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REFLECTION AND TRANSMISSION OF DIPOLE RADIATION IN PRESENCE OF AN EXTENDED BOUNDARY, SMOOTH AND ROUGH

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Technical Report
EE-159(69)NASA-027

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ABSTRACT

Several different forms of integral solutions exist to the problem of dipole radiation in presence of a half space with plane boundary, all of which can be easily shown to agree with one another by transformation. For the evaluation of these integrals, a number of assumptions are usually made on some parameters in the expressions such as the medium property, the wavelength and so on. Unfortunately, however, even with these simplifying (but meaningful) assumptions, practically useful results for the dipole radiation problem are still lacking. Also it appears that there is no theory of dipole radiation when the plane boundary is roughened.

This paper sets out to find steady state solutions in usable form for an arbitrarily oriented dipole source. The smooth boundary case is considered first, which is followed by a statistically rough boundary case. Use is made of a method which may be called a "plane wave approach." This method differs from other classical plane wave methods in that here the incident Hertzian wave is uniquely decomposed so that the reflected and the transmitted waves are found for each component Hertzian wave using the Fresnel coefficients. The integrals are evaluated in the geometrical optics approximation using the stationary phase method. The results are either new (the rough boundary case) or of new form (the smooth boundary case).

In the case of the smooth boundary, the applicability of such results is carefully reviewed. In both the vertical and the horizontal dipole cases for reflection as well as transmission, it is found that the source and the observation points cannot simultaneously approach the boundary. An exception to the above is the case of an horizontal dipole when the observation point lies on or near the dipole axis for the reflected field. The results also indicate that in the case of the horizontal dipole the total Hertz potential everywhere need not of necessity have a vertical component in addition to the component in the direction of the dipole, as has been hitherto generally believed. In fact, one finds that a Hertz potential, in order to be a solution to the horizontal dipole problem, must have at least two components in the rectangular coordinate system, so that there are altogether four permissible resolutions of the Hertz potential.

For the formulation of Hertz potentials for the roughened plane boundary, the vector Helmholtz integral is utilized, of which we give a somewhat more general derivation. To accomodate the vector nature of scattering including the effect of polarizations, dyadic reflection and transmission coefficients are used at the boundary. The rough boundary is slightly rough and considered to be a stationary random process with a gaussian height distribution. The stationary phase method is applied with respect to the mean plane of the rough boundary for the evaluation of the integrals. This was motivated by the physical fact that the density of the stationary points

of such a rough boundary is the greatest in the neighborhood of the stationary point of the mean plane. Results are obtained for the expected values of Hertz potentials, electromagnetic fields, and power. Each of these results involves a factor representing the effect of roughness, which in the limit of a smooth plane boundary correctly reduces to unity. A method for experimentally determining the r.m.s. slope of a class of natural surfaces by using overflight data is outlined.

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INTRODUCTION

1.1 Motivation and Objectives of the Present Study

The problem of radiation of an oscillating dipole source in presence of earth has been of great interest academically as well as practically since the early part of this century. The first investigation of the problem was made in 1909 when Sommerfeld [1909] published his classical work on the effect of a finitely conducting plane on the radiation of a dipole located on the plane. Since that time spectacular progress has been made by many researchers toward the solution and better understanding of essentially the same problem: what is the electromagnetic field everywhere due to radiation of a dipole (electric or magnetic, vertical or horizontal) which is located above the earth, or even embedded within the earth? The reason for particular interest in this problem is obvious, if one notes that any source of electromagnetic radiation can be represented in terms of a distribution of electric and magnetic dipoles and, if necessary, higher order poles.

It has been usually assumed with respect to the earth's geometry that the surface of the earth is an infinite plane. This assumption, being ideal, deviates from the more exact earth's surface in two major respects. In the first place, the true earth is of spherical shape rather than an infinite half space so that there is diffraction of waves, and reflection of waves occurs in a different manner than from a plane boundary. Thus if either or both of the source point and the observation point is sufficiently far from the ground, the assumption may become inapplicable. Secondly, the earth's surface in general is rough rather than smooth. If the scale of roughness is comparable to or greater than the wavelength of the radiation, the assumption may also become inapplicably poor.

In the past, many attempts have been made with fair success to solve the dipole-earth problem with the assumption of the spherical earth by such workers as Watson [1918], Epstein [1935] and Bremmer [1949].[1] However, no notable work has been reported on the study of the problem involving a rough ground. This is understandable in view of the extreme difficulties of solving boundary value problems associated with a rough (irregular) surface. Here the boundary conditions in terms of any fixed coordinate system vary from point to point along the surface.

The study of scattering of waves from a bounded region of rough surfaces was initiated in the late 19th century. In recent decades with the advent of radar, an enormous amount of research has been carried out on this subject. In most such investigations, it is assumed that the transmitter is so far removed from the finite area of the rough scattering surface that the incident wavefront could be considered plane. This is indeed the case for a great number of radar targets of interest, such as the airplane, the ship, and the moon. For a given surface roughness, the above two assumptions, namely, the finiteness of the scattering surface and the plane wave

[1]The year inside the bracket does not necessarily indicate the year the author contributed the first time; rather, it identifies the reference at the end of this paper.

incidence, considerably simplify the analysis in most theories and enable one to carry out the necessary integrals.

These two simplifying assumptions, however, are not applicable to a dipole radiation problem in presence of an infinitely extending rough interface. Thus a new method must be developed that is different from those employed in most scattering theories from rough surfaces so far. This problem is not only of an acaemic interest but also of great importance in ground-to-ground and ground-to-air communications, or vice versa. Often it is desirable to discriminate the background clutter (ground or sea) from a desired radar echo from a finite target. In reference to the space exploration programs, an important experiment involves a transmitter on an orbiting spacecraft around a planet body and a signal bounced off from it being received on earth. When the spacecraft is sufficiently near the planet, a theoretical modeling of the problem may require consideration of the scattering surface infinite and the incident wavefront curved.

No significant work, if any, has appeared on the problem of scattered transmission through a rough interface. The radiation of a dipole immersed in the conducting medium seems to have been first studied around 1950 [Moore, 1951]. Subsequently, fairly abundant work on this subject appeared. All of these, however, assume the interface to be a plane. The transmission theory also bears importance in numerous applications such as the air-to-submarine communications and vice versa. It is also necessary for problems of wave propagation through layered media with rough interfaces.

The objective of the present study is to obtain complete solutions of the (arbitrarily oriented) dipole problem in presence of an infinite boundary, both smooth and rough. We will derive integral expressions which represent Hertz potentials inside and outside the interface by uniquely decomposing the spherical Hertzian wave at the boundary and by satisfying the boundary condition in terms of electromagnetic plane waves. These integrals will be evaluated using the method of stationary phase. Thus the results are strictly valid within the limit of the geometrical optics approximation. The rough interface will be described by a random process with a gaussian height distribution, and subsequently we will find expected Hertz potentials as well as expected powers. We will assume the interface to be gently undulating (large correlation distance) and slightly rough (small variance). We will also assume that there is no shadowing of one part by another and that no multiple scattering exists between elements of the interface. These assumptions are necessary for the price of obtaining especially simple closed form results allowing us greater physical insights in the phenomena. Since we will be considering an arbitrarily oriented dipole source, the integrals as well as the results can be readily specialized to either the case of the vertical or the horizontal dipole,[1] making two separate formulations unnecessary.

We will consider only the steady state electromagnetic radiation arising from a steady state excitation of an

[1]The terms "vertical" and "horizontal" are with reference to the mean plane for the case of rough interface.

elementary dipole. Thus we do not consider such a special feature as transient phenomena. In principle, however, we can consider the steady state to be a member of the Fourier expansion of a non-steady state problem.

1.2 Previous Work on Radiation of Dipole Located Above an Infinite Plane Surface

The exact problem attacked by Sommerfeld in 1909 was that of a vertical dipole located at the interface of a finitely conducting plane earth. In view of the tremendous amount of influence this work has had on later work by others in radiation problems, we will outline it in a little more detail [Stratton, 1941].

The axis of the dipole coincides with the z-axis of a rectangular coordinate system, and the plane $z\equiv 0$ represents the earth's surface. The wave numbers are denoted k_2 and k_1 of the media above (the air) and below the interface, respectively, where $k_2=\omega/c$. The coordinate of the observation point is given as (r,ϕ,z) in the cylindrical coordinates and R is the radial distance, $R=\sqrt{r^2+z^2}$. These relations are illustrated in Figure 1-1.

In terms of the z-component of the total Hertz potentials the boundary condition is found as

$$\frac{\partial \Pi_1}{\partial z} = \frac{\partial \Pi_2}{\partial z} \quad , \quad k_1^2 \Pi_1 = k_2^2 \Pi_2 \ ,$$

where Π_1 and Π_2 are Hertz potentials within the earth and in the air, respectively, and each is given as the sum of the direct wave and the diffracted wave. Using the cylindrical

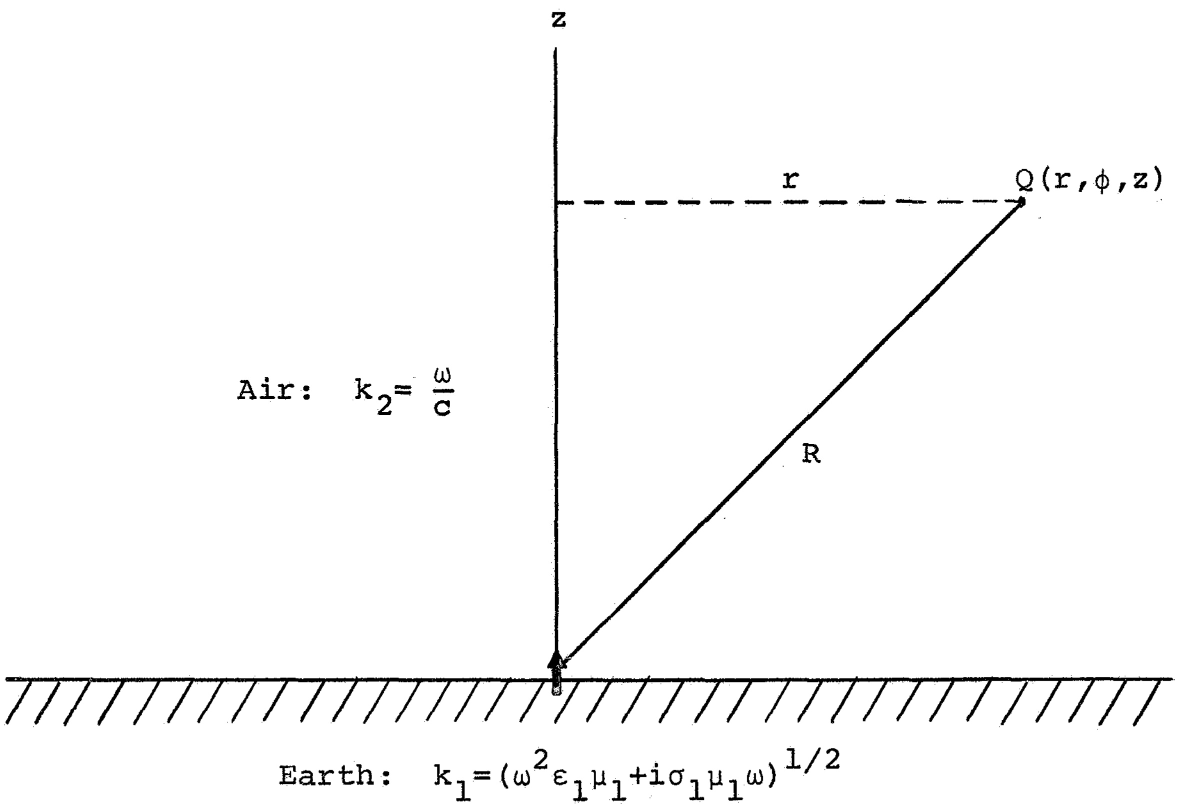

Figure 1-1. Geometry for Sommerfeld's Dipole Problem

wave representation of a spherical wave function

$$\frac{e^{ikR}}{R} = \int_0^\infty \frac{J_o(\lambda r)e^{-\sqrt{\lambda^2-k^2}\,|z|}}{\sqrt{\lambda^2-k^2}} \lambda d\lambda \quad ,$$

he is able to write

$$\Pi_1 = \int_0^\infty [\frac{\lambda}{\sqrt{\lambda^2-k_1^2}} + f_1(\lambda)]J_o(\lambda r)e^{\sqrt{\lambda^2-k_1^2}\,z} d\lambda \quad , \quad z<0$$

$$\Pi_2 = \int_0^\infty [\frac{\lambda}{\sqrt{\lambda^2-k_2^2}} + f_2(\lambda)]J_o(\lambda r)e^{-\sqrt{\lambda^2-k_2^2}\,z} d\lambda \quad , \quad z>0 \quad .$$

The functions $f_1(\lambda)$ and $f_2(\lambda)$ are determined from the boundary condition. Thus Sommerfeld finally obtains a formal solution in terms of infinite integrals:

$$\Pi_1 = 2k_2^2\int_0^\infty \frac{J_o(\lambda r)}{N} e^{\sqrt{\lambda^2-k_1^2}\,z} \lambda d\lambda \quad , \quad z<0$$

$$\Pi_2 = 2k_1^2\int_0^\infty \frac{J_o(\lambda r)}{N} e^{-\sqrt{\lambda^2-k_2^2}\,z} \lambda d\lambda \quad , \quad z>0$$

where

$$N = k_1^2 \sqrt{\lambda^2-k_2^2} + k_2^2 \sqrt{\lambda^2-k_1^2}$$

The vertical dipole problem has been later generalized by Weyl [1919] in the sense that the dipole was put a finite distance above the surface. His method is based on the plane wave expansion of a spherical function and Snell's law. The results naturally reduce to an identity with Sommerfeld's results as the elevation of the dipole is shrunk to zero. There are a few other methods which lead to the same results. Among these we mention in particular Niessen's [1933] and Brekhovskikh's [1960]. Niessen uses a surface integral and the Kirchoff-Huygens Principle, and Brekhovskikh applies essentially the same plane wave method as Weyl's, except, however, that he assumes the Fresnel coefficients to be known.

The solution to the horizontal dipole problem was apparently first obtained by Horschelmann [1912] and subsequently by Sommerfeld [1926], Frank and von Mises [1935], and Niessen [1938]. The steps leading to the integrals are shown to be similar to the vertical dipole case but more involved owing to the necessity of assuming two components for the Hertz potentials. In all treatments, the choice of the components for the Hertz potentials is the z-component in addition to the x-component (for an x-oriented dipole).

The case of a magnetic dipole has also been treated extensively. It has been shown that solving a magnetic dipole problem is a relatively simple matter once the solution to the corresponding electric dipole problem is obtained. This is understandable in view of the change of the roles of E and H, and as a result the polarization of the elementary plane waves into which the spherical wave is expanded.

Each solution to a dipole problem obtained from the aforementioned methods is first given in integral representations and as such is exact. The true difficulty has been the evaluation of the integrals to yield results which are general and simple enough to allow practical use of them. Sommerfeld [1909], Weyl [1919], and others have obtained results using techniques of complex integration. However, the results were not all consistent and there was some controversy over the possibility of resolving the terms into the sky wave and the surface wave. Baños [1966] concludes that Sommerfeld's surface wave indeed exists. At any rate the interest of the current investigation lies in the geometrical optics term of the sky wave, and not in the surface wave.

For the vertical dipole problem the geometrical optics results or the asymptotic results have been derived by Norton [1937], Brekhovskikh [1960] and Baños [1966]. In particular, Baños in his book [1966] reports complete work on the horizontal dipole problem including the results for the transmitted field.

1.3 Previous Work on Scattering from Rough Surfaces

The theory of scattering of electromagnetic waves from rough surfaces has been studied continuously since the late 19th century, especially in a large number of papers on the subject published in the last twenty years. Interest in the problem has been not only in determining the scattered field from a known surface, but also in the converse, namely, determining the surface characteristics of a body, such as the electromagnetic parameters (e.g., μ, ε or σ) or roughness, from a study of the scattered field. In connection with the radar astronomy, emphasis has been shifted to the latter problem.

In spite of the great amount of work done on scattering from (irregular) rough surfaces, it appears that there does not as yet exist a satisfactory theory simple enough for use and at the same time sufficiently rigorous. Most of the theoretical results also lack generality because of a number of assumptions and approximations that had to be introduced to obtain them. The assumptions are made according to the specifics of the problem, and some of the more important and frequently made are [Beckmann and Spizzichino, 1963]:

1. The dimensions of the scattering elements of the rough surface are taken as either much smaller or much greater than the wavelength of the incident radiation;
2. The radius of curvature of the scattering elements is taken to be much greater than the wavelength of the incident radiation;
3. Shadowing effects are neglected;
4. Only the far field is calculated;
5. Multiple scattering is neglected;
6. The density of irregularities (number of scatterers per unit length or area of the surface) is not considered;
7. The treatment is restricted to a particular model of surface roughness, e.g., sinusoidal or saw-tooth undulations, protrusions of definite shape in random positions, random variations in height given by their statistical distribution and correlation, etc.

Approximations are then applied in accordance with the set of particular assumptions.

In order to find the total field at the surface with gentle slopes (assumption 2 of the above), one of the most commonly made approximations is the tangent plane approximation.[1] In this approximation, the reflected field at each point on the surface is given either by multiplying the incident field by the Fresnel coefficient at that point [Barrick, 1965] or by approximating the surface current by the current on the tangent plane [Kodis, 1966]. There exists another method that circumvents the tangent plane approximation by using a new set of approximate boundary conditions directly in terms of total fields. This new set of boundary conditions was originally developed by Leontovich [1948] and is also called the impedance boundary condition. It is applicable when the refractive index of the scattering medium is large in addition to the gentleness of the surface slopes (compared with the skin depth) [Senior, 1960].

For the case of a slightly rough surface, Bass and Bocharov [1958] have developed a method which is essentially based on the concept of perturbations. In this method, the effect of the surface roughness is obtained in terms of an equivalent source distribution on the unperturbed surface, by expanding the perturbed field about the unperturbed surface and applying the boundary condition to each term. Rice's

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[1] Some authors also use for the same meaning the phrase "Kirchoff approximation." But it seems that there are discrepancies among people in the definition of these phrases.

