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FOURIER SPECTRA AND SHOCK SPECTRA FOR SIMPLE UNDAMPED SYSTEMS -- A GENERALIZED APPROACH

by

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SUMMARY

It is known that a relationship exists between the Fourier Spectrum of a transient excitation and the maximum residual response of an undamped mass-spring system to this excitation. This relationship is derived in detail and unified to cover all common forms of excitation and response of an undamped single-degree-of-freedom system. The method provides a simple design tool for application to many types of transient response problems. For a step-type transient excitation, the method specifies the overall transient response magnitude. For pulse-type transient excitations, the maximum residual response, or residual shock spectrum, specified by the method, is equal to the maximum response for a pulse whose duration T is less than about 1/2 the natural period of the system.

The Fourier Spectra of a variety of common transient excitations is presented in graphical form in the text. Analytical expressions for these spectra, expressed in a normalized form so that they are numerically equal to the corresponding residual response spectra, are given in the Appendix. In addition, the Appendix contains a summary of the expressions for the response time history of an undamped system and the primary shock spectra, or envelope, of maximum response during the transient excitation.

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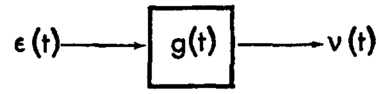
1.0 INTRODUCTION

The response of an undamped single-degree-of-freedom system to a transient excitation is a classical problem in dynamics which can be solved by a number of methods, including the use of the Duhamel (convolution) integral, Laplace Transform, numerical integration, and graphical phase plane methods (References 1, 2 and 3). In all cases, however, these methods lack a simple and direct means of establishing the peak response to the transient excitation. It is this latter quantity which is ordinarily of concern for engineering purposes. One direct method is available, however, for defining the peak residual response to a transient excitation (References 1 and 3). This residual response occurs after the end of the transient excitation and can be determined solely by a Fourier Spectrum of the excitation itself. This relationship between the Fourier Spectrum of a transient excitation and its residual shock spectrum is explored in detail in a unified form for most types of excitation and response of an undamped single-degree-of-freedom system.

2.0 BASIC THEORY



For any linear system, initially at rest, the response time history $\nu(t)$ to any input excitation $\epsilon(t)$ can be defined by the Duhamel integral

$$\nu(t) = \int_0^t g(t-\tau)\ \epsilon(\tau)\ d\tau \tag{1}$$

where

τ = dummy time variable $\leq t$

$g(t)$ = response of system to a unit impulse at time t, and

$\nu(t)$, $\epsilon(t)$ = generalized response and input variables to be defined.

A general equation of motion for the undamped single-degree-of-freedom system can be expressed in the form (Reference 1):

$$\frac{1}{\omega_0^2}\ \ddot{\nu}(t) + \nu(t) = \epsilon(t) \tag{2}$$

where $\epsilon(t)$ and $\nu(t)$ represent generalized variables for the specific forms of excitation and response which are listed in Table I and identified in Figure 1. The undamped natural frequency of the system is ω_0. The particular form for Equation 2 is chosen for reasons to be made clear later on.

It is sufficient, for now, to emphasize that the generalized excitation variable $\epsilon(t)$ can have any of the forms listed in the left-hand column of Table I. The corresponding generalized response variable $\nu(t)$ is listed in the middle column of Table I.

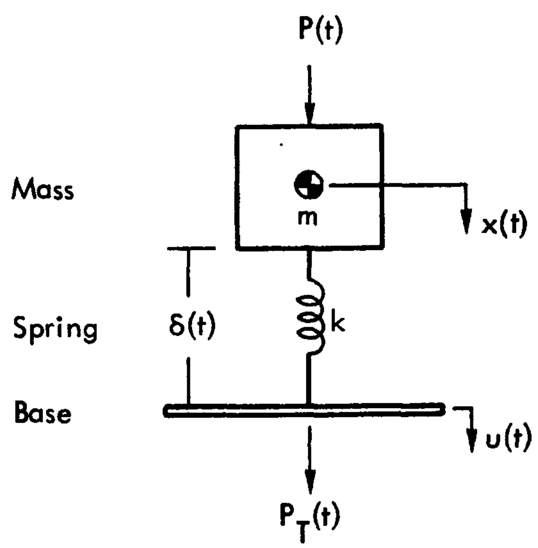
For example, if the undamped mass-spring system in Figure 1 has its base attached to a rigid foundation and the mass m is driven by a transient vertical force P(t), the usual form for the equation of motion of the mass would be

$$m\ \ddot{x}(t) + k\ x(t) = P(t)$$

TABLE I

EXCITATION AND RESPONSE VARIABLES FOR A FIXED-BASE OR MOVING-BASE UNDAMPED SINGLE-DEGREE-OF-FREEDOM SYSTEM

System	Excitation $\epsilon(t)$		Response $v(t)$	
Fixed Base	Force on Mass	$P(t)/k$ $P(t)$	$x(t) = \delta(t)$ $P_T(t)$	- Mass Displacement - Reaction Force on Base
Moving Base	Base Displacement	$u(t)$	$x(t)$	- Mass Displacement
	Base Velocity	$\dot{u}(t)$	$\dot{x}(t)$	- Mass Velocity
	Base Acceleration	$\ddot{u}(t)$ $-\ddot{u}(t)/\omega_0^2$ $m\,\ddot{u}(t)$	$\ddot{x}(t)$ $\delta(t)$ $P_T(t)$	- Mass Acceleration - Relative Displacement of Spring - Reaction Force on Base
	Base Jerk	$-\dddot{u}(t)/\omega_0^2$	$\dot{\delta}(t)$	- Relative Velocity of Spring



$$\omega_0^2 = k/m$$

Figure 1. General Model for Response and Excitation of Undamped Single Degree-of-Freedom System

Dividing through by the spring constant k and setting $m/k = 1/\omega_0^2$, this becomes

$$\frac{1}{\omega_0^2}\ddot{x}(t) + x(t) = \frac{P(t)}{k}$$

This is the same as the generalized form given by Equation 2 and identified by the first entry in Table I where the excitation $\epsilon(t)$ is $P(t)/k$ and the response $v(t)$ is $x(t)$.

The remaining entries in Table I are formed in a similar manner so that Equation 2 becomes a single general form for the equation of motion for all the forms of excitation and response listed.

2.1 Unit Impulse Response

The unit impulse can be defined, in general form, by the integral

$$\lim_{t \to 0} \int_0^t \epsilon(\tau)\, d\tau = 1 \qquad (3)$$

where τ is a dummy time variable of integration $\leq t$.

This represents an excitation whose duration is vanishingly small and whose integral with time is unity. Since it corresponds to an excitation with essentially zero duration, it may be treated as an initial condition to the solution of Equation 2 for free vibration where $\epsilon(t)$ is zero.

This general solution to Equation 2 for $\epsilon(t) = 0$ is

$$v(t) = A \cos \omega_0 t + B \sin \omega_0 t \qquad (4)$$

The initial value of the response variable $v(0)$ is

$$v\,(t \to 0) = A \qquad (5)$$

so that for a system starting at rest, $A = 0$. The initial rate of change of the response variable is

$$\dot{v}\,(t \to 0) = \omega_0 B \qquad (6)$$

However, Equation 2 can also be used to define the initial rate of change, in the limit, as $t \rightarrow 0$. Since the initial response magnitude is zero, then the initial rate of change is obtained by setting $v = 0$ in Equation 2, integrating $\ddot{v} = d\dot{v}/dt$ with time, and taking the limit as $t \rightarrow 0$. The result is

$$\dot{v}(t \rightarrow 0) = \lim_{t \rightarrow 0} \left[\int_0^t \frac{d\dot{v}}{d\tau} \cdot d\tau \right] = \lim_{t \rightarrow 0} \left[\omega_0^2 \int_0^t \epsilon(\tau)\, d\tau \right] \tag{7}$$

The left-hand side, $\dot{v}(t \rightarrow 0)$, is given by Equation 6 and the right side by ω_0^2 times Equation 3 for the unit impulse excitation, so that

$$\omega_0 B = \omega_0^2 \cdot 1$$

or

$$B = \omega_0 \tag{8}$$

Thus, the generalized unit impulse response for the form of the equation of motion in Equation 2 is (see Appendix A)

$$g(t) = \omega_0 \sin \omega_0 t \tag{9}$$

This can be shown to have the units corresponding to (units of v)/(units of $\epsilon \cdot$ time).

2.2 General Response Equation

If Equation 9 is inserted into Equation 1, and the sine function expanded, the general response equation for any input $\epsilon(t)$ becomes

$$v(t) = \int_0^t \omega_0 \sin \omega_0 (t-\tau)\, \epsilon(\tau)\, d\tau$$

$$= \omega_0 \sin \omega_0 t \int_0^t \epsilon(\tau) \cos \omega_0 \tau\, d\tau$$

$$- \omega_0 \cos \omega_0 t \int_0^t \epsilon(\tau) \sin \omega_0 \tau\, d\tau \tag{10}$$

Note that $v(t)$ is still a general response variable, not necessarily a displacement. The generalized rate of change is obtained by differentiating Equation 10 with respect to time t. This does not involve the integrals in Equation 10 since these are independent of time t. Thus,

$$\dot{v}(t) = \omega_0^2 \cos \omega_0 t \int_0^t \epsilon(\tau) \cos \omega_0 \tau \, d\tau$$

$$+ \omega_0^2 \sin \omega_0 t \int_0^t \epsilon(\tau) \sin \omega_0 \tau \, d\tau \qquad (11)$$

2.3 Residual Response and Fourier Spectrum

Equations 10 and 11 can now be used to define the response magnitude $v(T)$ and its rate of change $\dot{v}(T)$ at the end of a transient excitation of finite duration T. These will then become initial conditions for the free vibration after cessation of the transient. This is the period of residual vibration for which an envelope of maximum response as a function of ω_0 is desired. This will be shown to be directly related to the Fourier Spectrum of the excitation only, without requiring any knowledge of the forced response during the transient (Reference 3). The system itself is defined only by zero damping and its undamped natural frequency ω_0.

