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REFLECTION OF A SHOCK WAVE INTO A DENSITY GRADIENT

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by

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and

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ABSTRACT

The reflection of a shock wave at the end wall of a shock tube is examined theoretically for the case in which the flow behind the incident shock wave exhibits a linear density variation and the gas in the shock tube is ideal. The reflection process is shown to be unsteady, resulting in the development of a nonuniform flow field called an entropy layer. The character of the entropy layer displayed in this problem provides useful information for understanding the flow field formed when nonideal effects such as relaxation phenomena or radiative cooling are important behind an incident shock wave.

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I. INTRODUCTION

Calculations of the equilibrium gas state behind a reflected shock wave in a shock tube usually employ the assumption that the reflected shock wave is travelling at a constant speed into a uniform equilibrium flow. When relaxation phenomena are present, a shock wave actually reflects with a speed which varies during the time required for the reflected shock wave to travel through the nonuniform relaxation zone behind the incident shock wave. The gas particles which pass through the unsteady reflected shock wave experience different thermodynamic histories than those which pass through the later steady shock wave, and hence ultimately reach different equilibrium states than the state predicted by the steady-shock calculation. The spatial region containing this non-uniform equilibrium gas can be called an entropy layer; this terminology is consistent with that used to describe the region adjacent to the surface of a wedge in steady supersonic flow of a relaxing gas.

It can be shown that an entropy layer will always be formed in the region behind the reflected shock wave when the flow is nonuniform behind the incident shock wave, e.g., owing to relaxation phenomena, radiative cooling, side-wall boundary-layer effects, etc. Unfortunately, exact solutions for the thermodynamic properties throughout such entropy layers are difficult to obtain, and even the qualitative character of these regions is not immediately obvious.^{1,2} The object of this paper is to present an analysis for a simpler but illustrative case of shock-wave reflection in which the flow behind the incident shock wave exhibits a linear density variation and the gas is everywhere ideal. The density variation is assumed small so that all quantities behind both the incident and reflected shock waves vary only slightly from their unperturbed values, and linearized equations can be used to calculate the flow field.

II. ANALYSIS

In order to solve the linearized gas-dynamic equations behind the reflected shock wave, one must first determine the boundary (jump) conditions along the reflected-shock trajectory. The values of the flow-field properties along this trajectory differ from the unperturbed values due to two coupled effects: variations in the flow properties behind the incident shock wave, and variations in the speed of the reflected shock wave. The determination of these boundary conditions is straightforward, using standard shock-jump relations, since the variations in the incident-flow properties are all known (because of the specified density distribution) and the position of the reflected shock wave can be approximated by the unperturbed trajectory, even though the reflected-shock speed is allowed to vary. Once the boundary conditions are determined, the solution for the flow field behind the reflected shock wave is obtained by solving the elementary wave equation in the perturbed flow variables of pressure and particle velocity.

The space-time diagram representing the overall flow process is shown in Fig. 1, where x is the distance from the shock-tube end wall and t is the time after reflection. The density perturbation behind the incident shock wave will be represented by

$$\rho_2' / \rho_2 = \beta s, \quad (1)$$

where ρ_2 is the unperturbed value of the density, β is the density-gradient parameter, and s is the distance behind the incident shock wave. For a given density distribution, the variations in the values of other thermodynamic properties behind the incident shock wave are determined by the conservation equations of mass and momentum and the thermal equation of state. For example, since the pressure behind a constant-speed incident shock wave is given by

$$p_2 = p_1 + \rho_1 V_s^2 (1 - \rho_1 / \rho_2),$$

then the perturbation in pressure owing to a small variation in density is

$$p_2' = \rho_1 V_s^2 (\rho_1/\rho_2) (\rho_2'/\rho_2).$$

If one neglects p_1 in the previous equation, the last equation can be rewritten in dimensionless form as

$$p_2'/p_2 = \beta s / (\rho_{21} - 1),$$

where the abbreviated notation $\rho_{21} = \rho_2/\rho_1$ has been adopted. Thus the parameter β fixes the variations of all flow-field properties with distance behind the incident shock wave. See Appendix A for a derivation of the pertinent linearized relations in the incident-shock flow.

The boundary conditions for the flow in region 5 are determined from the standard shock-jump relations. For example, the relation for the density jump across the reflected shock wave is

$$\rho_5 = \rho_{52} \rho_2 = (\gamma + 1) M_r^2 \rho_2 / [(\gamma - 1) M_r^2 + 2],$$

where M_r represents the dimensionless speed of the reflected shock wave with respect to the flow in region 2 and is defined by

$$M_r = [V_r + V_s(1 - \rho_{12})] / a_2.$$

The quantity a_2 is the local speed of sound in region 2, i.e.,

$$a_2 = (\gamma R T_2)^{1/2}.$$

Retaining only first-order terms in the perturbed quantities, one finds from the shock-jump relation for density that

$$\hat{\rho}_5'/\rho_5 = \hat{\rho}_2'/\rho_2 + \hat{\rho}_{52}'/\rho_{52},$$

where

$$\hat{\rho}_{52}'/\rho_{52} = (4/G)(\hat{M}_r'/M_r),$$

$$G = (\gamma - 1) M_r^2 + 2,$$

and the caret symbol has been used to denote the value of the quantity along the reflected-shock trajectory.