[1951] method is also based on a similar idea of perturbation but uses the Fourier analysis. The Fourier coefficients of the field are found by using the divergence relation and the boundary condition. Aside from the problem of summing the infinite series, the advantage of the perturbation methods over that of the tangent plane approximation is obviously the fact that the total field at the surface in the perturbation method is exact.

The final stage of difficulties in most scattering theories is the one associated with the carrying out of the integration. Given the total field at the surface, the next usual procedure is to evaluate the Helmholtz integral or the Stratton-Chu integral. The Green's function in the integrand can be readily approximated if the surface is finite and the receiver is in the far-zone, which facilitates integration. Usually more approximations are needed that depend on the specifics of the problem, especially on the particular model of surface roughness. Where the high frequency approximation is applicable, the method of stationary phase has been used to obtain geometrical optics results [Semenov, 1964] that further simplify the evaluation of the integral.

CHAPTER 2

REFLECTION AND TRANSMISSION OF DIPOLE RADIATION DUE TO AN INFINITE PLANE INTERFACE

2.1 Hertz Potential Due to a Dipole Source

When the source currents have a common direction, the primary electromagnetic field due to such sources can be described in terms of a single scalar function, namely, a rectangular component of the Hertz potential in the direction of the currents. If the coordinate system for the problem is so chosen that the Hertz potential in the direction of the currents may be decomposed into more than one component in that coordinate system, each component Hertz potential can be considered separately. By the principle of superposition, the electromagnetic field due to the actual currents can then be found by adding the electromagnetic fields described by each component Hertz potential. Therefore, for a boundary value problem of an arbitrarily oriented dipole antenna above or on an infinite plane interface, it is sufficient to consider only two components, the vertical and the horizontal, for the incident Hertz potential, by choosing one of the coordinate planes to coincide with the plane interface.

When the source currents consist of small current loops whose axes have a common direction, due to the symmetry between $\vec{E}$ and $\vec{H}$ in Maxwell's equations, we can consider another kind of Hertz potential which has the same use for the analysis of the problem as the Hertz potential for linear current sources. The linear current sources and the Hertz potential, usually denoted $\vec{\Pi}$, are called "electric type," while the loop current sources and the Hertz potential, $\vec{\Pi}^*$, are called "magnetic type."

In the present study, the case of the magnetic type sources will not be treated separately, since in most cases the aforementioned symmetry between $\vec{E}$ and $\vec{H}$ in Maxwell's equations enables us to immediately write the results for the case of magnetic type sources by simple inspection of the results obtained for the case of electric type sources. However, the results for the case of a vertical dipole and those for the case of a horizontal dipole are considerably different--usually the former are simpler than the latter. The reason can be easily seen by noting that the geometry of the vertical dipole is cylindrically symmetrical, whereas that of the horizontal dipole is not.

Therefore, at the outset we will assume an arbitrary orientation for the dipole and obtain results for this general case so that results corresponding to the vertical dipole or the horizontal dipole can be obtained by specializing the dipole polarization in the general results. The medium in which the dipole is situated is assumed to be the air, which is homogeneous and isotropic and occupies the upper half space above the infinite plane interface. The lower medium is arbitrary; it could be the earth, the sea, or some gaseous medium such as the ionosphere. To distinguish the symbols for the medium parameters between the upper and the lower medium, subscript 2 will be used to refer to the upper medium and subscript 1 to refer to the lower medium.

We will now derive an expression for the incident, or primary, Hertz potential for a dipole oriented in an arbitrary direction. For current sources distributed in a volume V with density given by $\vec{J}=\vec{J}_o e^{-i\omega t}$, it can be shown that the primary Hertz potential is given by [Stratton, p. 431]

$$\vec{\Pi}_i = \frac{i}{4\pi\omega\varepsilon_2} \int_V \vec{J}_o \frac{e^{ik_2 R}}{R} dv \quad , \tag{2-1}$$

where R is the distance between the source point and the observation point. As usual the time dependence of $\vec{\Pi}_i$ is neglected by writing $\vec{J}_o$ instead of $\vec{J}$ as the source function. If the source is a dipole, which may be physically considered as a straight filament of wire with an infinitesimal length, say, $d\ell$, then on replacing $\vec{J}_o dv$ by $Id\ell\vec{a}_\pi$, where I is the current in the filament and $\vec{a}_\pi$ is the unit vector in the direction of the current, and dropping the integral sign, we obtain from (2-1)

$$\vec{\Pi}_i = \frac{iId\ell}{4\pi\omega\varepsilon_2} \frac{e^{ik_2 R}}{R} \vec{a}_\pi \quad . \tag{2-2}$$

The factor multiplying the spherical function in (2-1) characterizes the strength of the wave function and is irrelevant as far as investigations of reflection and transmission from and through interfaces are concerned. Therefore for simplicity we may drop this factor and write the incident Hertz potential as

$$\vec{\Pi}_i = \frac{e^{ik_2 R}}{R} \vec{a}_\pi \quad . \tag{2-3}$$

As we are going to use a "plane wave approach" as the basis for our formulation, we need to express (2-3) in terms of a family of plane waves. This can be done by using the formula [Brekhovskikh, 1960, p. 239],

$$\frac{e^{ik_2R}}{R} = \frac{i}{2\pi}\iint_{-\infty}^{\infty} \frac{e^{ik_{2x}x + ik_{2y}y + ik_{2z}(h-z)}}{k_{2z}} dk_{2x}dk_{2y} \qquad (2\text{-}4)$$

where $k_{2z}=\sqrt{k_2^2-k_{2x}^2-k_{2y}^2}$, $R=\sqrt{x^2+y^2+(h-z)^2}$ and h is the height to the dipole from the interface z≡0, so that the incident Hertz potential can be expressed as

$$\vec{\Pi}_i(x,y,z) = \frac{i}{2\pi}\iint_{-\infty}^{\infty} \frac{e^{ik_{2x}x+ik_{2y}y+ik_{2z}(h-z)}}{k_{2z}} \vec{a}_\pi dk_{2x}dk_{2y} \quad . \qquad (2\text{-}5)$$

If we choose the x-coordinate in the direction of the projection of the dipole on the horizontal plane, then the dipole lies in the xz-plane and $\vec{a}_\pi$ can be resolved as

$$\vec{a}_\pi = (\vec{a}_\pi\cdot\vec{a}_x)\vec{a}_x + (\vec{a}_\pi\cdot\vec{a}_z)\vec{a}_z \quad . \qquad (2\text{-}6)$$

The symbol $\vec{a}$ will be used for the unit vector with a subscript indicating the direction of the unit vector; for example, the unit vectors in the x-, y- and z-directions are given by $\vec{a}_x$, $\vec{a}_y$ and $\vec{a}_z$, respectively.

2.2 Reflection and Transmission Coefficients for Plane Hertzian Waves

Considering a typical member of the Hertzian plane waves given in the integrand of (2-5) (except for the factor $\frac{i}{2\pi k_{2z}}$),

$$\vec{\pi}_i(x,y,z) = e^{ik_{2x}x+ik_{2y}y+ik_{2z}(h-z)}\vec{a}_\pi \quad , \tag{2-7}$$

we now introduce a new rectangular coordinate system (x',y',z) with the z-coordinate unchanged so that the y'z-plane coincides with the plane of incidence of $\vec{\pi}_i$. Then the x'-direction is perpendicular to the plane of incidence. With respect to the new coordinate system, (2-7) can be written as

$$\vec{\pi}_i(x,y,z) = \{(\vec{a}_\pi\cdot\vec{a}_{x'})\vec{a}_{x'}+(\vec{a}_\pi\cdot\vec{a}_{y'})\vec{a}_{y'} + (\vec{a}_\pi\cdot\vec{a}_z)\vec{a}_z\}e^{ik_{2y'}y'+ik_{2z}(h-z)} \tag{2-8}$$

where $k_{2y'}=\sqrt{k_{2x}^2+k_{2y}^2}$ and $y'=\sqrt{x^2+y^2}$. Thus each component in the (x',y',z) system is given by

$$\pi_{ix'}(x,y,z) = (\vec{a}_\pi\cdot\vec{a}_{x'})e^{ik_{2y'}y'+ik_{2z}(h-z)} \quad , \tag{2-9}$$

$$\pi_{iy'}(x,y,z) = (\vec{a}_\pi\cdot\vec{a}_{y'})e^{ik_{2y'}y'+ik_{2z}(h-z)} \quad , \tag{2-10}$$

$$\pi_{iz}(x,y,z) = (\vec{a}_\pi\cdot\vec{a}_z)e^{ik_{2y'}y'+ik_{2z}(h-z)} \quad . \tag{2-11}$$

Next we will obtain the reflection and transmission coefficients for $\pi_{ix'}$, $\pi_{iy'}$, π_{iz} by converting these into electromagnetic waves through the equations,

$$\vec{E} = \nabla(\nabla\cdot\vec{\Pi}) + k^2\vec{\Pi} \quad ,$$
$$\vec{H} = -i\omega\varepsilon\nabla\times\vec{\Pi} \quad , \tag{2-12}$$

and requiring that the resulting $\vec{E}$ and $\vec{H}$ satisfy the boundary condition. It can be easily shown that each of $\pi_{ix'}$, $\pi_{iy'}$

and π_{iz} also represents a plane electromagnetic wave, and therefore it is sufficient to consider either $\vec{E}$ or $\vec{H}$ alone in determining the reflection or the transmission coefficients for the Hertzian waves. It will be also assumed that the lower medium is homogeneous and isotropic as we assumed for the air, and its conductivity is either finite or infinite. Let us now consider the following distinct cases of polarization of the incident $\vec{\pi}$-wave (see Figure 2-1).

1) $\pi_i = \pi_{ix'}$

This is the case where the vector direction of the Hertzian wave is directed perpendicular to the plane of incidence. Putting (2-9) into the first of (2-12) and carrying out the indicated vector operations, we immediately obtain

$$\vec{E}_i = k_2^2 \pi_{ix'} \vec{a}_{x'} \quad . \tag{2-13}$$

Thus $\vec{E}_i$ is also directed in the same direction as $\vec{\pi}_i$, and is hence perpendicular to the plane of incidence. For such $\vec{E}_i$ the boundary condition is satisfied only if [Stratton, 1941, p. 493]

$$\vec{E}_r = R^{\perp} \vec{E}_i \quad , \quad \vec{E}_t = T^{\perp} \vec{E}_i \tag{2-14}$$

where $R^{\perp}$ and $T^{\perp}$ are the Fresnel reflection and transmission coefficients, respectively, for perpendicularly polarized waves;

$$R^{\perp}(\alpha) = \frac{\mu_r \cos\alpha - \sqrt{n^2 - \sin^2\alpha}}{\mu_r \cos\alpha + \sqrt{n^2 - \sin^2\alpha}} \quad ,$$

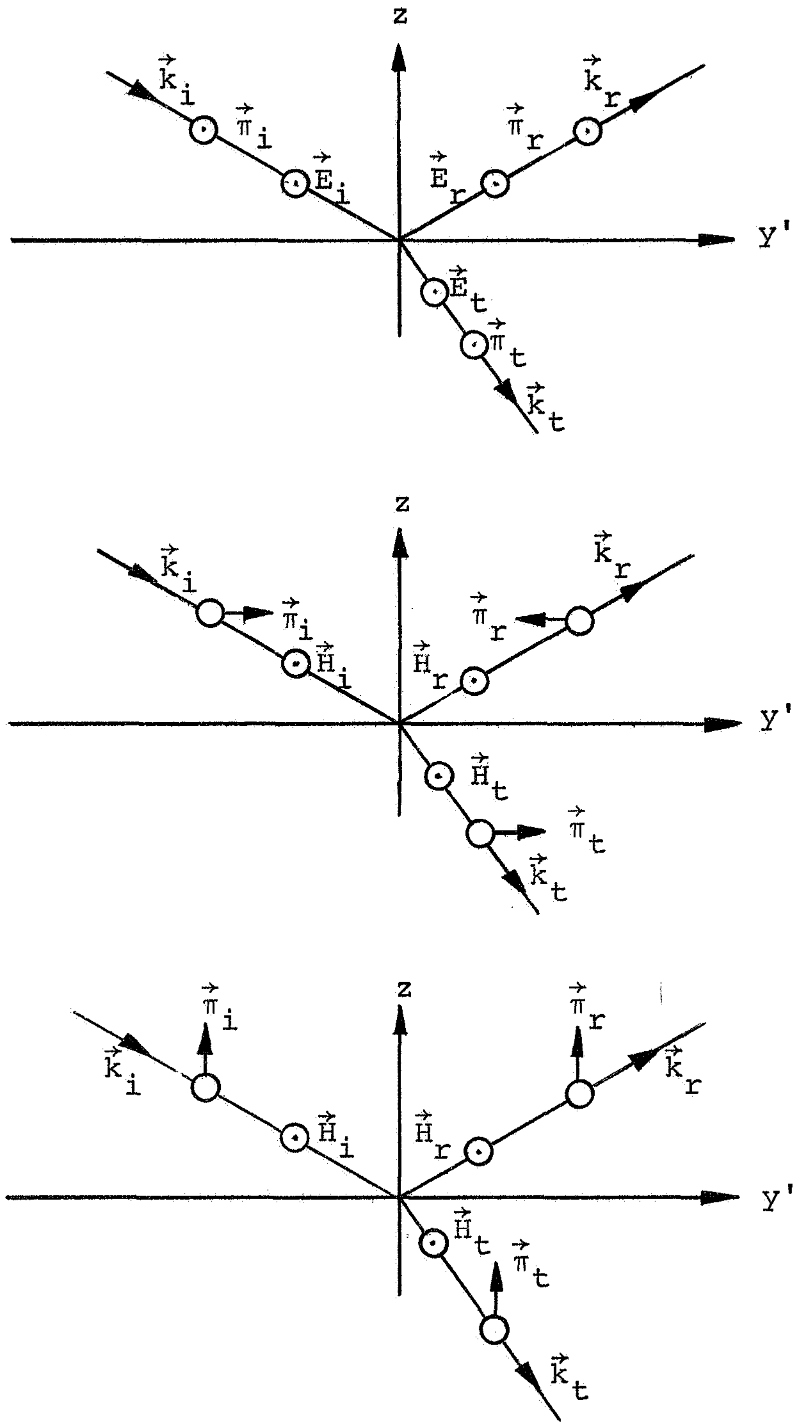

Figure 2-1. Reflection and Transmission Coefficients for Hertzian Plane Waves

$$T^{\perp}(\alpha) = \frac{2\mu_r\cos\alpha}{\mu_r\cos\alpha + \sqrt{n^2-\sin^2\alpha}} \quad , \tag{2-15}$$

where n is the index of refraction; $n=k_1/k_2$, and $\mu_r=\mu_1/\mu_2$. In analogy to (2-13), we can write

$$\vec{E}_r = k_2^2\pi_{rx'}\vec{a}_{x'} \quad , \quad \vec{E}_t = k_1^2\pi_{tx'}\vec{a}_{x'} \quad . \tag{2-16}$$

Finally combining (2-13), (2-14) and (2-16) yields the reflection and the transmission coefficients for the Hertzian wave, $\pi_{ix'}$;

$$\pi_{rx'} = R^{\perp}\pi_{ix'} \quad , \quad \pi_{tx'} = \frac{T^{\perp}}{n^2}\pi_{ix'} \quad . \tag{2-17}$$

2) $\pi_i = \pi_{iy'}$

In this case the Hertz vector lies in the plane of incidence and parallel to the interface. Inserting (2-10) into the second of (2-12) and manipulating give

$$\vec{H}_i = -\omega\varepsilon_2 k_{2z}\pi_{iy'}\vec{a}_{x'} \quad . \tag{2-18}$$

This time $\vec{H}_i$ is in the direction perpendicular to the plane of incidence, and therefore $\vec{E}_i$ is in the plane of incidence. Denoting respectively the Fresnel reflection and transmission coefficients for parallel polarized waves by R'' and T'', so that

$$R''(\alpha) = \frac{n^2\cos\alpha - \mu_r\sqrt{n^2-\sin^2\alpha}}{n^2\cos\alpha + \mu_r\sqrt{n^2-\sin^2\alpha}}$$

$$T''(\alpha) = \frac{2n^2\cos\alpha}{n^2\cos\alpha + \mu_r\sqrt{n^2-\sin^2\alpha}} \quad , \tag{2-19}$$

we have [Stratton, 1941, p. 494]

$$\vec{H}_r = R''\vec{H}_i \quad , \quad \vec{H}_t = T''\vec{H}_i \quad . \tag{2-20}$$

Since, by (2-18),

$$\vec{H}_r = \omega\varepsilon_2 k_{2z}\pi_{ry'}\vec{a}_{x'} \quad , \quad \vec{H}_t = -\omega\varepsilon_1 k_{1z}\pi_{ty'}\vec{a}_{x'} \quad , \tag{2-21}$$

where the plus sign in $\vec{H}_r$ is because of the change in sign of k_{2z} on reflection, we finally obtain

$$\pi_{ry'} = -R''\pi_{iy'} \quad , \quad \pi_{ty'} = \frac{\varepsilon_2 k_{2z}}{\varepsilon_1 k_{1z}} T''\pi_{iy'} \quad . \tag{2-22}$$

In the above ε_1 is the complex dielectric constant, $\varepsilon_1 = \varepsilon_1^o + (i\sigma_1/\omega)$. Since $k_1^2 = \omega^2\mu_1\varepsilon_1$, (2-22) can be alternatively written as

$$\pi_{ry'} = -R''\pi_{iy'} \quad , \quad \pi_{ty'} = \frac{\mu_r k_{2z}}{n^2 k_{1z}} T''\pi_{iy'} \quad . \tag{2-23}$$

3) $\pi_i = \pi_{iz}$

For this case the Hertz vector is directed perpendicular to the interface. Using the second of (2-12) on (2-11) gives

$$\vec{H}_i = \omega\varepsilon_2 k_{2y'}\pi_{iz}\vec{a}_{x'} \quad , \tag{2-24}$$

which shows that $\vec{E}_i$ is again in the plane of incidence. Thus combining (2-20) and (2-24) along with

$$\vec{H}_r = \omega\varepsilon_2 k_{2y'} \pi_{rz} \vec{a}_{x'} \quad , \quad \vec{H}_t = \omega\varepsilon_1 k_{1y'} \pi_{tz} \vec{a}_{x'} \quad , \tag{2-25}$$

we obtain

$$\pi_{rz} = R'' \pi_{iz} \quad , \quad \pi_{tz} = \frac{\mu_r T''}{n^2} \pi_{iz} \quad . \tag{2-26}$$

For easier reference, we summarize the results of the reflection and the transmission coefficients for each component plane Hertzian wave:

$$\pi_{rx'} = R^{\perp} \pi_{ix'} \quad , \quad \pi_{ry'} = -R'' \pi_{iy'} \quad , \quad \pi_{rz} = R'' \pi_{iz} \quad ;$$

$$\pi_{tx'} = \frac{T^{\perp}}{n^2} \pi_{ix'} \quad , \quad \pi_{ty'} = \frac{\mu_r k_{2z}}{n^2 k_{1z}} T'' \pi_{iy'} \quad ,$$

$$\pi_{tz} = \frac{\mu_r T''}{n^2} \pi_{iz} \quad . \tag{2-27}$$

We also list for comparison the reflection and the transmission coefficients for each component of $\vec{E}$:

$$\vec{E}_{rx'} = R^{\perp} \vec{E}_{ix'} \quad , \quad \vec{E}_{ry'} = -R'' \vec{E}_{iy'} \quad , \quad \vec{E}_{rz} = R'' \vec{E}_{iz} \quad ;$$

$$\vec{E}_{tx'} = T^{\perp} \vec{E}_{ix'} \quad , \quad \vec{E}_{ty'} = \frac{k_{1z}}{n k_{2z}} T'' \vec{E}_{iy'} \quad , \quad \vec{E}_{tz} = \frac{T''}{n} \vec{E}_{iz} \quad . \tag{2-28}$$

It is interesting to note that the reflection coefficients for each corresponding component of $\vec{\pi}$ and $\vec{E}$ are identical whereas the transmission coefficients are not.

