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THE RESTORATION OF CONSTRAINTS IN NONHOLONOMIC PROBLEMS

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Abstract. This paper considers a system described by n differential equations of the first order involving n state variables and m control variables. It is assumed that a nominal state and a nominal control, not satisfying all the equations, but consistent with the boundary conditions, are given. An iterative procedure is developed leading to a varied state and a varied control consistent with all the equations and the boundary conditions. The procedure involves quasilinearization with an added optimality condition, namely, the requirement of least-square change of the control. Several numerical examples are supplied.

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1. Introduction

In problems described by nonholonomic equations[4], a nominal state and a nominal control approximating a solution (but not satisfying all the equations exactly) may be available. Starting from this nominal state and control, one may wish to determine a varied state and a varied control, close to the nominal and satisfying all the equations exactly. This situation arises in some of the iterative algorithms for minimizing functionals of variables subject to nonholonomic constraints, namely, first variation methods and second variation methods. In this paper, which is a continuation of the research on holonomic problems presented in Ref. 1, a systematic procedure is developed to change the state and the control in an optimal way: this is the requirement that the constraints be restored with the least-square change of the control.

[4] The adjective nonholonomic is employed in this paper in the sense of nonentire, that is, differential.

2. Preliminary Considerations

Consider a system described by the nonholonomic equation

$$\dot{x} = \varphi(x,u,t) \tag{1}$$

where φ is a scalar function of the scalar arguments x (state variable), u (control variable), and t (independent variable, time); the dot denotes the derivative with respect to the time t. Assume that the state x is subject to the boundary conditions

$$x(0) = \alpha\ , \qquad x(\tau) = \beta \tag{2}$$

where α, β, τ are prescribed scalar quantities.

Suppose that nominal functions x(t), u(t) satisfying the boundary conditions (2), but not consistent with the differential constraint (1), are available. Let $\tilde{x}(t)$, $\tilde{u}(t)$ denote varied functions related to the nominal functions as follows:

$$\tilde{x}(t) = x(t) + \delta x(t)\quad , \qquad \tilde{u}(t) = u(t) + \delta u(t) \tag{3}$$

where $\delta x(t)$, $\delta u(t)$ denote the perturbations of x, u about the nominal functions. If quasi-linearization is employed, Eq. (1) is approximated by

$$\delta\dot{x} = \delta\varphi - (\dot{x} - \varphi) \tag{4}$$

where

$$\delta\varphi = \varphi_x\,\delta x + \varphi_u\,\delta u \tag{5}$$

denotes the first-order change of φ. From (4) and (5), one obtains the relation

$$\delta\dot{x} = \varphi_x \delta x + \varphi_u \delta u - (\dot{x} - \varphi) \tag{6}$$

which is subject to the boundary conditions

$$\delta x(0) = 0 \,, \quad \delta x(\tau) = 0 \tag{7}$$

In Eq. (6), the time-dependent coefficients φ_x, φ_u, and $(\dot{x} - \varphi)$ are computed for the nominal functions x(t), u(t). Equation (6) is the relation to be satisfied by the corrections $\delta x(t)$, $\delta u(t)$ in order to restore the constraint (1) to first order. Since we have one equation and two unknown functions, the solution of the restoration problem is nonunique. However, a unique solution can be obtained if we impose an additional condition on the system of variations.

If the functions x(t), u(t) are an approximation to an interesting solution, one may wish to restore the constraint (1) while causing the least-square change of the control[5]. Therefore, we minimize the functional

$$J = \frac{1}{2} \int_0^\tau (\delta u)^2 \, dt \tag{8}$$

subject to the linearized constraint (6) and the boundary conditions (7). Standard methods of the calculus of variations (see, for instance, Chapter 2 of Ref. 2) show that the fundamental function of this problem is given by

$$F = \frac{1}{2} (\delta u)^2 + \lambda(\delta\dot{x} - \varphi_x \delta x - \varphi_u \delta u + \dot{x} - \varphi) \tag{9}$$

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[5] For example, the nominal functions x(t), u(t) could be those obtained at the end of any gradient phase of a minimization algorithm. In this case, one may wish to restore the constraint (1) before starting the next gradient phase.