The perturbation in the reflected-shock Mach number results from variations in ρ_2 , a_2 , and V_r . However, the variations in a_2 and ρ_2 are related by

$$a_2'/a_2 = -[(\rho_{21} - 2)/2(\rho_{21} - 1)]\rho_2'/\rho_2$$

so that the perturbation in Mach number becomes simply

$$\hat{M}_r'/M_r = H(\hat{\rho}_2'/\rho_2) + \rho_{25}(V_r'/V_r).$$

The quantity H is defined by

$$H = M_2/M_r + (\rho_{21} - 2)/2(\rho_{21} - 1),$$

where M_2 is the dimensionless speed of the flow in region 2 with respect to the incident shock wave, i.e.

$$M_2 = V_s/\rho_{21}a_2.$$

Finally, one can write the boundary condition for density as

$$\hat{\rho}_5'/\rho_5 = [1 + 4H/G](\hat{\rho}_2'/\rho_2) + [4/G\rho_{52}](V_r'/V_r).$$

Consistent with the use of first-order perturbation theory, the jump conditions across the reflected shock wave can be satisfied along the unperturbed shock trajectory rather than along the actual trajectory. The perturbed values for the flow properties immediately upstream of the reflected shock wave are thus given simply by substituting

$$s = (V_r + V_s)t$$

into the appropriate equations for the flow properties in region 2.

Using Eq. (1), the density perturbation $\hat{\rho}_5'/\rho_5$ is given by

$$\hat{\rho}_5'(t)/\rho_5 = [1 + 4H/G]Lt + [4/G\rho_{52}](V_r'/V_r), \quad (2)$$

where

$$Lt = \beta(V_r + V_s)t.$$

For additional details regarding this derivation see Appendix B. Similar analyses for pressure and particle velocity (see Appendices C and D, respectively) yield the boundary conditions

$$\hat{p}'_5(t)/p_5 = \left[\frac{4\gamma M_r^2 H}{p_{52}(\gamma+1)} + \frac{1}{\rho_{21} - 1} \right] Lt + \left[\frac{4\gamma M_r V_r}{p_{52}(\gamma+1)a_2} \right] (V'_r/V_r) \quad (3)$$

and

$$\hat{u}'_5(t)/V_r = [4H/G - V_s/\rho_{51}V_r]Lt + [1 + \rho_{25}(4/G - 1)](V'_r/V_r). \quad (4)$$

All boundary conditions thus become functions of both time and the unknown quantity V'_r .

These boundary conditions (Eqs. (2) through (4)) can be rewritten in dimensional form (using the identity $p_5 = \rho_5 a_5^2/\gamma$) as

$$\hat{u}'_5(t) = ALt + BV'_r, \quad (5)$$

$$\hat{p}'_5(t)/\rho_5 a_5 = CLt + DV'_r, \quad (6)$$

and

$$\hat{p}'_5(t) = ELt + FV'_r, \quad (7)$$

where the coefficients A through F are directly related to the bracketed quantities shown above. Equations (5) through (7) provide the boundary conditions for the flow-field solution in region 5; the coefficients A through F are related to known (unperturbed) quantities since M_s and γ are assumed given.

Because of the unusual boundary conditions, care must be exercised in employing the linearized gas-dynamic equations to solve for the flow field in region 5. This can best be seen by considering the following derivation. The continuity, momentum, and energy equations for the

perturbed variables in region 5 are, to first order,

$$\partial \rho'_5 / \partial t + \rho_5 \partial u_5 / \partial x = 0,$$

$$\rho_5 \partial u_5 / \partial t + \partial p'_5 / \partial x = 0,$$

and

$$\rho_5 c_p \partial T'_5 / \partial t - \partial p'_5 / \partial t = 0.$$

Using the linearized equation of state,

$$p'_5 / p_5 = \rho'_5 / \rho_5 + T'_5 / T_5,$$

and the continuity equation, one can rewrite the energy equation in terms of p'_5 and u_5 , i.e.,

$$\partial p'_5 / \partial t + \gamma p_5 \partial u_5 / \partial x = 0.$$

The momentum equation and the second form of the energy equation can now be combined to yield elementary wave equations in the perturbed variables p'_5 and u_5 , e.g.,

$$\partial^2 p'_5 / \partial t^2 - a_5^2 \partial^2 p'_5 / \partial x^2 = 0. \quad (8)$$

Note, however, for this problem, that no combination of the conservation equations will yield an elementary wave equation in either of the perturbed variables ρ'_5 or T'_5 . This is a result of the fact that the entropy varies between adjacent particle paths in region 5 (non-homentropic flow), so the usual adiabatic state relation for pressure as a function of density cannot be invoked to replace $\partial p'_5 / \partial x$ in the momentum equation with $a_5^2 \partial \rho'_5 / \partial x$. The difference in entropy across the reflected flow field is generated, of course, by the mechanism which produced the initial perturbations in region 2, and by the varying strength of the reflected shock wave.

After applying the no-flow boundary condition at the end wall, one can write the general solutions of the wave equation for u_5 and p'_5 as

$$u_5 = f(x + a_5 t) - f(-x + a_5 t)$$

and

$$-p_5'/\rho_5 a_5 = f(x + a_5 t) + f(-x + a_5 t),$$

where f is an arbitrary function. Applying the boundary conditions along the unperturbed shock trajectory (Eqs. (5) and (6)), one can then show that the function f is given by the linear relation

$$f(\xi) = -\frac{1}{2} KL\xi,$$

where

$$K = (BC - DA)/(Ba_5 + DV_r).$$

The perturbed velocity and pressure distributions are therefore

$$u_5/V_r = -KLx/V_r = -(Kx/\hat{x})Lt, \quad (9)$$

and

$$p_5'/p_5 = (\gamma K)Lt. \quad (10)$$

The quantity $\hat{x} = V_r t$ was introduced in Eq. (9) in order to provide a common form with the solutions shown below where \hat{x} appears as a natural variable. The perturbation in the reflected-shock velocity is given by

$$V_r'/V_r = -JLt, \quad (11)$$

where

$$J = (C + Aa_5/V_r)/(DV_r + Ba_5).$$

A more detailed analysis of these wave-equation solutions is provided in Appendix E.

The density distribution can be established by integrating the continuity equation, i.e.,

$$\rho_5'/\rho_5 = KLt + g(x),$$

where $g(x)$ is a function which can be determined by satisfying the boundary condition for ρ_5' at the shock wave, Eq. (7). This last step yields

$$\rho_5'/\rho_5 = [K + I(x/\hat{x})]Lt, \quad (12)$$

where

$$I = (1 + 4H/G - 4J/G\rho_{52} - K).$$

For additional details regarding the steps involved in the density solution, see Appendix F.