2.3 Formulation and Evaluation of Reflected and Transmitted Hertz Potentials

Now that the reflection and the transmission coefficients for a typical Hertzian plane wave are found, we are able to write integral expressions for the reflected and the transmitted Hertz potential owing to the original incident Hertz potential (2-5) as

$$\vec{\Pi}_r(x_2,y_2,z_2) = \frac{i}{2\pi}\iint_{-\infty}^{\infty} \{R^+(\vec{a}_\pi\cdot\vec{a}_{x'})\vec{a}_{x'} - R''(\vec{a}_\pi\cdot\vec{a}_{y'})\vec{a}_{y'} + R''(\vec{a}_\pi\cdot\vec{a}_z)\vec{a}_z\} \frac{e^{ik_{2y'}y_2'+ik_{2z}(h+z_2)}}{k_{2z}} dk_{2x}dk_{2y} ,$$

$$\vec{\Pi}_t(x_1,y_1,z_1) = \frac{i}{2\pi}\iint_{-\infty}^{\infty} \{\frac{T^+}{n^2}(\vec{a}_\pi\cdot\vec{a}_{x'})\vec{a}_{x'} + \frac{\mu_r k_{2z}}{n^2 k_{1z}} T''(\vec{a}_\pi\cdot\vec{a}_{y'})\vec{a}_{y'} + \frac{\mu_r}{n^2} T''(\vec{a}_\pi\cdot\vec{a}_z)\vec{a}_z\} \frac{e^{ik_{2y'}y_1'+ik_{2z}h-ik_{1z}z_1}}{k_{2z}} dk_{2x}dk_{2y} \tag{2-29}$$

where we have used the fact that $k_{1y'}=k_{2y'}$ in writing the phase function of the exponential in the expression for $\vec{\Pi}_t$. On writing the integrands of $\vec{\Pi}_r$ and $\vec{\Pi}_t$ in terms of the components in the primary system (x,y,z), (2-29) becomes

$$\vec{\Pi}_r(x_2,y_2,z_2) = \frac{i}{2\pi}\iint_{-\infty}^{\infty} [\{R^+(\vec{a}_\pi\cdot\vec{a}_{x'})(\vec{a}_{x'}\cdot\vec{a}_x) - R''(\vec{a}_\pi\cdot\vec{a}_{y'})(\vec{a}_{y'}\cdot\vec{a}_x)\}\vec{a}_x + \{R^+(\vec{a}_\pi\cdot\vec{a}_{x'})(\vec{a}_{x'}\cdot\vec{a}_y) - R''(\vec{a}_\pi\cdot\vec{a}_{y'})(\vec{a}_{y'}\cdot\vec{a}_y)\}\vec{a}_y + R''(\vec{a}_\pi\cdot\vec{a}_z)\vec{a}_z] \frac{e^{ik_{2x}x_2+ik_{2y}y_2+ik_{2z}(h+z_2)}}{k_{2z}} dk_{2x}dk_{2y} ,$$

$$\vec{\Pi}_t(x_1,y_1,z_1) = \frac{i}{2\pi}\iint_{-\infty}^{\infty} [\{\frac{T^{\perp}}{n^2}(\vec{a}_\pi\cdot\vec{a}_{x'})(\vec{a}_{x'}\cdot\vec{a}_x)$$

$$+ \frac{\mu_r k_{2z}}{n^2 k_{1z}} T''(\vec{a}_\pi\cdot\vec{a}_{y'})(\vec{a}_{y'}\cdot\vec{a}_x)\}\vec{a}_x + \{\frac{T^{\perp}}{n^2}(\vec{a}_\pi\cdot\vec{a}_{x'})(\vec{a}_{x'}\cdot\vec{a}_y)$$

$$+ \frac{\mu_r k_{2z}}{n^2 k_{1z}} T''(\vec{a}_\pi\cdot\vec{a}_{y'})(\vec{a}_{y'}\cdot\vec{a}_y)\}\vec{a}_y + \frac{\mu_r}{n^2} T''(\vec{a}_\pi\cdot\vec{a}_z)\vec{a}_z]$$

$$\cdot \frac{e^{ik_{2x}x_1+ik_{2y}y_1+ik_{2z}h-ik_{1z}z_1}}{k_{2z}} dk_{2x}dk_{2y} \quad . \qquad (2\text{-}30)$$

Here we have to keep in mind that the unit vectors $\vec{a}_{x'}$ and $\vec{a}_{y'}$, as well as the Fresnel coefficients $R^{\perp}$, R'', $T^{\perp}$ and T'', are all functions of the variables of integration, k_{2x} and k_{2y}.

Before we attempt to evaluate the integrals given in (2-30), we mention an interesting observation. If we let $\vec{a}_\pi=\vec{a}_x$ (the horizontal dipole problem), then the z-components of both $\vec{\Pi}_r$ and $\vec{\Pi}_t$ vanish and all three vectors $\vec{\Pi}_i$, $\vec{\Pi}_r$ and $\vec{\Pi}_t$ become purely horizontal. Since the total Hertz potential above and below the interface can be given by, respectively,

$$\vec{\Pi}_2 = \vec{\Pi}_i + \vec{\Pi}_r \quad , \quad \vec{\Pi}_1 = \vec{\Pi}_t \quad , \qquad (2\text{-}31)$$

the result is that the total Hertz potential everywhere contains only the horizontal components. At first sight, this seems embarrassing since, according to most previous works on the horizontal dipole problem including those by Sommerfeld [1949, p. 257], the Hertz potential everywhere is shown to require a vertical component in addition to the horizontal component in the direction of the dipole orientation. A rather important task of resolving this apparent difficulty will be postponed until Section 2.6, so that we may presently proceed with our original problem of evaluating $\vec{\Pi}_r$ and $\vec{\Pi}_t$.

Returning to the integrals (2-30), we will use the stationary phase method for evaluation [Carrier, et al., 1966]. This method makes use of the self cancelling oscillations of an exponential factor in an integrand so that the evaluation of the integral is carried out by neglecting contributions of the integrand everywhere except in the neighborhood of a critical point where the phase of the exponential is stationary. This is the point at which the law of reflection is satisfied (the specular point) between the incident and the reflected rays in the case of reflection, and the point at which the law of refraction is satisfied between the incident and the refracted rays in the case of transmission. Thus the results obtained through the application of the stationary phase method are those of the geometrical optics approximation.

With respect to the integrals (2-30), it is not difficult to see what happens physically. When the observation point is far from the source point, the phase of a plane wave reaching the observation point after reflection (or transmission) oscillates rapidly as the direction of the incident wave on the interface changes. As a result, most plane waves virtually cancel one another, for each neighborhood of incident wave direction, as they are superposed at the observation point. However, in the neighborhood of a particular direction for which the phase is stationary, the phase changes slowly from one wave to another despite the great distance the waves travel between the source and the observation point, and hence these waves add up and amplify themselves.

Consider the following integral:

$$\vec{I} = \iint_{-\infty}^{\infty} \vec{g}\, \frac{e^{i\phi}}{k_{2z}}\, dk_{2x} dk_{2y} \tag{2-32}$$

where we first let, for the purpose of evaluating $\vec{\Pi}_r$,

$$\phi = k_{2x}x_2 + k_{2y}y_2 + k_{2z}(h+z_2) \quad ,$$

$$\begin{aligned}\vec{g} = & \{R^{\perp}(\vec{a}_\pi \cdot \vec{a}_{x'})(\vec{a}_{x'} \cdot \vec{a}_x) - R''(\vec{a}_\pi \cdot \vec{a}_{y'})(\vec{a}_{y'} \cdot \vec{a}_x)\}\vec{a}_x \\ & + \{R^{\perp}(\vec{a}_\pi \cdot \vec{a}_{x'})(\vec{a}_{x'} \cdot \vec{a}_y) - R''(\vec{a}_\pi \cdot \vec{a}_{y'})(\vec{a}_{y'} \cdot \vec{a}_y)\}\vec{a}_y + R''(\vec{a}_\pi \cdot \vec{a}_z)\vec{a}_z \quad .\end{aligned} \tag{2-33}$$

For the applicability of the stationary phase method, we assume k_2R_T to be large where $R_T = R_o + R_2 = \sqrt{x_2^2 + y_2^2 + (h+z_2)^2}$. The geometry is illustrated in Figure 2-2. To find the values of k_{2x} and k_{2y} at which point ϕ is stationary, which we will call $\tilde{k}_{2x}$ and $\tilde{k}_{2y}$, we put

$$\frac{\partial \phi}{\partial k_{2x}} = 0 \quad , \quad \frac{\partial \phi}{\partial k_{2y}} = 0 \quad , \tag{2-34}$$

obtaining

$$\tilde{k}_{2x} = k_2 \frac{x_2}{R_T} \quad , \quad \tilde{k}_{2y} = k_2 \frac{y_2}{R_T} \quad , \quad \tilde{k}_{2z} = k_2 \frac{h+z_2}{R_T} \quad . \tag{2-35}$$

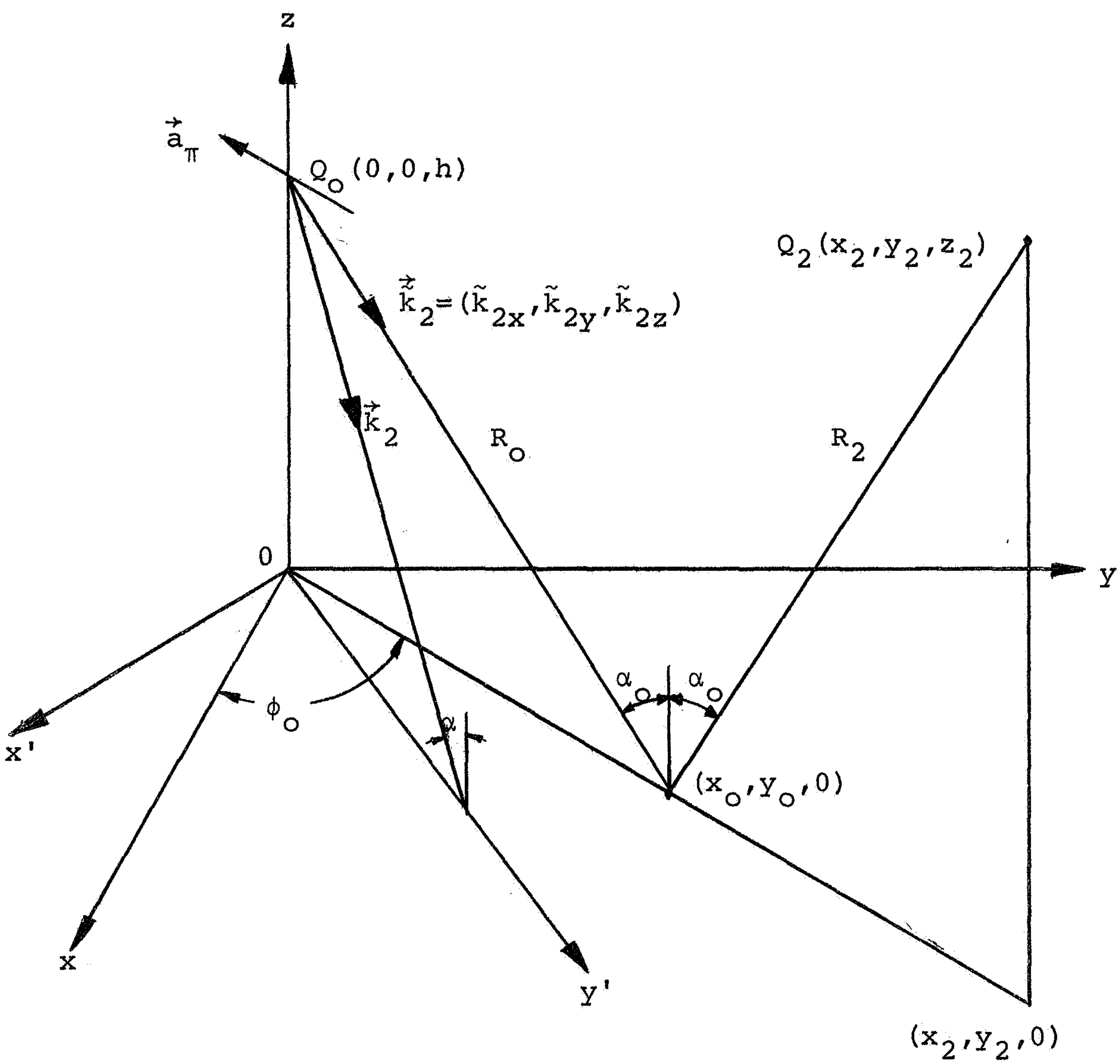

Figure 2-2. Symbols and Geometry for Reflection

On expanding ϕ in a power series at the stationary point up to the second order,

$$\phi = [\phi]_{ev} + \frac{1}{2}\left[\frac{\partial^2\phi}{\partial k_{2x}^2}\right]_{ev}(k_{2x}-\tilde{k}_{2x})^2 + \frac{1}{2}\left[\frac{\partial^2\phi}{\partial k_{2y}^2}\right]_{ev}(k_{2y}-\tilde{k}_{2y})^2$$

$$+ \left[\frac{\partial^2\phi}{\partial k_{2x}\partial k_{2y}}\right]_{ev}(k_{2x}-\tilde{k}_{2x})(k_{2y}-\tilde{k}_{2y}) \quad , \qquad (2\text{-}36)$$

where the symbol $[\,]_{ev}$ indicates evaluation of the function inside the bracket at the stationary point, and replacing $\vec{g}$ and k_{2z} in the factor by their values evaluated at the stationary point, (2-32) becomes

$$I = \left[\frac{\vec{g}}{k_{2z}}\right]_{ev} e^{i[\phi]_{ev}} I_1 \quad , \qquad (2\text{-}37)$$

where

$$I_1 = \iint_{-\infty}^{\infty} \exp\left[\frac{i}{2}\left\{\left[\frac{\partial^2\phi}{\partial k_{2x}^2}\right]_{ev}(k_{2x}-\tilde{k}_{2x})^2 + \left[\frac{\partial^2\phi}{\partial k_{2y}^2}\right]_{ev}(k_{2y}-\tilde{k}_{2y})^2\right.\right.$$

$$\left.\left. + 2\left[\frac{\partial^2\phi}{\partial k_{2x}\partial k_{2y}}\right]_{ev}(k_{2x}-\tilde{k}_{2x})(k_{2y}-\tilde{k}_{2y})\right\}\right] dk_{2x}dk_{2y} \quad . \qquad (2\text{-}38)$$

It is easy to show that (2-38) integrates out to

$$I_1 = \frac{2\pi}{i}\left(\left[\frac{\partial^2\phi}{\partial k_{2x}^2}\right]_{ev}\left[\frac{\partial^2\phi}{\partial k_{2y}^2}\right]_{ev} - \left[\frac{\partial^2\phi}{\partial k_{2x}\partial k_{2y}}\right]^2_{ev}\right)^{-1/2} . \qquad (2\text{-}39)$$

With reference to Figure 2-2, we have at the stationary point

$$R^{+} = R^{+}(\alpha_o) \quad , \quad R'' = R''(\alpha_o) \quad ;$$

$$\vec{a}_{x'}\cdot\vec{a}_x = \sin\phi_o \quad , \quad \vec{a}_{y'}\cdot\vec{a}_x = \cos\phi_o \quad ,$$

$$\vec{a}_{x'}\cdot\vec{a}_y = -\cos\phi_o \quad , \quad \vec{a}_{y'}\cdot\vec{a}_y = \sin\phi_o \quad . \tag{2-40}$$

Therefore the bracketed quantities in (2-36) can be given as

$$[\phi]_{ev} = k_2 R_T \quad ,$$

$$\left[\frac{\partial^2\phi}{\partial k_{2x}^2}\right]_{ev} = -\frac{R_T}{k_2}\left\{1+\left(\frac{x_2}{h+z_2}\right)^2\right\} = -\frac{R_T}{k_2}(1+\tan^2\alpha_o\cos^2\phi_o) \quad ,$$

$$\left[\frac{\partial^2\phi}{\partial k_{2y}^2}\right]_{ev} = -\frac{R_T}{k_2}\left\{1+\left(\frac{y_2}{h+z_2}\right)^2\right\} = -\frac{R_T}{k_2}(1+\tan^2\alpha_o\sin^2\phi_o) \quad ,$$

$$\left[\frac{\partial^2\phi}{\partial k_{2x}\partial k_{2y}}\right]_{ev} = -\frac{R_T x_2 y_2}{k_2(h+z_2)^2} = -\frac{R_T}{k_2}\tan^2\alpha_o\cos\phi_o\sin\phi_o \quad . \tag{2-41}$$

