ask under what conditions the minimal norm selection in (5.1) is differentiable. A simple example shows that even under very nice circumstances the selection in (5.1) is not differentiable. Define $\Omega: R \to \mathscr{B}(R)$ by the relation $\Omega(t) = t + [0,1]$, $t \in R$: Then Ω is Hukuhara differentiable (a fortiori conically differentiable and π-

differentiable) at each $t \in R$. However, the selection defined in (5.1) is $f(t) = t$ for $t \geq 0$, $f(t) = 0$, $-1 \leq t < 0$, and $f(t) = t+1$, $t < -1$, which is not differentiable at 0 and -1. The next theorem provides some partial information on this problem.

<u>Theorem 5.2</u>. Let F be a uniformly convex Banach space, and let $\Omega: R \to \mathscr{B}(F)$ be a continuous multifunction. Let f be a continuous selection, $f: R \to F$, $f(t) \in \Omega(t)$, for each $t \in R$. Let $t_o \in R$ be a point at which the following conditions are satisfied 1°) Ω is conically differentiable at t_o, 2°) There is a $\delta > 0$ such that for each t satisfying $|t-t_o| < \delta$ either $\Omega(t) \overset{h}{-} \Omega(t_o)$ exists and $f(t)-f(t_o) \in \Omega(t) \overset{h}{-} \Omega(t_o)$ or $\Omega(t_o) \overset{h}{-} \Omega(t)$ exists and $f(t_o)-f(t) \in \Omega(t_o) \overset{h}{-} \Omega(t)$. Then f is differentiable at t_o, and if $\hat{\Omega}'(t_o)(\Delta t) = \Delta t < A,0 >$, then $f'(t_o)(\Delta t) \in \Delta t A$.

<u>Remark</u>. In the example immediately preceding this theorem one sees that the f,Ω satisfy all the hypotheses of this theorem except 2°) and this fails to be fulfilled only on $[-1,0]$. However, in this example f is a continuous selection which is differentiable on $(-1,0)$ even though 2°) of Theorem 5.2 is not satisfied in $(-1,0)$. This shows that in general condition 2°) of Theorem 5.2 is not necessary in order that a continuous selection be differentiable.

<u>Proof of Theorem 5.2</u>. In view of Lemma 5.2, the relation, $\|f(t)-f(t_o)-(t-t_o)z(t)\| = \min(\|f(t)-f(t_o)-(t-t_o)x\| \mid x \in A)$, $z(t) \in A$, $t \in R$, defines a unique continuous function $z: R \to F$. We first establish that

$$(5.6) \qquad \|f(t)-f(t_o)-(t-t_o)z(t)\| \le \|\hat{\Omega}(t)-\hat{\Omega}(t_o)-\hat{\Omega}'(t_o)(t-t_o)\|$$

for $|t-t_o| \le \delta$. In order to do this observe that

$$(5.7) \qquad \|\hat{\Omega}(t)-\hat{\Omega}(t_o)-\hat{\Omega}'(t_o)(t-t_o)\| = d_H(\Omega(t),\Omega(t_o)+(t-t_o)A), \quad t > t_o,$$

and

$$(5.7') \qquad \|\hat{\Omega}(t)-\hat{\Omega}(t_o)-\hat{\Omega}'(t_o)(t-t_o)\| = d_H(\Omega(t)+(t_o-t)A,\Omega(t_o)), \quad t < t_o.$$

We only verify (5.6) for $t > t_o$; the proof for $t < t_o$ is similar. If $\delta > t-t_o > 0$, then 2°) implies that either $f(t)-f(t_o) \in \Omega(t) \overset{h}{-} \Omega(t_o)$ or $f(t_o)-f(t) \in \Omega(t_o) \overset{h}{-} \Omega(t)$. If $f(t)-f(t_o) \in \Omega(t) \overset{h}{-} \Omega(t_o)$, then one obtains (5.6) immediately from (5.7) and the identity,

$$d_H(\Omega(t),\Omega(t_o)+(t-t_o)A) = d_H(\Omega(t) \overset{h}{-} \Omega(t_o),(t-t_o)A).$$