Using the linearized equation of state, one can now show that the temperature distribution is given by

$$T_5'/T_5 = [(\gamma - 1)K - I(x/\hat{x})]Lt, \quad (13)$$

while the entropy perturbation becomes (see Appendix G)

$$S_5'/R = -[\gamma IL/(\gamma - 1)V_r]x. \quad (14)$$

III. DISCUSSION OF RESULTS

Numerical results for the dimensionless coefficients I, J, and K, which appear in Eqs. (9) through (14), are presented in Fig. 2. These coefficients vary only slightly for $M_s \geq 3$, and therefore one can approximate them with constants.

An inspection of Eq. (9) shows that the velocity perturbation is a function of x only; the negative sign indicates that the motion is toward the end wall. The pressure perturbation (Eq. (10)), on the contrary, is not a function of x , and it rises uniformly with time for all the fluid particles in region 5. It is of interest that, for either a monatomic or diatomic gas, the pressure perturbation p'_5/p_5 is nearly 8% at the instant the reflected shock wave intercepts a density variation $\hat{\rho}'_2/\rho_2$ of 10% (recall that $\hat{\rho}'_2/\rho_2 = Lt$ along the reflected-shock trajectory). This last result is physically reasonable when one recognizes that any density increase in region 2 represents a proportionate increase in dynamic pressure as seen by the reflected shock wave, and hence a similar increase in static pressure must arise in region 5.

The solution for the reflected-shock speed, given by Eq. (11), indicates a perturbation which is negative and proportional to time so that the perturbed shock trajectory becomes parabolic. Although the speed of the reflected shock wave decreases with time, the Mach number with respect to the incident flow actually increases because of the decreasing value of the speed of sound in region 2. The result of the increasing shock strength, and of the increasing value of mass density in region 2, is the development of a large positive perturbation in density immediately behind the reflected shock wave (see Eq. (12)). Owing to compressive effects, the density perturbation at the end wall is also positive, but smaller than the perturbation at the shock wave. The unsteady reflection process thus creates a negative gradient in density toward the end wall.

The existence of an entropy layer is confirmed by Eq. (14) which displays a functional dependence on x alone. This result verifies the formulation of the problem for region 5 wherein the compression process along a particle path was assumed to be isentropic and differences in entropy could exist only between adjacent particle paths.

The solution for the temperature distribution is perhaps the most important and interesting result of this analysis. An examination of Eq. (13) shows that the temperature perturbation just behind the reflected shock front is negative and becomes more negative with time, while the perturbation at the end wall is positive and increases with time. In fact, there is a growing region of gas near the end wall, of width $[(\gamma - 1)K/I]\hat{x}$, wherein the temperature perturbation is positive and increases steadily with time. This result is shown graphically in Fig. 3. In the strong-shock limit, $(\gamma - 1)K/I$ is about 0.4 for a monatomic gas, and 0.3 for a diatomic gas, which shows that an important fraction of the gas in region 5 actually experiences a positive temperature perturbation.

This unexpected result for the temperature can be explained as follows. Recall that no mention has been made of the source of the density perturbation in region 2. The results obtained thus far have not required such a specification. Implicit in the conservation equations for steady, one-dimensional flow of an ideal gas, however, is the requirement that the flow in region 2 lose thermal energy in order for the density to increase. Thus some nonadiabatic process must be present to extract energy from the gas and cause a temperature decrease, if the density is to increase. This energy loss is also responsible for the negative temperature perturbation immediately behind the reflected shock wave, as shown in Fig. 3. Whatever the mechanism for energy extraction from region 2, however, the momentum of this gas (in laboratory coordinates) is increased, and this increase in momentum manifests itself as a pressure rise in region 5. The net effect is that compressive work is continually being done on the gas in region 5, thus adding thermal energy along each particle path. Furthermore, the rate of energy addition is the same along all particle paths since the pressure rises uniformly. The energy of a particular gas element at any given time thus depends on the initial thermal energy of the element upon entering region 5 and the length of time spent in region 5.

A very interesting observation can be made by comparing the rate at which internal energy is gained by a unit mass of gas in region 5 with the rate at which internal energy is lost by a unit mass of gas in region 2. Since an ideal gas is being considered, Eq. (13) is also an expression

for the perturbation in the internal energy, i.e.,

$$e'_5/e_5 = [(\gamma - 1)K - I(x/\hat{x})]Lt.$$

The rate of change of internal energy following a fluid particle is therefore

$$De'_5/Dt \approx \partial e'_5/\partial t = (\gamma - 1)Ke_5L \quad (15)$$

since the fluid velocity u_5 is small. Note that this quantity has the same value for all of the particles in region 5 (i.e., it is not a function of x or t).

For the assumed density distribution, the perturbation in internal energy in region 2 is given approximately by

$$e'_2/e_2 = -[(\rho_{21} - 2)/(\rho_{21} - 1)](\rho'_2/\rho_2).$$

The rate of change of internal energy following a fluid particle in region 2 is therefore (see Appendix H)

$$De'_2/Dt = -[(\rho_{21} - 2)(\rho_{52} - 1)/(\rho_{21} - 1)(\rho_{51} - 1)]e_2L, \quad (16)$$

which is a constant for all of the particles in region 2. The ratio of the rates of energy change for the fluid particles in the two regions thus becomes

$$\left| \frac{De'_5/Dt}{De'_2/Dt} \right| = (\gamma - 1)K_{T52} \frac{(\rho_{21} - 1)(\rho_{51} - 1)}{(\rho_{21} - 2)(\rho_{52} - 1)}. \quad (17)$$

Figure 4 presents a plot of this ratio for a range of Mach numbers and for two values of γ . The results show that for strong shocks in a diatomic gas the rate at which a unit mass of gas in region 5 gains internal energy is more than four times the rate at which a unit mass of gas in region 2 loses internal energy; the rate is more than six times as great in a monatomic gas.

