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SD 69-17

FINITE-THRUST TRANSFER IN THE TWO- AND THREE-BODY PROBLEMS

FINAL REPORT

24 January 1969

Prepared by

G.A. McCue
D.F. Bender
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J.R. Radbill

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C.R. Faulders, Manager
Flight Technology
Science and Technology

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FOREWORD

This report constitutes the final technical document required under Contract NAS8-21077, Optimum Finite Thrust Orbital Transfer Study. The work described herein was performed by the authors at the Space Division of North American Rockwell Corporation during the period commencing on February 7, 1967 and ending on March 8, 1969.

Please note that this report and its attachments supersede all previous reports under the contract. These constitute a summary of extensive contractual studies which are documented in detail in the following four reports:

SD 69-4	Quasilinearization Program for Determining Optimum Finite-Thrust Transfers Between Inclined Orbits
SD 69-3	Program for Optimization of Two-Impulse Transfers by Contouring and Steep Descent
SD 68-1055	Generalized Quasilinearization Routines for Solving Nonlinear Boundary-Value Problems
SD 68-309	Two-Impulse Transfer in the Three-Body Problem

ABSTRACT

The results obtained under the study contract NAS8-21077 are summarized. These studies have involved the use of the quasilinearization technique for the solution of two-point boundary value problems concerned with orbital transfer in the two- and three-body problems. It was possible to obtain optimal finite-thrust transfers between any two elliptical non-coapsidal inclined orbits about a single attracting center. The procedure has been completely checked out in the FORTRAN H programming language using an IBM System 360, Model 65, digital computer. It was found that successful quasilinearization solution of this "bang-bang" control problem, in most cases, required a good a priori knowledge of the complete time histories of the state variables and Lagrange multipliers. Methods were therefore developed for predicting the primer vector time history with an accuracy of about one percent.

The quasilinearization technique was also applied to the computation of impulsive transfers between given terminals in a given time in the three-body problem. The resulting computer program is in an essentially completed state and is available for studies. The description of the program and the details for operating it are also documented and available.

In order to extend the above work, the authors obtained preliminary results concerning the development of a program for computing finite-thrust transfer in the three-body problem. For the three-body case, the problem has been formulated in rotating coordinates, and the Jacobian matrix has been derived and programmed in preparation for use in a three-body finite-thrust quasilinearization optimization computation.

CONTENTS

ILLUSTRATIONS

I. INTRODUCTION

During the past ten years, the authors have participated in a series of contractual studies of optimal orbital transfer and rendezvous. Under the initial contract, NAS8-4, work consisted of formulation and parameter studies involving coplanar two-impulse transfer (References 1, 2, and 3). The second contracted effort (NAS8-1582) was devoted to developing numerical methods for finding the absolute minimum two-impulse transfers between arbitrary non-coplanar non-coapsidal elliptical orbits. This work, which is documented in References 4 through 13, led to several computational methods for solving such problems. The third contract in this series (NAS8-5211) produced several refinements to the previously successful numerical techniques (References 14 and 15). It also led to the development of a steep-descent numerical optimization program (References 16 and 17). Using this numerical program, it was possible to conduct a number of studies which have now been published (References 17 through 20). This contract also produced a variational formulation of the finite-thrust optimum orbital transfer problem (Reference 21). The formulation was programmed for solution using an ordinary Newton-Raphson convergence technique which was later found to be inadequate for this extremely sensitive problem. The results of this third contract are summarized in Reference 22.

Under the fourth contract (NAS8-20238), effort was concentrated upon solving the two-dimensional finite-thrust orbital transfer problem and upon an investigation of impulsive transfer in the three-body problem. During the course of the study, both of these problems were satisfactorily solved by employing a mathematical technique known as quasilinearization (References 23 through 26). The results of this fourth contract are summarized in Reference 27.

This report summarizes the results of the latest contract in this series (NAS8-21077). Under this contract, work has been primarily concerned with the development of numerical programs for computing optimal finite-thrust transfers in the two-body problem, two-impulse transfers in the three-body problem, and optimal finite-thrust transfers in the three-body problem (References 28 through 31).

In order to make the complete computer program documentation beneficial to a wide number of diverse users, the authors have chosen to divide it into the following four reports:

Quasilinearization Program for Determining Optimum Finite-Thrust Transfers Between Inclined Orbits, SD 69-4 (Reference 28).

Program for Optimization of Two-Impulse Transfers by Contouring and Steep Descent, SD 69-3 (Reference 29).

Generalized Quasilinearization Routines for Solving Nonlinear Boundary-Value Problems, SD 68-1055 (Reference 30).

Two-Impulse Transfer in the Three-Body Problem, SD 68-309 (Reference 31).

Because of this logical subdivision, it is possible to offer prospective users of specific parts of the program only that documentation that they require. For instance, a user who wishes to apply the generalized quasilinearization subroutine to a heat transfer problem will only require Reference 30.

Similarly, the subroutine package used to provide data for the initial iterate computation to the finite-thrust program may be employed separately to study two-impulse orbital transfer.

II. OPTIMAL FINITE-THRUST TRANSFERS IN THE TWO-BODY PROBLEM

INTRODUCTION

The objective of this task was to investigate methods for computing finite-thrust transfers between any two orbits about a single attracting center in three-dimensional space. The authors are pleased to report the successful completion of this task, and the development of an IBM System 360 FORTRAN H digital computer program for computing the optimal trajectories. The complete program has been documented in References 28, 29, and 30.

For a number of years, the authors have addressed themselves to the general problem of obtaining numerical solutions to the two-point boundary value problems associated with optimal orbital transfer and rendezvous. Significant new results, which are reported here, involve techniques for obtaining solutions to the set of 14 discontinuous non-linear differential equations which specify the optimal one- or two-burn fixed-time finite-thrust transfer between any two orbits about a single attracting center in three-dimensional space. It has been found that solution of this "bang-bang" control problem requires the power of the quasilinearization technique, and, in most cases, requires also a good a priori estimate of the complete time histories of the state variables and Lagrange multipliers. Methods have therefore been developed for predicting the primer vector time history with an accuracy of about one percent. This information, in turn, allows a priori specification of the proper control variables for vehicle steering and thrust initiation and termination. Of critical importance to the above formulation is an accurate knowledge of an optimum two-impulse maneuver. The computational techniques for obtaining the first iterate and for controlling the computational procedure are of prime importance to the effective utilization of the quasilinearization procedure. The numerous equations and mathematical details of the complete method are not included in this descriptive summary. However, they may be found in complete detail in Reference 28, which constitutes the finite-thrust computer program documentation.

SIGNIFICANT NEW RESULTS

The initial iterates for the Lagrange multipliers are derived from an optimum two-impulse transfer which is obtained using the program of Reference 29. The multipliers that correspond to velocity and position, i.e., the primer vector and its derivative, are obtained for the coasting arc of the transfer orbit by assuming that the Hamiltonian of the system is zero (time-open problem) and that the direction of the primer gives the impulsive thrust directions used for both burns. The values of the Lagrange multipliers for position and velocity along the initial and final orbits then are inferred from the continuity conditions required at the corners (impulse points). These are joined over the two-burn intervals by a straight line or by a cubic. This portion of the procedure must determine the initial time histories of the very sensitive control parameters. The procedure for extracting this information, therefore, contains several innovations which are reported in Reference 28.