where $\lambda(t)$ denotes an undetermined, variable Lagrange multiplier. The Euler equations of this problem are

$$\dot{\lambda} = -\varphi_x \lambda , \quad \delta u = \varphi_u \lambda \tag{10}$$

and are to be solved in combination with Eq. (6) and the boundary conditions (7). Upon eliminating δu from (6) and (10-2), we obtain the differential system

$$\delta\dot{x} = \varphi_x \delta x + \varphi_u^2 \lambda - (\dot{x} - \varphi) , \quad \dot{\lambda} = -\varphi_x \lambda \tag{11}$$

which must be integrated subject to the boundary conditions (7). Once the functions $\delta x(t)$ and $\lambda(t)$ are known, the function $\delta u(t)$ can be computed from (10-2).

Since Eqs. (11) are linear in δx and λ, any of the methods for solving linear equations with variable coefficients can be employed. For example, let the method of particular solutions be used (Ref. 3). Denote by[6]

$$\delta x_1 = \delta x_1(t) , \quad \lambda_1 = \lambda_1(t) \tag{12}$$

the particular solution of the system (11) subject to the initial conditions

$$\delta x_1(0) = 0 , \quad \lambda_1(0) = 1 \tag{13}$$

Next, denote by[6]

$$\delta x_2 = \delta x_2(t) , \quad \lambda_2 = \lambda_2(t) \tag{14}$$

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[6] The subscripts 1 and 2 denote the first and second integration, respectively.

the particular solution of the system (11) subject to the initial conditions

$$\delta x_2(0) = 0 \ , \quad \lambda_2(0) = 0 \tag{15}$$

Then, the linear combinations

$$\delta x(t) = k_1 \delta x_1(t) + k_2 \delta x_2(t) \ , \qquad \lambda(t) = k_1 \lambda_1(t) + k_2 \lambda_2(t) \tag{16}$$

satisfy the differential equations (11) and the boundary conditions (7) providing the constants k_1 and k_2 are chosen as follows:

$$k_1 + k_2 = 1 \quad , \quad k_1 \delta x_1(\tau) + k_2 \delta x_2(\tau) = 0 \tag{17}$$

This means that

$$k_1 = \frac{\delta x_2(\tau)}{\delta x_2(\tau) - \delta x_1(\tau)} \quad , \quad k_2 = - \frac{\delta x_1(\tau)}{\delta x_2(\tau) - \delta x_1(\tau)} \tag{18}$$

The previous algorithm can be summarized as follows: (a) assume nominal functions $x(t)$, $u(t)$; (b) compute the variable coefficients φ_x, φ_u, and $(\dot{x} - \varphi)$; (c) determine the particular solution $\delta x_1(t)$, $\lambda_1(t)$ by forward integration of Eqs. (11) subject to the initial conditions (13); also, determine the particular solution $\delta x_2(t)$, $\lambda_2(t)$ by forward integration of Eqs. (11) subject to the initial conditions (15); (d) compute the constants k_1 and k_2 from Eqs. (18); (e) determine the correction $\delta x(t)$ with Eq. (16-1), the function $\lambda(t)$ with Eq. (16-2), and the correction $\delta u(t)$ with Eq. (10-2); and (f) compute the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$ with Eqs. (3). In this way, the first iteration is completed. Next, the functions $\tilde{x}(t)$, $\tilde{u}(t)$ given by Eqs. (3) are employed as the nominal functions $x(t)$, $u(t)$ for the second iteration, and the procedure is repeated until a desired degree of accuracy is obtained, that is, until the inequality[7]

$$P = \int_0^T (\dot{x} - \varphi)^2 dt \le \epsilon \tag{19}$$

is satisfied, where ϵ is a small number.

Remark 2.1. If the nominal functions $x(t)$, $u(t)$ satisfy the constraint (1), the forcing term $(\dot{x} - \varphi)$ is missing in Eq. (11-1). Therefore, Eqs. (11) become homogeneous and, for the boundary conditions (7), admit the solutions

$$\delta x(t) = 0 \ , \quad \lambda(t) = 0 \tag{20}$$

with the implication that

$$\delta u(t) = 0 \tag{21}$$

everywhere.