On the other hand, if $f(t_o)-f(t) \in \Omega(t_o) \overset{h}{-} \Omega(t)$, then (5.6) results from (5.7) and the identity,

$$d_H(\Omega(t),\Omega(t_o)+(t-t_o)A) = d_H(0,\Omega(t_o) \overset{h}{-} \Omega(t)+(t-t_o)A).$$

We also have the inequality,

$$\|f(t)-f(t_o)-(t-t_o)z(t_o)\| \le \|f(t)-f(t_o)-(t-t_o)z(t)\|+|t-t_o|\,\|z(t)-z(t_o)\|,$$

and this combined with (5.6), hypothesis 1°), and the continuity of z at t_o yield the conclusion that $\|f(t)-f(t_o)-(t-t_o)z(t_o)\| = o(|t-t_o|)$. Hence f is differentiable at t_o and $f'(t_o)(\Delta t) \in \Delta t A$, $\Delta t \in R$.

As was mentioned in Section 2 the completeness of the reflexive normed linear space F does not imply that the corresponding normed linear space $\mathfrak{B}(F)$ is complete, and this presents a minor difficulty when discussing the integration of multifunctions [D-1]. Let $I = [a,b]$ be a compact interval of real numbers and let m denote Lebesgue measure on I. We want to discuss briefly some applications of the differential calculus for multifunctions to integrals of multifunctions, $\Omega: I \to \mathcal{S}(F)$, where F is a reflexive Banach space. The integral as defined by Debreu in [D-1] is essentially what will be used here. However, in [D-1] Debreu is assuming that the multifunctions are compact valued while requiring F to be only a Banach space. The main results of section 6 of Debreu's paper are needed here, and, indeed those results are true for the situation which is discussed here. In fact Debreu's proofs can be used virtually without change. We indicate below how the results we need can be obtained in a little more direct manner. First we shall consider $\mathfrak{B}(F)$ to be a subspace of its completion $\bar{\mathfrak{B}}(F)$, which is a Banach space. Thus we say $\Omega: I \to \mathcal{S}(F)$ is <u>integrable</u> (Lebesgue measure m on I is understood) if $\hat{\Omega}$: $I \to \bar{\mathfrak{B}}(F)$ is integrable in the sense of [D-3, Chapter III], and the integral of $\hat{\Omega}$ is denoted by $\int_I \hat{\Omega}(t)dt$ or $\int_a^b \hat{\Omega}(t)dt$.

<u>Lemma 5.4</u>. Let F be a reflexive Banach space, and let $\Omega: I \to \mathcal{S}(F)$ be integrable. Then $\int_I \hat{\Omega}(t)dt$ belongs to the convex cone $m(I)\hat{\mathcal{S}}(F) = \hat{\mathcal{S}}(F)$.

Proof. As we mentioned in Section 2, the completeness of F implies $\mathcal{B}(F)$ is complete. Thus $\hat{\mathcal{B}}(F)$ is a closed convex cone in $\bar{\mathcal{B}}(F)$, and the lemma follows from the convexity theorem [B-3, pg. 203].

Lemma 5.5. Let F be a reflexive Banach space, and let $\Omega: I \to \mathcal{B}(F)$ be integrable. Then there is a sequence of measurable simple functions $\hat{S}_n: I \to \hat{\mathcal{B}}(F)$ such that $\hat{S}_n(t) \to \hat{\Omega}(t)$ a.e. on I and $\|\hat{S}_n(t)\| \leq \|\hat{\Omega}(t)\|$ for every $t \in I$. Moreover $\int_I \|\hat{S}_n(t) - \hat{\Omega}(t)\| dt \to 0$ as $n \to \infty$.

Proof. The first part is an immediate consequence of Corollary 1 (and its proof) appearing in [B-3, pg. 178]. The last result follows from the preceding one and the Lebesgue dominated convergence theorem (see [B-3, pg. 137] or [D-3, pg. 151]).

In view of the lemmas it makes sense to define $\int_I \Omega(t)dt$ to be the $A \in \mathcal{B}(F)$ such that $\int_I \hat{\Omega} dt = \langle A, 0 \rangle$. The integral $\int_I \Omega(t)dt$ is connected to Aumann's integral [A-1], $\int_I^{\#} \Omega(t)dt \equiv \{\int_I f(t)dt \mid f: I \to F,\ f(t) \in \Omega(t),\ \forall t \in I,\ f \text{ integrable}\}$, by the following lemma:

Lemma 5.6. If F is a reflexive Banach space, and if $\Omega: I \to \mathcal{B}(F)$ is integrable, then $\int_I \Omega(t)dt = \int_I^{\#} \Omega(t)dt$.