Thus, by replacing t in Equations 10 and 11 with the pulse duration T, the initial response magnitude and rate of change at the end of the transient input are determined. Consider now the two integrals in each of these equations. The Fourier Spectrum $F(j\omega)$ of the excitation $\epsilon(t)$ is

$$F(j\omega) = \int_0^T \epsilon(t) \, e^{-j\omega t} \, dt \qquad (12)$$

or, in trigonometric form,

$$F(j\omega) = \int_0^T \epsilon(t) \cos \omega t \, dt - j \int_0^T \epsilon(t) \sin \omega t \, dt \qquad (13)$$

$$= \mathscr{R}[F(j\omega)] + j \mathscr{I}[F(j\omega)]$$

where

$$\mathscr{R}[F(j\omega)] = \int_0^T \epsilon(t) \cos \omega t \, dt = \text{Real Part of Fourier Spectrum}$$

$$\mathscr{I}[F(j\omega)] = - \int_0^T \epsilon(t) \sin \omega t \, dt = \text{Imaginary Part of Fourier Spectrum}$$

Comparing Equation 13 with the integrals in Equations 10 and 11, it is clear that these are identical providing t becomes T, τ becomes t, and ω_0 becomes ω. The first transformation has already been established as a requirement to define conditions at the end of the transient excitation. This allows τ to become the actual time t during the transient, or $\tau = t < T$. Finally, the natural frequency of the system ω_0 can take on any value since a specific system has not yet been defined, only its mathematical model, so that ω_0 can become any frequency ω, or vise versa.

Thus, Equations 10 and 11 can be modified to specify the response magnitude and its rate of change at the end of the *transient* excitation by the form

$$v(\omega_0, T) = \omega_0 \left\{ \mathscr{R}[F(j\omega_0)] \sin \omega_0 T + \mathscr{I}[F(j\omega_0)] \cos \omega_0 T \right\} \tag{14}$$

$$\dot{v}(\omega_0, T) = \omega_0^2 \left\{ \mathscr{R}[F(j\omega_0)] \cos \omega_0 T - \mathscr{I}[F(j\omega_0)] \sin \omega_0 T \right\} \tag{15}$$

For this period of free vibration after the transient excitation, the residual response $v_r(t) = v(t > T)$ can now be defined by returning to Equations 4, 5 and 6 to give

$$v_r(t) = v(\omega_0, T) \cos \omega_0 t + \frac{\dot{v}(\omega_0, T)}{\omega_0} \sin \omega_0 t \tag{16}$$

This is, of course, a pure sinusoidal motion with a frequency ω_0 and has a magnitude

$$v_{r\,max} = \left\{ [v(\omega_0 T)]^2 + \left[\frac{\dot{v}(\omega_0, T)}{\omega_0} \right]^2 \right\}^{\frac{1}{2}} \tag{17}$$

Inserting Equations 14 and 15 into Equation 17, the cross-product terms in the square cancel and, since $\sin^2 x + \cos^2 x = 1$, one obtains

$$v_{r\,max} = \omega_0 \left\{ \mathscr{R}^2 \; [F(j\omega_0)] + \mathscr{I}^2 \; [F(j\omega_0)] \right\}^{\frac{1}{2}} \tag{18}$$

However, the square root term is simply the absolute value of the Fourier Spectrum of the excitation so that Equation 17 becomes:

$$\nu_{r\,max} = \omega_0 \left| F(j\omega_0) \right| \tag{19}$$

Thus, a simple expression is obtained which relates the maximum amplitude of the residual vibration in terms of the absolute value of the Fourier Spectra of the excitation and the resonance frequency of the system.

Consider now the units of the right side of Equation 19. From Equation 12, the units of $F(j\omega)$ are (units of excitation variable ϵ) x (time). Thus, the units of $\omega\,|F(j\omega)|$ will have the same units as the excitation. This is also obvious from Equations 2 and 19. Clearly, if the transient excitation is always normalized by its maximum value ϵ_{max} then one can write Equation 19 in the dimensionless form,

$$\frac{\nu_{r\,max}}{\epsilon_{max}} = \frac{\omega_0\,|F(j\omega_0)|}{\epsilon_{max}} \tag{20}$$

This is the key result which dictates the reason for choosing the particular form of the equation of motion given by Equation 2. In other words, for any given type of excitation ϵ, a nondimensional plot can be made of the quantity $\omega\,|F(j\omega)|/\epsilon_{max}$ which specifies the ratio of $\nu_{r\,max}$ to ϵ_{max}. Even more generality is provided however by the fact that the right-hand side of Equation 20 will always reduce to a function of only a single nondimensional quantity, $\omega_0 T$, for any given form of transient excitation where T is the duration (or some characteristic time proportional to duration) of the transient excitation.

Equation 20 can now be combined with Table I to summarize the maximum residual response for specific response variables, as shown in Table II. The first column is the specific form of the general excitation variable $\epsilon(t)$. It can be given by the various forms ranging from input force P(t) to the mass to various derivatives of ground motion.

The second column represents the Fourier Spectrum of this excitation, nondimensionalized, according to the right side of Equation 20, by the maximum value of the transient excitation. Note that this quantity is totally independent of the responding system.

The third column, equal to the second column, term by term, is the nondimensional form of the maximum residual response, as specified by Equation 20. This parameter is, of course, determined for a specific frequency ω equal to the natural frequency ω_0 of the system.

TABLE II

MAXIMUM RESIDUAL SHOCK RESPONSE OF UNDAMPED SINGLE DEGREE-OF-FREEDOM SYSTEM IN FIGURE 1 RELATED TO NORMALIZED FOURIER SPECTRUM OF TRANSIENT EXCITATION

	Generalized Excitation $\epsilon(t)$		Normalized Fourier Spectra $\omega\,\lvert F(j\omega)\rvert/\epsilon_{max}$ (1)	Normalized Maximum Residual Response $\nu_{r\,max}/\epsilon_{max}$
Fixed Base	Force on Mass	$P(t)/k$	$\omega\,\lvert F(j\omega)\rvert/X_s$ (2)	$X_{r\,max}/X_s$(2)
		$P(t)$	$\omega\,\lvert F(j\omega)\rvert/P_{max}$	$P_{T\,max}/P_{max}$
Moving Base	Base Displacement $u(t)$		$\omega\,\lvert F(j\omega)\rvert/U_{max}$	$X_{r\,max}/U_{max}$
	Base Velocity $\dot{u}(t)$		$\omega\,\lvert F(j\omega)\rvert/\dot{U}_{max}$	$\dot{X}_{r\,max}/\dot{U}_{max}$
	Base Acceleration	$\ddot{u}(t)$	$\omega\,\lvert F(j\omega)\rvert/\ddot{U}_{max}$	$\ddot{X}_{r\,max}/\ddot{U}_{max}$
		$-\ddot{u}(t)/\omega_0^2$	$\omega\,\lvert F(j\omega)\rvert/\ddot{U}_{max}$	$\omega_0^2\,\delta_{r\,max}/\ddot{U}_{max}$
		$m\ddot{u}(t)$	$\omega\,\lvert F(j\omega)\rvert/\ddot{U}_{max}$	$P_{T\,max}/m\,\ddot{U}_{max}$ (3)
	Base Jerk $-\dddot{u}(t)/\omega_0^2$		$\omega\,\lvert F(j\omega)\rvert/\dddot{U}_{max}$	$\omega_0^2\,\delta_{r\,max}/\dddot{U}_{max}$

(1) $F(j\omega)$ = Fourier Spectrum of $\epsilon(t)$

(2) X_s = P_{max}/k, Static Displacement to Peak Force.

(3) $P_{T\,max}$ = *Maximum Dynamic* Reaction Force on Base for *Maximum* Ground Acceleration $\ddot{U}_{max}$.

3.0 APPLICATION TO PREDICTION OF RESIDUAL SHOCK SPECTRA FOR TRANSIENT EXCITATION

The preceeding results may now be used to define the maximum response amplitude of an undamped single-degree-of-freedom system following the end of a transient excitation. The envelope of this response, plotted versus the natural frequency of the system, is the Residual Response Spectra.

3.1 Pulse-Type Transient Excitation

For a pulse-type excitation, the peak response generally occurs after the end of the excitation (i.e., residual response period) whenever the characteristic duration (T) of the shock is less than 1/2 the natural period ($2\pi/\omega_0$) of the single degree-of-freedom system. This condition applies in a large number of practical cases of interest in shock design.

The value of the residual response spectra is given, in normalized form, by Equation 20. The right hand side of Equation 20 has been evaluated for a number of common types of pulse excitation and the results are plotted in Figure 2. The analytical expressions used to derive these plots are summarized in Appendix A.

Example - As an example, consider a 10 millisecond half-sine pulse excitation with a peak amplitude of 100 g's applied to the base of an undamped system with a natural frequency of 46 Hz. The following parameters can be defined:

$\epsilon(t) = \ddot{u}(t)$, the generalized excitation

$\epsilon_{max} = \ddot{U}_{max} = 100g$, the peak excitation

$T = 0.01$ sec.

$\omega_0 = 2\pi(46) = 290$ radius/sec.

$\omega_0 T = 2.9$

This case meets the criteria that $T \leq (2\pi/\omega_0)$ so that the peak response of the system will occur after the end of the half-sine pulse.

Referring to Figure 2, for $\omega T = \omega_0 T = 2.9$, the normalized Fourier spectrum, $\omega \left| F(j\omega) \right|$ for a unit excitation is 1.50. Thus, according to Table II, the maximum residual acceleration response is defined as follows:

$$\frac{\omega \left| F(j\omega_0) \right|}{\epsilon_{max}} = \frac{\ddot{X}_{r\,max}}{\ddot{U}_{max}} = 1.5$$

or $$\ddot{X}_{r\,max} = 1.5 \cdot \ddot{U}_{max} = 1.5 \cdot 100 = 150 \text{ g's}.$$