The technique of quasilinearization, as documented in Reference 30, is now used to find initial boundary conditions which also satisfy the required final boundary conditions. An important aspect of the computer program is the management of the extremely sensitive switching times. Full changes in these times as called for by the program are not allowed until the multipliers contain more meaningful information than the initial iterate. (See Reference 26 or 28.)

As an example of the excellent starting iterates obtained by the above techniques, the reader is directed to Figures 1 through 7. Figure 1 depicts an optimal finite-thrust transfer between two inclined elliptical orbits. The table at the top of Figure 1 lists the parameters of the two-impulse transfer which was used as a starting iterate.

Figures 2 through 7 are a series of computer-generated graphs which show the type of results obtained. On each of the graphs there is a solid line which represents the converged solution for the variable. The initial time history is plotted on the same graph as a series of points indicated by the symbol "+." Note the close agreement. Especially, note the accuracy of the initial iterate in Figures 3, 4, and 5. These particular parameters are extremely sensitive to small deviation from the optimal impulsive solution.

As one might expect, these superior initial iterates led to convergence properties which were significantly improved over those reported in Reference 26. In fact, certain cases which were previously troublesome or non-convergent responded quite favorably. These results further substantiate the power of quasilinearization for solving extremely complicated and sensitive multipoint boundary-value problems. However, they also indicate the care and detail that is required for consistent success.

The above work led to the development of a computer program which was used to generate a number of interesting new comparisons of the results

*** FINAL TRANSFER CONDITIONS *** CENTER IS EARTH , UNIT SYSTEM...MILES, FT/SEC

IMPULSE	X COMPONENT	Y COMPONENT	Z COMPONENT	MAGNITUDE
FIRST VECTOR	-0.26828421294D 04	0.95107456929D 03	0.35347669241D 03	0.28682974913D 04
SECOND VECTOR	-0.95340475006D 03	0.76483369258D 03	0.16531246588D 04	0.20343537324D 04
TOTAL				0.49026512239D 04

	INITIAL ORBIT	FINAL ORBIT	TRANSFER ORBIT
P	0.50000000000D 04	0.60000000000D 04	0.66448493661D 04
E	0.20000000000D 00	0.20000000000D 00	0.10433053296D 00
I (DEG)	0.30000000000D 01	0.0	0.52452458751D 01
R A NODE (DEG)	0.0	0.0	0.75567912386D 01
ARG PER (DEG)	-0.90000000000D 02	0.30000000000D 02	0.21239662287D 02
A	0.52083333333D 04	0.62500000000D 04	0.67179735686D 04
PERIOD	0.76369637072D 04	0.10039049504D 05	0.11187421642D 05
TRAN PT RAD	0.61868017359D 04	0.73806236329D 04	
PHI-1, PHI-2, P	0.73815161751D 02	0.18755679124D 03	0.66448493661D 04