Using (2-41) we can simplify (2-39), which then becomes

$$I_1 = \frac{2\pi k_2\cos\alpha_o}{iR_T} \quad , \tag{2-42}$$

so that I, (2-37), can be given by

$$I = \frac{2\pi}{i}[\vec{g}]_{ev}\frac{e^{ik_2R_T}}{R_T} \quad . \tag{2-43}$$

If we let $\vec{a}_\pi\cdot\vec{a}_x=\sin\theta_o$ and $\vec{a}_\pi\cdot\vec{a}_z=\cos\theta_o$ in (2-6), then we can write

$$\vec{a}_\pi = \sin\theta_o\vec{a}_x + \cos\theta_o\vec{a}_z \quad , \tag{2-44}$$

and therefore

$$\vec{a}_\pi\cdot\vec{a}_{x'} = \sin\theta_o\sin\phi_o \quad , \quad \vec{a}_\pi\cdot\vec{a}_{y'} = \sin\theta_o\cos\phi_o \quad . \tag{2-45}$$

By (2-40) and (2-45), $\vec{g}$ given by (2-33) becomes at the stationary point

$$[\vec{g}]_{ev} = \{R^{\perp}(\alpha_o)\sin^2\phi_o - R''(\alpha_o)\cos^2\phi_o\}\sin\theta_o\vec{a}_x$$
$$-\{R^{\perp}(\alpha_o) + R''(\alpha_o)\}\sin\theta_o\cos\phi_o\sin\phi_o\vec{a}_y$$
$$+ R''(\alpha_o)\cos\theta_o\vec{a}_z \quad . \tag{2-46}$$

Since

$$\vec{\Pi}_r = \frac{i}{2\pi}\vec{I} \quad , \tag{2-47}$$

we finally obtain the reflected Hertz potential,

$$\vec{\Pi}_r = [\{R^{\perp}(\alpha_o)\sin^2\phi_o - R''(\alpha_o)\cos^2\phi_o\}\sin\theta_o\vec{a}_x - \{R^{\perp}(\alpha_o)$$
$$+R''(\alpha_o)\}\sin\theta_o\cos\phi_o\sin\phi_o\vec{a}_y + R''(\alpha_o)\cos\theta_o\vec{a}_z]\frac{e^{ik_2R_T}}{R_T} \quad . \tag{2-48}$$

Next, for the purpose of evaluating $\vec{\Pi}_t$, we put in (2-32)

$$\phi = k_{2x}x_1 + k_{2y}y_1 + k_{2z}h - k_{1z}z_1 \quad ,$$

$$\vec{g} = \frac{1}{n^2}[\{T^{\perp}(\vec{a}_\pi\cdot\vec{a}_{x'})(\vec{a}_{x'}\cdot\vec{a}_x) + \frac{\mu_r k_{2z}}{k_{1z}}T''(\vec{a}_\pi\cdot\vec{a}_{y'})(\vec{a}_{y'}\cdot\vec{a}_x)\}\vec{a}_x$$
$$+\{T^{\perp}(\vec{a}_\pi\cdot\vec{a}_{x'})(\vec{a}_{x'}\cdot\vec{a}_y) + \frac{\mu_r k_{2z}}{k_{1z}}T''(\vec{a}_\pi\cdot\vec{a}_{y'})(\vec{a}_{y'}\cdot\vec{a}_y)\}\vec{a}_y$$
$$+\mu_r T''(\vec{a}_\pi\cdot\vec{a}_z)\vec{a}_z] \quad . \tag{2-49}$$

For this case we assume $k_2R_o+k_1R_1$ to be large for the validity of the stationary phase method. The stationary point can be found again through (2-34) (see Figure 2-3);

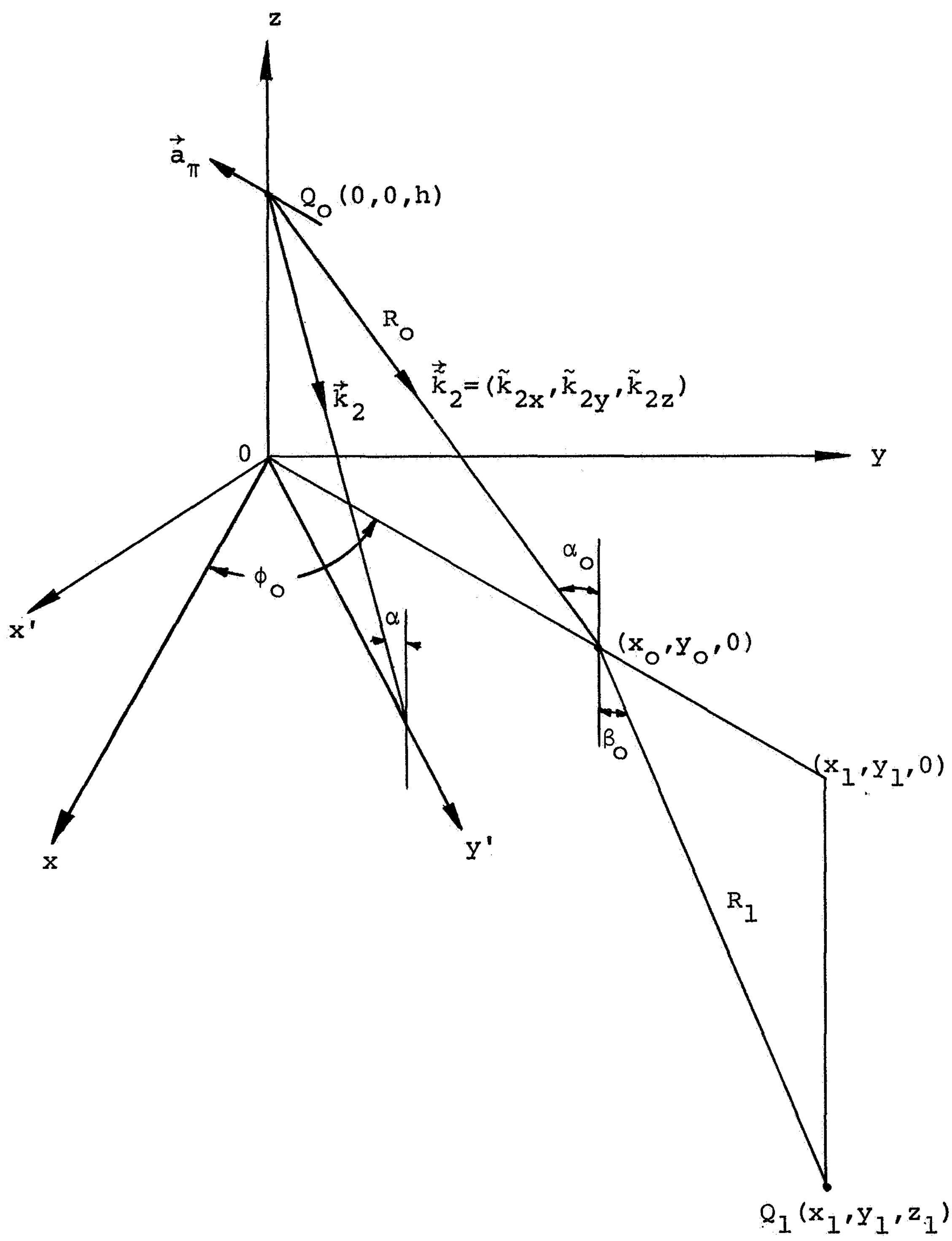

Figure 2-3. Symbols and Geometry for Transmission

$$k_{2x} = k_{1x} = \begin{cases} k_2 \sin\alpha_o \cos\phi_o \\ k_1 \sin\beta_o \cos\phi_o \end{cases} ,$$

$$k_{2y} = k_{1y} = \begin{cases} k_2 \sin\alpha_o \sin\phi_o \\ k_1 \sin\beta_o \sin\phi_o \end{cases} ,$$

$$k_{2z} = k_2 \cos\alpha_o \quad , \quad k_{1z} = k_1 \cos\beta_o \quad . \tag{2-50}$$

Using (2-50), we obtain

$$[\phi]_{ev} = k_2(R_o + nR_1) \quad ,$$

$$\left[\frac{\partial^2\phi}{\partial k_{2x}^2}\right]_{ev} = -\frac{R_o}{k_2}(1+\tan^2\alpha_o\cos^2\phi_o) - \frac{R_1}{k_1}(1+\tan^2\beta_o\cos^2\phi_o) \quad ,$$

$$\left[\frac{\partial^2\phi}{\partial k_{2y}^2}\right]_{ev} = -\frac{R_o}{k_2}(1+\tan^2\alpha_o\sin^2\phi_o) - \frac{R_1}{k_1}(1+\tan^2\beta_o\sin^2\phi_o) \quad ,$$

$$\left[\frac{\partial^2\phi}{\partial k_{2x}\partial k_{2y}}\right]_{ev} = -\frac{R_o}{k_2}\tan^2\alpha_o\cos\phi_o\sin\phi_o - \frac{R_1}{k_1}\tan^2\beta_o\cos\phi_o\sin\phi_o \quad , \tag{2-51}$$

which on inserting into (2-39) and reducing now give

$$I_1 = \frac{2\pi}{i}\frac{k_1\cos\alpha_o\cos\beta_o}{\sqrt{(nR_o+R_1)(nR_o\cos^2\beta_o+R_1\cos^2\alpha_o)}} \quad . \tag{2-52}$$

$\vec{g}$ given in (2-49) at the stationary point becomes

$$[\vec{g}]_{ev} = \frac{1}{n^2}[\{T^{\perp}(\alpha_o)\sin^2\phi_o + \frac{\mu_r}{n}T''(\alpha_o)\frac{\cos\alpha_o}{\cos\beta_o}\cos^2\phi_o\}\sin\theta_o\vec{a}_x - \{T^{\perp}(\alpha_o)$$
$$- \frac{\mu_r}{n}T''(\alpha_o)\frac{\cos\alpha_o}{\cos\beta_o}\}\sin\theta_o\cos\phi_o\sin\phi_o\vec{a}_y + \mu_r T''(\alpha_o)\cos\theta_o\vec{a}_z] \quad . \tag{2-53}$$

The transmitted Hertz potential can be given by

$$\vec{\Pi}_t = [\frac{\vec{g}}{k_{2z}}]_{ev}\, I_1 \quad , \tag{2-54}$$

and hence

$$\vec{\Pi}_t = [\{T^{\perp}(\alpha_o)\cos\beta_o\sin^2\phi_o + \frac{\mu_r}{n}\, T''(\alpha_o)\cos\alpha_o\cos^2\phi_o\}\sin\theta_o\vec{a}_x$$
$$-\{T^{\perp}(\alpha_o)\cos\beta_o - \frac{\mu_r}{n}\, T''(\alpha_o)\cos\alpha_o\}\sin\theta_o\cos\phi_o\sin\phi_o\vec{a}_y$$
$$+\mu_r T''(\alpha_o)\cos\beta_o\cos\theta_o\vec{a}_z]\ \frac{e^{ik_2(R_o+nR_1)}}{n\sqrt{(nR_o+R_1)(nR_o\cos^2\beta_o+R_1\cos^2\alpha_o)}} \quad . \tag{2-55}$$

$\vec{\Pi}_r$ and $\vec{\Pi}_t$ given in (2-48) and (2-55) are the desired results for the reflected and the transmitted Hertz potentials due to a dipole of general orientation and an infinite plane interface.

2.4 Some Special Cases

The previous results have been obtained under the geometrical optics approximation, and hence their validity is limited by the requirement of far-zone observation as we have assumed in applying the stationary phase method. However, in some limiting cases, they become exact and lead to well known results, namely, when $n \to 1$ or $n \to \infty$. The limit $n \to 1$ corresponds to the case where there is no interface (an imaginary interface). For $n=1$ we easily see from (2-48) and (2-55) that

$$\vec{\Pi}_r \equiv 0 \quad , \quad \vec{\Pi}_t = (\sin\theta_o \vec{a}_x + \cos\theta_o \vec{a}_z) \frac{e^{ik_2(R_o+R_1)}}{R_o+R_1} \qquad (2\text{-}56)$$

where now $R_o+R_1=\sqrt{x_1^2+y_1^2+(h-z_1)^2}$. As we should expect, no reflection occurs and the transmitted Hertz potential is just the same as the incident Hertz potential at that point. On the other hand, the limit $n\to\infty$ corresponds to the case of the lower medium being perfectly conducting. In this limit, from (2-15) and (2-19)

$$R''(\alpha) \equiv -R^{\perp}(\alpha) \equiv 1 \quad ,$$

$$T^{\perp}(\alpha) \equiv T''(\alpha) \equiv 0 \quad , \qquad (2\text{-}57)$$

and thus (2-48) and (2-55) reduce to

$$\vec{\Pi}_r = (-\sin\theta_o \vec{a}_x + \cos\theta_o \vec{a}_z)\frac{e^{ik_2R_T}}{R_T} \quad , \quad \vec{\Pi}_t \equiv 0 \qquad (2\text{-}58)$$

in agreement with what we already know from the theory of images for a dipole in presence of a perfectly conducting half space: the image of a vertical dipole is a dipole of the same magnitude and orientation, while that of a horizontal dipole is a dipole of the same magnitude but of opposite orientation, both images located at their mirror image points with respect to the interface.

There is one situation in which the geometrical optics results even for a far-zone observation become inapplicable. This situation arises in both cases of the vertical dipole and the horizontal dipole when the source point and the observation point simultaneously approach the interface, so that the incident wave (at the stationary point) becomes close to grazing. We will examine this separately for the two cases as we now specialize the general results to the vertical dipole case and the horizontal dipole case.

a) The Vertical Dipole

By (2-3) the incident wave can be described by

$$\Pi_{iz} = \frac{e^{ik_2 R}}{R} \quad , \quad \Pi_{ix} \equiv \Pi_{iy} \equiv 0 \quad , \tag{2-59}$$

and the reflected and the transmitted waves are given by letting $\theta_o=0$ in (2-48) and (2-55), yielding

$$\Pi_{rz} = R''(\alpha_o) \frac{e^{ik_2 R_T}}{R_T} \quad , \quad \Pi_{rx} \equiv \Pi_{ry} \equiv 0 \quad ;$$

$$\Pi_{tz} = \frac{\mu_r T''(\alpha_o) \cos\beta_o e^{ik_2(R_o+nR_1)}}{n\sqrt{(nR_o+R_1)(nR_o\cos^2\beta_o+R_1\cos^2\alpha_o)}} \quad , \quad \Pi_{tx} \equiv \Pi_{ty} \equiv 0 \ . \tag{2-60}$$

The above expression for $\vec{\Pi}_r$ is well known and has been obtained by Wise [1929], Norton [1937] and others. It is also identical to the first term of equation (19.36) of Brekhovskikh [1960, p. 255] which was obtained by using the saddle point method. Using the energy flux method, Brekhovskikh also derives an expression for $\vec{\Pi}_t$ in the geometrical optics approximation, his equation (23-8), which can be easily shown to be identical to the above $\vec{\Pi}_t$ by rearranging.

To see whether the geometrical optics approximation is applicable when the incident wave is near grazing, we look at the total Hertz potentials, given by adding $\vec{\Pi}_i$ and $\vec{\Pi}_r$ in the upper medium and by $\vec{\Pi}_t$ alone in the lower medium. They then become

$$\Pi_{2z} = \frac{e^{ik_2R}}{R} + R''(\alpha_o)\,\frac{e^{ik_2R_T}}{R_T} \quad , \quad \Pi_{2x} \equiv \Pi_{2y} \equiv 0 \quad ;$$

$$\Pi_{1z} = \frac{\mu_r T''(\alpha_o)\cos\beta_o e^{ik_2(R_o+nR_1)}}{n\sqrt{(nR_o+R_1)(nR_o\cos^2\beta_o+R_1\cos^2\alpha_o)}} \quad , \quad \Pi_{1x} \equiv \Pi_{1y} \equiv 0 \quad . \qquad (2\text{-}61)$$

If the incident wave is sufficiently close to grazing, we have $\alpha_o \approx \pi/2$, and hence

$$R''(\alpha_o) \approx -1 \quad , \quad T''(\alpha_o) \approx 0 \quad , \quad R_T \approx R \qquad (2\text{-}62)$$

so that (2-61) becomes

$$\Pi_{2z} \approx 0 \quad , \quad \Pi_{2x} \equiv \Pi_{2y} \equiv 0 \quad ;$$

$$\Pi_{1z} \approx 0 \quad , \quad \Pi_{1x} \equiv \Pi_{1y} \equiv 0 \quad . \qquad (2\text{-}63)$$

Thus the geometrical optics results become vanishingly small, in which case the terms neglected in the approximation because of the use of the stationary phase method are no longer negligible and become either comparable to or greater than the geometrical optics results. The geometrical optics approximation for both $\vec{\Pi}_r$ and $\vec{\Pi}_t$ is therefore inapplicable for the grazing incidence. The rate of the approximation getting poor as to the grazing incidence of course depends on the value of k_1 or n. From (2-19) and (2-61) it can be seen that as n increases, the region of α_o for which the geometrical optics approximation is applicable also increases.

b) The Horizontal Dipole

By replacing $\vec{a}_\pi$ by $\vec{a}_x$, we obtain from (2-3)

$$\Pi_{ix} = \frac{e^{ik_2 R}}{R} \quad , \quad \Pi_{iy} \equiv \Pi_{iz} \equiv 0 \quad ,$$

and with $\theta_o=\pi/2$, (2-48) and (2-55) yield

$$\Pi_{rx} = \{R^{\perp}(\alpha_o)\sin^2\phi_o - R''(\alpha_o)\cos^2\phi_o\} \frac{e^{ik_2 R_T}}{R_T}$$

$$\Pi_{ry} = -\{R^{\perp}(\alpha_o)+R''(\alpha_o)\}\sin\phi_o\cos\phi_o \frac{e^{ik_2 R_T}}{R_T} \quad , \quad \Pi_{rz} \equiv 0 \quad ;$$

$$\Pi_{tx} = \frac{nT^{\perp}(\alpha_o)\cos\beta_o\sin^2\phi_o+\mu_r T''(\alpha_o)\cos\alpha_o\cos^2\phi_o}{n^2\sqrt{(nR_o+R_1)(nR_o\cos^2\beta_o+R_1\cos^2\alpha_o)}} e^{ik_2(R_o+nR_1)} \quad ,$$

$$\Pi_{ty} = -\frac{\{nT^{\perp}(\alpha_o)\cos\beta_o-\mu_r T''(\alpha_o)\cos\alpha_o\}\sin\phi_o\cos\phi_o}{n^2\sqrt{(nR_o+R_1)(nR_o\cos^2\beta_o+R_1\cos^2\alpha_o)}} e^{ik_2(R_o+nR_1)} \quad ,$$

$$\Pi_{tz} \equiv 0 \quad . \qquad (2\text{-}64)$$

An expression for $\vec{\Pi}_r$ in the case of a horizontal dipole and a finitely conducting plane interface is also obtained in Brekhovskikh [1960, p. 259], which, however, is different from our $\vec{\Pi}_r$ shown in (2-64) in magnitude as well as in the vector direction. The apparent discrepancy between our

Hertz potentials and those obtained previously using the resolution of $\vec{\Pi}=(\Pi_x, 0, \Pi_z)$ was already mentioned earlier when we derived the integral formulas for $\vec{\Pi}_r$ and $\vec{\Pi}_t$, namely, (2-30). These two distinctively different $\vec{\Pi}_r$'s for the horizontal dipole problem, although both are approximations, will serve as a particular example when we demonstrate the nonuniqueness of the Hertz potential in the horizontal dipole problem in Section 2.6. This apparently has been overlooked by all previous workers in this area and is believed to bear great significance. The expression for $\vec{\Pi}_t$ given in (2-64) is new. In fact, this seems to be the first time that a closed form result for $\vec{\Pi}_t$ in the horizontal dipole problem is derived.