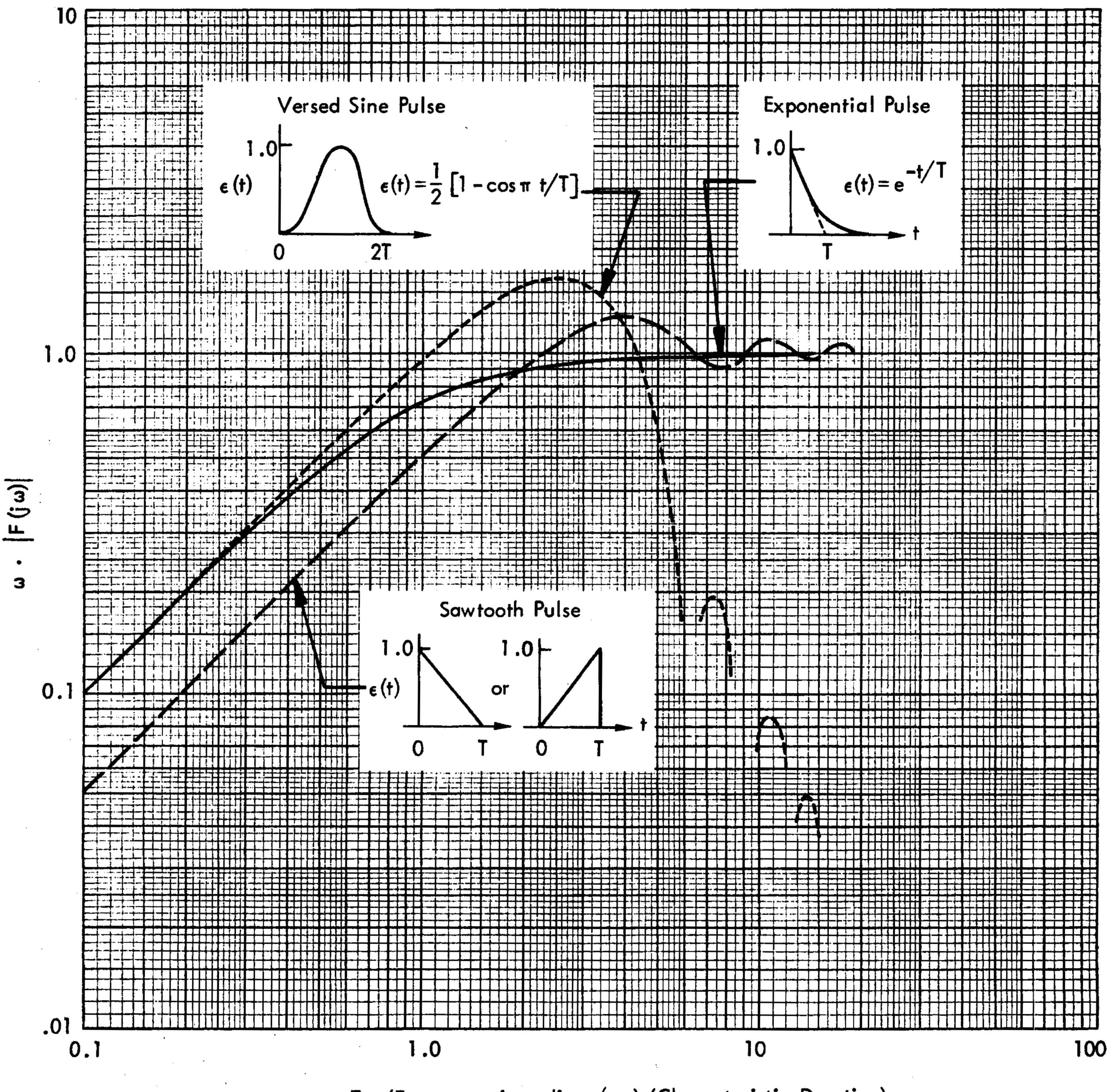


Figure 2a. Normalized Fourier Spectrum of 6 Pulse-Type Shocks

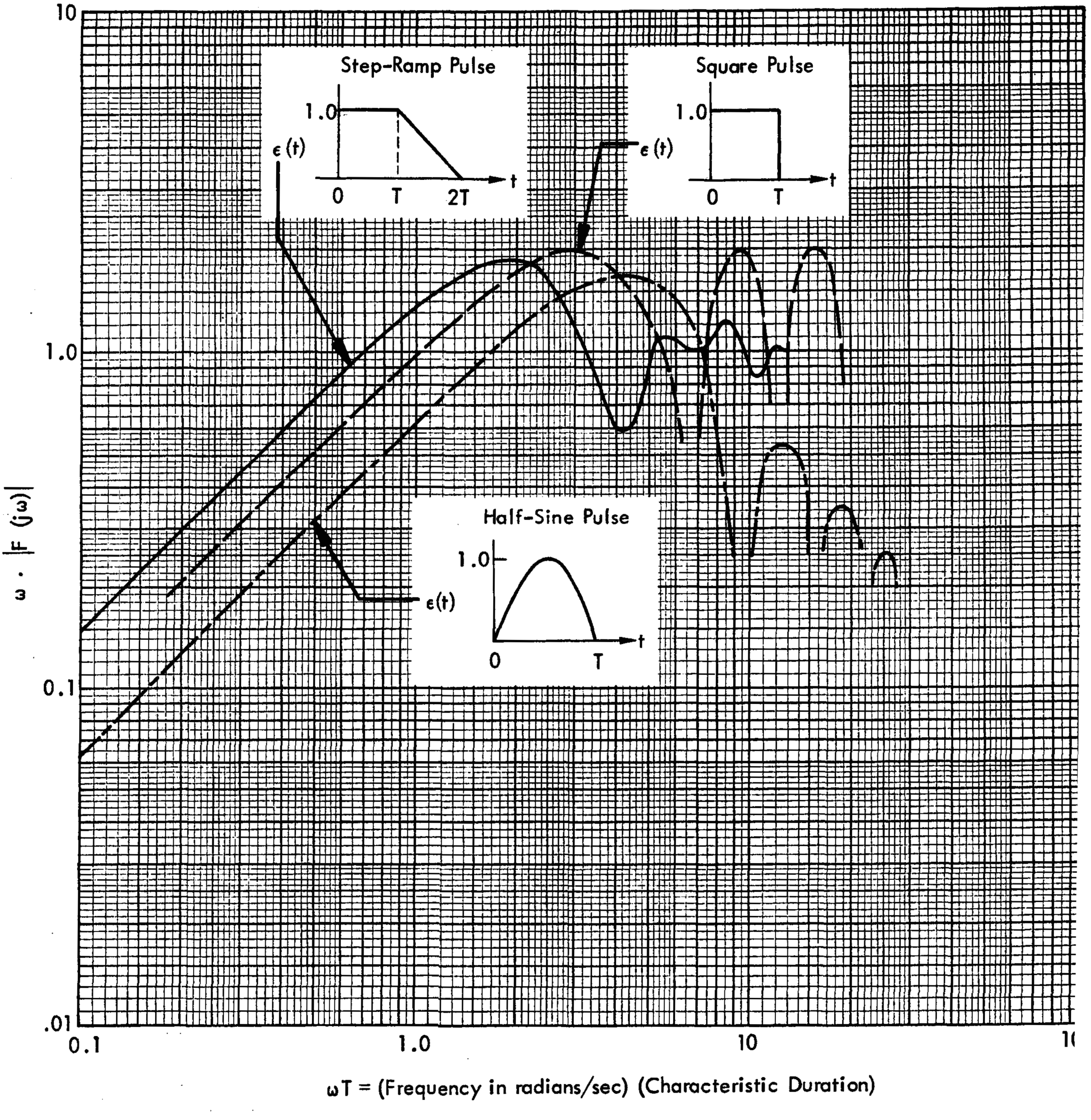


Figure 2b. Normalized Fourier Spectrum of 6 Pulse-Type Shocks

Other response variables, such as the maximum relative displacement $\delta_{r\,max}$ or reaction force $P_{T\,max}$ can be defined in a similar manner by using the relationships in Table II between the desired excitation and response variables.

3.2 Step-Type Excitation

If the transient excitation consists of a step-type input, the maximum transient response ν_{max} occurs after the excitation has reached its maximum value ϵ_{max} and is given by (Reference 1)

$$\nu_{max} = \nu_{r\,max} + \epsilon_{max} \tag{21}$$

as shown in Figure 3.

Thus, for this type of excitation, the maximum residual relative response, added to the maximum excitation defines the maximum total response ν_{max}. In this case, combining Equations 20 and 21, the maximum total response, in nondimensionalized form, is simply

$$\frac{\nu_{max}}{\epsilon_{max}} = \frac{\omega_0 \left|F(j\omega_0)\right|}{\epsilon_{max}} + 1 \tag{22}$$

The first term in Equation 22 has been evaluated for several common types of step-type excitation (see Appendix A) and the results are shown in Figure 4.