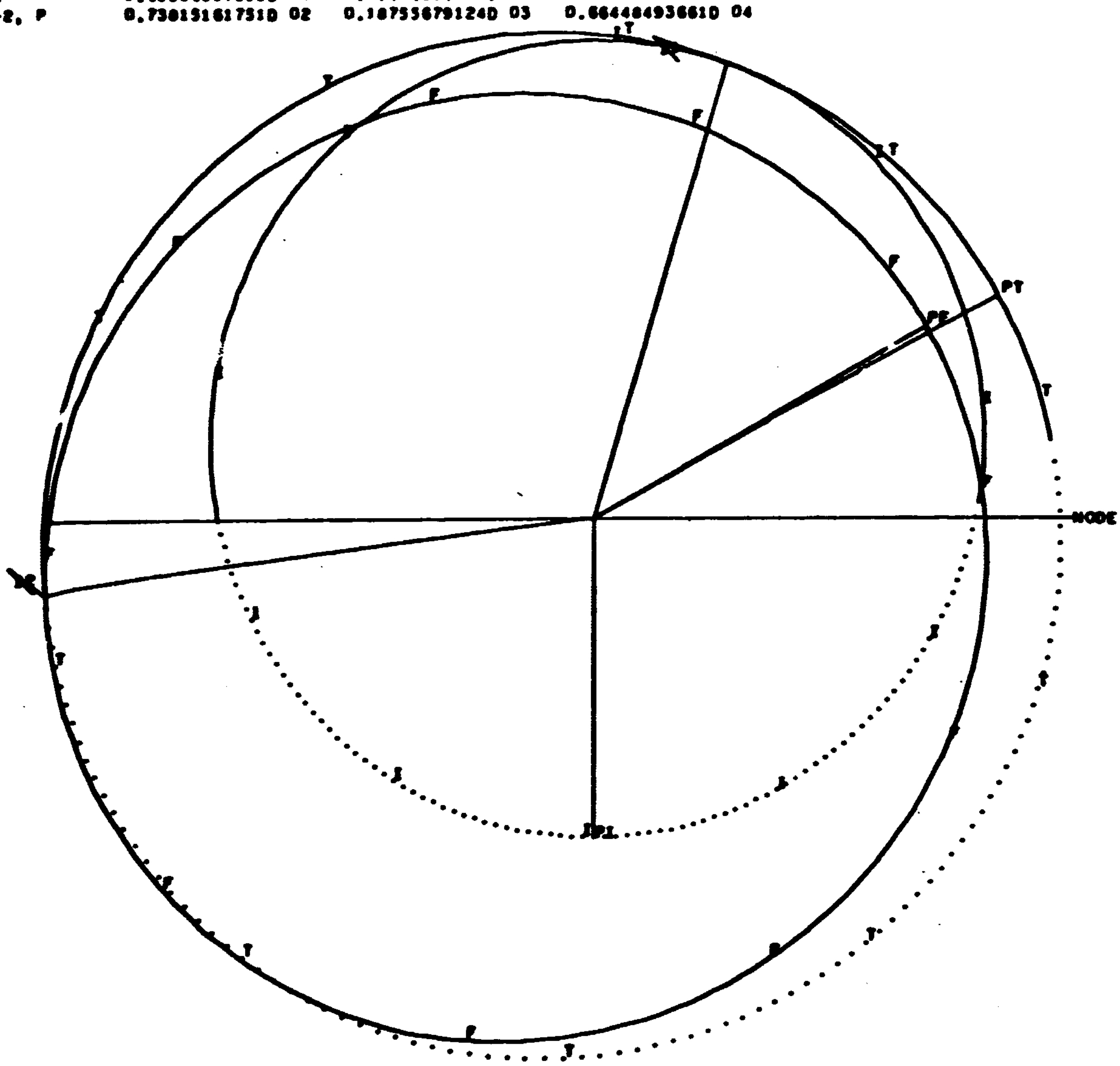

Figure 1. Schematic Diagram of the Orbits

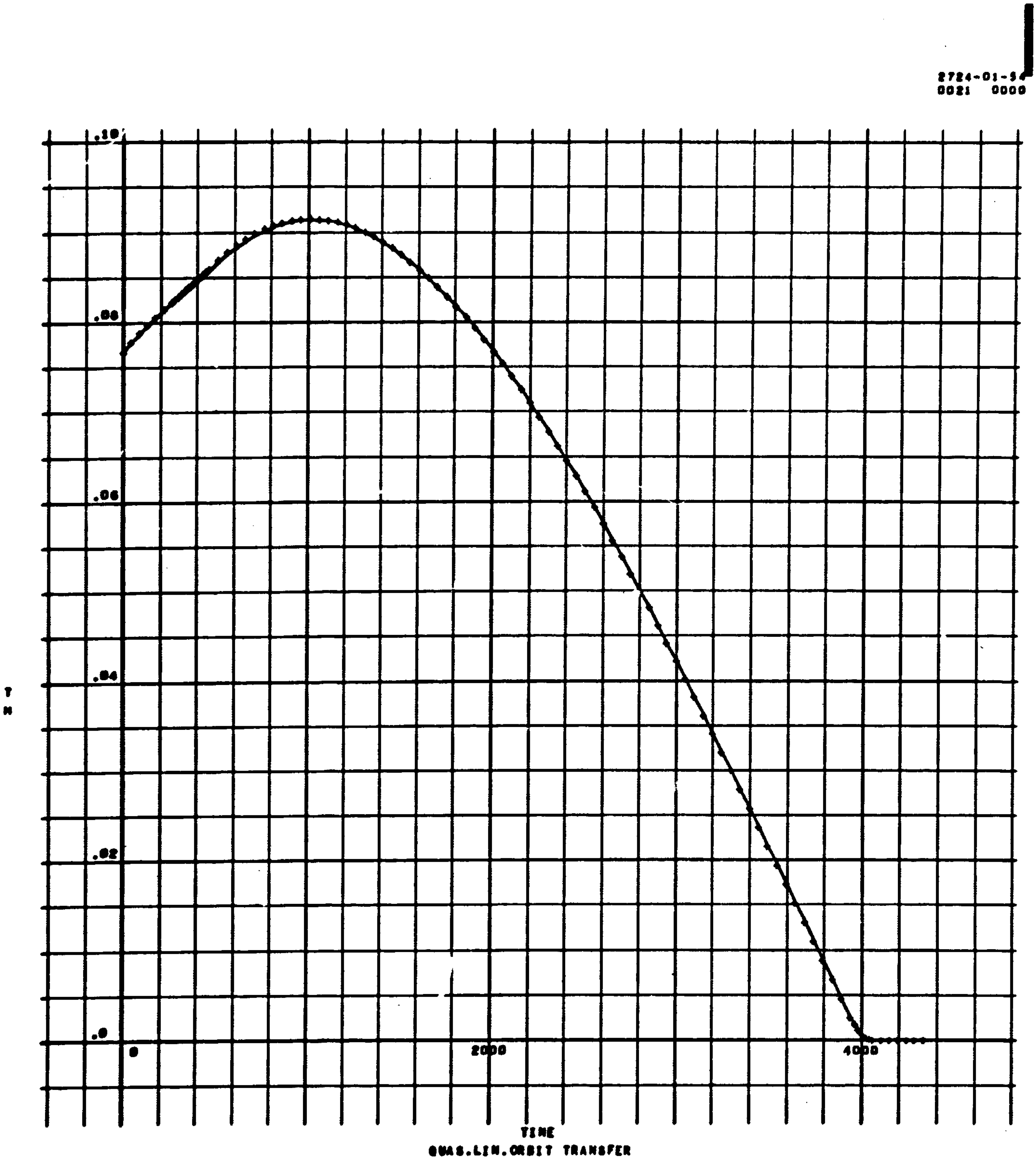

Figure 2. Derivative of the Out-of-Plane Angle - Initial and Final Values Compared