Next we will examine the applicability of (2-64) in the special case when the incident wave is grazing. Using (2-62) and the following approximate values for the other Fresnel coefficients

$$R^+(\alpha_o) \approx -1 \quad , \quad T^+(\alpha_o) \approx 0 \quad , \tag{2-65}$$

we obtain in terms of the total Hertz potentials

$$\Pi_{2x} \approx 2\cos^2\phi_o \frac{e^{ik_2R}}{R} \quad , \quad \Pi_{2y} \approx 2\sin\phi_o\cos\phi_o \frac{e^{ik_2R}}{R} \quad , \quad \Pi_{2z} \equiv 0 \quad ;$$

$$\Pi_{1x} \approx 0 \quad , \quad \Pi_{1y} \approx 0 \quad , \quad \Pi_{1z} \approx 0 \quad . \tag{2-66}$$

By the same argument as presented in the case of the vertical dipole, here the geometrical optics approximation of $\vec{\Pi}_1$ for the grazing incidence is also seen inapplicable. It is interesting to note that for $\vec{\Pi}_2$, however, whether or not the approximation for the grazing incidence is inapplicable is dependent on the value of ϕ_o. For example, we obtain for $\phi_o=0$

$$\Pi_{2x} \approx 2 \frac{e^{ik_2R}}{R} \quad , \quad \Pi_{2y} \approx 0 \quad , \quad \Pi_{2z} \equiv 0 \quad , \tag{2-67}$$

and for $\phi_o=\pi/2$

$$\Pi_{2x} \approx 0 \quad , \quad \Pi_{2y} \approx 0 \quad , \quad \Pi_{2z} \equiv 0 \quad . \tag{2-68}$$

Thus considering both the dipole and the observation point located slightly above the interface, when the observation point is located in the direction of the dipole polarization, the reflected Hertz potential is about as large as the incident Hertz potential with the same sign so that the geometrical optics results for the total Hertz potential become even larger, making the approximation for this case quite valid. As the observaton point is moved further and further from this direction, the reflected Hertz potential first becomes smaller and smaller, and then after changing the sign it again grows until finally when the observation point lies in the direction normal to the dipole polarization, the incident and the reflected Hertz potentials almost cancel out each other making the approximation invalid. Actually the electromagnetic field derived from (2-67) is very close to zero in spite of the reenforcement in the Hertz potentials, which appears disturbing. But in this case the contribution from the neglected terms in $\vec{\Pi}_r$ is even smaller.

This is in contrast to the vertical dipole case where the total Hertz potential in the geometrical optics approximation at any point becomes invalid for the grazing geometry.

2.5 Derivation of Reflected and Transmitted Electromagnetic Fields

From the Hertz potentials obtained in Section 2.3 the reflected and the transmitted electromagnetic fields can be derived through the relation (2-12) in which the differential operators are to operate on the field point coordinates. In carrying out the indicated differentiations, we can assume the factors multiplying the exponential, including R^{-1}, to be approximately constant. This is justified since the factors change much more slowly in comparison to the exponential under the condition of far-zone observation. The far-zone approximation is equivalent to a high frequency approximation, which can be easily seen. With this assumption, the manipulation is straightforward, as each of the differential operators is replaced by

$$\frac{\partial}{\partial x_2} \approx ik_2 \sin\alpha_o \cos\phi_o \quad , \quad \frac{\partial}{\partial y_2} \approx ik_2 \sin\alpha_o \sin\phi_o \quad , \quad \frac{\partial}{\partial z_2} \approx ik_2 \cos\alpha_o \; ;$$

$$\frac{\partial}{\partial x_1} \approx \begin{cases} ik_2 \sin\alpha_o \cos\phi_o \\ ik_1 \sin\beta_o \cos\phi_o \end{cases} , \quad \frac{\partial}{\partial y_1} \approx \begin{cases} ik_2 \sin\alpha_o \sin\phi_o \\ ik_1 \sin\beta_o \sin\phi_o \end{cases} ,$$

$$\frac{\partial}{\partial z_1} \approx ik_1 \cos\beta_o \quad . \tag{2-69}$$

The results are tabulated in the following (Appendix A):

a) The Vertical Dipole

The reflected field:

$$E_{rx} = -k_2^2 R''(\alpha_o)\sin\alpha_o\cos\alpha_o\cos\phi_o \frac{e^{ik_2R_T}}{R_T} ,$$

$$E_{ry} = -k_2^2 R''(\alpha_o)\sin\alpha_o\cos\alpha_o\sin\phi_o \frac{e^{ik_2R_T}}{R_T} ,$$

$$E_{rz} = k_2^2 R''(\alpha_o)\sin^2\alpha_o \frac{e^{ik_2R_T}}{R_T} ;$$

$$H_{rx} = \frac{k_2^3}{\omega\mu_2} R''(\alpha_o)\sin\alpha_o\sin\phi_o \frac{e^{ik_2R_T}}{R_T} ,$$

$$H_{ry} = -\frac{k_2^3}{\omega\mu_2} R''(\alpha_o)\sin\alpha_o\cos\phi_o \frac{e^{ik_2R_T}}{R_T}$$

$$H_{rz} = 0 \quad . \qquad (2\text{-}70)$$

In view of the cylindrical symmetry of the dipole orientation with respect to the interface, (2-64) becomes particularly simple in terms of the cylindrical components:

$$E_{rr} = -k_2^2 R''(\alpha_o)\sin\alpha_o\cos\alpha_o \frac{e^{ik_2R_T}}{R_T} ,$$

$$E_{r\phi} = 0 ,$$

$$E_{rz} = k_2^2 R''(\alpha_o)\sin^2\alpha_o \frac{e^{ik_2R_T}}{R_T} ;$$

$$H_{rr} = 0 ,$$

$$H_{r\phi} = -\frac{k_2^3}{\omega\mu_2} R''(\alpha_o)\sin\alpha_o \frac{e^{ik_2R_T}}{R_T} ,$$

$$H_{rz} = 0 \quad . \qquad (2\text{-}71)$$

In writing the components for the transmitted fields, we denote for simplification

$$\frac{\cos\beta_o}{\sqrt{(nR_o+R_1)(nR_o\cos^2\beta_o+R_1\cos^2\alpha_o)}} = \frac{1}{P} \quad . \tag{2-72}$$

Then the transmitted field is given by

$$E_{tx} = -k_2^2\mu_r T''(\alpha_o)\sin\alpha_o\cos\beta_o\cos\phi_o \frac{e^{ik_2(R_o+nR_1)}}{R_T} \quad ,$$

$$E_{ty} = -k_2^2\mu_r T''(\alpha_o)\sin\alpha_o\cos\beta_o\sin\phi_o \frac{e^{ik_2(R_o+nR_1)}}{R_T} \quad ,$$

$$E_{tz} = k_2^2\mu_r T''(\alpha_o)\sin\alpha_o\sin\beta_o \frac{e^{ik_2(R_o+nR_1)}}{R_T} \quad ;$$

$$H_{tx} = \frac{k_2^3 n}{\omega\mu_2} T''(\alpha_o)\sin\alpha_o\sin\phi_o \frac{e^{ik_2(R_o+nR_1)}}{P} \quad ,$$

$$H_{ty} = -\frac{k_2^3 n}{\omega\mu_2} T''(\alpha_o)\sin\alpha_o\cos\phi_o \frac{e^{ik_2(R_o+nR_1)}}{P} \quad ,$$

$$H_{tz} = 0 \quad ; \tag{2-73}$$

which in the cylindrical components become

$$E_{tr} = -k_2^2\mu_r T''(\alpha_o)\sin\alpha_o\cos\beta_o \frac{e^{ik_2(R_o+nR_1)}}{P} \quad ,$$

$$E_{t\phi} = 0 \quad ,$$

$$E_{tz} = k_2^2\mu_r T''(\alpha_o)\sin\alpha_o\sin\beta_o \frac{e^{ik_2(R_o+nR_1)}}{P} \quad ;$$

$$H_{tr} = 0 \quad ,$$

$$H_{t\phi} = -\frac{k_2^3 n}{\omega\mu_2} T''(\alpha_o)\sin\alpha_o \frac{e^{ik_2(R_o+nR_1)}}{P} \quad ,$$

$$H_{tz} = 0 \quad . \tag{2-74}$$

b) The Horizontal Dipole

The reflected field:

$$E_{rx} = k_2^2\{R^+(\alpha_o)\sin^2\phi_o - R''(\alpha_o)\cos^2\alpha_o\cos^2\phi_o\}\frac{e^{ik_2R_T}}{R_T} \quad ,$$

$$E_{ry} = -k_2^2\{R^+(\alpha_o)+R''(\alpha_o)\cos^2\alpha_o\}\sin\phi_o\cos\phi_o \frac{e^{ik_2R_T}}{R_T} \quad ,$$

$$E_{rz} = k_2^2 R''(\alpha_o)\sin\alpha_o\cos\alpha_o\cos\phi_o \frac{e^{ik_2R_T}}{R_T} \quad ;$$

$$H_{rx} = \frac{k_2^3}{\omega\mu_2}\{R^+(\alpha_o)+R''(\alpha_o)\}\cos\alpha_o\sin\phi_o\cos\phi_o \frac{e^{ik_2R_T}}{R_T} \quad ,$$

$$H_{ry} = \frac{k_2^3}{\omega\mu_2}\{R^+(\alpha_o)\sin^2\phi_o - R''(\alpha_o)\cos^2\phi_o\}\cos\alpha_o \frac{e^{ik_2R_T}}{R_T} \quad ,$$

$$H_{rz} = -\frac{k_2^3}{\omega\mu_2} R^+(\alpha_o)\sin\alpha_o\sin\phi_o \frac{e^{ik_2R_T}}{R_T} \quad . \tag{2-75}$$

In the cylindrical coordinate system,

$$E_{rr} = -k_2^2 R''(\alpha_o)\cos^2\alpha_o\cos\phi_o \frac{e^{ik_2R_T}}{R_T} \quad ,$$

$$E_{r\phi} = -k_2^2 R^+(\alpha_o)\sin\phi_o \frac{e^{ik_2R_T}}{R_T} \quad ,$$

$$E_{rz} = k_2^2 R''(\alpha_o)\sin\alpha_o\cos\alpha_o\cos\phi_o \frac{e^{ik_2R_T}}{R_T} \quad ;$$

$$H_{rr} = \frac{k_2^3}{\omega\mu_2} R^{\perp}(\alpha_o)\cos\alpha_o\sin\phi_o \frac{e^{ik_2R_T}}{R_T} \quad ,$$

$$H_{r\phi} = -\frac{k_2^3}{\omega\mu_2} R''(\alpha_o)\cos\alpha_o\cos\phi_o \frac{e^{ik_2R_T}}{R_T} \quad ,$$

$$H_{rz} = -\frac{k_2^3}{\omega\mu_2} R^{\perp}(\alpha_o)\sin\alpha_o\sin\phi_o \frac{e^{ik_2R_T}}{R_T} \quad . \tag{2-76}$$

The transmitted field:

$$E_{tx} = k_2^2\{nT^{\perp}(\alpha_o)\sin^2\phi_o+\mu_r T''(\alpha_o)\cos\alpha_o\cos\beta_o\cos^2\phi_o\}\frac{e^{ik_2(R_o+nR_1)}}{P} \quad ,$$

$$E_{ty} = -k_2^2\{nT^{\perp}(\alpha_o)-\mu_r T''(\alpha_o)\cos\alpha_o\cos\beta_o\}\sin\phi_o\cos\phi_o \frac{e^{ik_2(R_o+nR_1)}}{P} \quad ,$$

$$E_{tz} = -k_2^2\mu_r T''(\alpha_o)\cos\alpha_o\sin\beta_o\cos\phi_o \frac{e^{ik_2(R_o+nR_1)}}{P} \quad ;$$

$$H_{tx} = \frac{k_2^3 n}{\omega\mu_1}\{nT^{\perp}(\alpha_o)\cos\beta_o-\mu_r T''(\alpha_o)\cos\alpha_o\}\sin\phi_o\cos\phi_o \frac{e^{ik_2(R_o+nR_1)}}{P} \quad ,$$

$$H_{ty} = \frac{k_2^3 n}{\omega\mu_1}\{nT^{\perp}(\alpha_o)\cos\beta_o\sin^2\phi_o+\mu_r T''(\alpha_o)\cos\alpha_o\cos^2\phi_o\}\frac{e^{ik_2(R_o+nR_1)}}{P} \quad ,$$

$$H_{tz} = -\frac{k_2^3 n^2}{\omega\mu_1} T^{\perp}(\alpha_o)\sin\beta_o\sin\phi_o \frac{e^{ik_2(R_o+nR_1)}}{P} \quad . \tag{2-77}$$

In the cylindrical coordinate system,

$$E_{tr} = k_2^2\mu_r T''(\alpha_o)\cos\alpha_o\cos\beta_o\cos\phi_o \frac{e^{ik_2(R_o+nR_1)}}{P} \quad ,$$

$$E_{t\phi} = -k_2^2 n T^+(\alpha_o)\sin\phi_o \frac{e^{ik_2(R_o+nR_1)}}{P} ,$$

$$E_{tz} = -k_2^2 \mu_r T''(\alpha_o)\cos\alpha_o \sin\beta_o \cos\phi_o \frac{e^{ik_2(R_o+nR_1)}}{P} ;$$

$$H_{tr} = \frac{k_2^3 n^2}{\omega\mu_1} T^+(\alpha_o)\cos\beta_o \sin\phi_o \frac{e^{ik_2(R_o+nR_1)}}{P} ,$$

$$H_{t\phi} = \frac{k_2^3 n}{\omega\mu_2} T''(\alpha_o)\cos\alpha_o \cos\phi_o \frac{e^{ik_2(R_o+nR_1)}}{P} ,$$

$$H_{tz} = -\frac{k_2^3 n^2}{\omega\mu_1} T^+(\alpha_o)\sin\beta_o \sin\phi_o \frac{e^{ik_2(R_o+nR_1)}}{P} . \qquad (2\text{-}78)$$

These results confirm that the radiation field is zero along the dipole axis.

2.6 Nonuniqueness of Hertz Potentials

In the classical horizontal dipole problem where the dipole is oriented in the x-direction of a rectangular coordinate system (x,y,z), and is located above or on an infinite plane interface (the xy-plane) of a finitely conducting earth, it is well known that the x-component of the Hertz potential, Π_x, alone cannot describe the electromagnetic field everywhere. This is because $\vec{E}$ and $\vec{H}$ derived from Π_x alone cannot satisfy the required boundary condition, namely, the continuity of the tangential components of $\vec{E}$ and $\vec{H}$. Sommerfeld [1949, p. 258] thus assumed a z-component, Π_z, in addition to Π_x, which then led to a set of boundary relations in terms of Π_x and Π_z that were consistent with the boundary conditions on $\vec{E}$ and $\vec{H}$. In view of his work, a question naturally arises whether Sommerfeld's $\vec{\Pi}$[1] is unique for the problem. It appears that most workers assume his choice of components for $\vec{\Pi}$ to be unique and the resolution $\vec{\Pi}=(\Pi_x, 0, \Pi_z)$ to be necessary. So does Brekhovskikh [1960, p. 259] in stating that: "It turns out that, in addition to Π_x, the reflected and refracted waves will also contain the component Π_z, since otherwise the four boundary conditions expressing the continuity of the field components, E_x, E_y, H_x and H_y across the interface could not be satisfied."

Thus assuming that $\vec{\Pi}_r=(\Pi_{rx}, 0, \Pi_{rz})$, Brekhovskikh [1960, p. 259] obtains a complete solution for Π_{rx} and Π_{rz} by using the saddle point method. The first terms of his expressions correspond to the usual geometrical optics results, which in the present notations become

$$\Pi_{rx} = R^+(\alpha_o)\frac{e^{ik_2 R_T}}{R_T} \quad ,$$

$$\Pi_{rz} = \{R^+(\alpha_o)+R''(\alpha_o)\}\cot\alpha_o\cos\phi_o \frac{e^{ik_2 R_T}}{R_T} \quad .^{2} \qquad (2\text{-}79)$$

Our Hertz potential, (2-64), shows for the same problem that $\vec{\Pi}_r=(\Pi_{rx}, \Pi_{ry}, 0)$ where

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[1] In Sommerfeld, the symbol $\vec{\Pi}$ denotes the total Hertz potential above the earth, but in this section $\vec{\Pi}$ will be used to mean the general term "the Hertz potential," whenever distinction between two Hertz potentials above and within the interface is not necessary.