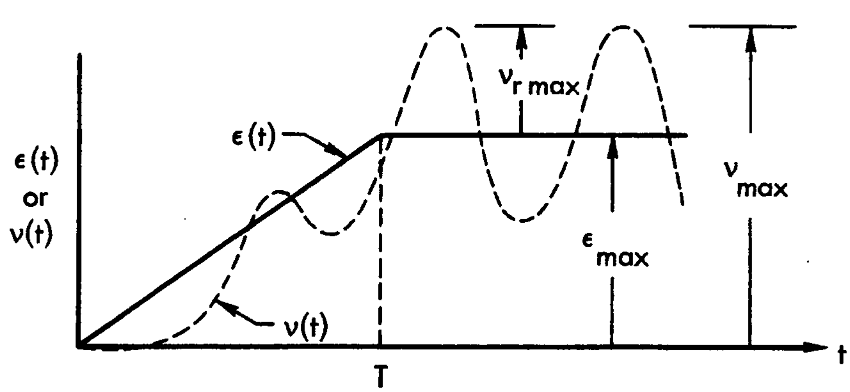


Figure 3. General Form of Excitation and Response to a Step-Type Transient

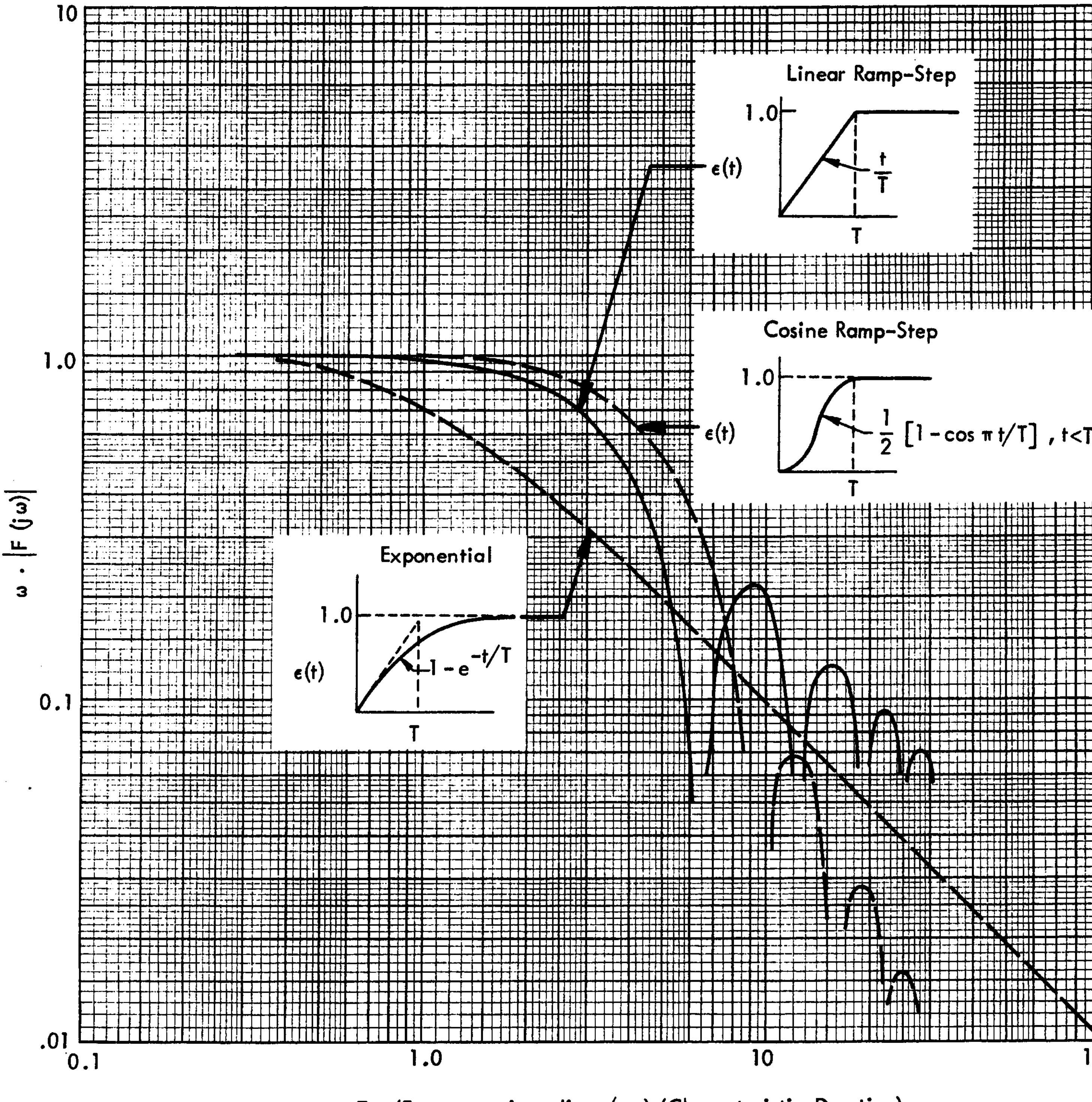


Figure 4. Normalized Fourier Transform for Step-Type Shocks

Example - For illustration, consider the case of a ramp-step force P(t) applied to the mass of a single degree-of-freedom system. The following parameters are assumed.

$\epsilon(t)$ = $P(t)/k$, generalized excitation

P_{max} = 10^4 lbs, maximum value of the step force

k = $2 \cdot 10^5$ lb/in., stiffness of the system

X_s = $P_{max}/k = 0.05$ in., static displacement to the maximum force

f_0 = $\omega_0/2\pi = 46$ Hz, natural frequency

T = 10 milliseconds, rise time

$\omega_0 T$ = $(2\pi)(46)(.01) = 2.9$

From Figure 4, for the ramp step excitation, the maximum relative residual response for $\omega T = 2.9$ is

$$\frac{\nu_{r\,max}}{\epsilon_{max}} = \frac{\omega \left| F(j\omega_0) \right|}{\epsilon_{max}} = 0.67$$

The maximum generalized excitation ϵ_{max} is simply $X_s = P_{max}/k$. According to Equation 21, the maximum total generalized response is

$$\nu_{max} = \nu_{r\,max} + \epsilon_{max}$$

Therefore, for this case, when $\nu_{max} = X_{max}$, $\nu_{r\,max} = X_{r\,max}$ and $\epsilon_{max} = X_s$, the maximum total displacement is

$$X_{max} = \left[\frac{X_{r\,max}}{X_s} + 1 \right] X_s = \left[0.67 + 1 \right] (0.05) = 0.083 \text{ in.}$$

Note that the relative displacement response, after the end of the ramp excitation, is purely sinusoidal. The velocity and acceleration response $\dot{X}(t)$ and $\ddot{x}(t)$ can be determined exactly by simply differentiating this relative displacement response $x_r(t)$. Thus, the following additional maximum response parameters can be defined.

Maximum Velocity of Mass

$$\dot{X}_{max} = \left.\frac{d\,x_r(t)}{d\,t}\right|_{max} = \omega_0 \cdot X_{r\,max}$$

or

$$\dot{X}_{max} = \omega_0 \cdot \left[\frac{X_{r\,max}}{X_s}\right] \cdot X_s$$

$$\dot{X}_{max} = 2\pi \cdot 46 \left[0.67\right] (0.05) = 9.7 \text{ in./sec}$$

Maximum Acceleration of Mass

$$\ddot{X}_{max} = \left.\frac{d^2\,x_r(t)}{d\,t^2}\right|_{max} = \omega_0^2\, X_{r\,max}$$

or

$$\frac{\ddot{X}_{max}}{g} = \omega_0^2 \cdot \left[\frac{X_{r\,max}}{X_s}\right] \frac{X_s}{g}$$

$$= (2\pi \cdot 46)^2\, (0.67)\, (0.05)/386 = 7.25 \text{ g's}$$

3.3 N-Wave and Decaying Sine Excitation

Two special forms of transient excitation are treated separately in this section because of their particular application for analyzing shock response to sonic booms, and response of equipment mounted on structure which is itself subjected to a transient pulse load.

The N-wave characterizes the ideal time hisotry of the free-field overpressure due to a sonic boom. A decaying sine is a useful approximation for the time history of the response of a building subjected to an impulse load such as that induced by an explosive blast. The normalized Fourier Spectra, as defined by Equation 20, are shown in Figures 5a and 6 for these two types of shock excitation.

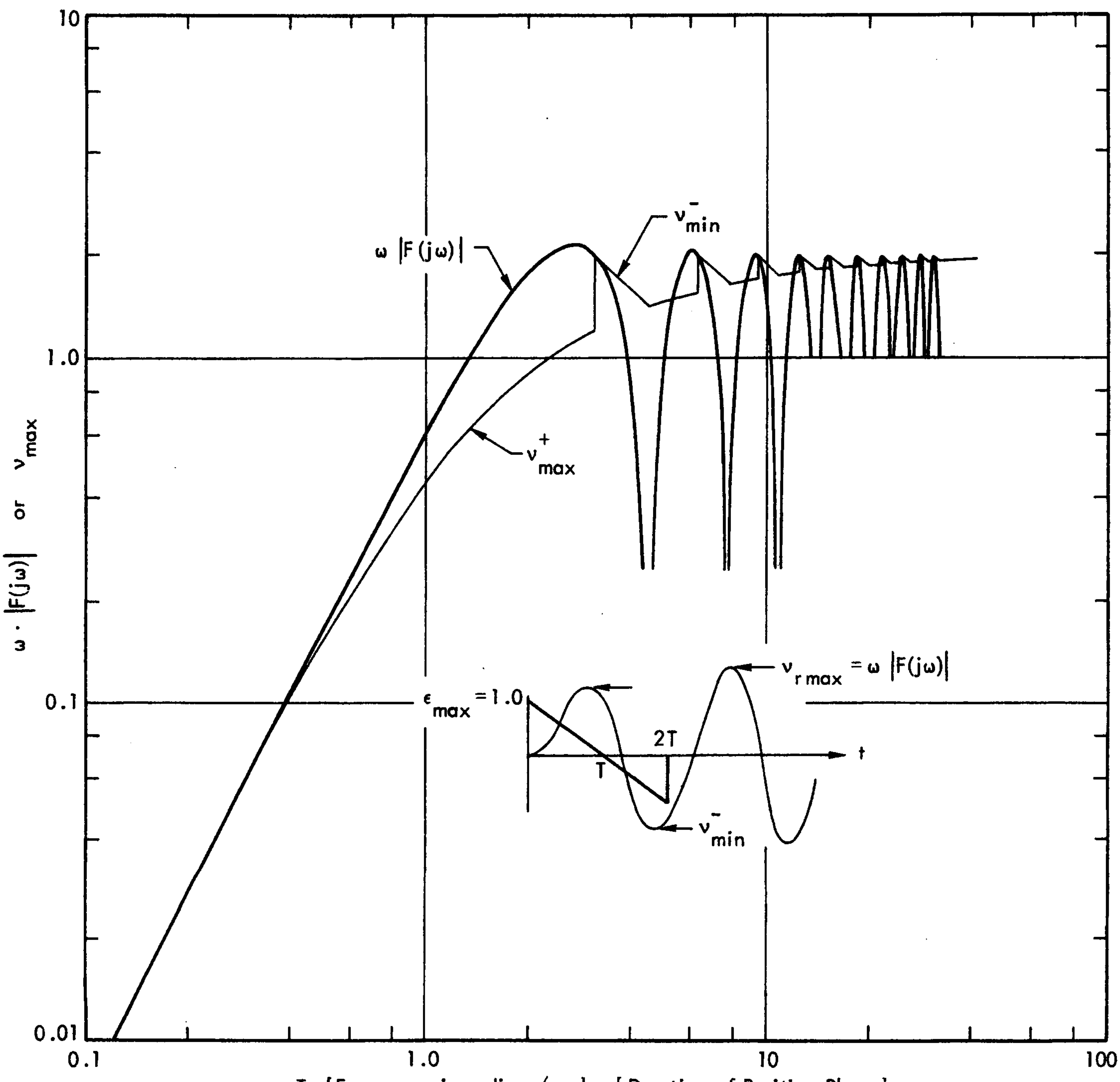


Figure 5a. Normalized Fourier Spectrum (Residual Shock Spectrum) and Primary Shock Spectrum for N-Wave of Unit Peak Amplitude