Remark 2.2. An interesting modification of the previous problem arises if the boundary conditions (2) are replaced by

$$x(0) = \alpha \ , \quad x(\tau) = \text{free} \tag{22}$$

Therefore, the restoration problem consists of minimizing the integral (8) subject to the linearized constraint (6) and the boundary conditions

$$\delta x(0) = 0 \ , \quad \delta x(\tau) = \text{free} \tag{23}$$

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[7] The symbol P denotes the performance index.

As in the fixed-endpoint problem, the optimum functions $\delta x(t)$, $\delta u(t)$, $\lambda(t)$ are described by the differential equations (6) and (10). However, the boundary conditions are different, in the sense that Eqs. (7) must be replaced by

$$\delta x(0) = 0 \,, \quad \lambda(\tau) = 0 \tag{24}$$

with condition (24-2) resulting from the transversality condition of the calculus of variations (see, for instance, Chapter 2 of Ref. 2). In the light of (24-2), Eq. (10-1) is solved by

$$\lambda(t) = 0 \tag{25}$$

and Eq. (10-2) by

$$\delta u(t) = 0 \tag{26}$$

with the implication that

$$\tilde{u}(t) = u(t) \tag{27}$$

and that $\dot{J} = 0$. Under these conditions, Eq. (6) reduces to

$$\delta \dot{x} = \varphi_x \delta x - (\dot{x} - \varphi) \tag{28}$$

and, in combination with the initial condition (24-1), supplies the correction $\delta x(t)$ to first-order terms. Once $\delta x(t)$ is known, the varied function $\tilde{x}(t)$ can be computed with Eq. (3-1). In theory, this procedure must be employed iteratively until the converged solution is obtained. In practice, one can bypass the linearized equation (28) and restore the constraint (1) directly: the function $\tilde{x}(t)$ is obtained by forward integration of Eq. (1) subject to the initial condition (22-1) and the control law (27).

3. General Theory

Consider a system described by the nonholonomic equations

$$
\begin{aligned}
\dot{x}^1 &= \varphi^1(x^1, x^2, \ldots, x^n, u^1, u^2, \ldots, u^m, t) \\
\dot{x}^2 &= \varphi^2(x^1, x^2, \ldots, x^n, u^1, u^2, \ldots, u^m, t) \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\dot{x}^n &= \varphi^n(x^1, x^2, \ldots, x^n, u^1, u^2, \ldots, u^m, t)
\end{aligned}
\tag{29}
$$

where $\varphi^1, \varphi^2, \ldots, \varphi^n$ denote scalar functions of the scalar arguments $x^1, x^2, \ldots, x^n$ (state variables), $u^1, u^2, \ldots, u^m$ (control variables), and t (independent variable, time). In vector-matrix notation, Eqs. (29) can be rewritten in the form

$$\dot{x} = \varphi(x, u, t) \tag{30}$$

where x denotes an n-vector, u an m-vector, and φ an n-vector, respectively defined as follows:

$$
x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix}, \qquad u = \begin{bmatrix} u^1 \\ u^2 \\ \vdots \\ u^m \end{bmatrix}, \qquad \varphi(x,u,t) = \begin{bmatrix} \varphi^1(x,u,t) \\ \varphi^2(x,u,t) \\ \vdots \\ \varphi^n(x,u,t) \end{bmatrix}
\tag{31}
$$

Assume that the state variables are subject to the end conditions

$$\begin{aligned} x^1(0) &= \alpha^1 \quad , \quad x^1(\tau) = \beta^1 \\ x^2(0) &= \alpha^2 \quad , \quad x^2(\tau) = \beta^2 \\ &\cdots\cdots\cdots\cdots \\ x^n(0) &= \alpha^n \quad , \quad x^n(\tau) = \beta^n \end{aligned} \tag{32}$$

where $\alpha^1, \alpha^2, \ldots, \alpha^n$ and $\beta^1, \beta^2, \ldots, \beta^n$ denote prescribed scalar quantities. In vector-matrix notation, Eqs. (32) can be rewritten in the form

$$x(0) = \alpha \quad , \quad x(\tau) = \beta \tag{33}$$

where α and β denote n-vectors, respectively defined as follows:

$$\alpha = \begin{bmatrix} \alpha^1 \\ \alpha^2 \\ \vdots \\ \alpha^n \end{bmatrix} \quad , \quad \beta = \begin{bmatrix} \beta^1 \\ \beta^2 \\ \vdots \\ \beta^n \end{bmatrix} \tag{34}$$