[2] Brekhovskikh's equation (19.49) appears to be in error in that the azymuthal dependence factor $\cos\phi_o$ is missing. Multiplying (19.49) by $\cos\phi_o$ and rearranging gives Π_{rz} in (2-79).

$$\Pi_{rx} = \{R^+(\alpha_o)\sin^2\phi_o - R''(\alpha_o)\cos^2\phi_o\}\frac{e^{ik_2R_T}}{R_T} ,$$

$$\Pi_{ry} = -\{R^+(\alpha_o)+R''(\alpha_o)\}\sin\phi_o\cos\phi_o \frac{e^{ik_2R_T}}{R_T} . \quad (2\text{-}80)$$

It can be seen that (2-79) and (2-80) are two distinctively different Hertz potentials. We have already calculated $\vec{E}_r$ and $\vec{H}_r$ in (2-75) corresponding to $\vec{\Pi}_r$ of (2-80). If we do the same for $\vec{\Pi}_r$ of (2-79) using the same approximation (2-69), then it is found that the result is identical to (2-75). This indicates that there are at least two solutions for $\vec{\Pi}_r$ for this problem. In fact we will show in what follows that there exists an infinite number of solutions for the Hertz potential, the reflected as well as the transmitted, for the horizontal dipole problem. Furthermore it will be seen that the existence of infinite solutions for $\vec{\Pi}$ is not only true for the horizontal dipole problem but actually for any boundary value problem.

We first note that the requirement on $\vec{\Pi}_r$ is that $\vec{E}_r$ and $\vec{H}_r$ derived from $\vec{\Pi}_r$ satisfy the boundary condition and that $\vec{\Pi}_r$ be any solution of the homogeneous vector wave equation [Stratton, p. 430]

$$\nabla \times \nabla \times \vec{C} - \nabla(\nabla\cdot\vec{C}) - k_2^2\vec{C} = 0 . \quad (2\text{-}81)$$

Let us suppose that $\vec{\Pi}_r$ is a solution of (2-81), and let $\vec{E}_r$ and $\vec{H}_r$ be the corresponding electric and magnetic fields that satisfy the boundary condition. If we add to $\vec{\Pi}_r$ the gradient of any solution Φ of the homogeneous Helmholtz equation

$$\nabla^2\Psi + k_2^2\Psi = 0 \quad , \tag{2-82}$$

then $\vec{\Pi}_r+\nabla\Phi$ is also a solution of (2-81) but in view of the operations indicated in (2-12), $\vec{E}_r$ and $\vec{H}_r$ do not change, regardless of the magnitude and the particular form of Φ. The specific factor in $\vec{\Pi}_r$ must of course be determined according to the form of the incident wave and the boundary condition. However, the boundary condition is actually given in terms of $\vec{E}$ and $\vec{H}$ and not directly in terms of $\vec{\Pi}$; hence, insofar as the derived boundary relations in terms of the components of $\vec{\Pi}$ are compatible with each other, all such $\vec{\Pi}$'s correctly describe one and the same electromagnetic field. The nonuniqueness of the acceptable boundary relations in terms of $\vec{\Pi}$ can be verified immediately. The boundary relations for $\vec{\Pi}_2=(\Pi_{2x},\Pi_{2y},0)$ and $\vec{\Pi}_1=(\Pi_{1x},\Pi_{1y},0)$ become

$$\frac{\partial}{\partial x}(\nabla\cdot\vec{\Pi}_2-\nabla\cdot\vec{\Pi}_1) = k_1^2\Pi_{1x}-k_2^2\Pi_{2x} \quad , \quad k_2^2\,\frac{\partial\Pi_{2x}}{\partial z} = k_1^2\,\frac{\partial\Pi_{1x}}{\partial z} \quad ,$$

$$\frac{\partial}{\partial y}(\nabla\cdot\vec{\Pi}_2-\nabla\cdot\vec{\Pi}_1) = k_1^2\Pi_{1y}-k_2^2\Pi_{2y} \quad , \quad k_2^2\,\frac{\partial\Pi_{2y}}{\partial z} = k_1^2\,\frac{\partial\Pi_{1y}}{\partial z} \quad . \tag{2-83}$$

For comparison, we list the boundary relations for Sommerfeld's resolution, i.e., for $\vec{\Pi}_2=(\Pi_{2x},0,\Pi_{2z})$ and $\vec{\Pi}_1=(\Pi_{1x},0,\Pi_{1z})$:

$$k_2^2\Pi_{2x} = k_1^2\Pi_{1x} \quad , \quad k_2^2\,\frac{\partial\Pi_{2x}}{\partial z} = k_1^2\,\frac{\partial\Pi_{1x}}{\partial z} \quad ,$$

$$k_2^2\Pi_{2z} = k_1^2\Pi_{1z} \quad , \quad \frac{\partial\Pi_{2z}}{\partial z} - \frac{\partial\Pi_{1z}}{\partial z} = \frac{\partial\Pi_{1x}}{\partial z} - \frac{\partial\Pi_{2x}}{\partial z} \quad . \tag{2-84}$$

It can be seen that the four equations in (2-83) are mutually compatible just as are those in (2-84). Since (2-82) has an infinite number of solutions, so does (2-81).

For illustration, let $\vec{\Pi}_r=(\Pi_{rx},\Pi_{ry},0)$ with Π_{rx} and Π_{ry} given by (2-80) and choose

$$\Phi = \frac{\{R^{\perp}(\alpha_o)+R''(\alpha_o)\}\cos\phi_o}{ik_2\sin\alpha_o}\,\frac{e^{ik_2R_T}}{R_T} \quad . \tag{2-85}$$

Using the approximations (2-69), we can easily show that Φ is a solution of (2-82), and also that

$$\nabla\Phi = \left\{\left(R^{\perp}(\alpha_o)+R''(\alpha_o)\right)\cos^2\phi_o\,\frac{e^{ik_2R_T}}{R_T}\right. ,$$
$$\left(R^{\perp}(\alpha_o)+R''(\alpha_o)\right)\sin\phi_o\cos\phi_o\,\frac{e^{ik_2R_T}}{R_T} ,$$
$$\left.\left(R^{\perp}(\alpha_o)+R''(\alpha_o)\right)\cos\alpha_o\cos\phi_o\,\frac{e^{ik_2R_T}}{R_T}\right\} . \tag{2-86}$$

Adding (2-80) and (2-86) gives a new solution $\vec{\Pi}_r'$:

$$\vec{\Pi}_r' = \left\{R^{\perp}(\alpha_o)\frac{e^{ik_2R_T}}{R_T} \quad , \quad 0 \quad , \quad \{R^{\perp}(\alpha_o)+R''(\alpha_o)\}\cot\alpha_o\cos\phi_o\,\frac{e^{ik_2R_T}}{R_T}\right\} \tag{2-87}$$

which is none other than Brekhovskikh's reflected Hertz potential, (2-79). Thus in this illustration the particular choice of the factor for Φ in (2-85) was made in order to eliminate the y-component of (2-80) and obtain Brekhovskikh's expression. It may be observed that, by choosing the factor in $\vec{\Pi}$ properly, we can actually eliminate any one component of $\vec{\Pi}_r$, but not two components simultaneously, or leave all three components finite; hence there are all four possible resolutions for $\vec{\Pi}_r$.

$\vec{\Pi}_t$ also satisfies an equation of the same form as equation (2-81) in addition to the boundary condition on $\vec{E}_t$ and $\vec{H}_t$ derived from $\vec{\Pi}_t$. Therefore, by the same argument as in the above, it can be inferred that $\vec{\Pi}_t$ is also not unique.

Since both $\vec{\Pi}_r$ and $\vec{\Pi}_t$ are not unique independently, it can be further said that the resolution of $\vec{\Pi}_r$ and that of $\vec{\Pi}_t$ need not be the same in a problem; for example, in terms of total Hertz potentials we can let $\vec{\Pi}_2=(\Pi_{2x},0,\Pi_{2z})$ and $\vec{\Pi}_1=(\Pi_{1x},\Pi_{1y},0)$, if such a combination of different resolutions is desired. Needless to say, the derived boundary relations are compatible with each other.

It may be worthwhile in connection with the preceding discussions on the nonuniqueness of $\vec{\Pi}_r$ and $\vec{\Pi}_t$ to examine whether the incident Hertz potential can be given uniquely. $\vec{\Pi}_i$ satisfies the nonhomogeneous equation

$$\nabla \times \nabla \times \vec{C} - \nabla(\nabla\cdot\vec{C}) - k_2^2\vec{C} = \frac{i}{\omega\varepsilon_2}\vec{J} \quad . \tag{2-88}$$

We have used the particular solution of the above equation, given in (2-1), as our incident Hertz potential. From the mathematical standpoint, the complete solution includes also the general solution of the homogeneous equation. We have seen that (2-88) has nontrivial homogeneous solutions.

Therefore according to the Fredholm theory [Epstein, 1962], there exists an infinite number of solutions for (2-88). However, on the physical grounds, all the homogeneous solutions are interpreted to represent the contributions from the sources other than $\vec{J}$, and hence must be zero in the absence of any other sources than $\vec{J}$. Thus there is a unique choice for the incident potential, namely, that given by (2-1).

CHAPTER 3

INTEGRAL FORMULATION OF REFLECTED AND TRANSMITTED HERTZ POTENTIALS FOR AN INFINITE ROUGH INTERFACE

3.1 Vector Helmholtz Integral

As was shown in the preceding chapter, when the interface was a smooth plane, the spherical function of the incident Hertz potential was first expanded into a family of plane waves. Then the boundary condition on each of the plane waves in the integrand could be prescribed in a manner independent of the spatial coordinates. Thus spatial integrations were not necessary in the formulation of the reflected or the transmitted Hertz potential for a smooth plane boundary. However, this is no longer the case when the interface becomes rough and this roughness is to be considered. Obviously, spatial integrations now become necessary in addition to the integrations with respect to wave numbers in order to account for all contributions from each differential area of the interface with varying height and orientation.

For this purpose it is convenient to use the vector Helmholtz integrals given in terms of the Hertz potentials as the wave functions. The derivation of the vector Helmholtz integral is usually made by first deriving the scalar Helmholtz integral for each rectangular component of the wave function using Green's theorem. Each resulting component Helmholtz integral is then recombined to yield the vector Helmholtz integral. Alternatively, if we carry out this recombination of each rectangular component for Green's theorem instead of the Helmholtz integral we then obtain "vector Green's theorem." Namely, for twice differentiable functions, G and $\vec{C}$,

$$\int_V \{(\nabla^2 G)\vec{C} - (\nabla^2\vec{C})G\}dv = \int_S \vec{a}_{on}\cdot\{(\nabla G)\vec{C} - (\nabla\vec{C})G\}ds \tag{3-1}$$

where S is a closed surface whose volume is V and $\vec{a}_{on}$ is the outward unit normal of S. A more general derivation of (3-1) using some dyadic identities is given by Ezell, Erteza, Doran and Park [1968]. The following are the dyadic identities:

$$\nabla\cdot(\vec{A}\vec{B}) = (\nabla\cdot\vec{A})\vec{B} + \vec{A}\cdot\nabla\vec{B} \quad ,$$
$$\nabla\cdot(G\hat{A}) = \nabla G\cdot\hat{A} + G\nabla\cdot\hat{A} \quad , \tag{3-2}$$

where $\vec{A}$ and $\vec{B}$ are any two vectors, $\hat{A}$ is a dyadic, and G is a scalar function. In (3-2) let $\vec{A}=\nabla G$, $\vec{B}=\vec{C}$, and $\hat{A}=\nabla\vec{C}$, then (3-2) becomes

$$\nabla\cdot\{(\nabla G)\vec{C}\} = (\nabla\cdot\nabla G)\vec{C} + \nabla G\cdot\nabla\vec{C} \quad ,$$
$$\nabla\cdot(G\nabla\vec{C}) = \nabla G\cdot\nabla\vec{C} + G\nabla\cdot\nabla\vec{C} \quad . \tag{3-3}$$

Finally subtracting the top equation from the bottom equation and applying the divergence theorem immediately gives (3-1).

The vector Helmholtz integral follows from (3-1). To show this, suppose $\vec{C}$ satisfies the homogeneous vector Helmholtz equation

$$\nabla^2\vec{C} + k^2\vec{C} = -\vec{L} \tag{3-4}$$

and let G be the Green's function defined by

$$\nabla^2 G + k^2 G = -4\pi\delta(\vec{r}-\vec{r}\,') \qquad (3\text{-}5)$$

where $\delta(\vec{r}-\vec{r}\,')$ is the dirac delta function, $G(\vec{r},\vec{r}\,') = \frac{e^{ikR}}{R}$, and $R=|\vec{r}-\vec{r}\,'|$. Then just as in the scalar case, we can obtain from (3-1)

$$\vec{C}(\vec{r}\,') = \frac{1}{4\pi}\int_V \vec{L}(\vec{r})G(\vec{r},\vec{r}\,')dv + \frac{1}{4\pi}\int_S \vec{a}_{on}\cdot[\{\nabla\vec{C}(\vec{r})\}G(\vec{r},\vec{r}\,') - \{\nabla G(\vec{r},\vec{r}\,')\}\vec{C}(\vec{r})]ds \quad . \qquad (3\text{-}6)$$

If we let $\vec{L}\equiv 0$, i.e., if $\vec{C}$ satisfies the homogeneous Helmholtz equation

$$\nabla^2\vec{C} + k^2\vec{C} = 0 \qquad (3\text{-}7)$$

then the first term of (3-6) vanishes, leaving the vector Helmholtz integral

$$\vec{C} = \frac{1}{4\pi}\int_S \vec{a}_{on}\cdot\{(\nabla\vec{C})G - (\nabla G)\vec{C}\}ds \qquad (3\text{-}8)$$

where for conciseness we left out the arguments for the functions $\vec{C}$ and G.

The applicability of (3-8) for the purpose of evaluating $\vec{\Pi}_r$ and $\vec{\Pi}_t$ in the dipole problem is obvious, since they also satisfy the homogeneous equation (3-7):

$$\nabla^2\vec{\Pi}_r + k_2^2\vec{\Pi}_r = 0 \quad , \quad \nabla^2\vec{\Pi}_t + k_1^2\vec{\Pi}_t = 0 \quad . \qquad (3\text{-}9)$$

If we define the Green's functions in the two media by G_r and G_t so that

$$\nabla^2 G_r + k_2^2 G_r = -4\pi\delta(\vec{r}-\vec{r}') \quad , \quad \nabla^2 G_t + k_1^2 G_t = -4\pi\delta(\vec{r}-\vec{r}') \quad , \tag{3-10}$$

then, making proper replacement in (3-8), we obtain

$$\vec{\Pi}_r = \frac{1}{4\pi} \int \vec{a}_n \cdot \{(\nabla G_r)\vec{\Pi}_r - (\nabla\vec{\Pi}_r)G_r\} ds \quad ,$$

$$\vec{\Pi}_t = -\frac{1}{4\pi} \int \vec{a}_n \cdot \{(\nabla G_t)\vec{\Pi}_t - (\nabla\vec{\Pi}_t)G_t\} ds \quad , \tag{3-11}$$

where the minus sign in front of the integral for $\vec{\Pi}_t$ is because of our definition of $\vec{a}_n$, which is directed from the lower medium (medium 1) to the upper medium (medium 2).

3.2 Derivation of Integral Formulas for Reflected and Transmitted Hertz Potentials

We define the coordinate system for the case of the rough interface the same way as in the case of the smooth plane interface except that here the plane $z \equiv 0$ is chosen to coincide with the mean plane of the interface, the height of which from the mean plane is described by the function

$$z = \zeta(x,y) \quad . \tag{3-12}$$

The definition of the medium properties above and below the interface and the convention regarding the use of the subscripts are the same as in the previous chapter. To reduce the Helmholtz integrals given in (3-11) to the forms from which we can readily evaluate $\vec{\Pi}_r$ and $\vec{\Pi}_t$ at any point with specification of the interface function $\zeta(x,y)$, it is first necessary to evaluate $\vec{\Pi}_r$ and $\vec{\Pi}_t$ and their gradients at the interface. To do so, however, some form of approximation is necessary, since the interface, being irregularly rough, cannot be separated in any coordinate system. A simple and by far the most frequently used approximation is the tangent plane approximation. Here, we suppose that the incident plane wave at a point $P(x,y,\zeta)$ on the interface is reflected and transmitted as if the interface were a plane tangent to the true interface at P. Thus it can be seen that the tangent plane approximation requires the roughness of the interface to be locally gentle everywhere relative to the wavelength. Brekhovskikh [1952] shows the criterion for the applicability of this approximation in terms of the local radius of curvature as

$$4\pi r_c \cos\alpha >> \lambda \qquad (3\text{-}13)$$

where r_c is the smaller of the two principal radii of curvature and α is the local angle of incidence.

Assuming this is the case for the present interface, we again are required to expand the incident Hertz potential in terms of the plane waves. At $P(x,y,\zeta)$ on the interface, we obtain from (2-5)

$$\vec{\Pi}_i(P) = \frac{i}{2\pi} \iint_{-\infty}^{\infty} \frac{e^{ik_{2x}x+ik_{2y}y+ik_{2z}(h-\zeta)}}{k_{2z}} \vec{a}_\pi dk_{2x}dk_{2y} \quad . \qquad (3\text{-}14)$$

Since the phase function of the exponential does not change on reflection, we can write the reflected Hertz potential at P due to (3-14) as

$$\vec{\Pi}_r(P) = \frac{i}{2\pi} \iint_{-\infty}^{\infty} \frac{e^{ik_{2x}x+ik_{2y}y+ik_{2z}(h-\zeta)}}{k_{2z}} \hat{R}\cdot\vec{a}_\pi dk_{2x}dk_{2y} \qquad (3\text{-}15)$$

where $\hat{R}$ is the dyadic reflection coefficient. The assumption of dyadic nature for $\hat{R}$ is necessary because the reflection coefficients for each component of the incident plane wave are different and also because in general there are cross polarizations present on reflection. Similarly, assuming a dyadic transmission coefficient $\hat{T}$, the transmitted Hertz potential at P can be given by

$$\vec{\Pi}_t(P) = \frac{i}{2\pi} \iint_{-\infty}^{\infty} \frac{e^{ik_{2x}x+ik_{2y}y+ik_{2z}(h-\zeta)}}{k_{2z}} \hat{T}\cdot\vec{a}_\pi dk_{2x}dk_{2y} \quad . \qquad (3\text{-}16)$$

The components of $\hat{R}$ and $\hat{T}$ are functions of the Fresnel coefficients and parameters of the interface $\zeta(x,y)$. The explicit expressions for these will be derived in Section 3.3.