Proof. Debreu's proof of 6.5 in [D-1] can be applied essentially without change.

Theorem 5.3. If F is a reflexive Banach space, and if $\Omega: I \to \mathcal{B}(F)$ is integrable, then the function $t \to F(t) \equiv \int_a^t \Omega(s)ds$, $t \in I$ is conically differentiable almost everywhere on I. Moreover, if $\hat{F}(t) = \int_0^t \hat{\Omega}(s)ds$, then $\hat{F}'(t_o)(\Delta t) = \Delta t\hat{\Omega}(t_o)$ for almost every $t_o \in I$.

__Remark.__ Finite dimensional versions of this theorem have been given by Bridgland [B-4], Hermes [H-2], and Hukuhara [H-4].

__Proof of Theorem 5.3.__ The result can now be easily obtained from Theorem 8 in [D-3, pg. 217].

__Theorem 5.4.__ Let E be a normed linear space, and let F be a reflexive Banach space. Let $\Omega: I \times E \to \mathscr{B}(F)$ be a multifunction such that: 1°) For each fixed $x \in E$, the multifunction $t \to \Omega(t,x)$, $t \in I$ is integrable, 2°) There is a set $N \subset I$, $m(N) = 0$ such that for each fixed $t \in I \setminus N$ the mapping $x \to \Omega(t,x)$ is π-differentiable, and 3°) There is an integrable function $\varphi: I \to R$ such that $\|\frac{\partial\hat{\Omega}(t,x)}{\partial x}\| \leq \varphi(t)$ for each $x \in E$, and $t \in I\setminus N$. Then the function $x \to G(x) \equiv \int_I \Omega(t,x)dt$ is π-differentiable and $\hat{G}'(x_o)(\Delta x) = \int_I \frac{\partial\hat{\Omega}}{\partial x}(t,x)(\Delta x)dt$, for each $x_o, \Delta x \in E$.

__Proof.__ We first verify that for fixed $x, x_o \in E$, the mapping $t \to \frac{\partial\hat{\Omega}}{\partial x}(t,x_o)(x-x_o)$ is integrable. The assumption that for fixed $t \in I\setminus N$, $x \to \Omega(t,x)$ is π-differentiable implies that $x \to \hat{\Omega}(t,x)$ is Gateaux differentiable, i.e.,

$$(5.8) \qquad \lim_{n \to \infty} \frac{\hat{\Omega}(t,x_o+\tau_n(x-x_o))-\hat{\Omega}(t,x_o)}{\tau_n} = \frac{\partial\hat{\Omega}}{\partial x}(t,x_o)(x-x_o), \quad t \in I\setminus N,$$

for any sequence $\tau_n \neq 0$, $\tau_n \to 0$ as $n \to \infty$. Thus (5.8) and 1°), 2°), 3°) imply that for fixed $x, x_o \in E$, $t \to \frac{\partial\hat{\Omega}}{\partial x}(t,x_o)(x-x_o)$ is integrable. From the mean value theorem [D-2, pg. 156] one obtains that

$$(5.9)\quad \|\hat{\Omega}(t,x) - \hat{\Omega}(t,x_o) - \frac{\partial\hat{\Omega}}{\partial x}(t,x_o)(x-x_o)\| \leq$$

$$\|x-x_o\| \sup\{\|\frac{\partial\hat{\Omega}}{\partial x}(t,\xi)-\frac{\partial\hat{\Omega}}{\partial x}(t,x_o)\| \mid \xi \in co\{x_o,x\}\} \leq 2\|x-x_o\|\varphi(t),\ t \in I\setminus N.$$