Next, suppose that <u>nominal functions</u> $x(t)$, $u(t)$ satisfying the boundary conditions (33), but not consistent with the differential constraint (30), are available. Let $\tilde{x}(t)$, $\tilde{u}(t)$ denote <u>varied functions</u> related to the nominal functions as follows:

$$\tilde{x}(t) = x(t) + \delta x(t) \quad , \quad \tilde{u}(t) = u(t) + \delta u(t) \tag{35}$$

If quasilinearization is employed, Eq. (30) is approximated by[8]

$$\delta\dot{x} = A^T \delta x + B^T \delta u - (\dot{x} - \varphi) \tag{36}$$

[8] The superscript T denotes the transpose of a matrix.

where A denotes the n x n matrix

$$A(x,u,t) = \begin{bmatrix} \partial\varphi^1/\partial x^1 & \partial\varphi^2/\partial x^1 & \ldots\ldots & \partial\varphi^n/\partial x^1 \\ \partial\varphi^1/\partial x^2 & \partial\varphi^2/\partial x^2 & \ldots\ldots & \partial\varphi^n/\partial x^2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial\varphi^1/\partial x^n & \partial\varphi^2/\partial x^n & \ldots\ldots & \partial\varphi^n/\partial x^n \end{bmatrix} \tag{37}$$

and B denotes the m x n matrix

$$B(x,u,t) = \begin{bmatrix} \partial\varphi^1/\partial u^1 & \partial\varphi^2/\partial u^1 & \ldots\ldots & \partial\varphi^n/\partial u^1 \\ \partial\varphi^1/\partial u^2 & \partial\varphi^2/\partial u^2 & \ldots\ldots & \partial\varphi^n/\partial u^2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial\varphi^1/\partial u^m & \partial\varphi^2/\partial u^m & \ldots\ldots & \partial\varphi^n/\partial u^m \end{bmatrix} \tag{38}$$

Note that the jth column of the matrix (37) is the gradient of the function φ^j with respect to the vector x; analogously, the jth column of the matrix (38) is the gradient of the function φ^j with respect to the vector u. The boundary conditions (33) become

$$\delta x(0) = 0 \quad , \quad \delta x(\tau) = 0 \tag{39}$$

If the functions x(t), u(t) are an approximation to an interesting solution, one may wish to restore the constraint (30), while causing the least-square change of the control. Therefore, we minimize the functional

$$J = \frac{1}{2} \int_0^{\tau} \delta u^T \delta u \, dt \tag{40}$$

subject to the linearized constraint (36) and the boundary conditions (39).

Standard methods of the calculus of variations (see, for instance, Chapter 2 of Ref. 2) show that the fundamental function of this problem is given by

$$F = \frac{1}{2}\,\delta u^T \delta u + \lambda^T(\delta\dot{x} - A^T\delta x - B^T\delta u + \dot{x} - \varphi) \tag{41}$$

where $\lambda(t)$, an n-vector, denotes the undetermined, variable Lagrange multiplier

$$\lambda = \begin{bmatrix} \lambda^1 \\ \lambda^2 \\ \vdots \\ \lambda^n \end{bmatrix} \tag{42}$$

The Euler equations of this problem are

$$\dot{\lambda} = -A\lambda \quad , \quad \delta u = B\lambda \tag{43}$$

and are to be solved in combination with Eq. (36) and the boundary conditions (39). Upon eliminating δu from (36) and (43-2), we obtain the differential system

$$\delta\dot{x} = A^T\delta x + (B^T B)\lambda - (\dot{x} - \varphi) \quad , \quad \dot{\lambda} = -A\lambda \tag{44}$$

which must be integrated subject to the boundary conditions (39). Once the functions $\delta x(t)$ and $\lambda(t)$ are known, the function $\delta u(t)$ can be computed from (43-2).