In order to evaluate (3-15) and (3-16) in the geometrical optics approximation, we assume that the source point is sufficiently far removed (compared with the wavelength) from the interface according to the requirement of far-zone observation. Thus the geometrical optics results for $\vec{\Pi}_r$ and $\vec{\Pi}_t$ become

$$\vec{\Pi}_r(P) = [\hat{R}]_{ev}\cdot\vec{a}_\pi \frac{e^{ik_2R_i}}{R_i} \quad ,$$

$$\vec{\Pi}_t(P) = [\hat{T}]_{ev}\cdot\vec{a}_\pi \frac{e^{ik_2R_i}}{R_i} \quad , \qquad (3\text{-}17)$$

where $R_i=\sqrt{x^2+y^2+(h-\zeta)^2}$, and $[\hat{R}]_{ev}$ and $[\hat{T}]_{ev}$ are $\hat{R}$ and $\hat{T}$ evaluated at the stationary point

$$\tilde{k}_{2x} = k_2 \frac{x}{R_i} \quad , \quad \tilde{k}_{2y} = k_2 \frac{y}{R_i} \quad , \quad \tilde{k}_{2z} = k_2 \frac{h-\zeta}{R_i} \quad . \tag{3-18}$$

$\tilde{k}_{2z}$ was obtained by definition $k_{2z}=\sqrt{k_2^2-k_{2x}^2-k_{2y}^2}$. See Figures 3-1 and 3-2 for related symbols and geometry.

For the evaluation of the gradients of $\vec{\Pi}_r$ and $\vec{\Pi}_t$ at the interface, we note that, in the integrands of (3-15) and (3-16), the factors $\hat{R}\cdot\vec{a}_\pi$ and $\hat{T}\cdot\vec{a}_\pi$ vary much more slowly compared to e^{iw}, where $w=k_{2x}x+k_{2y}y+k_{2z}(h-\zeta)$, along the interface, and hence we can approximately set

$$\nabla\vec{\Pi}_r(P) \approx \frac{i}{2\pi} \iint_{-\infty}^{\infty} (\nabla e^{iw}) \frac{\hat{R}\cdot\vec{a}_\pi}{k_{2z}}\, dk_{2x}dk_{2y} \quad ,$$

$$\nabla\vec{\Pi}_t(P) \approx \frac{i}{2\pi} \iint_{-\infty}^{\infty} (\nabla e^{iw}) \frac{\hat{T}\cdot\vec{a}_\pi}{k_{2z}}\, dk_{2x}dk_{2y} \quad . \tag{3-19}$$

On denoting the reflected and the transmitted wave vectors corresponding to the incident wave vector $\vec{k}_2=(k_{2x},k_{2y},k_{2z})$, by $\vec{k}_r$ and $\vec{k}_t$, respectively, it can be easily shown that

$$\nabla e^{iw} = \begin{cases} i\vec{k}_r e^{iw} & \text{for } \nabla\vec{\Pi}_r(P) \\ i\vec{k}_t e^{iw} & \text{for } \nabla\vec{\Pi}_t(P) \end{cases} \tag{3-20}$$

so that (3-19) becomes

$$\nabla\vec{\Pi}_r(P) = \frac{i}{2\pi} \iint_{-\infty}^{\infty} i\vec{k}_r \hat{R}\cdot\vec{a}_\pi \frac{e^{iw}}{k_{2z}}\, dk_{2x}dk_{2y} \quad ,$$

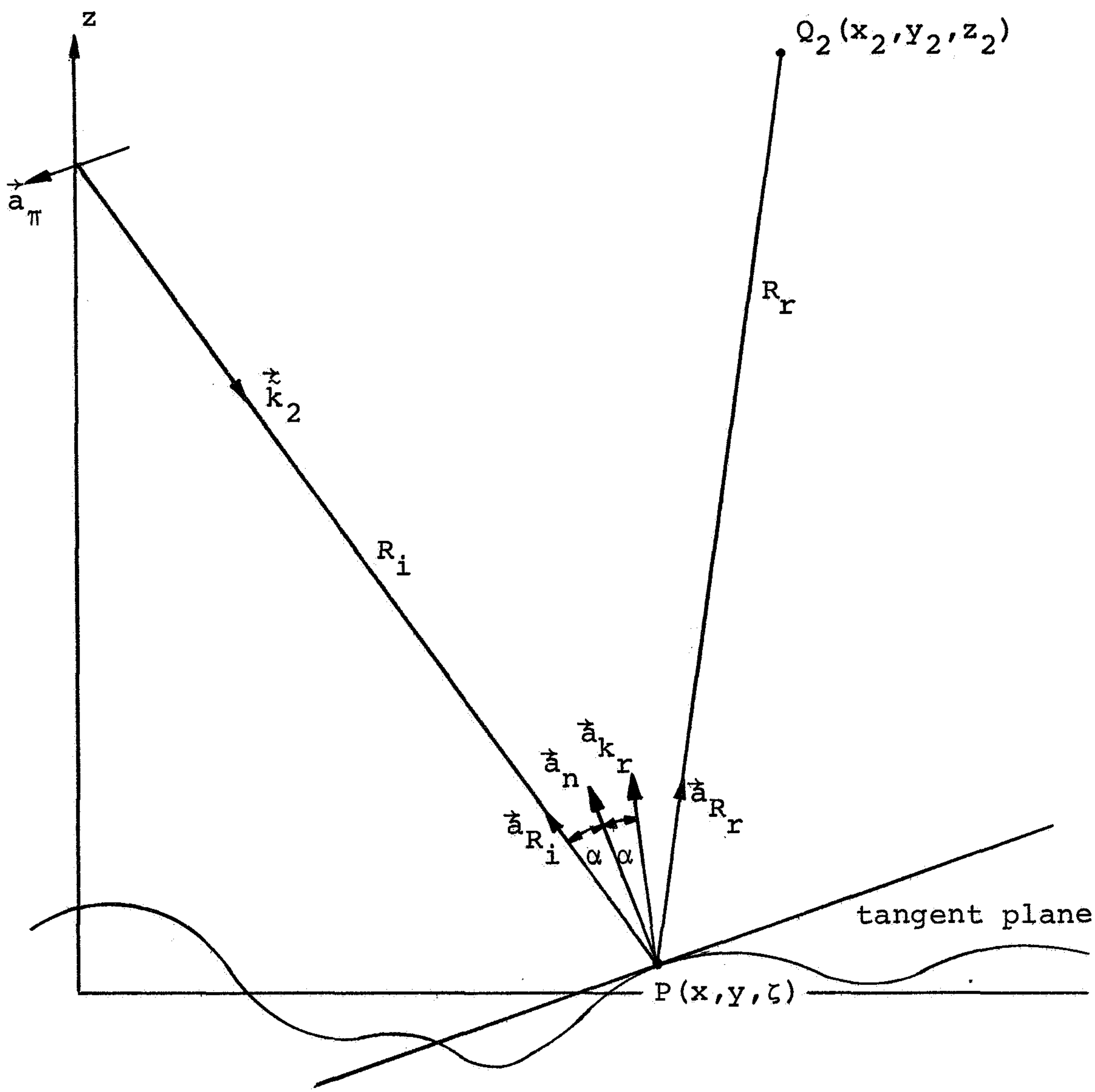

Figure 3-1. Symbols and Geometry for Reflection from a Rough Boundary

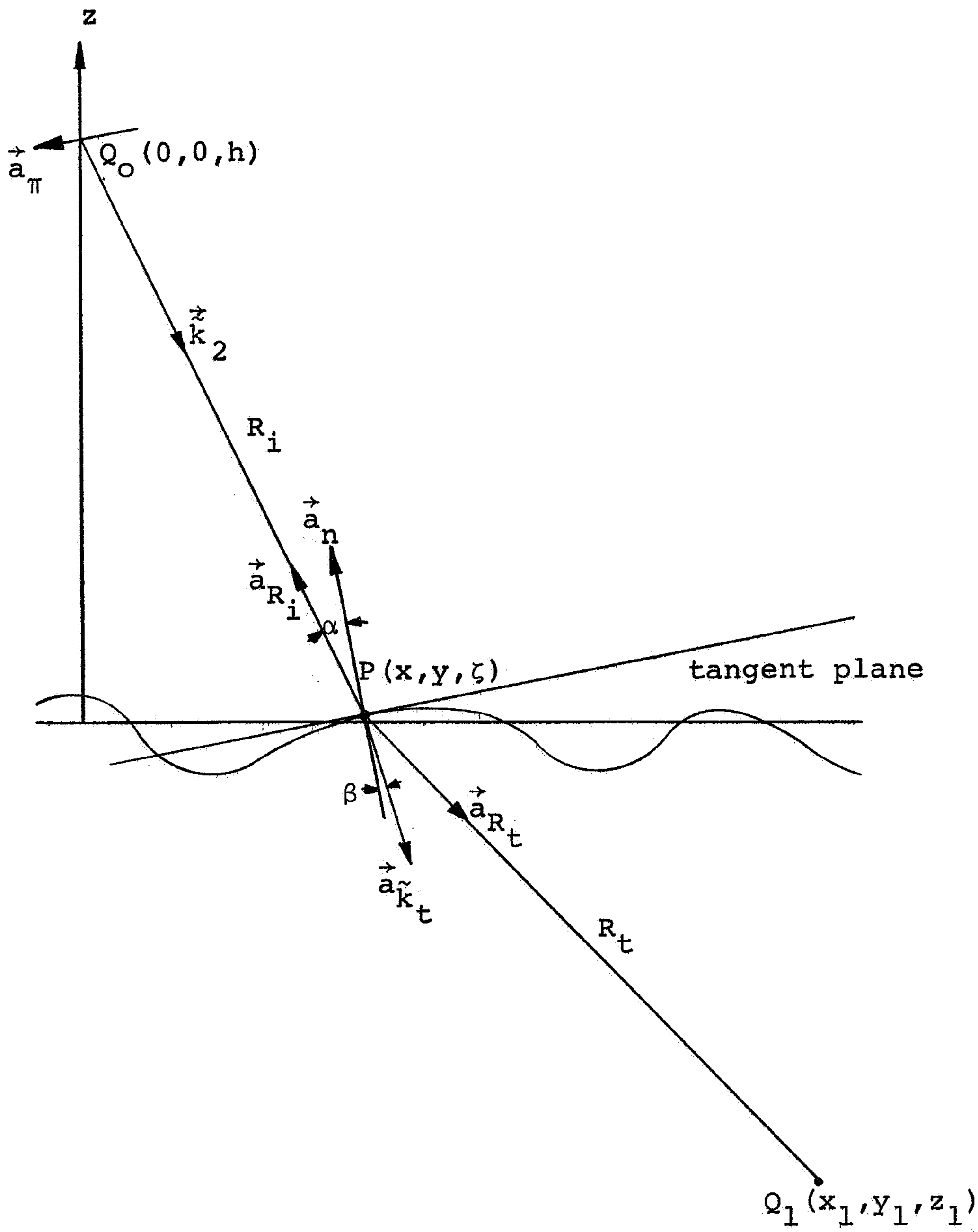

Figure 3-2. Symbols and Geometry for Transmission through a Rough Boundary

$$\nabla\vec{\Pi}_t(P) = \frac{i}{2\pi}\iint_{-\infty}^{\infty} i\vec{k}_t\hat{T}\cdot\vec{a}_\pi \frac{e^{iw}}{k_{2z}} dk_{2x}dk_{2y} \quad . \tag{3-21}$$

The results for the above integrals in the geometrical optics approximation can be simply written down, in analogy to (3-17):

$$\vec{\Pi}_r(P) = i[\vec{k}_r\hat{R}]_{ev}\cdot\vec{a}_\pi \frac{e^{ik_2R_i}}{R_i} \quad ,$$

$$\vec{\Pi}_t(P) = i[\vec{k}_t\hat{T}]_{ev}\cdot\vec{a}_\pi \frac{e^{ik_2R_i}}{R_i} \quad . \tag{3-22}$$

The Green's functions in (3-11) for the observation points of $Q_2(x_2,y_2,z_2)$ and $Q_1(x_1,y_1,z_1)$ are explicitly

$$G_r = \frac{e^{ik_2R_r}}{R_r} \quad ,$$

$$G_t = \frac{e^{ik_1R_t}}{R_t} \quad , \tag{3-23}$$

where $R_r=\sqrt{(x_2-x)^2+(y_2-y)^2+(z_2-\zeta)^2}$ and $R_t=\sqrt{(x_1-x)^2+(y_1-y)^2+(z_1-\zeta)^2}$. We have already assumed that the source point is sufficiently far removed from the interface. Now supposing that the observation point is also sufficiently far away from the interface, so that $k_2R_r>>1$ and $k_1R_t>>1$, we have approximately

$$\nabla G_r \approx ik_2 \frac{e^{ik_2R_r}}{R_r} \nabla R_r = -ik_2 \frac{e^{ik_2R_r}}{R_r} \vec{a}_{R_r} \quad ,$$

$$\nabla G_t \approx ik_1 \frac{e^{ik_1R_t}}{R_t} \nabla R_t = -ik_1 \frac{e^{ik_1R_t}}{R_t} \vec{a}_{R_t} \quad . \tag{3-24}$$

Inserting (3-17), (3-22), (3-23), and (3-24) into the corresponding equation of (3-11), we finally obtain the desired formulas,

$$\vec{\Pi}_r(Q_2) = \frac{k_2}{4\pi i} \int \vec{a}_n \cdot (\vec{a}_{R_r} + \vec{a}_{R_i}) [\hat{R}]_{ev} \cdot \vec{a}_\pi \frac{e^{ik_2 R_i}}{R_i} \frac{e^{ik_2 R_r}}{R_r} ds \quad ,$$

$$\vec{\Pi}_t(Q_1) = \frac{-k_1}{4\pi i} \int \vec{a}_n \cdot (\vec{a}_{R_t} + \vec{a}_{\tilde{k}_t}) [\hat{T}]_{ev} \cdot \vec{a}_\pi \frac{e^{ik_2 R_i}}{R_i} \frac{e^{ik_1 R_t}}{R_t} ds \quad , \quad (3\text{-}25)$$

where we have used that

$$\vec{a}_n \cdot [\vec{k}_r]_{ev} = k_2 \vec{a}_n \cdot \vec{a}_{\tilde{k}_r} = k_2 \vec{a}_n \cdot \vec{a}_{R_i} \quad , \quad \vec{a}_n \cdot [\vec{k}_t]_{ev} = k_1 \vec{a}_n \cdot \vec{a}_{\tilde{k}_t} \quad . \quad (3\text{-}26)$$

The integrals (3-25) have the familiar form of an integral in the physical optics, which is not surprising since (3-25) is a generalization of the physical optics integral for the case of a rough interface and an arbitrary medium property below the interface.

Making use of new symbols, we may write in place of (3-25)

$$\vec{\Pi}_r(Q_2) = \int \hat{\sigma}_r(Q_2,P) \cdot \vec{\Pi}_i(P) ds \quad ,$$

$$\vec{\Pi}_t(Q_1) = \int \hat{\sigma}_t(Q_1,P) \cdot \vec{\Pi}_i(P) ds \quad , \quad (3\text{-}27)$$

where the new dyadic quantities are

$$\hat{\sigma}_r(Q_2,P) = \frac{k_2}{4\pi i} \vec{a}_n \cdot (\vec{a}_{R_r} + \vec{a}_{R_i}) [\hat{R}]_{ev} \frac{e^{ik_2 R_r}}{R_r} \quad ,$$

$$\hat{\sigma}_t(Q_1,P) = \frac{-k_1}{4\pi i}\vec{a}_n\cdot(\vec{a}_{R_t}+\vec{a}_{\tilde{k}_t})[\hat{T}]_{ev}\frac{e^{ik_1R_t}}{R_t} \quad . \tag{3-28}$$

From (3-27), it is seen that, at each point on the interface, the contribution to the total reflected or transmitted Hertz potential is given by multiplying the incident Hertz potential by a functional, $\hat{\sigma}_r$ or $\hat{\sigma}_t$. This form for $\vec{\Pi}_r$ in terms of $\hat{\sigma}_r$ was first introduced by Erteza, et al. [1965] through a use of a different method. Although their $\hat{\sigma}_r$, which is called "the differential reflectivity," differs slightly from that given in (3-28), the agreement of their results after evaluating the integrals in the geometrical optics approximation can be easily noted. By the same reason for calling $\hat{\sigma}_r$ the differential reflectivity, $\hat{\sigma}_t$ may be called "the differential transmissivity."[1]

The evaluation of the integrals (3-25) can be again carried out by using the method of stationary phase. Since there are in general more than one stationary point for rough interfaces, we first calculate the contribution from each stationary point considering one stationary point at a time. The total reflected or transmitted Hertz potential can then be given by summing the contributions from each stationary point, provided that the total number of stationary points and their locations are known. For arbitrary

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[1]The terms "reflectivity" and "transmissivity" are sometimes used to refer to the ratio of energies of reflected or transmitted over incident. See, for example, Born and Wolf [1964, p.4]. These two uses should not be confused.

rough interfaces (including random rough interfaces), however, the number of stationary points as well as their respective locations are usually not known, or at least are extremely difficult to determine.

In Chapter 4, we will assume the rough interface to be a stationary random process and will be able to approximately evaluate the integrals without resorting to knowledge of individual stationary points.

3.3 Calculation of Dyadic Reflection and Transmission Coefficients

In order to derive explicit expressions for each component of dyadic quantities $\hat{R}$ and $\hat{T}$ in the primary coordinate system (x,y,z), it will prove useful to use the matrix notation.