Some of the results obtained here for an ideal gas are also of use in understanding shock-wave reflection in non-ideal gases. Although the

numerical results may differ, many of the important physical concepts are retained. For example, one must conclude that radiative cooling behind an incident shock wave, a non-adiabatic effect which would increase the density in region 2, should also cause an entropy layer similar to that discussed above. If radiative cooling occurs in region 2, one may thus expect a trend toward higher temperatures near the end wall and lower temperatures near the reflected shock wave. In a real case, of course, when energy is being lost by means of radiation from region 2, region 5 will also be losing energy, and this fact would have to be included in any meaningful calculation. Nevertheless, it is quite clear that the effect discussed here can be significant (as exhibited by the magnitude of the ratio in Eq. (17)) and should be considered if one is to interpret correctly the results of an experiment involving, say, the measurement of radiative intensity from region 5.

The results for an ideal gas are also useful as an aid in explaining the overall character of the reflected-shock flow field when vibrational or chemical relaxation are important behind an incident shock wave. This is possible since, in general, the perturbing effects of relaxation in region 2 can be isolated from those effects owing to relaxation in region 5. That is, one can reasonably assume that the gas in the reflected-shock flow field relaxes extremely fast (because of the higher temperature) and simply passes through a series of local equilibrium states as the flow field is perturbed by the nonuniformities behind the incident shock wave. In this case, the boundary conditions along the reflected-shock trajectory would correspond to a state of immediate local equilibration for the new gas which passes into region 5. One major difference between the ideal gas studied here and relaxing flows, however, is that thermal energy is not actually lost from gas which relaxes in region 2, but simply transferred from the translational energy mode into internal energy modes, or into changing the chemical composition of the gas. As a result, the energy perturbation in region 5, along the reflected-shock trajectory, is slightly positive instead of negative, and the reflected shock wave accelerates slightly rather than slows down. However, relaxation in region 2 must still cause the pressure in region 5 to increase with time, much the same as for the ideal-gas case, since the effect of increasing

density in region 2 is always to increase the pressure in region 5. Furthermore, as a result of the pressure rise, work is done on the gas, thus causing a nonuniform increase in the energy content of all the particles in region 5 and a perturbation of the flow variables throughout region 5 similar to that found for the ideal gas.

There is another difference between the idealized flow examined above and relaxing gas flows which should be mentioned. For cases involving relaxation, only a limited density change occurs in region 2 so that the resulting perturbations in region 5 do not continue to grow with time. Neglecting transport processes, the reflected shock must eventually move away from the end wall at a steady speed, leaving behind a growing region of gas with a uniform equilibrium state and a stationary entropy layer of finite thickness adjacent to the end wall. For such cases, the "unperturbed" flow conditions in the reflected region should be considered to be the uniform equilibrium state, rather than the state present at the instant of reflection. The thickness of the finite entropy layer scales directly with the length of the relaxation zone behind the incident shock wave, and is essentially the relaxation length compressed by the density ratio across the reflected shock wave.

It is of interest to note that more detailed analyses of shock-wave reflection in relaxing gases have recently been reported, and the results of those studies confirm the existence of finite entropy layers of the same type discussed here.^{1,2} The simple model employing an ideal gas is therefore useful in providing, with a minimum of mathematical effort, an understanding of the character of reflected flow fields when perturbed by relaxation or radiative-cooling phenomena behind incident shock waves.

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APPENDIX A

LINEARIZED EQUATIONS FOR INCIDENT-SHOCK FLOW

The conservation equations of mass and momentum for one-dimensional, inviscid flow across a constant-speed normal shock wave are, in shock-fixed coordinates,

$$\rho_1 V_s = \rho_2 u_2, \quad (\text{A-1})$$

and

$$p_1 + \rho_1 V_s^2 = p_2 + \rho_2 u_2^2. \quad (\text{A-2})$$

The subscript 1 refers to the constant properties upstream of the incident shock wave while subscript 2 refers to the properties downstream of the shock wave.

Rearranging these equations, one can show that the pressure downstream of the shock wave is given by

$$p_2 = p_1 + \rho_1 V_s^2 (1 - \rho_1/\rho_2). \quad (\text{A-3})$$

A small perturbation in pressure owing to a small perturbation in density is, accordingly,

$$p_2' = \rho_1 V_s^2 (\rho_1/\rho_2) (\rho_2'/\rho_2), \quad (\text{A-4})$$

where the prime symbol denotes a perturbation quantity and the unprimed variables refer to the unperturbed values of the properties at the shock front. Neglecting p_1 in Eq. (A-3) (a reasonable approximation for $M_s \gg 3$; e.g., $p_2/p_1 = 10.3$ for $M_s = 3$ and $\gamma = 1.4$), one can write the dimensionless variation in pressure as

$$p_2'/p_2 = (\rho_2'/\rho_2)/(\rho_2/\rho_1 - 1). \quad (\text{A-5})$$

The thermal equation of state for an ideal gas is

$$p = \rho R t.$$

The linearized form of this equation, valid for small perturbations away from a reference condition, is

$$p'/p = \rho'/\rho + T'/T. \quad (\text{A-6})$$

Assuming that the density perturbation is specified, we see that the temperature perturbation is given directly from Eqs. (A-5) and (A-6), i.e.,

$$T'_2/T_2 = -[(\rho_{21} - 2)/(\rho_{21} - 1)]\rho'_2/\rho_2. \quad (\text{A-7})$$

Since the speed of sound is defined by

$$a = (\gamma RT)^{1/2},$$

it follows that the dimensionless perturbation in this quantity, at any position in region 2, is

$$a'_2/a_2 = \frac{1}{2}(T'_2/T_2) = -[(\rho_{21} - 2)/2(\rho_{21} - 1)](\rho'_2/\rho_2). \quad (\text{A-8})$$