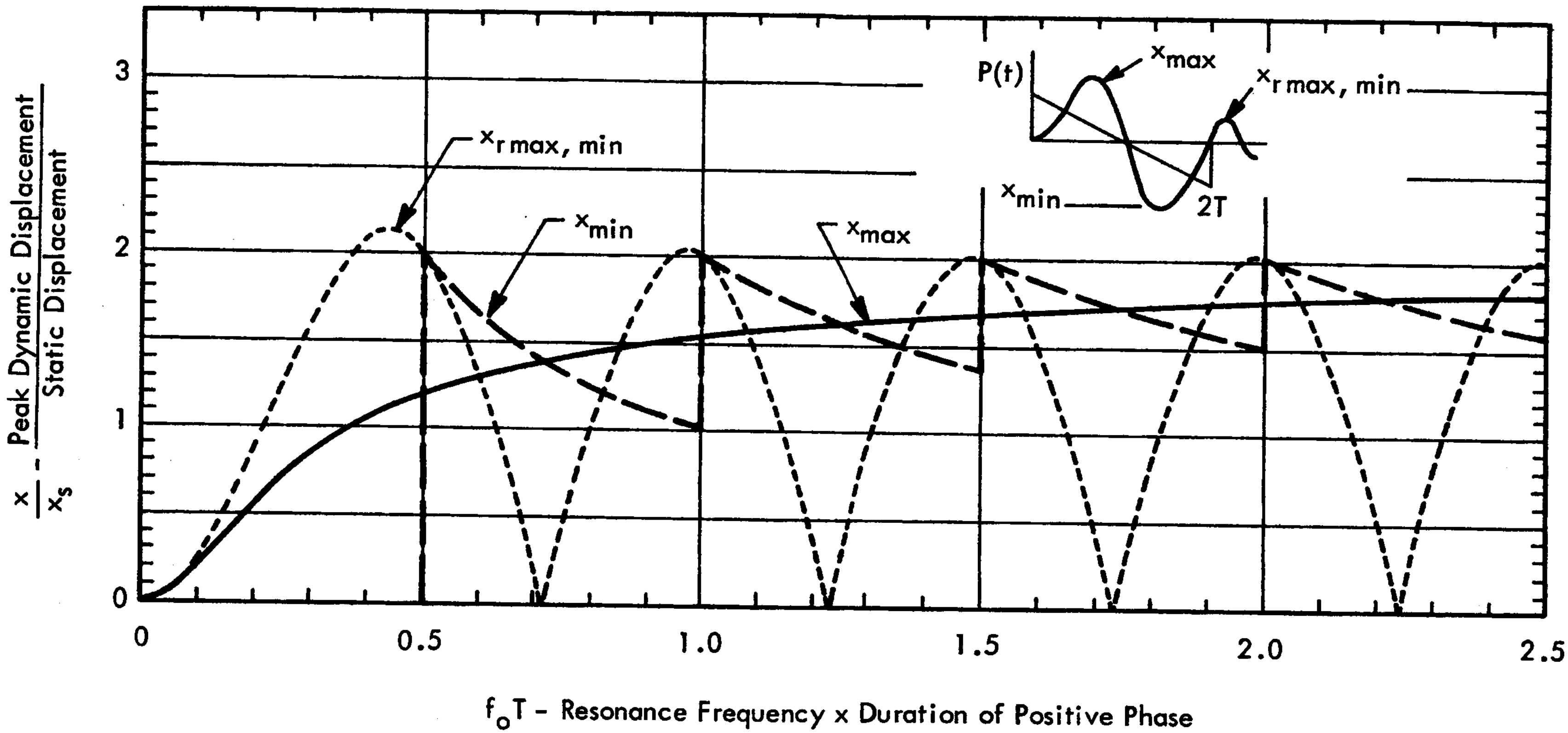


Figure 5b. Normalized Displacement Shock Spectrum for Ideal Sonic Boom N-Wave Excitation of Undamped Mass-Spring System

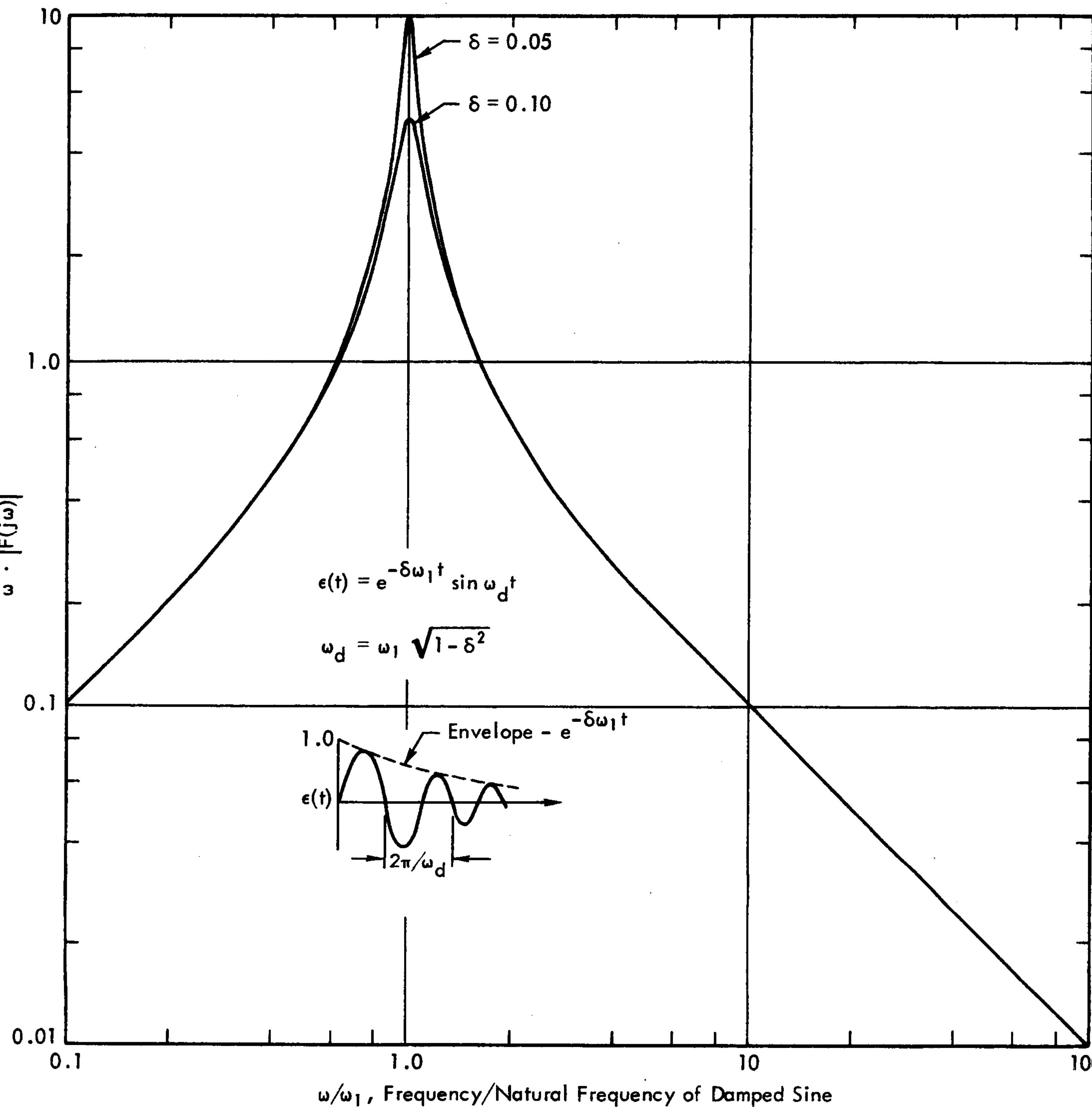


Figure 6. Normalized Fourier Spectrum for Decaying Sine

Response for N-Wave

Note that the N-wave has a zero net impulse and the normalized Fourier Spectra is proportional to $(\omega T)^2$ for small values of ωT. This low-frequency approximation for the normalized Fourier Spectra is characteristic of all double shocks with a net impulse of zero. In contrast, the spectra shown earlier in Figure 2 for single pulse-type shocks, vary as ωT for low values of ωT. See Appendix A for a more complete analysis of this.

For comparison with the residual shock spectrum for the N-wave, the primary shock spectra for the N-wave is also shown in Figure 5a. This defines the envelope of the maximum or minimum response peaks during the forced response period (i.e., $0 < t < 2T$). Both a positive (ν^+_{max}) and negative (ν^-_{min}) primary shock spectra must be defined for this type of pulse. For simplicity, only the upper envelope of these two combined primary shock spectra are shown in Figure 5a. The analytical expressions which define all of these shock spectra for the N-wave, as well as the actual time history of response for an undamped system are summarized in Appendix A.

Example

For application of these generalized response spectra to a particular case, the relationships between the generalized and specific excitation response variables in Table II, page 9, may be used. For example, if the peak overpressure of a sonic boom is P_{max} - lb/in.^{2} and the stiffness of the responding system is k(lb/in.^{2})/in., the corresponding peak excitation is

$$\epsilon_{max} = P_{max}/k = X_s, \quad \text{in.}$$

where X_s = static displacement to the peak load, P_{max}, and the corresponding value of the peak response, ν_{max}, is the peak displacement of the mass X_{max}. The residual shock spectra, $\nu_{r\,max}/\epsilon_{max} = X_{r\,max}/X_s$ is given by the normalized Fourier Spectra $\omega\,|F(j\omega)|$ in Figure 5a for ϵ_{max} = unity. The forced response peaks, X^+_{max}/X_s or X^-_{min}/X_s are found from the plot of the corresponding primary shock spectra $\nu^+_{max}/\epsilon_{max}$ or $\nu^-_{min}/\epsilon_{max}$ in Figure 5a. These shock spectra are replotted in Figure 5b in a more convenient form for the particular variables just defined. In this case, frequency ω is specified on the abscissa as $f_0 = \omega_0/2\pi$; where f_0 = natural frequency of the undamped system in Hz.

Response for Decaying Sine

The decaying sine shock has no finite duration so that a discrete residual shock spectrum does not exist. However, the maximum response of an undamped system to this type of excitation tends to occur near the end of the transient for systems with a natural frequency ω_0 equal to or less than about 1.5 times the frequency ω_d of the damped oscillation. For this condition, the normalized Fourier Spectrum, plotted in Figure 6, is a good approximation for the overall shock response spectra of an undamped system. For the case where $\omega_0 > 1.5\,\omega_d$, the peak response occurs near the beginning of the damped oscillation, and the normalized Fourier Spectrum is no longer applicable for estimating the shock spectra. Approximate values for this range of ω_0/ω_1 are given in Appendix A.

It is clear, from Figure 6, that this normalized Fourier Spectra has a maximum value at a frequency ω equal to the frequency ω_1 which is the limiting value for the frequency ω_d of the damped sine pulse when the decay constant approaches zero. That is,

$$\omega\ |F(j\omega)| \rightarrow \text{maximum} \quad \text{for} \quad \omega = \omega_1 = \omega_d \Big/ \sqrt{1-\delta^2}$$

where

$$F(j\omega) = \int_{-\infty}^{+\infty} \left[e^{-\delta\omega_1 t} \sin \omega_d t\right] e^{-j\omega t}\, dt$$

This normalized Fourier Spectrum is identical to one form of the sinusoidal frequency response function for a damped single-degree-of-freedom system. In this case, however, it approximates the peak response of an undamped system to a damped sine wave for $\omega_0/\omega_d \leq 1.5$. The analytical expressions which define this normalized Fourier Spectra as well as the response time history for any value of ω_0/ω_d are are summarized in Appendix A.