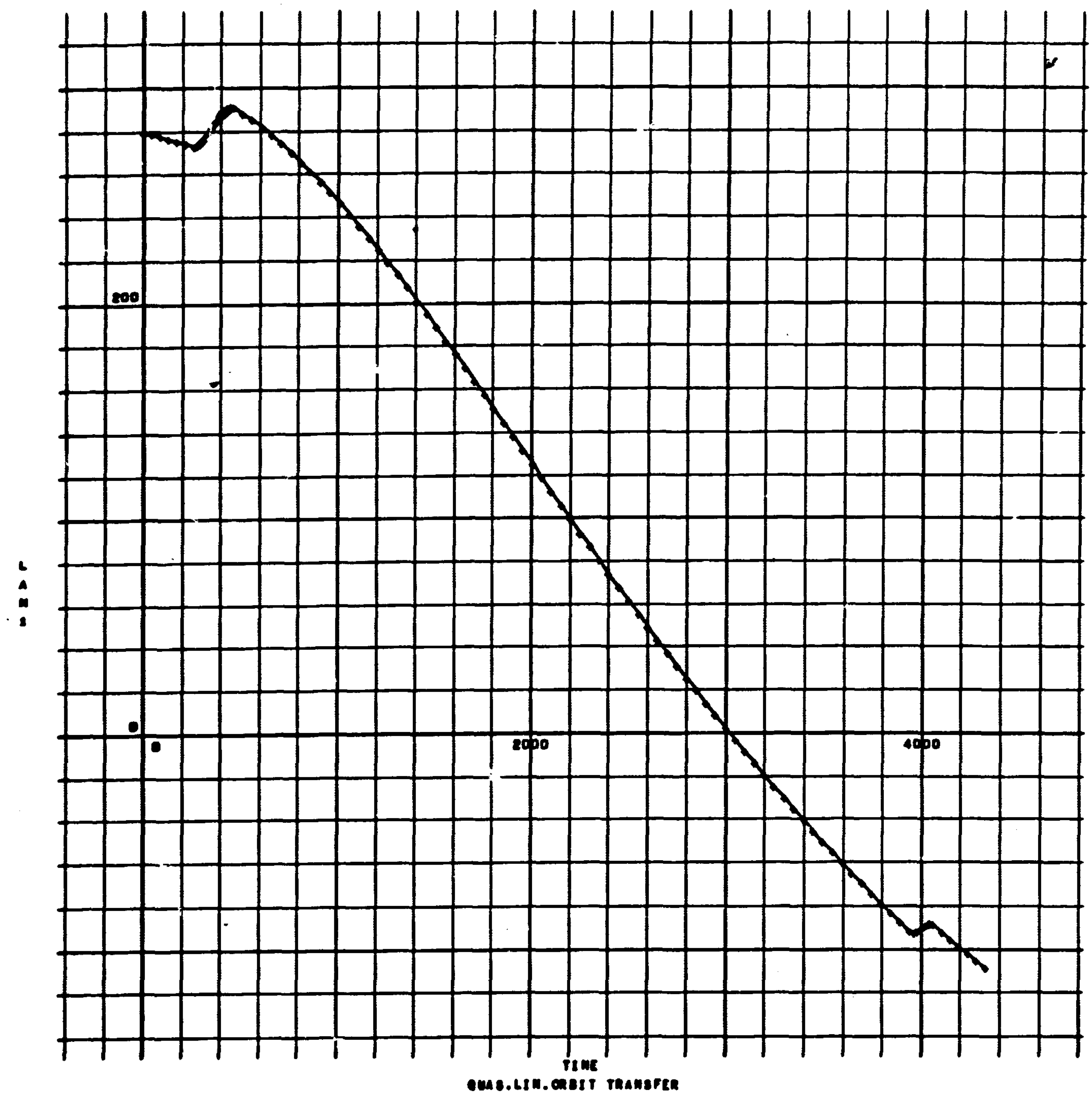

Figure 3. Langrange Multiplier λ_1 - Initial and Final Values Compared

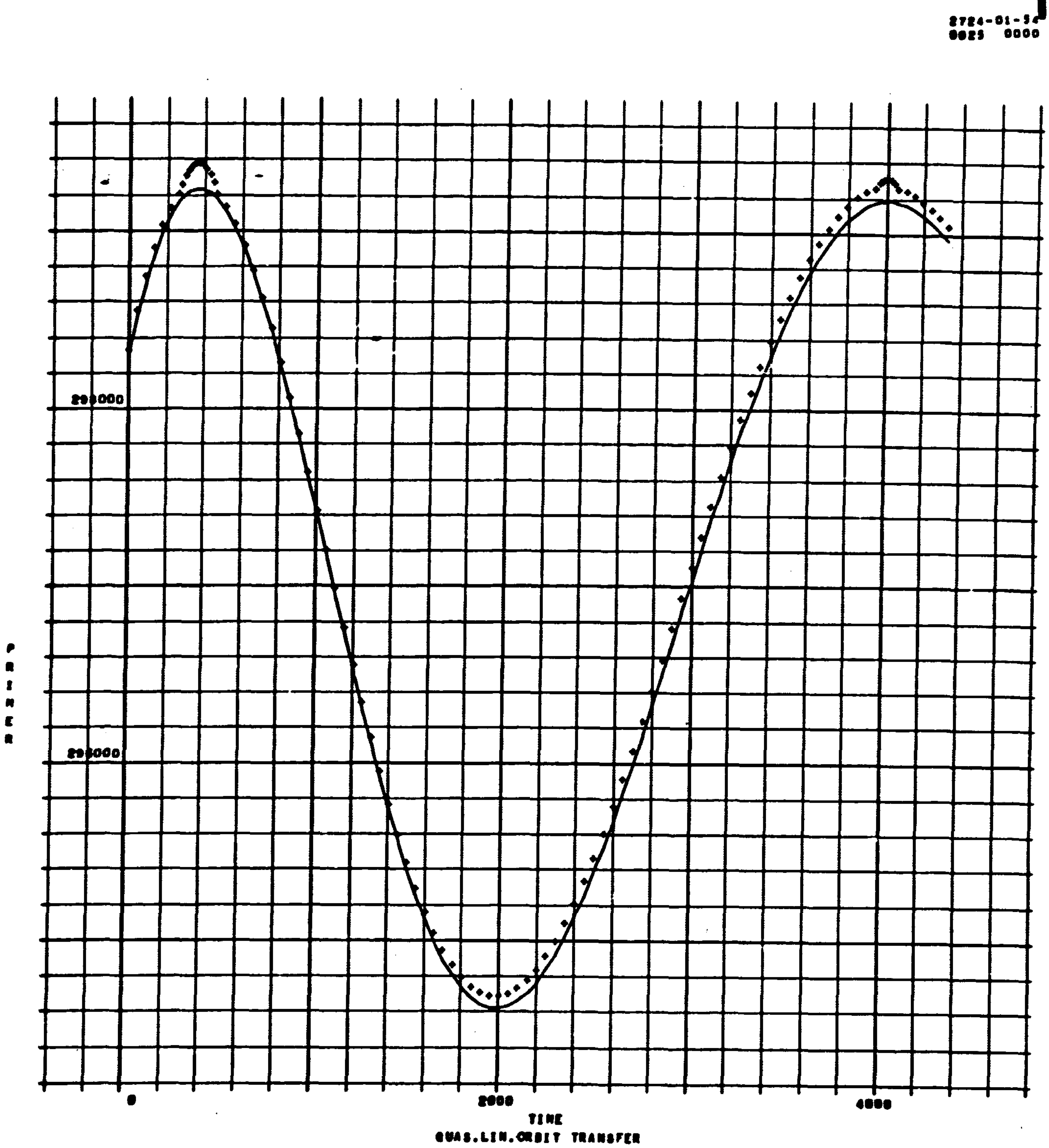

Figure 4. Primer Vector Magnitude - Initial and Final Values Compared
(Primer Components are λ_4, λ_5, λ_6 sec θ/r)