APPENDIX B

BOUNDARY CONDITION ON DENSITY ALONG THE REFLECTED-SHOCK TRAJECTORY

The shock-jump relation for the density behind a reflected shock wave is given by

$$\rho_5 = \rho_{52} \rho_2 = \frac{(\gamma+1)M_r^2}{(\gamma-1)M_r^2 + 2} \rho_2, \quad (\text{B-1})$$

where M_r is the reflected-shock Mach number defined by

$$M_r = [V_s(1-\rho_{12}) + V_r]/a_2.$$

Retaining the first-order terms in the perturbed quantities, one finds from the density-jump relation that

$$\hat{\rho}'_5/\rho_5 = \hat{\rho}'_{52}/\rho_{52} + \hat{\rho}'_2/\rho_2, \quad (\text{B-2})$$

where

$$\hat{\rho}'_{52}/\rho_{52} = (4/G)\hat{M}'_r/M_r,$$

$$G = (\gamma-1)M_r^2 + 2,$$

and the caret symbol has been used to denote the value of the quantity along the reflected-shock trajectory.

Although variations in M_r are caused by variations in ρ_2 , a_2 , and V_r , the variations in ρ_2 and a_2 are related by (see Appendix A, Eq. (A-8))

$$a'_2/a_2 = -[(\rho_{21} - 2)/2(\rho_{21} - 1)](\rho'_2/\rho_2)$$

so that

$$\begin{aligned} \hat{M}'_r/M_r &= (\partial M_r / \partial V_r)_{\rho_2, a_2} V'_r/M_r + (\partial M_r / \partial \rho_2)_{a_2, V_r} \hat{\rho}'_2/M_r \\ &\quad + (\partial M_r / \partial a_2)_{V_r, \rho_2} \hat{a}'_2/a_2 \end{aligned}$$

can be written simply as

$$\hat{M}'_r/M_r = [M_2/M_r + (\rho_{21} - 2)/2(\rho_{21} - 1)]\hat{\rho}'_2/\rho_2 + \rho_{25} V'_r/V_r. \quad (B-3)$$

The quantity M_2 represents the dimensionless speed of the flow in region 2 with respect to the incident shock wave and is defined by

$$M_2 = V_s/\rho_{21} a_2.$$

Substituting this last result into Eq. (B-2) yields the density boundary condition

$$\rho'_5/\rho_5 = (1+4H/G)\rho'_2/\rho_2 + (4/G\rho_{52})V'_r/V_r, \quad (B-4)$$

where

$$H = M_2/M_r + (\rho_{21} - 2)/2(\rho_{21} - 1).$$

This boundary condition can be evaluated along the unperturbed shock trajectory ($\hat{x} = V_r t$) by substituting

$$s = (V_s + V_r)t$$

into the expression for the incident-flow density variation, Eq. (1) in the text, i.e.,

$$\hat{\rho}'_2/\rho_2 = \beta(V_s + V_r)t = Lt.$$

The final result is

$$\hat{\rho}'_5/\rho_5 = (1+4H/G)Lt + (4/G\rho_{52})V'_r/V_r, \quad (B-5)$$

which is Eq. (2) in the text.

APPENDIX C

BOUNDARY CONDITION ON PRESSURE ALONG THE REFLECTED-SHOCK TRAJECTORY

The shock-jump relation for the pressure behind a reflected shock wave is given by

$$p_5 = p_{52} p_2 = [1 + 2\gamma(M_r^2 - 1)/(\gamma+1)] p_2. \quad (C-1)$$

Retaining first-order terms in the perturbed quantities, one finds

$$\hat{p}'_5/p_5 = \hat{p}'_{52}/p_{52} + \hat{p}'_2/p_2, \quad (C-2)$$

where

$$\hat{p}'_{52}/p_{52} = [4\gamma M_r^2/(\gamma+1)p_{52}] \hat{M}'_r/M_r.$$

Recalling Eq. (B-3) from Appendix B,

$$\hat{M}'_r/M_r = H(\hat{\rho}'_2/\rho_2) + \rho_{25}(V'_r/V_r),$$

and Eq. (A-5) from Appendix A,

$$p'_2/p_2 = (\rho'_2/\rho_2)/(\rho_{21} - 1),$$

one can easily show that

$$\hat{p}'_5/p_5 = \left[\frac{4\gamma M_r^2 H}{p_{52}(\gamma+1)} + \frac{1}{\rho_{21} - 1} \right] (\hat{\rho}'_2/\rho_2) + \left[\frac{4\gamma M_r V_r}{p_{52}(\gamma+1)a_2} \right] (V'_r/V_r). \quad (C-3)$$

If this last result is evaluated along the unperturbed reflected-shock trajectory, one obtains

$$\hat{p}'_5/p_5 = \left[\frac{4\gamma M_r^2 H}{p_{52}(\gamma+1)} + \frac{1}{\rho_{21} - 1} \right] Lt + \left[\frac{4\gamma M_r V_r}{p_{52}(\gamma+1)a_2} \right] (V'_r/V_r), \quad (C-4)$$

which is Eq. (3) in the text.