Since Eqs. (44) are linear in δx and λ, any of the methods for solving linear equations with variable coefficients can be employed. For example, let the method of particular solutions be used (Ref. 3). To this effect, we integrate Eqs. (44) forward n + 1 times from $t = 0$ to $t = \tau$ using $n + 1$ different sets of initial conditions and the stopping condition $t = \tau$. From these integrations, we obtain the pairs of functions[9]

$$\delta x_i = \delta x_i(t) \quad , \quad \lambda_i = \lambda_i(t) \quad , \qquad i = 1, 2, \ldots, n+1 \tag{45}$$

each of which is a particular solution of (44). In each integration, the prescribed initial condition (39-1) is employed. That is, $\delta x_i(0)$ is such that

$$\delta x_i(0) = 0 \quad , \qquad i = 1, 2, \ldots, n+1 \tag{46}$$

We note that, for each i, Eq. (46) is equivalent to n scalar conditions. Since 2n initial conditions are needed for each integration, Eq. (46) must be completed by the relation

$$\lambda_i(0) = \gamma_i \quad , \qquad i = 1, 2, \ldots, n+1 \tag{47}$$

where, for each i, γ_i denotes the n-vector

$$\gamma_i = \begin{bmatrix} \delta_{i1} \\ \delta_{i2} \\ \vdots \\ \delta_{in} \end{bmatrix} \quad , \qquad i = 1, 2, \ldots, n+1 \tag{48}$$

The elements of (48) are Kronecker deltas, such that

$$\begin{aligned} \delta_{ij} &= 1 \quad , \qquad i = j \\ \delta_{ij} &= 0 \quad , \qquad i \neq j \end{aligned} \tag{49}$$

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[9] The subscript i denotes the generic integration.

In the light of (48)-(49), the explicit form of (47) is the following:

$$\lambda_1(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \lambda_2(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \ldots\ldots, \lambda_n(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \lambda_{n+1}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{50}$$

Next, we introduce the n + 1 undetermined, scalar constants k_i and form the linear combinations

$$\delta x(t) = \sum_{i=1}^{n+1} k_i \delta x_i(t), \qquad \lambda(t) = \sum_{i=1}^{n+1} k_i \lambda_i(t) \tag{51}$$

Then, we inquire whether, by an appropriate choice of the constants, these linear combinations can satisfy the differential equations (44) and the end conditions (39). As shown in Ref. 3, this is precisely the case if the constants k_i are determined as follows:

$$\sum_{i=1}^{n+1} k_i = 1, \qquad \sum_{i=1}^{n+1} k_i \delta x_i(\tau) = 0 \tag{52}$$

Equations (52) are equivalent to the n + 1 scalar equations

$$\begin{aligned}
&k_1 + k_2 + \ldots\ldots + k_{n+1} = 1 \\
&k_1 \delta x_1^1(\tau) + k_2 \delta x_2^1(\tau) + \ldots\ldots + k_{n+1} \delta x_{n+1}^1(\tau) = 0 \\
&k_1 \delta x_1^2(\tau) + k_2 \delta x_2^2(\tau) + \ldots\ldots + k_{n+1} \delta x_{n+1}^2(\tau) = 0 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&k_1 \delta x_1^n(\tau) + k_2 \delta x_2^n(\tau) + \ldots\ldots + k_{n+1} \delta x_{n+1}^n(\tau) = 0
\end{aligned} \tag{53}$$

which are linear and supply the constants $k_1, k_2, \ldots, k_{n+1}$.

The previous algorithm can be summarized as follows: (a) assume nominal functions $x(t)$, $u(t)$; (b) compute the variable matrices A, B, $(\dot{x} - \varphi)$; (c) determine the $n + 1$ particular solutions $\delta x_i(t)$, $\lambda_i(t)$ by integrating Eqs. (44) subject to the initial conditions (46)-(47); (d) compute the $n + 1$ constants k_i from Eqs. (53); (e) determine the correction $\delta x(t)$ with Eq. (51-1), the function $\lambda(t)$ with Eq. (51-2), and the correction $\delta u(t)$ with Eqs. (43-2); and (f) compute the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$ with Eqs. (35). In this way, the first iteration is completed. Next, the functions $\tilde{x}(t)$, $\tilde{u}(t)$ given by Eqs. (35) are employed as the nominal functions $x(t)$, $u(t)$ for the second iteration, and the procedure is repeated until a desired degree of accuracy is obtained, that is, until the inequality

$$P = \int_0^T (\dot{x} - \varphi)^T (\dot{x} - \varphi)\, dt \leq \epsilon \tag{54}$$

is satisfied, where ϵ is a small number.