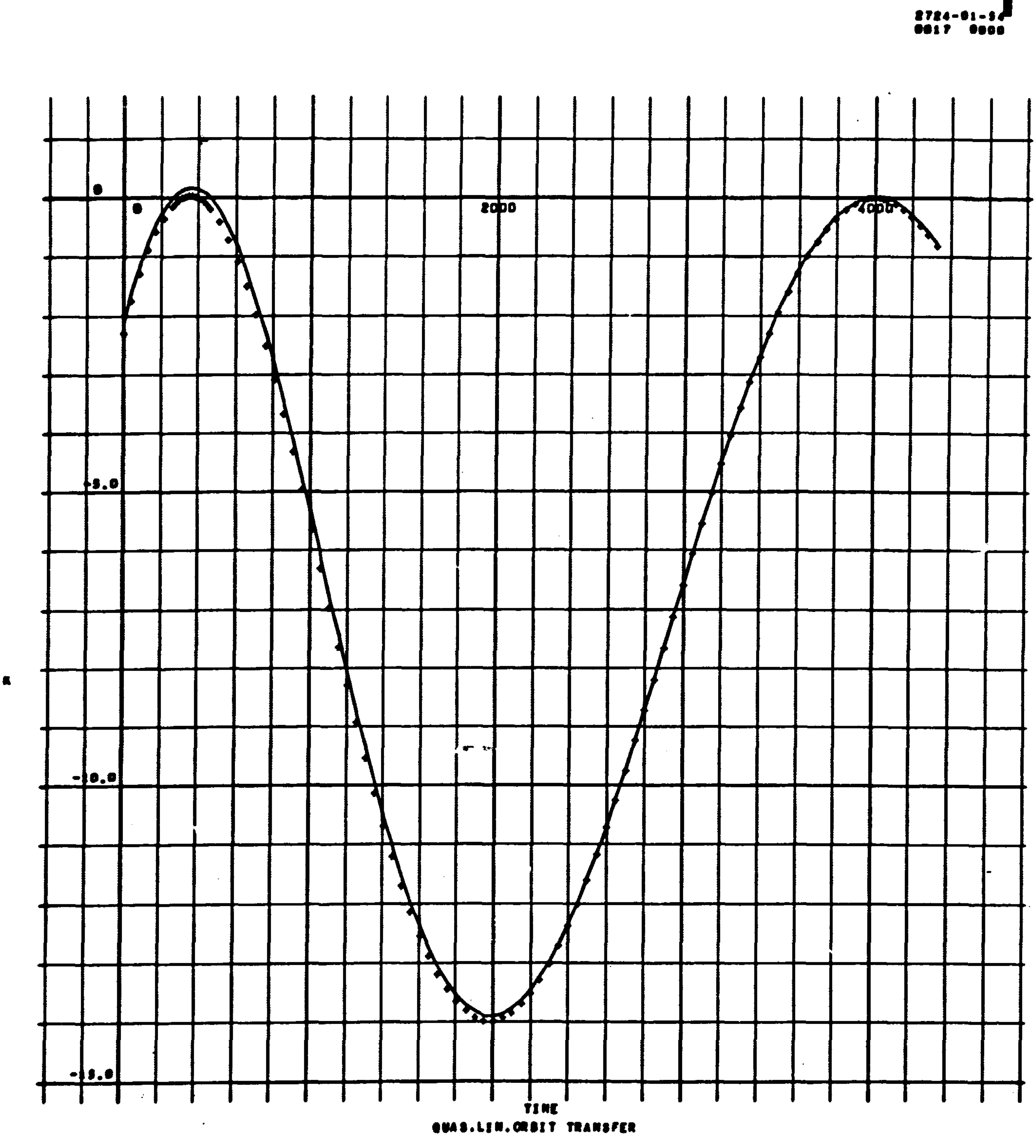

Figure 5. Switching Function - Initial and Final Values Compared

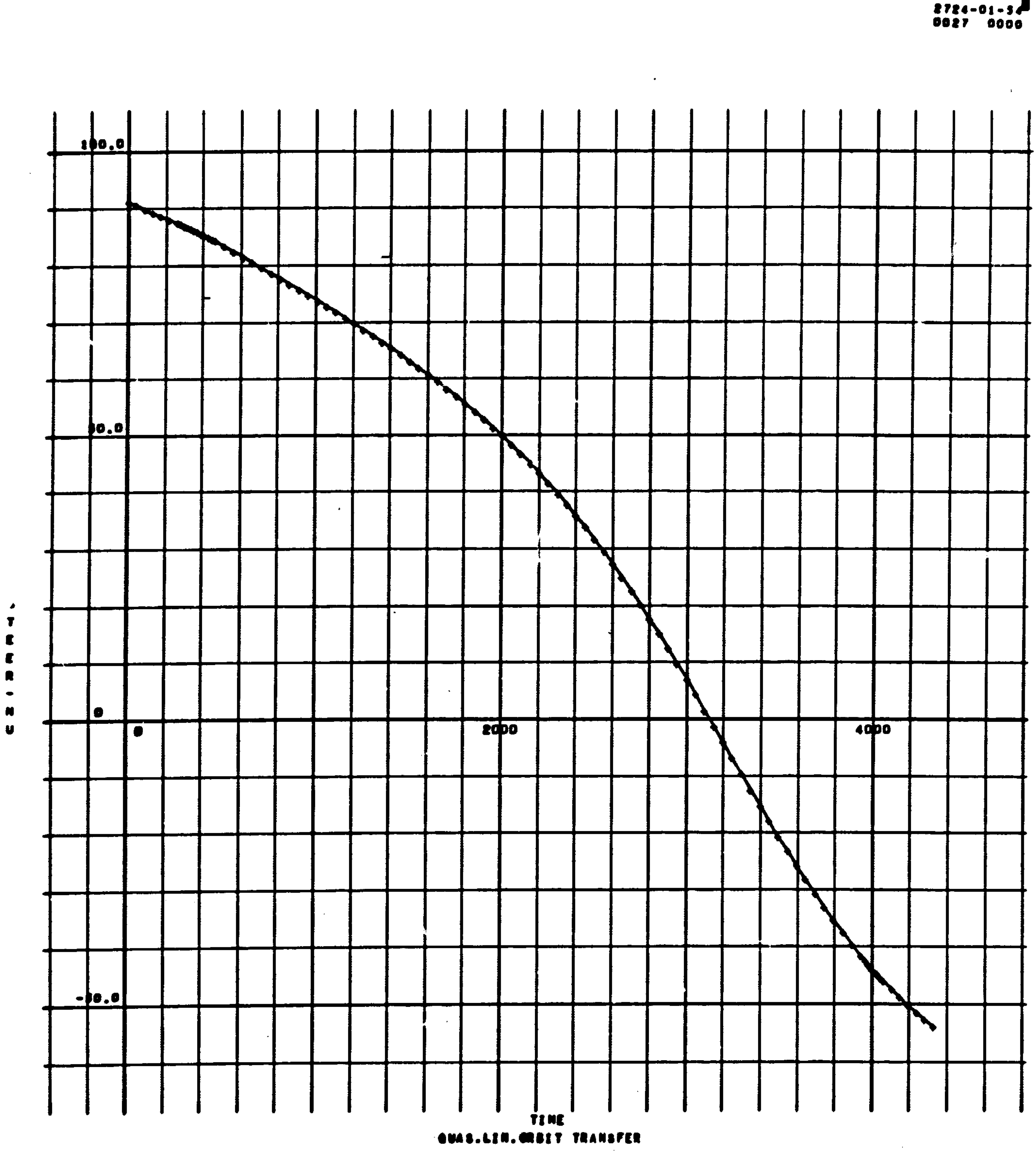

Figure 6. In-Plane Steering Angle - Initial and Final Values Compared

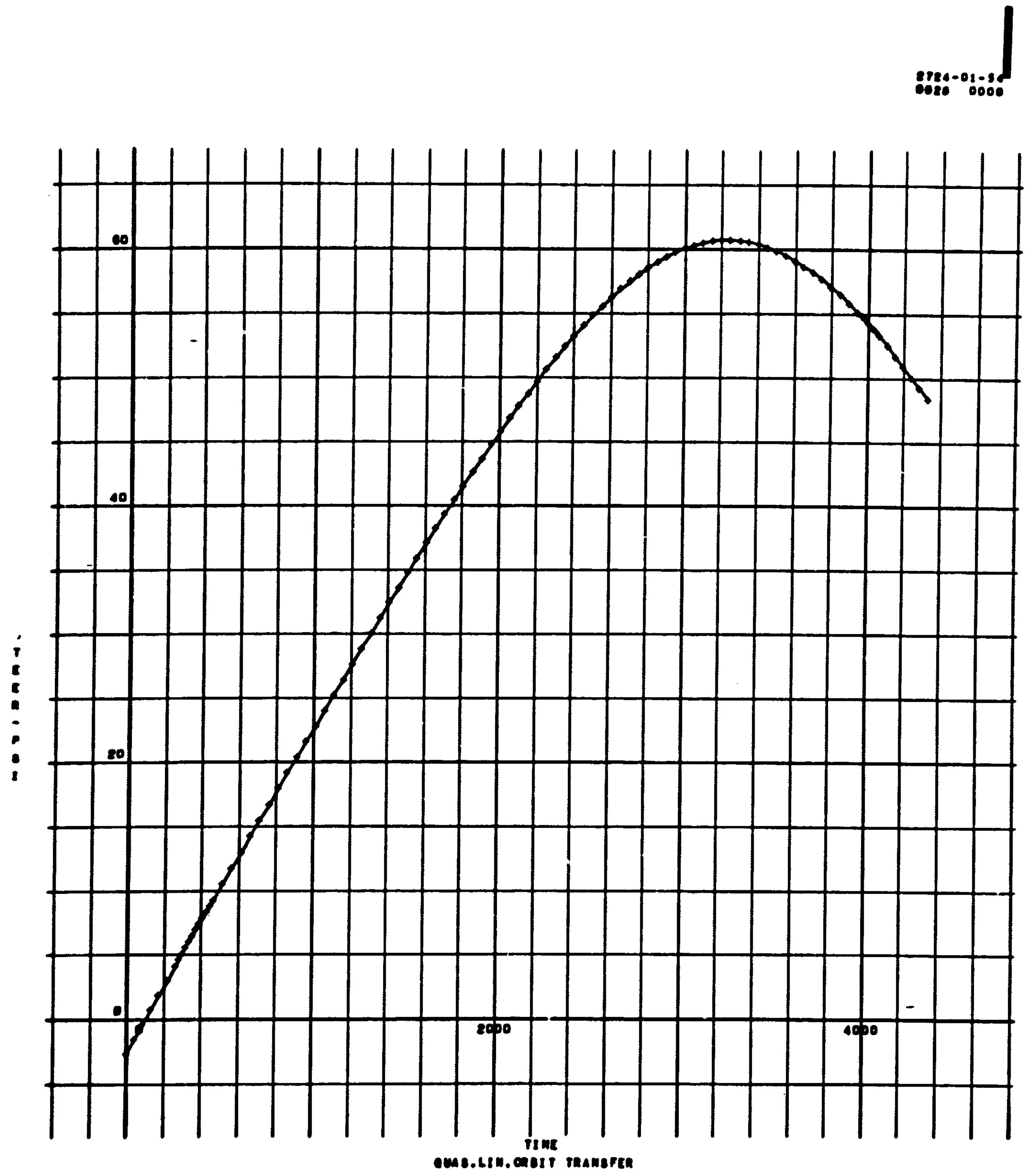

Figure 7. Out-of-Plane Steering Angle - Initial and Final Values Compared

of finite-thrust and impulsive orbital transfer. Good comparison of ΔV requirements were obtained for a wide range of initial thrust-to-weight ratios—including very-low-thrust cases. For example, good results were obtained for an Earth-to-Mars transfer when the thrusting time exceeded 10 days. These very brief parameter studies exposed some of the differences and similarities between finite-thrust and impulsive transfer between inclined orbits (inclination varied from zero to 80 degrees).

Investigations of two-impulse minima that were not absolute minima indicated that some could be replaced with three or four finite-duration burns which required less total ΔV to complete the transfer. When this situation occurred, its existence was always indicated by the behavior of the primer vector in the initial iterate. This was a new, significant, and rather unexpected result. As a consequence, it is clear that the behavior of the primer length in the initial iterate, or of the switching function which is derived directly from it, is an indicator of whether or not the program will find a finite-thrust optimum. When improper behavior is found, the search by the quasilinearization technique can be stopped.

DOCUMENTATION OF THE TWO-BODY FINITE-THRUST PROGRAM

Because of the extreme complexity of the two-body finite-thrust program, the computer program documentation was divided into three separate logical segments. The first segment contains all of the detailed two-impulse orbital transfer optimization calculations which are used to obtain initial conditions for the finite-thrust program. These routines were segregated into a separate computer program that can be used by numerous individuals who wish to study only the two-impulse transfer maneuver. Accordingly, this collection of subroutines has been documented in Reference 29, Program for Optimization of Two-Impulse Transfers by Contouring and Steep Descent.

Similarly, the authors' generalized quasilinearization subroutine, which forms a vital part of the finite-thrust program, has numerous applications to problems outside the field of orbital mechanics. For this reason, it and its related subroutines are documented separately in Reference 30, Generalized Quasilinearization Routines for Solving Non-Linear Boundary-Value Problems. Because of this logical subdivision, the quasilinearization program can be made available as a significant example of technology utilization. It will therefore find uses in many diverse areas outside direct NASA applications.

The third segment contains the details of the finite-thrust computation procedure. That report, Quasilinearization Program for Determining Optimum Finite-Thrust Transfers Between Inclined Orbits, documents the detailed equations and mathematical procedures required to solve the finite-thrust problem. It also contains a complete description of the computer program, an example of the program's use, and the detailed FORTRAN H program listings.