We also have that if x_n is any sequence in E such that $x_n \to x_o$ with $x_n \neq x_o$, then $\|\hat{\Omega}(t,x_n)-\hat{\Omega}(t,x_o)-\frac{\partial\hat{\Omega}}{\partial x}(t,x_o)(x_n-x_o)\|/\|x_n-x_o\|$ is a null sequence for $t \in I\setminus N$. From the inequality,

$$(5.10)\quad \|\hat{G}(x)-\hat{G}(x_o)-\int_I \frac{\partial\hat{\Omega}}{\partial x}(t,x)(x-x_o)dt\| \leq$$

$$\int_I \|\hat{\Omega}(t,x)-\hat{\Omega}(t,x_o)-\frac{\partial\hat{\Omega}}{\partial x}(t,x_o)(x-x_o)\|dt,$$

inequality (5.9), and the Lebesgue dominated convergence theorem [D-3, pg. 151], we infer that $\int_I[\|\hat{\Omega}(t,x_n)-\hat{\Omega}(t,x_o)-\frac{\partial\hat{\Omega}}{\partial x}(t,x_o)(x_n-x_o)\|/\|x_n-x_o\|]dt$ converges to 0 as $n \to \infty$. Therefore the right hand side of inequality (5.10) is $o(\|x-x_o\|)$, and the conclusions of the theorem follow.

We turn now to a few more examples.

Example 5.1. This example comes from linear control theory. Consider the control system,

$$(5.11)\qquad \dot{x} = A(t)x + B(t)u,$$

where the $n \times n$ matrix function, $t \to A(t)$, $t \in [0,T]$, and the $n \times m$ matrix function, $t \to B(t)$, $t \in [0,T]$ are bounded and measurable on

$[0,T]$. If $u: [0,T] \to R^m$ is integrable, then there corresponds a unique absolutely continuous function (response) $x(\cdot,u): [0,T] \to R^n$ satisfying (5.11) a.e. on $[0,T]$ and the initial condition

$$x(0,u) = x_0.$$

Moreover, there is an absolutely continuous fundamental matrix solution of $\dot{X} = A(t)X$, $X(0) = I = n \times n$ identity matrix, and the variation of parameters formula gives

$$(5.12) \qquad x(t,u) = X(t)[x_0 + \int_0^t X^{-1}(\xi)B(\xi)u(\xi)d\xi].$$

Let a^j, b^j, $j = 1,2,\ldots,m$ be functions mapping $[0,T] \times R^p$ into R^m and such that $a^j \leq b^j$, $j = 1,2,\ldots,m$. Then let $\Omega: [0,T] \times R^p \to \mathscr{B}(R^m)$ be the multifunction defined by

$$\Omega(t,\lambda) = \prod_{j=1}^{m} [a^j(t,\lambda), b^j(t,\lambda)], \ (t,\lambda) \in [0,T] \times R^p.$$

Let $C([0,T],R^n)$ be the Banach space of all continuous functions from $[0,T]$ into R^n with the norm of uniform convergence on $[0,T]$. In optimal control problems the <u>admissible</u> integrable controls $u: [0,T] \to R^m$ are often constrained by side conditions of the form $u(t) \in \Omega(t,\lambda)$, $t \in [0,T]$ or $u(t) \in \Omega(t,x(t,u))$ (with $p = n$), $t \in [0,T]$, or, indeed, by combinations of these two types of side conditions. It is of some interest then to consider the multifunction $F: [0,T] \times R^p \to \mathscr{B}(R^n)$ defined formally by the relation,

$$(5.13)\qquad \hat{F}(t,\lambda) = X(t)[x_0 + \int_0^t X^{-1}(\xi)B(\xi)\Omega(\xi,\lambda)d\xi],$$

where the integral (if it exists) is understood in Debreu's sense described above, and likewise with $p = n$ we consider the induced multifunction $\hat{F}: [0,T] \times C([0,T],R^n) \to \mathscr{B}(R^n)$ defined formally by the relation,

$$(5.14)\qquad \hat{F}(t,x) = X(t)[x_0 + \int_0^t X^{-1}(\xi)B(\xi)\Omega(\xi,x(\xi))d\xi].$$