APPENDIX D

BOUNDARY CONDITION ON PARTICLE VELOCITY ALONG THE
REFLECTED-SHOCK TRAJECTORY

The application of the relation for the conservation of mass immediately across the perturbed reflected shock wave yields

$$(\rho_2 + \hat{\rho}'_2)(u_p + \hat{u}'_p + V_r + V'_r) = (\rho_5 + \hat{\rho}'_5)(V_r + V'_r - \hat{u}'_5), \quad (D-1)$$

where u_p is the particle velocity in region 2 in laboratory coordinates, i.e.,

$$u_p = V_s(1 - \rho_{12}). \quad (D-2)$$

Retaining only the first-order terms, one finds

$$\rho_2(u_p + \hat{u}'_p + V_r + V'_r) + \hat{\rho}'_2(u_p + V_r) = \rho_5(V_r + V'_r - \hat{u}'_5) + \hat{\rho}'_5 V_r. \quad (D-3)$$

However, since

$$\rho_2(u_p + V_r) = \rho_5 V_r,$$

from the conservation relation applied across the unperturbed shock wave, Eq. (D-3) can be simplified to read

$$\hat{u}'_5/V_r = (1 - \rho_{25})(V'_r/V_r) + (\hat{\rho}'_5/\rho_5 - \hat{\rho}'_2/\rho_2) - \hat{u}'_p/\rho_{52} V_r. \quad (D-4)$$

From Eq. (D-2) it follows that

$$u'_p = + V_s \rho_{12} (\rho'_2/\rho_2)$$

so that

$$u'_p/\rho_{52} V_r = (V_s/\rho_{51} V_r)(\rho'_2/\rho_2)$$

and Eq. (D-4) becomes

$$\hat{u}_5/v_r = \hat{\rho}_5/\rho_5 - (1 + v_s/\rho_{51}v_r)(\hat{\rho}'_2/\rho_2) + (1 - \rho_{25})v'_r/v_r.$$

Substituting the known boundary condition for density, Eq. (B-4) of Appendix B, this last result becomes

$$\hat{u}_5/v_r = [4H/G - v_s/\rho_{51}v_r]\hat{\rho}'_2/\rho_2 + [1 + \rho_{25}(4/G - 1)]v'_r/v_r. \quad (D-5)$$

Finally, evaluating this boundary condition along the unperturbed reflected-shock trajectory, one obtains

$$\hat{u}_5/v_r = [4H/G - v_s/\rho_{51}v_r]1 + [1 + \rho_{25}(4/G - 1)]v'_r/v_r, \quad (D-6)$$

which is Eq. (4) in the text.

APPENDIX E

WAVE-EQUATION SOLUTIONS IN REGION 5

The wave equations in particle velocity and pressure in region 5 are

$$\partial^2 u_5 / \partial t^2 - a_5^2 \partial^2 u_5 / \partial x^2 = 0 \quad (\text{E-1})$$

and

$$\partial^2 p_5' / \partial t^2 - a_5^2 \partial^2 p_5' / \partial x^2 = 0. \quad (\text{E-2})$$

The boundary conditions are given by

$$u_5(x = 0, t) \equiv 0, \quad (\text{E-3})$$

$$\hat{u}_5(t) = A L t + B V_r', \quad (\text{E-4})$$

and

$$\hat{p}_5'(t) / \rho_5 a_5 = C L t + D V_r', \quad (\text{E-5})$$

where, in the notation used in the text,

$$A = 4 H V_r' / G - V_s / \rho_{51},$$

$$B = 1 - \rho_{25} + 4 \rho_{25} / G,$$

$$C = \frac{4 M_r^2 H a_5}{p_{52} (\gamma + 1)} + \frac{a_5}{\gamma (\rho_{21} - 1)},$$

and

$$D = \frac{4 M_r a_5}{p_{52} (\gamma + 1) a_2}.$$

The general solutions of Eqs. (E-1) and (E-2) are

$$u_5 = f(x + a_5 t) + g(x - a_5 t) \quad (\text{E-6})$$

and

$$-p'_5/\rho_2 a_5 = f(x + a_5 t) - g(x - a_5 t). \quad (\text{E-7})$$

Applying the boundary condition at the end wall, one finds

$$g(\xi) = -f(-\xi)$$

so that Eqs. (E-6) and (E-7) can be rewritten as

$$u_5 = f(x + a_5 t) - f(-x + a_5 t) \quad (\text{E-8})$$

and

$$-p'_5/\rho_5 a_5 = f(x + a_5 t) + f(-x + a_5 t). \quad (\text{E-9})$$

Applying the boundary conditions along the unperturbed shock trajectory, Eqs. (E-4) and (E-5), one can write

$$ALT + BV'_r = f[(V_r + a_5)t] - f[(-V_r + a_5)t] \quad (\text{E-10})$$

and

$$CLT + DV'_r = -f[(V_r + a_r)t] - f[(-V_r + a_5)t]. \quad (\text{E-11})$$

Eliminating V'_r from these last two equations, one finds that

$$(DA-BC)Lt = (B+D)f[(V_r + a_5)t] + (B-D)f[(-V_r + a_5)t], \quad (\text{E-12})$$

which yields the solution

$$f(\xi) = -\frac{1}{2} KL\xi \quad (\text{E-13})$$

where

$$K = \frac{BC - DA}{Ba_5 + DV'_r}.$$

The fact that this solution is unique can be seen by expanding the function f in a Taylor series,

$$f(\xi) = f(0) + f'(0)\xi + f''(0)\xi^2/2 \dots,$$

and then substituting the expression into Eq. (E-12). The zeroth-order derivative of Eq. (E-12) evaluated at $t = 0$ yields

$$f(0) = 0,$$

while the first-order derivative establishes the relation

$$(DA-BC)L = [(B+D)(V_r + a_5) + (B-D)(-V_r + a_5)]f'(0)$$

so that

$$f'(0) = -\frac{1}{2} \frac{BC - DA}{Ba_5 + DV_r} L.$$

The expressions for the second- and all higher-order derivatives yield

$$f''(0) = 0, \quad f'''(0) = 0, \dots,$$

so that the solution for $f(\xi)$ is given by Eq. (E-13).

The solution for the particle velocity now follows from Eq. (E-8), i.e.,

$$u_5 = -\frac{1}{2} KL[(x + a_5 t) - (-x + a_5 t)],$$

which simplifies to Eq. (9) in the text,

$$u_5/V_r = -KLx/V_r = -(Kx/\hat{x})Lt. \tag{E-14}$$

By similar substitution in Eq. (E-9), one can show that the pressure solution, Eq. (10) in the text, is given by

$$p'_5/p_5 = (\gamma K)Lt. \tag{E-15}$$

The solution for the perturbation in the reflected-shock velocity can be obtained from Eq. (E-10), i.e.,

$$ALt + BV'_r = -\frac{1}{2} KL[(V_r + a_5)t - (-V_r + a_5)t],$$

so that

$$V'_r/V_r = - \left(\frac{C + Aa_5/V_r}{Ba_5 + DV_r} \right) \text{ (Lt)}. \quad (\text{E-16})$$

This last result is Eq. (11) in the text.