SUGGESTED PARAMETRIC STUDIES

As noted, the program has been thoroughly verified by computing a number of critical and/or limiting cases. The program has also been checked by performing several parametric studies of the effect of changing the thrust-to-weight ratio and of varying the inclination. The results of these initial applications suggested that the program is indeed a valuable tool for investigating finite-thrust transfers between orbits and that it should be employed extensively for numerous parameter studies.

For the most part, parametric studies can be made easily with minimal changes to the data cards between computer runs. An outline of possible studies is given below. It is divided into two parts: the primary parametric study types that might be considered, and the orbit pair type that might be considered for each study type. In addition, there is a brief discussion of the one problem that is known to arise.

Outline for Suggested Finite-Thrust Transfer Studies

A. Primary parametric studies of significance

   1. Effects of varying the initial thrust-to-weight ratio on a wide range of orbit-pair types

   2. Effects of varying inclination at very low values, moderate values and high values including retrograde cases

B. Classification scheme for orbit pairs to be considered for each of these studies

   1. Circle-to-circle

   2. Circle-to-ellipse

      a. Intersecting type (if rotated to a single plane)

      b. Non-intersecting (if rotated to a single plane)

      c. Large, highly eccentric ellipses (e.g., translunar orbit)

3. Ellipse-to-ellipse

   a. Intersecting type (if rotated to a single plane)

   b. Non-intersecting type (if rotated to a single plane)

4. Special mathematical cases

   a. Nearly tangent orbits with one-impulse transfer as optimum (coplanar)

   b. Non-coplanar cases where one impulse is very much smaller than the other

5. Special physical cases

   a. Finite-thrust transfers from earth's orbit to that of another planet or an asteroid and vice versa

   b. Finite-thrust transfer from earth orbit to a translunar orbit

Significant Unanswered Questions

The problem of a very small burn at either the first or second impulse point remains unresolved if the orbits are inclined (item 4b). For some pairs of inclined orbits for which the impulsive program finds that one of the impulses is negligible compared to the other, no converging solutions were found. The initial primer behavior was unsatisfactory; that is, it appeared to call for a burn midway between those of the impulsive points selected. The possibility of varying this initial primer history by varying the impulsive starting point or ending point has been investigated. Some changes have been observed in the tests to date, but no conclusions have been reached. Another possible answer is that the thrust direction of the negligible impulse, which is required for the multiplier initial estimate procedure, and which is almost immaterial to the impulse computation, may be in error. In any case, the results appear to indicate, once more, the requirement for very good impulsive optimization data if optimum finite-thrust transfers are to be found for specially sensitive cases.