We shall show that under suitable conditions $\frac{\partial\hat{F}}{\partial t}(t,\lambda)$ (resp. $\frac{\partial\hat{F}}{\partial t}(t,x)$) exist a.e. on $[0,T]$ for each fixed $\lambda \in R^p$ (resp. for each fixed $x \in C([0,T],R^n)$, $p = n$), and $\frac{\partial\hat{F}}{\partial\lambda}(t,\lambda)$ (resp. $\frac{\partial\hat{F}}{\partial x}(t,x)$) exist for all $t \in [0,T]$ and all $\lambda \in R^p$ (resp. all $x \in C([0,T],R^n)$, $p = n$). We give the assumptions only for the case of the multifunction $(t,x) \to \hat{F}(t,x)$, $(t,x) \in [0,T] \times C([0,T],R^n)$ since the hypotheses needed to obtain the desired differentiability result for the multifunction in (5.13) are entirely similar. Taking $p = n$ we require 1°) For each fixed $\lambda \in R^n$ the functions $t \to a^j(t,\lambda)$, $t \to b^j(t,\lambda)$, $t \in [0,T]$ are integrable; 2°) There is an $N \subset [0,T]$, $m(N) = 0$ such that for each fixed $t \in [0,T]\backslash N$, $\lambda \to a^j(t,\lambda)$, $\lambda \to b^j(t,\lambda)$, $\lambda \in R^n$, $j = 1,2,\dots,n$ are differentiable, and 3°) There is an integrable function $\Psi: [0,T] \to R$ such that $|\frac{\partial a^j}{\partial\lambda}(t,\lambda)|$, $|\frac{\partial b^j(t,\lambda)}{\partial\lambda}| \le \Psi(t)$, $t \in [0,T]\backslash N$, $\lambda \in R^n$. Hence in view of Example 3.3, $\Omega: [0,T] \times R^n \to \mathscr{B}(R^m)$ satisfies conditions 1°), 2°), and 3°) of Theorem 5.4. Now consider the multifunction $f_\Omega(t,x) = \Omega(t,x(t))$, $(t,x) \in [0,T] \times C([0,T],R^n)$. One readily verifies that f_Ω also satisfies the conditions 1°), 2°), and 3°) of Theorem 5.4. Let us define

Φ: $[0,T] \times R^n \to \mathscr{B}(R^n)$ by $\Phi(t,\lambda) = X^{-1}(t)B(t)\Omega(t,\lambda)$. Thus if we invoke Corollary 5.1 and Theorem 5.4 we obtain

$$(5.15)\quad \frac{\partial\hat{F}}{\partial x}(t,x)(\Delta x) = X(t)\int_0^t \frac{\partial\hat{\Phi}}{\partial\lambda}(\xi,x(\xi))(\Delta x(\xi))d\xi,\quad x,\Delta x \in C([0,T],R^n),$$
$$t \in [0,T].$$

By Theorem 5.3 we have that for each fixed $x \in C([0,T],R^n)$ the multifunction $t \to \int_0^t \Phi(\xi,x(\xi))d\xi$, $t \in [0,T]$ is π-differentiable a.e. on $[0,T]$, and the π-derivative can be calculated with the formula in Theorem 5.3. If $X(t) = (x_{ij}(t))$ is such that each of the functions $t \to x_{ij}(t)$, $t \in [0,T]$, $i,j = 1,2,\dots,n$ changes signs only on a subset of $[0,T]$ of measure zero, then the multifunction $t \to X(t)\int_0^t \Phi(\xi,x(\xi))d\xi$, $t \in [0,T]$ is also π-differentiable a.e. on $[0,T]$. The partial derivative $\frac{\partial\hat{F}}{\partial t}(t,x)$ can easily be calculated, but the formula is tedious because of the complications discussed in Example 3.2, and we therefore omit the expression for $\frac{\partial\hat{F}}{\partial t}(t,x)$. The condition that each x_{ij} change signs only on a subset of $[0,T]$ of measure zero can be met if, for example, $t \to A(t)$, $t \in [0,T]$ is analytic, since in this case, $t \to X(t)$, $t \in [0,T]$ is analytic.

<u>Example 5.2</u>. Consider the differential inequality,

$$(5.16)\qquad \|\dot{x}-g(t,x)\| \le f(t,x),$$

where g: $R \times R^n \to R^n$ and f: $R \times R^n \to R$, $f \ge 0$. Clearly, solving (5.16) is equivalent to solving the contingent equation

$$(5.16') \qquad \dot{x}(t) \in F(t,x(t))$$

where $F(t,x) \equiv \{g(t,x)\} + f(t,x)\sigma_n$, where σ_n is the closed unit ball in R^n. Thus according to Examples 3.1 and 3.2 the multifunction F arising in the contingent equation (5.16') is π-differentiable (actually conically differentiable) on $R \times R^n$ if both g and f are differentiable on $R \times R^n$.