Remark 3.1. If the nominal functions $x(t)$, $u(t)$ satisfy the constraints (30) exactly, the forcing term $(\dot{x} - \varphi)$ is missing in Eq. (44-1). Therefore, Eqs. (44) become homogeneous and, for the boundary conditions (39), admit the solutions

$$\delta x(t) = 0 \quad , \quad \lambda(t) = 0 \tag{55}$$

with the implication that

$$\delta u(t) = 0 \tag{56}$$

everywhere.

Remark 3.2. An interesting modification of the previous problem arises if the boundary conditions (33) are replaced by

$$x(0) = \alpha \quad , \quad x(\tau) = \text{free} \tag{57}$$

Therefore, the restoration problem consists of minimizing the integral (40) subject to the linearized constraint (36) and the boundary conditions

$$\delta x(0) = 0 \quad , \quad \delta x(\tau) = \text{free} \tag{58}$$

As in the fixed-endpoint problem, the optimum functions $\delta x(t)$, $\delta u(t)$, $\lambda(t)$ are described by the differential equations (36) and (43). However, the boundary conditions are different in the sense that Eqs. (39) must be replaced by

$$\delta x(0) = 0 \quad , \quad \lambda(\tau) = 0 \tag{59}$$

with condition (59-2) resulting from the transversality condition of the calculus of variations (see, for instance, Chapter 2 of Ref. 2). In the light of (59-2), Eq. (43-1) is solved by

$$\lambda(t) = 0 \tag{60}$$

and Eq. (43-2) by

$$\delta u(t) = 0 \tag{61}$$

with the implication that

$$\tilde{u}(t) = u(t) \tag{62}$$

and that $J = 0$. Under these conditions, Eq. (36) reduces to

$$\delta\dot{x} = A^T \delta x - (\dot{x} - \varphi) \tag{63}$$

and, in combination with the initial condition (59-1), supplies the correction $\delta x(t)$ to first-order terms. Once $\delta x(t)$ is known, the varied function $\tilde{x}(t)$ can be computed with Eq. (35-1). In theory, this procedure must be employed iteratively until the converged solution is obtained. In practice, one can bypass the linearized equation (63) and restore the constraint (30) directly: the function $\tilde{x}(t)$ is obtained by forward iteration of Eq. (30) subject to the initial condition (57-1) and the control law (62).

Remark 3.3. In the previous sections, we considered two limiting cases: (a) the case where the n state variables are all given at the final point and (b) the case where the n state variables are all free at the final point. For case (a), the optimum control change is $\delta u(t) \neq 0$. For case (b), the optimum control change is $\delta u(t) = 0$; this corresponds to the customary way in which the constraints are restored in the gradient method.

Between the previous limiting cases a great variety of intermediate situations can be imagined. For example, p state variables may be given and q may be free at the final point, with $p + q = n$. Only q Lagrange multipliers vanish at the final point, and the remaining p Lagrange multipliers must be determined by solving the two-point boundary value problem. Since the n Lagrange multipliers do not vanish simultaneously at the final point, the optimum control change is $\delta u(t) \neq 0$.

4. Numerical Examples

In order to illustrate the theory, several numerical examples are now supplied. For simplicity, the symbols employed in this section denote scalar quantities.

Example 4.1. Consider the differential constraint

$$\dot{x} = x^2 + u \tag{64}$$

subject to the boundary conditions

$$x(0) = 1 \quad , \quad x(1) = 1 \tag{65}$$

Assume the following nominal functions:

$$x(t) = 1 \quad , \quad u(t) = 0 \tag{66}$$

Clearly, these functions satisfy the boundary conditions (65) but not the differential constraint (64). To restore the constraint, the algorithm of Section 3 is employed and is repeated until Ineq. (54) is satisfied for $\epsilon = 10^{-10}$. Computations performed with a Burroughs B-5500 computer in double-precision arithmetic are summarized in Tables 1 and 2, where N denotes the iteration number.