Example

Assume a ground shock excites a single mode of a structure so that the building response can be described approximately by

$$\dot{u}(t) = \dot{U}_{max}\, e^{-\delta\omega_1 t} \sin\left(\omega_1 \sqrt{1-\delta^2}\, t\right)$$

where

$\dot{U}_{max}$ = peak velocity of the envelope of the decaying sine extrapolated back to $t = 0$

δ = decay constant for decaying sine response

ω_1 = undamped natural frequency of building mode

ω_d = damped natural frequency of building.

According to Table II, page 9, if the excitation $\epsilon(t)$ to an equipment item in the building is the base velocity $\dot{u}(t)$, the peak "residual" response velocity $\dot{X}_{r\,max}$ is given by

$$\dot{X}_{r\,max} = \dot{U}_{max} \cdot \left[\omega \cdot |F(j\omega)|\right]$$

The bracketed term is the normalized Fourier Spectrum plotted in Figure 6, for unit excitation, and is evaluated at the natural frequency ω_0 of the system.

To illustrate a specific case, assume the following parameters.

$\dot{U}_{max}$ = 20 in./sec

$\omega_1/2\pi$ = 3 Hz

δ = 0.1

$\omega_0/2\pi$ = 2.5 Hz, natural frequency of equipment to be analyzed.

The following response parameters can now be determined.

$$\frac{\omega}{\omega_1} = \frac{\omega_0}{\omega_1} = \frac{2.5}{3} = 0.833$$

From Figure 6,

$$\omega\,|F(j\omega)| = 2.4$$

Peak Velocity Response, $\dot{X}_{max} \simeq (2.4)(20) = 48$ in./sec

Peak Acceleration Response, $\dfrac{\ddot{X}_{max}}{g} = \dfrac{\omega_0 \dot{X}_{max}}{g} = \dfrac{(2\pi)(2.5)(48)}{386} = 1.95$ g's

Peak Displacement (Approx.), $X_{max} \simeq \dfrac{\dot{X}_{max}}{\omega_0} = \dfrac{(48)}{(2\pi)(2.5)} = 1.53$ in.

It should be emphasized that these calculated response peaks are close approximations to the true values since $\omega_0/\omega_d = 2.5/3\sqrt{1-(.1)^2} \simeq 0.83 < 1.5$. In this case, the peak vibration response occurs after the damped sine excitation has decayed to a low value so that the response is very nearly one of free sinusoidal vibration.

There is one important limitation of the normalized Fourier Transform for predicting response of a single-degree-of-freedom system to a damped sine excitation. Damping of the responding system is very influential in limiting the peak response for values of ω_0/ω_d from 2/3 to 3/2. In this range, the natural frequency of the responding system is close to the frequency of the damped sine excitation. The resonant response build-up is therefore very sensitive to damping in the responding system. This is not true for the other types of shocks considered earlier. For these cases, the peak response of a damped system to the shock excitation is only slightly less than the undamped response for the usual amount of damping encountered in structure (i.e., $\delta \leq 0.1$).

REFERENCES

1. Ayre, R. S., Shock and Vibration Handbook, Chap. 8, C. M. Harris, ed., McGraw-Hill Book Co., Inc., New York (1961).

2. Thompson, W. T., Vibration Theory and Applications, Prentice-Hall, Inc., New Jersey (1965).

3. Rubin, S., Shock and Vibration Handbook, Chap. 23, C. M. Harris, ed., McGraw-Hill Book Co., Inc., New York (1961).

APPENDIX A

NON-DIMENSIONAL FOURIER AND SHOCK RESPONSE SPECTRA FOR SEVERAL TYPES OF TRANSIENT EXCITATIONS

Fourier Spectra

The Fourier transform at a function $\epsilon(t)$ is

$$F(j\omega) = \int_{-\infty}^{+\infty} \epsilon(t)\ e^{-j\omega t}\ dt \qquad (A1)$$

The absolute value of $F(j\omega)$ can be expressed in non-dimensional form by

$$\frac{\omega \left| F(j\omega) \right|}{\epsilon_{max}} \qquad (A2)$$

where ϵ_{max} is the maximum value of $\epsilon(t)$.

This non-dimensional normalized form is listed in Column 3 of Table III for several types of transient excitation identified in the first two columns.

For step type pulses which have a finite value at $t = 0$, it is necessary to modify Equation A1 by adding a decay term $e^{-\alpha t}$ to the integral and take the limit as $\alpha \longrightarrow 0$ so that Equation A1 becomes

$$F(j\omega) = \lim_{\alpha \to 0} \int_{0}^{+\infty} \epsilon(t)\ e^{-(\alpha + j\omega)t}\ dt \qquad (A3)$$

This insures that the integration can be performed since the function $\epsilon(t)\, e^{-\alpha t}$ approaches zero for $t \longrightarrow \infty$. The normalized Fourier Spectra given by Equation A2 have been plotted for a variety of pulse and step transients in Figures 2, 4, 5 and 6.

Transient Response

The response time history $v(t)$ to these transient excitations can be determined by well known methods such as the Duhamel integral solution given by Equation 10 in the text. Known solutions for these response time histories were obtained from Reference 1 or derived independently where necessary. The resulting solutions are listed in Column 4 of Table III.

TABLE III

SHOCK EXCITATION, FOURIER SPECTRA, RESPONSE TIME HISTORIES AND SHOCK SPECTRA FOR UNDAMPED SINGLE-DEGREE-OF-FREEDOM SYSTEMS

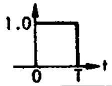
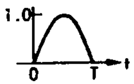
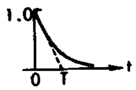
Column 1	Column 2	Column 3	Column 4	Column 5
Shock Excitation[1]		Normalized Fourier Spectrum (Residual Shock Spectrum or $v_{r\,max}$ - After Excitation)[2]	Generalized Response Time History v(t)	Primary Shock Spectrum v_{max} (During Excitation)
Time History	$\epsilon(t)$			
Unit Impulse 	$\lim_{t\to 0}\int_0^t \epsilon(\tau)\,d\tau = 1$	$\omega\cdot\lvert F(j\omega)\rvert = 1\cdot\omega_0$	$v(t) = 1\cdot\omega_0 \sin\omega_0 t$	
Unit Pulse 	1.0, $t \le T$ 0. $t > T$	$\lvert 2\sin(\omega T/2)\rvert$	$v(t) = 1 - \cos\omega_0 t$, $t \le T$ $v_r(t) = 2\sin\left(\frac{\omega_0 T}{2}\right)\sin\omega_0\left(t - \frac{T}{2}\right)$, $t \ge T$	2, $\omega_0 T \ge \pi$
Half-Sine Pulse 	$\sin\pi t/T$, $t \le T$ 0, $t > T$	$\left\lvert \frac{2\omega T/\pi}{1-(\omega T/\pi)^2}\cos(\omega T/2)\right\rvert$	$v(t) = \frac{\sin(\pi t/T) - (\pi/\omega_0 T)\sin(\omega_0 t)}{(\pi/\omega_0 T)^2 - 1}$, $t \le T$ $v_r(t) = \frac{(2\omega_0 T/\pi)\cos(\omega_0 T/2)\sin\omega_0(t - T/2)}{1-(\omega_0 T/\pi)^2}$, $t \ge T$	$\frac{\sin\left[2\pi n\Big/\left(1+\frac{\omega_0 T}{\pi}\right)\right]}{1-(\pi/\omega_0 T)}$, $\omega_0 T > \pi$, $n = 1,2,3$ $\pi/2$, $\omega_0 T = \pi$
1-Cosine Pulse 	$\frac{1}{2}[1-\cos(\pi t/T)]$, $t \le 2T$ 0, $t > 2T$	$\left\lvert \frac{\sin\omega T}{1-(\omega T/\pi)^2}\right\rvert$	$v(t) = \frac{1}{2}\left[1 + \frac{(\pi/\omega_0 T)^2\cos(\omega_0 t) - \cos(\pi t/T)}{1-(\pi/\omega_0 T)^2}\right]$, $t < 2T$ $v_r(t) = \frac{\sin(\omega_0 T)\sin\omega_0(t-T)}{1-(\omega_0 T/\pi)^2}$, $t > 2T$	(No Closed Form Solution) Maximum occurs when $\sin(\pi t/T) = (\pi/\omega_0 T)\sin\omega_0 t$ v_{max}: 1.73 | 1.35 | 1.0 $\omega_0 T/2\pi$: 0.45 | 1.0 | $\ge$ 1.5
Exponential Pulse 	$e^{-t/T}$, $t > 0$	$\frac{\omega T}{\sqrt{1+(\omega T)^2}}$	$v(t) = \frac{(1/\omega_0 T)\sin\omega_0 t - \cos\omega_0 t + e^{-t/T}}{1+(1/\omega_0 T)^2}$	$\to 2$, $\omega_0 T \gg \pi$

TABLE III (Continued)