The problem of a very small burn at either the first or second impulse point is, essentially, the problem of the existence of optimal one-burn transfers between inclined orbits. Although some techniques have been developed to handle this problem, no successful solutions are known to have been obtained. In fact, the present computational results suggest that optimal single-burn transfers between inclined orbits may not exist. The numerical experiments used to test this hypothesis involved finding an optimal single-impulse transfer and then attempting to converge upon its finite-thrust counterpart. As noted above, this was not successful. However, it may be possible to isolate such maneuvers from initial conditions obtained in some other manner. This is an area that requires further investigation.

III. TWO-IMPULSE TRANSFER IN THE THREE-BODY PROBLEM

Suppose it is desired that a spaceship on some orbit in earth-moon space transfer to a new orbit by means of a two-impulse maneuver. While this problem is topologically similar to two-impulse transfer in the two-body problem, there are two significant differences from the computational and analytic points of view. In the first place, the terminal orbits are not generally cyclic, and points along them cannot be represented by five orbital elements and an angle. The six components of position and velocity are used instead. The second major difference is that one cannot describe the orbits between two fixed points as a known function of any one parameter. In general, there is a single infinity of orbits through two given points in a single-attracting-center gravitational field; therefore, in the two-body case one can choose, for example, the semilatus rectum of the transfer orbit as the parameter to generate the orbits. For this case, it is possible to compute the orbit, the velocities at both ends, and the impulses.

In the three-body case, it was decided to span the infinity of orbits by using time-to-transfer as a parameter. One must specify also the general shape of the path; that is, for example, counterclockwise around the earth to clockwise around the moon, as in Figure 8. Just as in the two-body case, multiple orbits in the system are not considered. It is a major problem to obtain the orbit, and the velocities and impulses at both ends. In fact, the aim of this portion of the contractual work was to provide a computer program to obtain these quantities. That is, from the given departure point (B_O in Figure 8), the given arrival point (B_T in Figure 8), the given orbit shape and the given time interval (T), the program was to determine the transfer orbit trajectory, the velocities at both ends, and the impulses. The type of motion to be described is known as that of the reduced three-body problem. That is, the third body whose motion is being investigated does not affect the motion of the two primary bodies which may move in either circular or elliptical orbits about their common center of mass.

The above summarized formulation has resulted in a double-precision computer program which uses quasilinearization to find impulsive transfers between given terminals in a given time in the reduced three-body problem (Reference 31). The program accepts a wide variety of input trajectory shapes and coordinate systems, is designed to be used in impulsive transfer parameter studies, and has been shown to give accurate solutions with considerable speed. In a typical usage it has been shown to yield meaningful results quickly when applied to the problem of reaching a Lagrange point from a translunar trajectory or from an earth orbit.

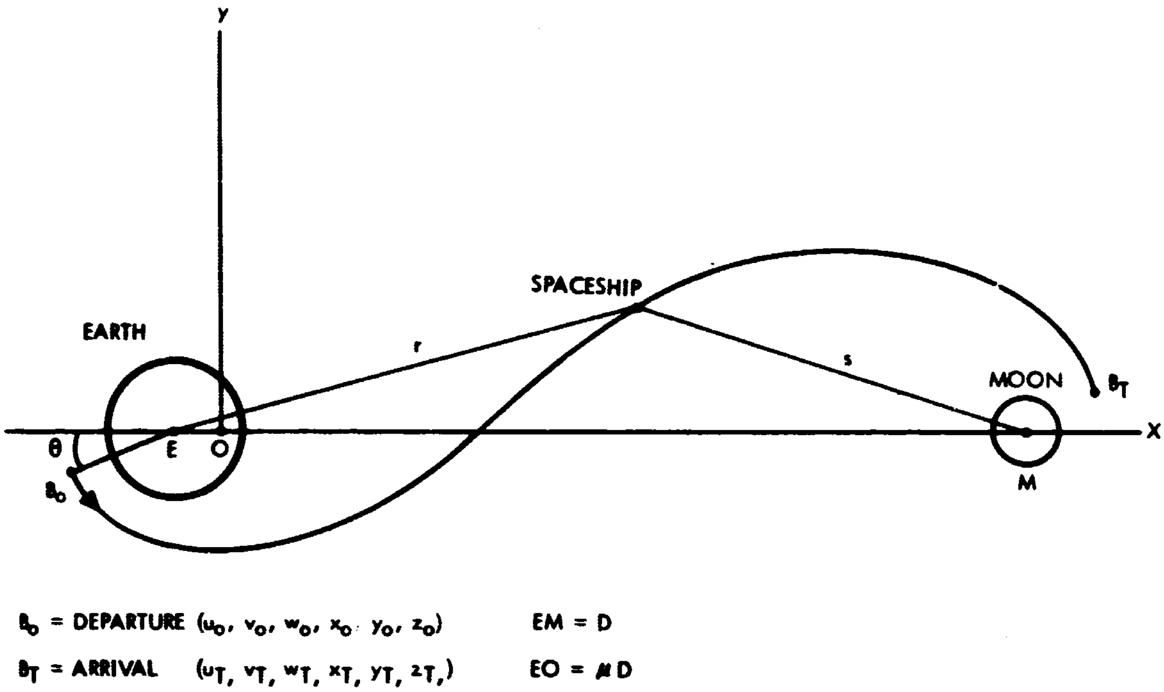

Figure 8. Rotating Coordinate System

Because of its length and the fact that it is really a separate program, it has been documented separately in Reference 31, Two-Impulse Transfer in the Three-Body Problem. In that document, the program and its usage are described in detail. Some preliminary results showing excellent convergence properties and illustrating its application to an impulsive transfer problem are also presented.

This double-precision computer program represents the first necessary step in a longer-range problem which is the numerical analysis of two-impulse and two-burn transfers in earth-moon space. Now that it has been developed, systematic studies of two-impulse transfers can be undertaken. The program will also provide the vital initial approximations required for the computation of optimal finite-thrust transfers in the three-body problem.

IV. OPTIMUM FINITE-THRUST TRANSFERS IN THE THREE-BODY PROBLEM

INTRODUCTION

The task of adapting the quasilinearization technique for finding optimum finite-thrust transfers between three-dimensional orbits around a single attracting center has been successfully completed. It is therefore proper to begin to develop computer programs which will use quasilinearization to find optimal finite-thrust maneuvers in the reduced three-body problem. It is believed that this can be accomplished in a straightforward manner and that it will be advantageous to make use of a single set of equations for the entire trajectory. For this reason it seems best to use the rotating system of coordinates which is centered at the barycenter and which has the earth-moon orbit plane as the xy plane with the x-axis directed toward the moon at all times (Figure 8). The formulation of the equations for the optimum finite-thrust maneuver in this system is given in the following section. It is assumed that the earth-moon system is in an elliptic orbit, but most of the checkout of the computer program will deal with the circular-orbit case. The new program will be built up in sections which optimize individual fragments of the total trajectory, but it is planned that the sections will eventually be combined into a single procedure.

There are three essential parts to the quasilinearization procedure as applied to an optimization program such as this which utilizes the Denbow formulation. One is the obtaining and programming of the Jacobian matrix, the partial derivatives of the differential equations of motion. This has been accomplished, but the program has not been tested.