Example 4.2. Consider the differential constraint

$$\dot{x} = x^4 + u \tag{67}$$

subject to the boundary conditions

$$x(0) = 0 \quad , \quad x(1) = 1 \tag{68}$$

Assume the following nominal functions:

$$x(t) = t \quad , \quad u(t) = 0.5 \tag{69}$$

Clearly, these functions satisfy the boundary conditions (68) but not the differential constraint (67). To restore the constraint, the algorithm of Section 3 is employed and is repeated until Ineq. (54) is satisfied for $\varepsilon = 10^{-10}$. Computations performed with a Burroughs B-5500 computer in double-precision arithmetic are summarized in Tables 3 and 4, where N denotes the iteration number.

<u>Example 4.3.</u> Consider the differential constraints

$$\dot{x} = y \ , \quad \dot{y} = 2 \sin u - 1 \tag{70}$$

subject to the boundary conditions

$$x(0) = 0 \quad , \quad y(0) = 0 \quad , \quad x(1) = 0.3 \quad , \quad y(1) = 0 \tag{71}$$

Assume the following nominal functions:

$$x(t) = 0.3t \quad , \quad y(t) = 0 \quad , \quad u(t) = 0 \tag{72}$$

Clearly, these functions satisfy the boundary conditions (71) but not the differential constraints (70). To restore the constraints, the algorithm of Section 3 is employed and is repeated until Ineq. (54) is satisfied for $\varepsilon = 10^{-10}$. Computations performed with a Burroughs B-5500 computer in double-precision arithmetic are summarized in Tables 5 and 6, where N denotes the iteration number.

Table 1. Converged values of the functions x(t), u(t)

t	x(t)	u(t)
0.0	1.0000	-3.5233
0.1	0.7774	-2.5263
0.2	0.6147	-1.7152
0.3	0.5085	-1.0536
0.4	0.4539	-0.5128
0.5	0.4454	-0.0698
0.6	0.4782	0.2939
0.7	0.5492	0.5937
0.8	0.6574	0.8415
0.9	0.8054	1.0472
1.0	1.0000	1.2185

Table 2. The function P(N), J(N)

N	P	J
0	0.90×10^{1}	
1	0.32×10^{-1}	0.34×10^{1}
2	0.79×10^{-6}	0.97×10^{-2}
3	0.98×10^{-16}	0.21×10^{-6}

Table 3. Converged values of the functions x(t), u(t)

t	x(t)	u(t)
0.0	0.0000	0.9008
0.1	0.0900	0.9007
0.2	0.1801	0.9001
0.3	0.2703	0.8975
0.4	0.3608	0.8906
0.5	0.4520	0.8764
0.6	0.5448	0.8519
0.7	0.6408	0.8150
0.8	0.7430	0.7658
0.9	0.8579	0.7076
1.0	1.0000	0.6472

Table 4. The functions P(N), J(N)

N	P	J
0	0.16×10^{0}	
1	0.32×10^{-4}	0.61×10^{-1}
2	0.79×10^{-11}	0.61×10^{-5}

Table 5. Converged values of the functions x(t), y(t), u(t)

t	x(t)	y(t)	u(t)
0.0	0.0000	0.0000	1.5602
0.1	0.0049	0.0999	1.5193
0.2	0.0199	0.1990	1.4384
0.3	0.0447	0.2952	1.3087
0.4	0.0787	0.3828	1.1232
0.5	0.1206	0.4513	0.8788
0.6	0.1678	0.4847	0.5795
0.7	0.2158	0.4643	0.2387
0.8	0.2584	0.3761	-0.1207
0.9	0.2886	0.2178	-0.4711
1.0	0.3000	0.0000	-0.7871

Table 6. The functions P(N), J(N)

N	P	J
0	0.10×10^{1}	
1	0.79×10^{-1}	0.26×10^{0}
2	0.32×10^{-2}	0.29×10^{-1}
3	0.31×10^{-4}	0.28×10^{-2}
4	0.17×10^{-8}	0.21×10^{-4}
5	0.38×10^{-17}	0.97×10^{-9}

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