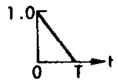
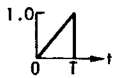
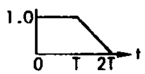
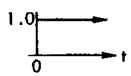
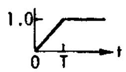
Column 1	Column 2	Column 3	Column 4	Column 5
Shock Excitation[1]		Normalized Fourier Spectrum (Residual Shock Spectrum or $v_{r\,max}$ - After Excitation)[2]	Generalized Response Time History $v(t)$	Primary Shock Spectrum v_{max} (During Excitation)
Time History	$\epsilon(t)$			
Triangular Pulse 	$1 - t/T, \quad t \le T$ $0, \quad t > T$	$\left[1 - \frac{2\sin\omega_0 T}{\omega_0 T} + \left(\frac{\sin\omega_0 T/2}{\omega_0 T/2}\right)^2\right]^{\frac{1}{2}}$	$v(t) = 1 - \frac{t}{T} - \cos\omega_0 t + \frac{\sin(\omega_0 t)}{\omega_0 T}, \quad t \le T$ $v_r(t) = \frac{2\sin(\omega_0 T/2)\cos\omega_0(t - T/2)}{\omega_0 T} - \cos\omega_0 t, \quad t \ge T$	$v^+_{max} = +2\left[1 - \frac{\tan^{-1}\omega_0 T}{\omega_0 T}\right]$ $v^-_{min} = -2\pi n/\omega_0 T, \quad n = 1,2,3$
Triangular Pulse 	$t/T, \quad t \le T$ $0, \quad t > T$	(Same as above)	$v(t) = t/T - \frac{\sin\omega_0 t}{\omega_0 T}, \quad t \le T$ $v_r(t) = \cos\omega_0(t - T) - \frac{2\sin(\omega_0 T/2)\cos\omega_0(t - T/2)}{\omega_0 T}, \quad t \ge T$	No peak occurs for $t < T$ Maximum amplitude is $1 - (\sin\omega_0 T)/\omega_0 T$ at $t = T$
Step-Ramp Pulse 	$1, \quad 0 < t < T$ $2 - t/T, \quad T < t < 2T$ $0, \quad t > 2T$	$\left[\left(\frac{\sin(\omega T/2)}{\omega T/2}\right)^2 - \frac{4}{\omega T}\sin\left(\frac{\omega T}{2}\right)\cos\left(\frac{3\omega T}{2}\right) + 1\right]^{\frac{1}{2}}$	$v(t) = 1 - \cos\omega_0 t, \quad 0 < t < T$ $v(t) = 2 - \frac{t}{T} - \cos\omega_0 t + \frac{\sin\omega_0(t - T)}{\omega_0 T}, \quad T < t < 2T$ $v_r(t) = \frac{2\sin(\omega_0 T/2)\cos\omega_0(t - 3T/2)}{\omega_0 T} - \cos\omega_0 t, \quad t > 2T$	$v_{max} = 2, \quad \omega_0 T \ge \pi$
Unit Step 	$1, \quad t > 0$	1	$1 - \cos\omega_0 t$	2
Ramp Step 	$t/T, \quad 0 < t < T$ $1, \quad t > T$	$\left\lvert\frac{\sin\omega T/2}{\omega T/2}\right\rvert$	$v(t) = t/T - \frac{\sin\omega_0 t}{\omega_0 T}, \quad t \le T$ $v_r(t) = 1 + \frac{\sin(\omega_0 T/2)}{\omega_0 T/2}\cos\omega_0(t - T/2), \quad t \ge T$	$1 + \left\lvert\frac{\sin\omega_0 T/2}{\omega_0 T/2}\right\rvert$

TABLE III (Continued)

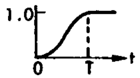
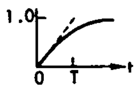
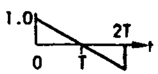
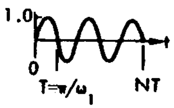
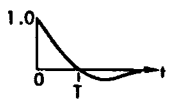
Column 1	Column 2	Column 3	Column 4	Column 5
Shock Excitation(1)		Normalized Fourier Spectrum (Residual Shock Spectrum or $\nu_{r\,max}$ - After Excitation)(2)	Generalized Response Time History $\nu(t)$	Primary Shock Spectrum ν_{max} (During Excitation)
Time History	$\varepsilon(t)$			
1-Cosine Step 	$\frac{1}{2}\,[1-\cos\pi t/T]\,,\quad t \le T$ $1,\quad t \ge T$	$\left\lvert \frac{\cos\omega T/2}{(\omega T/\pi)^2 - 1} \right\rvert$	$\nu(t) = \frac{1}{2}\left[1 + \frac{(\pi/\omega_0 T)^2 \cos(\omega_0 t) - \cos(\pi t/T)}{1-(\pi/\omega_0 T)^2}\right],\quad t \le T$ $\nu_r(t) = 1 + \frac{\cos(\omega_0 T/2)\cos\omega_0(t-T/2)}{(\omega_0 T/\pi)^2 - 1},\quad t \ge T$	$1 + \left\lvert \frac{\cos\omega_0 T/2}{(\omega_0 T/\pi)^2 - 1} \right\rvert$
Exponential Step 	$1 - e^{-t/T},\quad t>0$	$\frac{1}{\sqrt{1+(\omega T)^2}}$	$\nu(t) = 1 - \frac{(\omega_0 T)^2\, e^{-t/T} + (\omega_0 T)\sin\omega_0 t + \cos\omega_0 t}{(\omega_0 T)^2 + 1}$	$\simeq 1 + \frac{1}{\sqrt{1+(\omega_0 T)^2}}\,,\quad \omega_0 T > 1$
N-Wave 	$1 - t/T,\quad t \le 2T$ $0,\quad t \ge 2T$	$2\left\lvert \frac{\sin\omega T}{\omega T} - \cos\omega T \right\rvert$	$\nu(t) = 1 - \frac{t}{T} + \sqrt{1+(1/\omega_0 T)^2}\,\sin(\omega_0 t - \tan^{-1}\omega_0 T),\quad t \le 2T$ $\nu_r(t) = 2\left[\frac{\sin\omega_0 T}{\omega_0 T} - \cos\omega_0 T\right]\cos\omega_0(t-T),\quad t \ge 2T$	$\nu^+_{max} = +2\left[1 - \frac{\tan^{-1}\omega_0 T}{\omega_0 T}\right]$ $\nu^-_{min} = -2\pi n/\omega_0 T\,,\ n = 1,2,3,\ldots$
Damped Sine 	$e^{-\delta\omega_1 t}\sin(\omega_d t)$ where $\omega_d = \omega_1\sqrt{1-\delta^2}$	$(\omega_d/\omega)\cdot\lvert H(\omega)\rvert$ where $\lvert H(\omega)\rvert = [(1-(\omega_1/\omega)^2)^2 + (2\delta\omega_1/\omega)^2]^{-\frac{1}{2}}$	$\nu(t) = \lvert H(\omega_0)\rvert\left[e^{-\delta\omega_1 t}\sin(\omega_d t + \theta_1) - \frac{\omega_d}{\omega_0}\sin(\omega_0 t + \theta_2)\right]$ where $\theta_1 = \tan^{-1}\left[2\delta\left(\frac{\omega_1}{\omega_0}\right)^2\sqrt{1-\delta^2}\right]\Big/\left[1 - \left(\frac{\omega_1}{\omega_0}\right)^2(1-2\delta^2)\right]$ $\theta_2 = \tan^{-1}\,[2\,\delta\,\omega_1/\omega_0\,]/[1-(\omega_1/\omega_0)^2\,]$	$\nu_{max} \simeq \begin{cases} \to 1.0, & \omega_1 \ll \omega_0 \\ \sim 1.7, & \omega_1 = \frac{2}{3}\omega_0 \\ \frac{1}{2\delta}, & \omega_1 = \omega_0 \\ \omega_0/\omega_1, & \omega_1 \gg \omega_0 \end{cases}$

TABLE III (Continued)

Column 1	Column 2	Column 3	Column 4	Column 5
Shock Excitation[1]		Normalized Fourier Spectrum (Residual Shock Spectrum or $v_{r\,max}$ - After Excitation)[2]	Generalized Response Time History v(t)	Primary Shock Spectrum v_{max} (During Excitation)
Time History	$\epsilon(t)$			
Finite Sine Train 	$\sin\omega_1 t, \quad t \le NT$ $0, \quad t > NT$	$\dfrac{2\,\omega_1/\omega}{(\omega_1/\omega)^2 - 1} \begin{Bmatrix} \cos(N\pi\,\omega/2\omega_1) \\ \text{or} \\ \sin(N\pi\,\omega/2\omega_1) \end{Bmatrix} \begin{matrix} N,\ \text{odd} \\ \\ N,\ \text{even} \end{matrix}$	$v(t) = \dfrac{\sin\omega_1 t - (\omega_1/\omega_0)\sin\omega_0 t}{1 - (\omega_1/\omega_0)^2}, \quad t < NT$ $v_r(t) = \dfrac{2\omega_1/\omega_0}{(\omega_1/\omega_0)^2 - 1}\left[2 - 2\cos N\pi\cos\dfrac{N\pi\omega_0}{\omega_1}\right]^{\frac{1}{2}} \cos[\omega_0(t-NT)+\theta_3]$ $\theta_3 = \tan^{-1}\dfrac{\cos N\pi - \cos N\pi\,\omega_0/\omega_1}{\sin N\pi\,\omega_0/\omega_1}$	$v_{max} = \dfrac{\sin[2n\pi/(1+\omega_1/\omega_0)]}{(\omega_1/\omega_0) - 1}$ $n \le \dfrac{N}{2}(1 + \omega_0/\omega_1)$ For $\omega_1/\omega = 1$ $v_{max} = -\dfrac{n\pi}{2}\cos n\pi, \quad n \le N$
Blast Pulse 	$[1 - t/T]\,e^{-t/T}$	$\dfrac{(\omega T)^2}{1 + (\omega T)^2}$	$v(t) = \dfrac{(\omega_0 T)^2}{1 + (\omega_0 T)^2}\left[e^{-t/T}\left(\dfrac{(\omega_0 T)^2 - 1}{(\omega_0 T)^2 + 1} - \dfrac{t}{T}\right) + \cos(\omega_0 t - \theta_4)\right]$ $\theta_4 = 2\tan^{-1}\omega_0 T$	$v_{max} \rightarrow \begin{cases} (\omega_0 T)^2, & \omega_0 T < 1 \\ 2, & \omega_0 T >> 1 \end{cases}$

(1) In all cases, the excitation and response is zero for $t < 0$.

(2) For Residual Shock Spectrum ω is set equal to ω_0.

For most of the types of transient excitation, two expressions are required to define the maximum response - one for the response during the excitation and one for the residual response period.

Transient Response to an Ideal Impulse

An impulse excitation is defined by the limiting value of the time integral of the excitation $\epsilon(t)$ as

$$\lim_{t \to 0} \int_0^t \epsilon(\tau)\, d\tau = I \qquad \text{(A4)}$$

where τ = dummy time variable of integration, and $\epsilon(\tau)$ = generalized excitation defined in Table I, page 3.

For a unit impulse, $I = 1$, which is the first case treated in Table III. The generalized response $v(t)$ to impulse excitation is given by

$$v(t) = I\, g(t) \qquad \text{(A5)}$$

where $g(t) = \omega_0 \sin \omega_0 t$, the generalized response to a unit impulse when $\epsilon(t)$ has the form defined in Table I, page 3.

Thus, the amplitude of the response to an impulse is equal to the shock spectra and is given by

$$v_{r\,max} = I\, \omega_0 \qquad \text{(A6)}$$

As discussed in Section 2.1, page 4, the magnitude of the impulse I times ω_0^2 is equal to the initial rate of change $\dot{v}\,(t \to 0)$ of the generalized response variable $v(t)$. Examples of the specific value of this "initial condition" and the resulting expressions for the response time history are given in Table IV for several common types of impulse excitation.

Transient Response to Finite Single and Double Pulse

When the natural period $(2\pi/\omega_0)$ of a single-degree-of-freedom system is much greater than the duration T of a single shock pulse, the transient response to this pulse is classified as an impulse response. In this case, $\omega_0 T < 1$ and the expressions derived in the preceding paragraph for response to an ideal impulse define the actual response to a single, finite duration pulse to a close approximation.