Example 5.3. Let $a,b: R \to R$ be continuous nonnegative functions. Consider the scalar differential inequality

$$(5.17) \qquad 0 \le \dot{x}(t) \le a(t) + b(t)x(t), \quad t \ge t_o$$
$$x(t_o) = x_o \ge 0.$$

Thus (5.17) is equivalent to the contingent equation

$$(5.17') \qquad \dot{x}(t) \in F(t,x(t)), \quad t \ge t_o$$
$$x(t_o) = x_o \ge 0$$

where $F(t,x) \equiv [0,a(t)+b(t)x]$, $t \in R$, $x \ge 0$. We use $t \to x(t,t_o,x_o)$, $t \ge t_o$ to denote a solution of (5.17) or (5.17'). From Gronwall's inequality we have that

$$(5.18) \qquad x_o \le x(t,t_o,x_o) \le k(t,t_o,x_o), \; t \ge t_o, \; x_o \ge 0$$

where

$$(5.18') \quad k(t,t_o,x_o) \equiv x_o+\int_{t_o}^{t} a(s)ds + \int_{t_o}^{t} ds[(x_o+\int_{t_o}^{s} a(\xi)d\xi)b(s)]\exp(\int_{s}^{t}b(\xi)d\xi),$$

for $t \geq t_o$, $x_o \geq 0$. Let $\Phi(t,t_o,x_o) \equiv \{y = x(t,t_o,x_o) | t \to x(t,t_o,x_o)$ satisfies (5.17')$\}$, $t \geq t_o$, $x_o \geq 0$. From (5.18) and (5.18') we have that $\Phi(t,t_o,x_o) \subset [x_o,k(t,t_o,x_o)]$, $t \geq t_o$, $x_o \geq 0$. Conversely one can establish that x_o and $k(t,t_o,x_o)$ belong to $\Phi(t,t_o,x_o)$ for $t \geq t_o$, $x_o \geq 0$. The convexity of $\Phi(t,t_o,x_o)$ then implies the reverse inclusion $\Phi(t,t_o,x_o) \supset [x_o,k(t,t_o,x_o)]$, $t \geq t_o$, $x_o \geq 0$. Thus $\Phi(t,t_o,x_o) = [x_o,k(t,t_o,x_o)]$, $t \geq t_o$, $x_o \geq 0$. From the representation in (5.18') $(t,t_o,x_o) \to k(t,t_o,x_o)$, $t \geq t_o$, $x_o \geq 0$ is differentiable. Consequently $(t,t_o,x_o) \to \Phi(t,t_o,x_o)$, $t \geq t_o$, $x_o \geq 0$ is π-differentiable, and since $\Phi(t,t_o,x_o) = [x_o,k(t,t_o,x_o)]$ one can calculate $\hat{\Phi}'(t,t_o,x_o)(\Delta t,\Delta t_o,\Delta x_o)$ according to Theorem 3.2 of Example 3.3. We can summarize this example as follows: if $a,b: R \to R$ are continuous nonnegative functions, and $(t,t_o,x_o) \to \Phi(t,t_o,x_o)$, $t \geq t_o$, $x_o \geq 0$ is the multifunction defined above; then Φ is π-differentiable, and $\hat{\Phi}'$ can be calculated from Theorem 3.2.

BIBLIOGRAPHY

[A-1] R. J. Aumann, "Integrals of set-valued functions", J. Math. Anal. Appl., 12(1965), pp. 1-12.

[B-1] C. Berge, Topological Spaces, Macmillan, New York, 1963.

[B-2] N. Bourbaki, General Topology, Parts 1 and 2, Addison-Wesley, Reading, Massachusetts, 1966.

[B-3] ____________, Integration, Hermann, Paris, 1965, Chaps. 1-4.

[B-4] T. F. Bridgland, "Trajectory integrals of set valued functions". to appear.

[B-5] ____________, "Contributions to the theory of generalized differential equations", I and II, to appear in Math. Systems Theory.