APPENDIX F

DENSITY SOLUTION IN REGION 5

The solution for the density can be obtained through the continuity equation and the known solution for particle velocity. The linearized form of the continuity equation is

$$\partial \rho_5' / \partial t + \rho_5 \partial u_5 / \partial x = 0, \quad (\text{F-1})$$

where

$$\partial u_5 / \partial x = -KL$$

(from Eq. (9) in the text). Integrating the continuity relation, one finds

$$\rho_5' / \rho_5 = +KLt + g(x), \quad (\text{F-2})$$

where $g(x)$ is a function which can be determined from the boundary condition for density at the shock wave (Eq. (7) in the text),

$$\hat{\rho}_5' = ELt + FV_r' . \quad (\text{F-3})$$

Substituting the known values for E , F , and V_r' , one can rewrite Eq. (F-2) as

$$\hat{\rho}_5' / \rho_5 = +KLt + g(V_r' t) = (1 + 4H/G)Lt - (4/\rho_{52}G)JLt,$$

which yields the solution

$$g(x) = [1 + 4H/G - K - 4J/G\rho_{52}]Lx/V_r'. \quad (\text{F-5})$$

The solution for density is therefore

$$\rho_5' / \rho_5 = [K + I x/\hat{x}]Lt, \quad (\text{F-6})$$

where $I = (1 + 4H/G - 4J/G\rho_{52} - K)$. This result is also Eq. (12) in the text.

APPENDIX G

ENTROPY SOLUTION IN REGION 5

The entropy solution follows from the differential form of the combined First and Second Laws of Thermodynamics,

$$Tds = de - (p/\rho^2)d\rho. \quad (G-1)$$

Substituting for the internal-energy term,

$$de = [R/(\gamma-1)]dT,$$

one can rewrite Eq. (G-1) as

$$ds = [R/(\gamma-1)]dT/T - (p/\rho T)d\rho/\rho.$$

In dimensionless form, and for small perturbations, this last result becomes

$$s'/R = T'/(\gamma-1)T - \rho'/\rho. \quad (G-2)$$

Substituting the known solutions for temperature and density, Eqs. (12) and (13) in the text, one can write finally

$$s'/R = -[\gamma I/(\gamma-1)](x/\hat{x})L_t, \quad (G-3)$$

or, equivalently,

$$s'/R = -[\gamma IL/(\gamma-1)V_r]x, \quad (G-4)$$

which is Eq. (14) in the text.

APPENDIX H

ENERGY RATE-OF-CHANGE RELATION

The variation of the temperature perturbation with distance behind the incident shock wave is given by (see Eq. (A-7) in Appendix A)

$$T_2'/T_2 = - [(\rho_{21} - 2)/(\rho_{21} - 1)]\beta s. \quad (\text{H-1})$$

For an ideal gas this quantity is also equal to e_2'/e_2 where e_2 is the specific internal energy. The rate of change of internal energy following a fluid particle is therefore

$$\frac{De_2'}{Dt} = - [(\rho_{21} - 2)/(\rho_{21} - 1)]e_2\beta \frac{Ds}{Dt}, \quad (\text{H-2})$$

where

$$\frac{Ds}{Dt} = u_2 = V_s/\rho_{21}.$$

In order to write this last result in terms of L , one can multiply and divide by $(V_s + V_r)$, i.e.,

$$\frac{De_2'}{Dt} = - [(\rho_{21} - 2)/(\rho_{21} - 1)][e_2 L / (1 + V_r/V_s)\rho_{21}]. \quad (\text{H-3})$$

However, one can readily show that

$$V_r/V_s = (\rho_{21} - 1)/(\rho_{51} - \rho_{21})$$

so that Eq. (H-3) can be rewritten simply as

$$\frac{De_2'}{Dt} = - (\rho_{21} - 2)(\rho_{52} - 1)e_2 L / (\rho_{51} - 1)(\rho_{21} - 1), \quad (\text{H-4})$$

which is Eq. (16) in the text.