An example of this concept is illustrated in the next to last row in Table IV for the general case of response to a single rectangular pulse with a maximum amplitude ϵ_{max} and duration T. The impulse for this pulse is

$$I = \epsilon_{max} T$$

so that, according to Equation A5, the generalized response $\nu(t)$ is

$$\nu(t) = I \omega_0 \sin \omega_0 t = \epsilon_{max} \omega_0 T \sin \omega_0 t$$

and the normalized shock spectrum is

$$\frac{\nu_{max}}{\epsilon_{max}} = \omega_0 T \qquad \text{(A7)}$$

Thus, as discussed in Section 3.3, the normalized Fourier Spectrum for all single positive pulses, regardless of their shape, varies directly as $\omega_0 T$ for $\omega_0 T < 1$. For any shape other than a rectangular one, the area under the time history of the excitation $\epsilon(t)$ defines the impulse I. If the maximum value of $\epsilon(t)$ during the pulse is ϵ_{max}, the normalized shock spectrum is given by

$$\frac{\nu_{max}}{\epsilon_{max}} = \left[\frac{I/T}{\epsilon_{max}}\right] \omega_0 T \qquad \text{(A8)}$$

where I/T = the <u>average value</u> of $\epsilon(t)$ over the pulse duration T.

For a symmetrical double pulse of very short duration relative to the natural period of a responding system, a slightly different result is obtained. Consider the case, illustrated in the last row of Table IV, for a double rectangular pulse of duration 2T with equal positive and negative peak amplitudes $+\epsilon_{max}$ and $-\epsilon_{max}$. Using the Duhamel integral method, the generalized response to this pulse can be expressed as (see Equations 1 and 9)

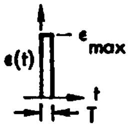
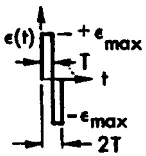
$$\nu(t) = \omega_0 \int_0^t \sin \omega_0 (t - \tau) \; \epsilon(\tau) \, d\tau$$

$$= \omega_0 \sin \omega_0 t \int_0^t \cos \omega_0 \tau \; \epsilon(\tau) \, d\tau$$

$$- \omega_0 \cos \omega_0 t \int_0^t \sin \omega_0 \tau \; \epsilon(\tau) \, d\tau$$

TABLE IV

SPECIFIC FORMS FOR IMPULSE EXCITATION AND RESPONSE OF UNDAMPED SYSTEM ILLUSTRATED IN FIGURE 1

Type of Impulse Excitation		Magnitude of Impulse	Initial Rate of Change of Response	Impulse Response				
				Mass Displ.	Mass Velocity	Mass Accel.	Spring Displ.	Rel. Spring Velocity
--	$\epsilon(t)$	$I = \int_0^{t \to 0} \epsilon(\tau)\, d\tau$	$\dot{v}(t \to 0) = \omega_0^2 I$	$x(t)$	$\dot{x}(t)$	$\ddot{x}(t)$	$\delta(t)$	$\dot{\delta}(t)$
Force on Mass (Fixed Base)	$P(t)/k$	$\frac{I_p}{k} = \frac{m\dot{X}_0}{k} = \frac{\dot{X}_0}{\omega_0^2}$, (Momentum $\underline{\text{Step}}$)/k	$\dot{X}_0$, Initial Mass Velocity	$\boxed{\frac{\dot{X}_0}{\omega_0} \sin \omega_0 t}$	$\underline{\dot{X}_0 \cos \omega_0 t}$	$-\dot{X}_0 \omega_0 \sin \omega_0 t$	$\frac{\dot{X}_0}{\omega_0} \sin \omega_0 t$	$\underline{\dot{X}_0 \cos \omega_0 t}$
Base Velocity	$\dot{u}(t)$	$\underline{U_0}$, Base Displacement $\underline{\text{Step}}$	$\omega_0^2 U_0$, Initial Accel. of Mass	$U_0 [1 - \cos \omega_0 t]$	$\boxed{U_0 \omega_0 \sin \omega_0 t}$	$\underline{U_0 \omega_0^2 \cos \omega_0 t}$	$\underline{U_0 \cos \omega_0 t}$	$\dot{u}(t) - U_0 \omega_0 \sin \omega_0 t$
Base Accel.	$\ddot{u}(t)$	$\underline{\dot{U}_0}$, Base Velocity $\underline{\text{Step}}$	$\omega_0^2 \dot{U}_0$, Initial Jerk of Mass	$\dot{U}_0 \left[t - \frac{\sin \omega_0 t}{\omega_0} \right]$	$\dot{U}_0 [1 - \cos \omega_0 t]$	$\boxed{\dot{U}_0 \omega_0 \sin \omega_0 t}$	$\frac{\dot{U}_0 \sin \omega_0 t}{\omega_0}$	$\underline{\dot{U}_0 \cos \omega_0 t}$
Rectangular Pulse with Finite Duration $T \to 0$		$I = \epsilon_{max} T$	$\dot{v}(t \to 0) = \epsilon_{max} \omega_0^2 T$; $\omega_0 T < 1$	$\boxed{v(t) \simeq \epsilon_{max} \omega_0 T \cdot \sin \omega_0 t}$; $\underline{\dot{v}(t) \simeq \epsilon_{max} \omega_0^2 T \cos \omega_0 t}$; $\ddot{v}(t) \simeq \epsilon_{max} \omega_0^3 T \sin \omega_0 t$ } $\omega_0 T < 1$				
Rectangular Double Pulse with Finite Duration $2T \to 0$		$I^+ = \epsilon_{max} T$; $I^- = -\epsilon_{max} T$; $I = I^+ + I^- = 0$	$\dot{v}(t \to 0) = 0$; Initial Response $v(t \to 0) = \epsilon_{max} (\omega_0 T)^2$; $\omega_0 T < 1$	$\boxed{v(t) \simeq \epsilon_{max} (\omega_0 T)^2 \cos \omega_0 t}$; $\dot{v}(t) \simeq -\epsilon_{max} \omega_0^3 T^2 \sin \omega_0 t$; $\underline{\ddot{v}(t) \simeq -\epsilon_{max} \omega_0^4 T^2 \cos \omega_0 t}$ } $\omega_0 T < 1$				

[dashed box] Identifies response variable related to input excitation by Equation A5.

[solid underline] Identifies excitation or response parameter which will undergo an instantaneous step change at t = 0 due to the impulse excitation.

In this case, $\omega_0 T << 1$ so that, for a first approximation,

$$\cos \omega_0 \tau \rightarrow 1$$

$$\sin \omega_0 \tau \rightarrow \omega_0 \tau$$

Since the excitation is zero for $t > 2T$ for this symmetrical pulse, the response for all values of $t > 2T$ is given by

$$v(t) \simeq \omega_0 \sin \omega_0 t \int_0^{2T} \epsilon(\tau)\, d\tau - \omega_0^2 \cos \omega_0 t \int_0^{2T} \tau \cdot \epsilon(\tau)\, d\tau \qquad \text{(A9)}$$

The first integral is zero for the symmetrical double pulse since its net impulse is zero. For the double rectangular pulse, the second integral is

$$\int_0^{2T} \tau\ \epsilon(\tau)\, d\tau = \int_0^{T} \tau \cdot \epsilon_{max} \cdot d\tau + \int_T^{2T} \tau\, (-\epsilon_{max})\, d\tau$$

$$= \epsilon_{max} \left[\frac{T^2}{2} - \left(\frac{4T^2}{2} - \frac{T^2}{2} \right) \right]$$

$$= -\epsilon_{max}\ T^2$$

Thus, to a close approximation for $\omega_0 T << 1$, the response to the double rectangular pulse is

$$v(t) \simeq \epsilon_{max}\ (\omega_0 T)^2 \cos \omega_0 t \qquad \text{(A10)}$$

and the corresponding normalized shock spectrum is

$$\frac{v_{max}}{\epsilon_{max}} = (\omega_0 T)^2 \qquad \text{(A11)}$$

For any double pulse shape with zero net impulse, and positive phase duration T, when $\omega_0 T << 1$, the normalized shock spectra is closely approximated by

$$\frac{\nu_{max}}{\epsilon_{max}} \simeq - (\omega_0 T)^2 \int_0^{T_0/T} \left(\frac{\tau}{T}\right) \left[\frac{\epsilon(\tau)}{\epsilon_{max}}\right] d\left(\frac{\tau}{T}\right) \qquad \text{(A12)}$$

where T_0 = total duration of double pulse.

The integral will normally be negative giving an overall positive value for the shock spectrum.

Shock Response Spectra

As discussed in the main body of the text, the Residual Response Spectra, or envelope, as a function of ω_0, of the maximum response amplitude after the transient excitation ends, is defined by the normalized Fourier Spectra. This has been given in Column 3 of Table III.

The envelope of maximum response amplitude during the transient excitation is called the Primary Response Spectra. Available solutions for the Primary Response Spectra have been obtained from Reference 1 or derived in the usual way by differentiating the response time history and solving for the time and amplitude when the rate of change of response is zero. These solutions are listed in the last column of Table III for the pulse-type excitations. For the ramp-type excitations, the Shock Response Spectra listed correspond to the total response spectra after the end of the ramp and are numerically equal to one plus the normalized Fourier Spectra listed in Column 3.

Application of Superposition Principle

The derivation of the response time history to a transient excitation is generally carried out most efficiently by applying the superposition principle. With this method, known solutions for the response to simple transient excitations can be utilized by defining a discrete shock as the summation of two or more elements for which the response time histories are known. This method was used for several of the entries in Table III.

As an example of this method, the residual response to a unit pulse of duration T can be defined as the sum of the response to a unit step starting at the t = 0 minus the response to a unit step starting at time t = T. The response of an undamped system to a unit step is

$$\nu(t) = 1 - \cos \omega_0 t$$

Thus, the residual response of an undamped system to a unit rectangular pulse is written down directly as

$$
\begin{aligned}
v(t) &= v(t)_{\text{unit step}} - v(t-T)_{\text{unit step}} \\
&= 1 - \cos \omega_0 t - [1 - \cos \omega_0 (t - T)] \\
&= \cos \omega_0 (t - T) - \cos \omega_0 t \\
&= 2 \sin \omega_0 T/2 \cdot \sin \omega_0 (t - T/2)
\end{aligned}
$$

This simple process of summation of known response solutions is generally much easier than the alternate method of defining the amplitude and rate of change of the response at the end of one forcing period to define initial conditions for the next period.