The second essential part consists of providing for the program a scheme for choosing an initial time history that is near enough to the final optimum so that the program will converge upon it. The experience in the two-body formulation indicates that this capability is important, if not crucial. It is likely that this phase of the problem for the three-body case will be the most difficult and time-consuming portion.

The third part involves the management of the boundary conditions at staging and switching points. It is believed that the introduction of staging into the procedure will offer no significant difficulties in programming. The ability to stop and to restart the integration procedure is a feature of the NR quasilinearization routine (Reference 30) which was introduced in order to allow changes in integration step size. In the two-body finite-thrust problem, techniques for using this feature at switching points were developed. The natures of switching and staging points appear to be computationally similar; thus, only minor additional programming complexity is required to include the additional corner conditions for staging as well as those for switching.

FORMULATION OF THE OPTIMUM FINITE-THRUST PROBLEM FOR THE REDUCED THREE-BODY SYSTEM

The equations of motion of the spacecraft may be taken from Equation 7 of Reference 31 and require only the addition of a thrust. Thus, we write

$$\dot{X} = g(X, t) + a$$

$$\dot{m} = -\beta$$

where

$$X^T = (\dot{x}, \dot{y}, \dot{z}, x, y, z) \text{ or } (u, v, w, x, y, z)$$

and

$$a^T = \frac{c\beta}{m}\left(q_1, q_2, q_3, 0, 0, 0\right)$$

Thus $\underline{q}$ is the unit vector along the thrust direction.

We choose to optimize the final mass, and we may write the Hamiltonian as

$$H = \lambda^T(g + a) - \lambda_7\beta$$

In the application of the Pontryagin Maximum Principle the switching function, k, is developed and it may be expressed as

$$k = \frac{c\Lambda}{m} - \lambda_7$$

where

$$\Lambda = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}.$$

The value of β is allowed to be either zero or β_m and the choice is governed by k, that is, for $k < 0$, $\beta = 0$, while for $k > 0$, $\beta = \beta_m$. In addition the vectors $\underline{\Lambda}$ and $\underline{q}$ are parallel. The situation $k = 0$ is assumed not to occur over any extended interval of time.

In developing the computation procedure, we have replaced the variable λ_7 with k as was done in the two-body cases. The differential equations for the 14 variables u, v, w, x, y, z, m, λ_1, λ_2, λ_3, λ_4, λ_5, λ_6, and k are:

$$\dot{u} = 2\omega v + \omega^2 x + \dot{\omega} y - \frac{\mu G (x - (1-\mu) D)}{s^3} - \frac{(1-\mu) G (x + \mu D)}{r^3} + \frac{c\beta\lambda_1}{m\Lambda}$$

$$\dot{v} = -2\omega u + \omega^2 y - \dot{\omega} x - \frac{\mu G y}{s^3} - \frac{(1-\mu) G y}{r^3} + \frac{c\beta\lambda_2}{\Lambda}$$

$$\dot{w} = -\frac{\mu G z}{s^3} - \frac{(1-\mu) G z}{r^3} + \frac{c\beta\lambda_3}{\Lambda}$$

$$\dot{x} = u$$

$$\dot{y} = v$$

$$\dot{z} = w$$

$$\dot{m} = -\beta$$

$$\dot{\lambda}_1 = -\lambda_4 + 2\omega\,\lambda_2$$

$$\dot{\lambda}_2 = -\lambda_5 - 2\omega\,\lambda_1$$

$$\dot{\lambda}_3 = -\lambda_6$$

$$\dot{\lambda}_4 = -\lambda_1 \left[\omega^2 - \frac{\mu G}{s^3} - \frac{(1-\mu) G}{r^3} + \frac{3\mu G}{s^5} (x - (1-\mu) D)^2 + \frac{3(1-\mu) G (x + \mu D)^2}{r^5} \right]$$
$$- \lambda_2 \left[\frac{3\mu G (x - (1-\mu) D) y}{s^5} + \frac{3(1-\mu) G (x + \mu D) y}{r^5} - \dot{\omega} \right]$$

$$-\lambda_3\left\{\frac{3\mu G[x-(1-\mu)D]z}{s^5}+\frac{3(1-\mu)G(x+\mu D)z}{r^5}\right\}$$

$$\dot{\lambda}_5 = -\lambda_1\left\{\frac{3\mu G[x-(1-\mu)D]y}{s^5}+\frac{3(1-\mu)G(x+\mu D)y}{r^5}+\dot{\omega}\right\}$$

$$-\lambda_2\left[\omega^2-\frac{\mu G}{s^3}-\frac{(1-\mu)G}{r^3}+\frac{3\mu Gy^2}{s^5}+\frac{3(1-\mu)Gy^2}{r^5}\right]$$

$$-\lambda_3\left[+\frac{3\mu Gyz}{s^5}+\frac{3(1-\mu)G\,yz}{r^5}\right]$$

$$\dot{\lambda}_6 = -\lambda_1\left[\frac{3\mu G}{s^5}(x-(1-\mu)D)z+\frac{3(1-\mu)G(x+\mu D)z}{r^5}\right]$$

$$-\lambda_2\left[\frac{3\mu G\,yz}{s^5}+\frac{3(1-\mu)G\,yz}{r^5}\right]$$

$$-\lambda_3\left[-\frac{\mu G}{s^3}-\frac{(1-\mu)G}{r^3}+\frac{3\mu G\,z^2}{s^5}+\frac{3(1-\mu)G\,z^2}{r^5}\right]$$

$$\dot{k} = -\frac{c}{m\Lambda}\left(\lambda_1\lambda_4+\lambda_2\lambda_5+\lambda_3\lambda_6\right)$$

The task of finding the Jacobian for this set of variables is somewhat tedious, but it is easily managed. It has been done, and the resulting equations have been programmed in a subroutine (JACØB) for future use in the three-body finite-thrust optimization procedure.

In case the earth-moon system is assumed to be in a circular orbit, ω is constant, the terms in $\dot{\omega}$ are zero, and the set of differential equations does not contain the time. The Hamiltonian, H, will be constant in this case. However, if the moon and earth are assumed to move in elliptical orbits, the $\dot{\omega}$ terms are required. H will not be constant. The computer formulation envisaged will not make use of the fact that H is constant for the circular orbit case; rather its constancy for circular orbits will be required as one check on the formulation.

V. CONCLUSIONS

This contracted effort resulted in the completion of several quasilinearization computer programs for investigating optimal impulsive and finite-thrust maneuvers in the two- and three-body problems. These valuable numerical tools have been fully documented and are now available for use in extensive parametric studies. It is believed that through judicious use of these programs, it will be possible to discover significant new properties of optimal transfers. For example, optimal transfers between the Lagrange points have never been studied in detail. The tools to perform such a study are now ready.

This prior experience with numerous computational techniques and the results presented here suggest that it would be wise to continue this work. Much could be done with the present programs, and of course they can be extended and modified to consider rendezvous and/or numerous more advanced problems. In particular, many of the principal ingredients for the complete development of the optimal finite-thrust three-body computer program are now ready for assembly and checkout to begin.

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