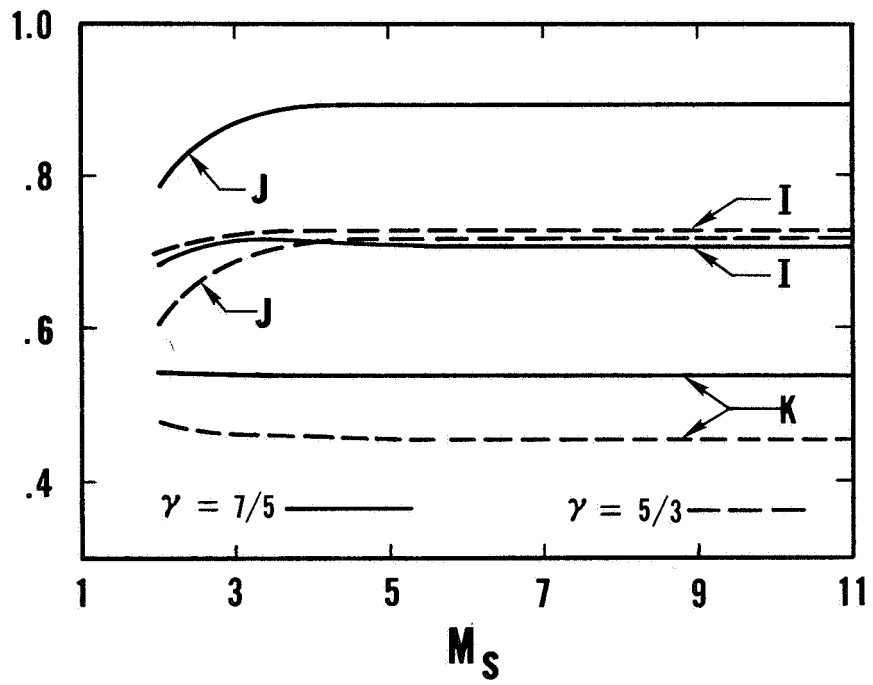


Figure 2. Dimensionless Coefficients I, J, and K

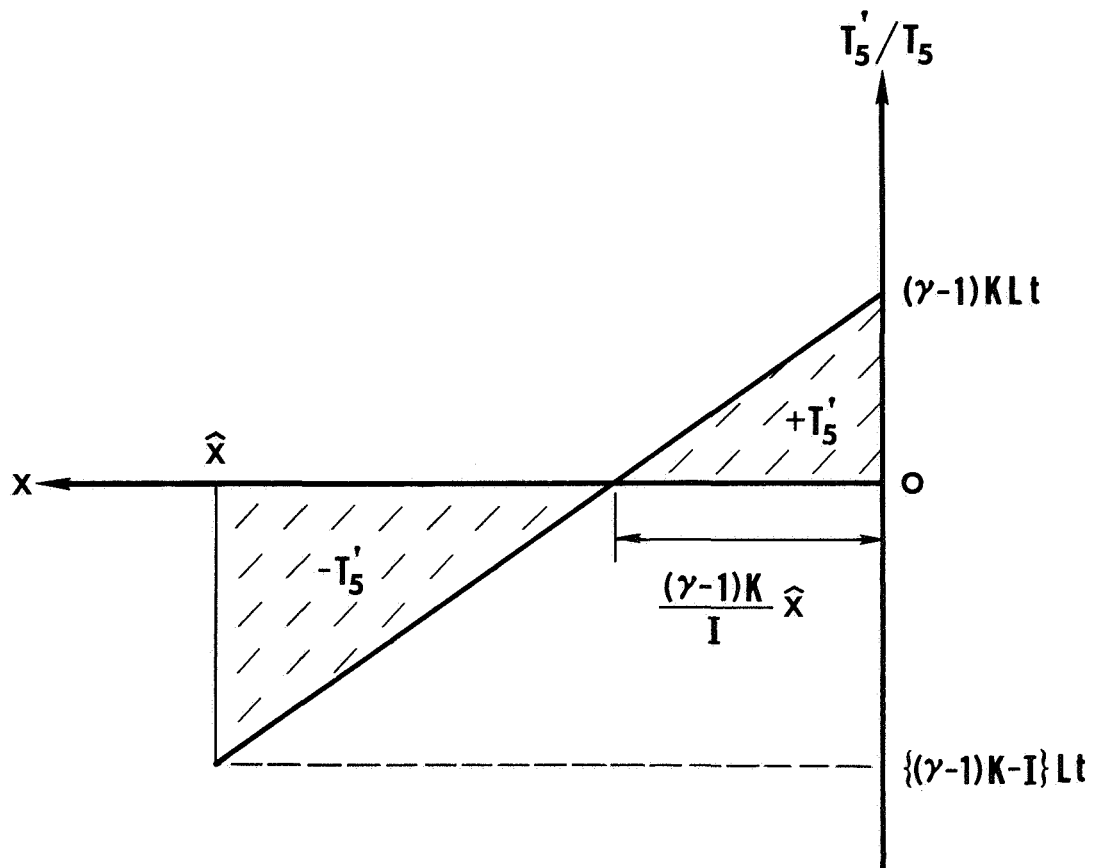


Figure 3. Temperature distribution

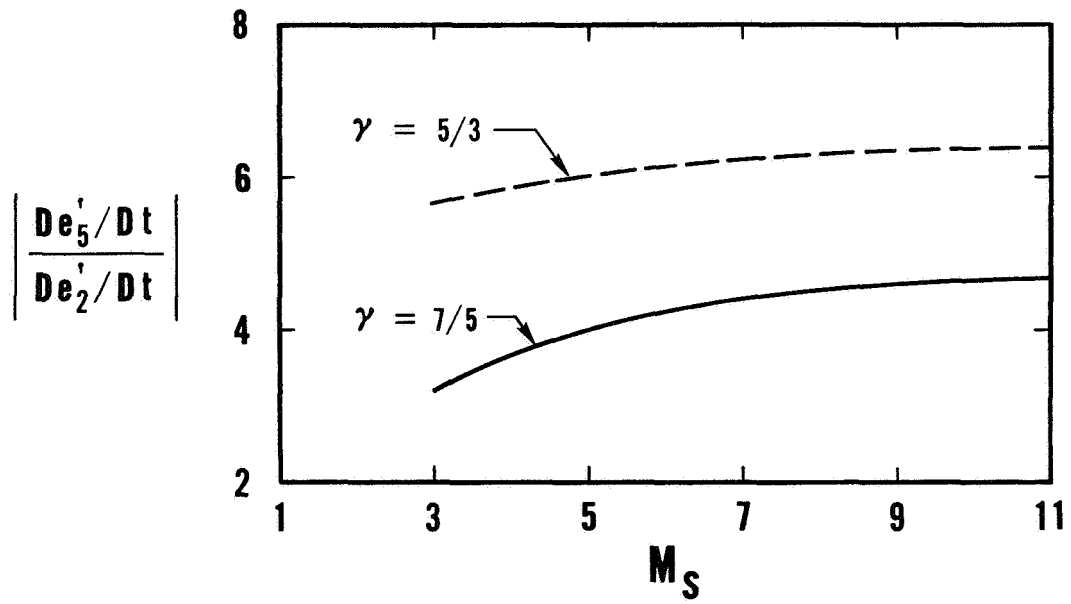


Figure 4. Internal energy gain-to-loss rate