[B-6] P. Brunovsky, "On the necessity of a certain convexity condition for lower closure of control problems", SIAM J. Control, 6(1968), pp. 174-185.

[C-1] C. Castaing, "Sur les multi-applications mesurables", Doctoral Dissertation, Universite de Caen, France, 1967.

[C-2] L. Cesari, "Existence theorems for weak and usual optimal solutions in Lagrange problems with unilateral constraints", I and II, Trans. Amer. Math. Soc. 124(1966), pp. 369-412, 413-429.

[C-3] ____________, "Existence theorems for optimal controls of the Mayer type", SIAM J. Control, 6(1968), pp. 517-552.

[D-1] G. Debreu, "Integration of correspondences", Proc. Fifth Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, 1966, pp. 351-372.

[D-2] J. Dieudonne, Foundations of Modern Analysis, Academic Press, New York, 1960.

[D-3] N. Dunford and J. Schwartz, Linear Operators, Interscience, New York, 1958.

[F-1] A. F. Filippov, "Classical solutions of differential equations with multivalued right-hand side", SIAM J. Control, 5(1967), pp. 609-621.

[H-1] H. Halkin and E. C. Hendricks, "Subintegrals of set-valued functions with semi-analytic graphs", Proc. Nat. Acad. Sci., U.S.A., 59(1968), pp. 365-367.

[H-2] H. Hermes, "Calculus of set valued functions and control", J. Math. Mech., 18(1968), pp. 47-60.

[H-3] ____________, "The generalized differential equation $\dot{x} \in R(t,x)$", to appear.

[H-4] M. Hukuhara, "Integration des applications mesurables dont la valeur es un compact convexe, "Funkcialaj Ekvacioj, 10(1967), pp. 205-223.

[J-1] M. Q. Jacobs, "Measurable multivalued mappings and Lusin's theorem", Trans. Amer. Math. Soc., 134(1968), pp. 471-481.

[J-2] ____________, "On the approximation of integrals of multivalued functions", SIAM J. Control, 7, No. 1 (1969), to appear.

[K-1] N. Kikuchi, "On contingent equations satisfying the Caratheodory type conditions", Publ. RIMS, Kyoto Univ., Ser. A, 3(1968), pp. 361-371.

[K-2] ____________, "Control problems of contingent equation", Publ. RIMS, Kyoto Univ., Ser. A, 3(1967), pp. 85-99.

[K-3] ____________, "On some fundamental theorems of contingent equations in connection with the control problems", Publ. RIMS, Kyoto Univ., Ser. A, 3(1967), pp. 177-201.

[K-4] C. Kuratowski, Topologie I, Monografie Matematyczne, Warsaw (1958), 4th edition.

[L-1] A. Lasota and C. Olech, "On the closedness of the set of trajectories of a control system", Bull. Acad. Pol. Sci., Ser. sci., math., astr. et phys., 14(1966), pp. 615-621.

[L-2] A. Lasota and Z. Opial, "An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations", Bull. Acad. Pol. Sc., Ser. sci., math., astr. et phys., 13 (1965), pp. 781-786.

[M-1] E. Michael, "Topologies on spaces of subsets", Trans. Amer. Math. Soc., 71(1951), 152-182.

[M-2] ____________, "Continuous selections I", Annals of Mathematics", 63(1956), pp. 361-381.

[M-3] J. J. Moreau, "Fonctionelles convexes", Sem. sur les Equations aux Derivees Partielles, II, 1966-1967, College de France.

[O-1] C. Olech, "A note concerning set-valued measurable functions", Bull. Acad. Polon. Sci. Ser. sci., math., astr. et phys., 13(1965), pp. 317-321.

[O-2] ___________, "Lexicographical order, range of integral and 'bang-bang' principle", Mathematical Theory of Control, Ed. A. V. Balakrishnan and L. W. Neustadt, Academic Press, New York, 1967, pp. 35-45.

[P-1] G. B. Price, "The theory of integration", Trans. Amer. Math. Soc., 47(1940), pp. 1-50.

[R-1] H. Radström, "An embedding theorem for spaces of convex sets", Proc. Amer. Math. Soc. 3(1952), pp. 165-169.

[W-1] A. Wilansky, Functional Analysis, Blaisdell, New York, 1964.