Again considering a point $P(x,y,\zeta)$ on the interface, we define an auxiliary coordinate system, namely, a local rectangular coordinate system $(\vec{a}_t,\vec{a}_p,\vec{a}_n)$ given by

$$\vec{a}_t = \frac{\vec{k}_2 \times \vec{a}_n}{|\vec{k}_2 \times \vec{a}_n|} \quad , \quad \vec{a}_p = \vec{a}_n \times \vec{a}_t \tag{3-39}$$

where $\vec{a}_n$ is the unit normal to the interface as defined earlier. The new coordinate system is illustrated in Figure 3-3. Then $\vec{a}_n$ is also normal to the tangent plane drawn at P, and both $\vec{a}_t$ and $\vec{a}_p$ lie on the tangent plane. Thus if we decompose the incident Hertzian plane wave at P into components in the $(\vec{a}_t,\vec{a}_p,\vec{a}_n)$ system, by (2-27) we can write

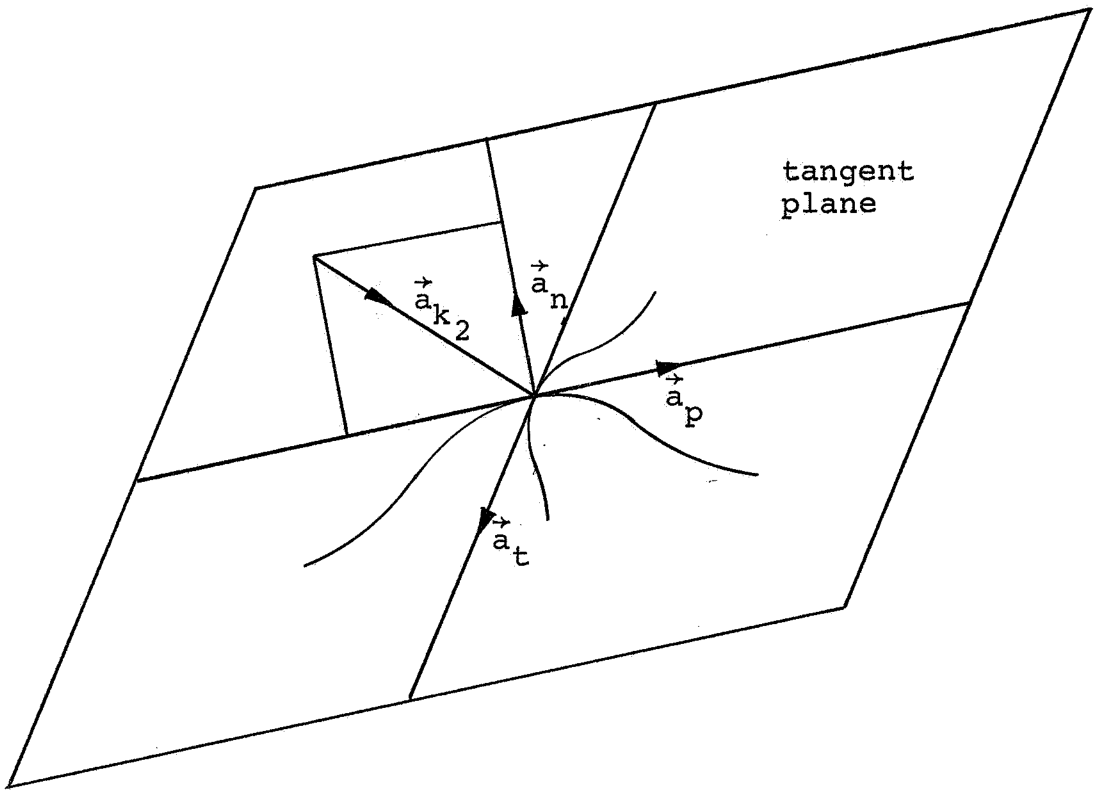

Figure 3-3. Local Coordinate System $(\vec{a}_t, \vec{a}_p, \vec{a}_n)$

$$\begin{pmatrix}\pi_{rt}\\ \pi_{rp}\\ \pi_{rn}\end{pmatrix} = (R_o)\begin{pmatrix}\pi_{it}\\ \pi_{ip}\\ \pi_{in}\end{pmatrix} ,$$

$$\begin{pmatrix}\pi_{tt}\\ \pi_{tp}\\ \pi_{tn}\end{pmatrix} = (T_o)\begin{pmatrix}\pi_{it}\\ \pi_{ip}\\ \pi_{in}\end{pmatrix} , \tag{3-30}$$

where (R_o) and (T_o) are diagonal matrices

$$(R_o) = \begin{pmatrix}R^{\perp} & 0 & 0\\ 0 & -R'' & 0\\ 0 & 0 & R''\end{pmatrix} ,$$

$$(T_o) = \frac{1}{n^2}\begin{pmatrix}T^{\perp} & 0 & 0\\ 0 & \frac{\mu_r k_{2n}}{k_{1n}} T'' & 0\\ 0 & 0 & \mu_r T''\end{pmatrix} . \tag{3-31}$$

If we denote the transformation matrix from the (x,y,z) system to the $(\vec{a}_t,\vec{a}_p,\vec{a}_n)$ system by (A) and that in the opposite direction by $(A)^{-1}$, we have, considering only the reflected waves for the time being,

$$\begin{pmatrix}\pi_{it}\\ \pi_{ip}\\ \pi_{in}\end{pmatrix} = (A)\begin{pmatrix}\pi_{ix}\\ \pi_{iy}\\ \pi_{iz}\end{pmatrix} ,$$

$$\begin{pmatrix}\pi_{rx}\\ \pi_{ry}\\ \pi_{rz}\end{pmatrix} = (A)^{-1}\begin{pmatrix}\pi_{rt}\\ \pi_{rp}\\ \pi_{rn}\end{pmatrix} , \tag{3-32}$$

where it can be easily shown that

$$(A) = \begin{pmatrix} \vec{a}_t\cdot\vec{a}_x & \vec{a}_t\cdot\vec{a}_y & \vec{a}_t\cdot\vec{a}_z \\ \vec{a}_p\cdot\vec{a}_x & \vec{a}_p\cdot\vec{a}_y & \vec{a}_p\cdot\vec{a}_z \\ \vec{a}_n\cdot\vec{a}_x & \vec{a}_n\cdot\vec{a}_y & \vec{a}_n\cdot\vec{a}_z \end{pmatrix} ,$$

$$(A)^{-1} = \begin{pmatrix} \vec{a}_t\cdot\vec{a}_x & \vec{a}_p\cdot\vec{a}_x & \vec{a}_n\cdot\vec{a}_x \\ \vec{a}_t\cdot\vec{a}_y & \vec{a}_p\cdot\vec{a}_y & \vec{a}_n\cdot\vec{a}_y \\ \vec{a}_t\cdot\vec{a}_z & \vec{a}_p\cdot\vec{a}_z & \vec{a}_n\cdot\vec{a}_z \end{pmatrix} . \tag{3-33}$$

Finally, inserting (3-32) into the top equation of (3-30) yields

$$\begin{pmatrix} \pi_{rx} \\ \pi_{ry} \\ \pi_{rz} \end{pmatrix} = (R) \begin{pmatrix} \pi_{ix} \\ \pi_{iy} \\ \pi_{iz} \end{pmatrix} \tag{3-34}$$

with the definition of $(R)=(A)^{-1}(R_o)(A)$. Defining the components of (R) by

$$(R) = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} , \tag{3-35}$$

we can obtain expressions for each component of (R) by carrying out multiplication of three matrices, $(A)^{-1}(R_o)(A)$, in a straightforward manner. If we do, it is found that

$$r_{11} = R^{+}(\vec{a}_t\cdot\vec{a}_x)^2 - R''(\vec{a}_p\cdot\vec{a}_x)^2 + R''(\vec{a}_n\cdot\vec{a}_x)^2$$

$$r_{12} = R^{+}(\vec{a}_t\cdot\vec{a}_x)(\vec{a}_t\cdot\vec{a}_y) - R''(\vec{a}_p\cdot\vec{a}_x)(\vec{a}_p\cdot\vec{a}_y) + R''(\vec{a}_n\cdot\vec{a}_x)(\vec{a}_n\cdot\vec{a}_y)$$

$$r_{13} = R^{+}(\vec{a}_t\cdot\vec{a}_x)(\vec{a}_t\cdot\vec{a}_z) - R''(\vec{a}_p\cdot\vec{a}_x)(\vec{a}_p\cdot\vec{a}_z) + R''(\vec{a}_n\cdot\vec{a}_x)(\vec{a}_n\cdot\vec{a}_z)$$

$$r_{21} = R^{+}(\vec{a}_t\cdot\vec{a}_y)(\vec{a}_t\cdot\vec{a}_x) - R''(\vec{a}_p\cdot\vec{a}_y)(\vec{a}_p\cdot\vec{a}_x) + R''(\vec{a}_n\cdot\vec{a}_y)(\vec{a}_n\cdot\vec{a}_x)$$

$$r_{22} = R^{\perp}(\vec{a}_t\cdot\vec{a}_y)^2 - R''(\vec{a}_p\cdot\vec{a}_y)^2 + R''(\vec{a}_n\cdot\vec{a}_y)^2$$

$$r_{23} = R^{\perp}(\vec{a}_t\cdot\vec{a}_y)(\vec{a}_t\cdot\vec{a}_z) - R''(\vec{a}_p\cdot\vec{a}_y)(\vec{a}_p\cdot\vec{a}_z) + R''(\vec{a}_n\cdot\vec{a}_y)(\vec{a}_n\cdot\vec{a}_z)$$

$$r_{31} = R^{\perp}(\vec{a}_t\cdot\vec{a}_z)(\vec{a}_t\cdot\vec{a}_x) - R''(\vec{a}_p\cdot\vec{a}_z)(\vec{a}_p\cdot\vec{a}_x) + R''(\vec{a}_n\cdot\vec{a}_z)(\vec{a}_n\cdot\vec{a}_x)$$

$$r_{32} = R^{\perp}(\vec{a}_t\cdot\vec{a}_z)(\vec{a}_t\cdot\vec{a}_y) - R''(\vec{a}_p\cdot\vec{a}_z)(\vec{a}_p\cdot\vec{a}_y) + R''(\vec{a}_n\cdot\vec{a}_z)(\vec{a}_n\cdot\vec{a}_y)$$

$$r_{33} = R^{\perp}(\vec{a}_t\cdot\vec{a}_z)^2 - R''(\vec{a}_p\cdot\vec{a}_z)^2 + R''(\vec{a}_n\cdot\vec{a}_z)^2 \quad . \tag{3-36}$$

We have now derived the reflection coefficient matrix (R) for the plane Hertzian waves in terms of the Fresnel reflection coefficients and the unit vectors of the primary and the auxiliary coordinate systems. The presence of finite nondiagonal terms in (R) indicates the existence of cross polarizations on reflection. Furthermore, we note that

$$r_{12} = r_{21} \quad , \quad r_{13} = r_{31} \quad , \quad r_{23} = r_{32} \tag{3-37}$$

which states that the effect of cross polarization from one component to another is the same as in the opposite case.

To further develop (3-36) to more explicit expressions, it is now necessary to write the local unit vectors, $\vec{a}_t$, $\vec{a}_p$ and $\vec{a}_n$ in terms of surface slopes and the incident wave vector. It is not difficult to show that $\vec{a}_n$ can be given by

$$\vec{a}_n = \frac{1}{J_n}(-\zeta_x, -\zeta_y, 1) \quad , \tag{3-38}$$

where ζ_x and ζ_y are partial derivatives of ζ with respect to x and y, respectively, and $J_n = \sqrt{\zeta_x^2 + \zeta_y^2 + 1}$. Following the definition and remembering that $k_2 = (k_{2x}, k_{2y}, k_{2z})$, we get

$$\vec{a}_t = \frac{1}{J_t} (k_{2z}\zeta_y+k_{2y}, -k_{2z}\zeta_x-k_{2x}, k_{2y}\zeta_x-k_{2x}\zeta_y) \quad , \tag{3-39}$$

where $J_t=\sqrt{(k_{2z}\zeta_y+k_{2y})^2+(k_{2z}\zeta_x+k_{2x})^2+(k_{2y}\zeta_x-k_{2x}\zeta_y)^2}$. The remaining unit vector $\vec{a}_p$ is found to be

$$\begin{aligned}\vec{a}_p = \frac{1}{J_p} (&k_{2x}+k_{2z}\zeta_x-k_{2y}\zeta_x\zeta_y+k_{2x}\zeta_y^2 \,, \\ &k_{2y}+k_{2z}\zeta_y-k_{2x}\zeta_x\zeta_y+k_{2y}\zeta_x^2 \,, \\ &k_{2x}\zeta_x+k_{2y}\zeta_y+k_{2z}\zeta_x^2+k_{2z}\zeta_y^2) \quad ,\end{aligned} \tag{3-40}$$

where $J_p=J_nJ_t$. Thus, on taking the indicated dot products, (3-36) becomes

$$r_{11} = R^{+}\left(\frac{k_{2z}\zeta_y+k_{2y}}{J_t}\right)^2 - R''\left(\frac{k_{2x}+k_{2z}\zeta_x-k_{2y}\zeta_x\zeta_y+k_{2x}\zeta_y^2}{J_p}\right) + R''\left(\frac{\zeta_x}{J_n}\right)^2$$

$$\begin{aligned}r_{12} = &-R^{+}\left(\frac{k_{2z}\zeta_y+k_{2y}}{J_t}\right)\left(\frac{k_{2z}\zeta_x+k_{2x}}{J_t}\right) \\ &-R''\left(\frac{k_{2x}+k_{2z}\zeta_x-k_{2y}\zeta_x\zeta_y+k_{2x}\zeta_y^2}{J_p}\right)\left(\frac{k_{2y}+k_{2z}\zeta_y-k_{2x}\zeta_x\zeta_y+k_{2y}\zeta_x^2}{J_p}\right) \\ &+R''\left(\frac{\zeta_x}{J_n}\right)\left(\frac{\zeta_y}{J_n}\right)\end{aligned}$$

$$r_{21} = r_{12}$$

$$r_{22} = R^{+}\left(\frac{k_{2z}\zeta_x+k_{2x}}{J_t}\right)^2 - R''\left(\frac{k_{2y}+k_{2z}\zeta_y-k_{2x}\zeta_x\zeta_y+k_{2y}\zeta_x^2}{J_p}\right) + R''\left(\frac{\zeta_y}{J_n}\right)^2$$

$$\begin{aligned}r_{13} = &-R^{+}\left(\frac{k_{2z}\zeta_y+k_{2y}}{J_t}\right)\left(\frac{-k_{2y}\zeta_x+k_{2x}\zeta_y}{J_t}\right) - R''\left(\frac{k_{2x}+k_{2z}\zeta_x-k_{2y}\zeta_x\zeta_y+k_{2x}\zeta_y^2}{J_p}\right) \\ &\cdot\left(\frac{k_{2x}\zeta_x+k_{2y}\zeta_y+k_{2z}\zeta_x^2+k_{2z}\zeta_y^2}{J_p}\right) - R''\left(\frac{\zeta_x}{J_n}\right)\left(\frac{1}{J_n}\right)\end{aligned}$$

$$r_{31} = r_{13}$$

$$r_{23} = R^{+}\left(\frac{k_{2z}\zeta_x+k_{2x}}{J_t}\right)\left(\frac{-k_{2y}\zeta_x+k_{2x}\zeta_y}{J_t}\right)$$

$$-R''\left(\frac{k_{2y}+k_{2z}\zeta_y-k_{2x}\zeta_x\zeta_y+k_{2y}\zeta_x^2}{J_p}\right)\left(\frac{k_{2x}\zeta_x+k_{2y}\zeta_y+k_{2z}\zeta_x^2+k_{2z}\zeta_y^2}{J_p}\right)$$

$$-R''\left(\frac{\zeta_y}{J_n}\right)\left(\frac{1}{J_n}\right)$$

$$r_{32} = r_{23}$$

$$r_{33} = R^{+}\left(\frac{-k_{2y}\zeta_x+k_{2x}\zeta_y}{J_t}\right)^2-R''\left(\frac{k_{2x}\zeta_x+k_{2y}\zeta_y+k_{2z}\zeta_x^2+k_{2z}\zeta_y^2}{J_p}\right)+R''\left(\frac{1}{J_n}\right)^2 \quad . \tag{3-41}$$

Now turning to the transmitted waves, we define the components of (T) by

$$(T) = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \quad , \tag{3-42}$$

where $(T)=(A)^{-1}(T_o)(A)$. Then each component of (T) can be given simply by replacing $(R^{+},-R'',R'')$ in the corresponding component of (3-41) with $(\frac{T^{+}}{n^2}, \frac{\mu_r k_{2n}}{n^2 k_{1n}} T'', \frac{\mu_r}{n^2} T'')$. But the factor $\frac{k_{2n}}{k_{1n}}$ in the middle term can be transformed first. Since

$$\frac{k_{2n}}{k_{1n}} = \frac{\vec{k}_2\cdot\vec{a}_n}{\vec{k}_1\cdot\vec{a}_n} \quad , \quad \vec{k}_1\cdot\vec{a}_n = \sqrt{k_1^2-(\vec{k}_1\cdot\vec{a}_p)^2} = \sqrt{k_1^2-(\vec{k}_2\cdot\vec{a}_p)^2} \tag{3-43}$$

we have

$$\frac{k_{2n}}{k_{1n}} = \frac{\vec{k}_2 \cdot \vec{a}_n}{\sqrt{k_1^2 - (\vec{k}_2 \cdot \vec{a}_p)^2}} \tag{3-44}$$

which, on noting that $\vec{k}_2 \cdot \vec{a}_p = |\vec{k}_2 \times \vec{a}_n| = J_t$, finally becomes

$$\frac{k_{2n}}{k_{1n}} = \frac{-k_{2x}\zeta_x - k_{2y}\zeta_y + k_{2z}}{J_n\sqrt{k_1^2 - J_t^2}} \quad . \tag{3-45}$$

Thus we have

$$t_{11} = \frac{T^{\perp}}{n^2}\left(\frac{k_{2y}+k_{2z}\zeta_y}{J_t}\right)^2 + \frac{\mu_r}{n^2}\,T''\left(\frac{-k_{2x}\zeta_x - k_{2y}\zeta_y + k_{2z}}{J_n\sqrt{k_1^2-J_t^2}}\right)$$

$$\cdot\left(\frac{k_{2x}+k_{2z}\zeta_x - k_{2y}\zeta_x\zeta_y + k_{2x}\zeta_y^2}{J_p}\right)^2 + \frac{\mu_r}{n^2}\,T''\left(\frac{\zeta_x}{J_n}\right)^2$$

$$t_{12} = -\frac{T^{+}}{n^2}\left(\frac{k_{2y}+k_{2z}\zeta_y}{J_t}\right)\left(\frac{k_{2x}+k_{2z}\zeta_x}{J_t}\right)$$

$$+ \frac{\mu_r}{n^2}\,T''\left(\frac{-k_{2x}\zeta_x - k_{2y}\zeta_y + k_{2z}}{J_n\sqrt{k_1^2-J_t^2}}\right)\left(\frac{k_{2x}+k_{2z}\zeta_x - k_{2y}\zeta_x\zeta_y + k_{2x}\zeta_y^2}{J_p}\right)$$

$$\cdot\left(\frac{k_{2y}+k_{2z}\zeta_y - k_{2x}\zeta_x\zeta_y + k_{2y}\zeta_x^2}{J_p}\right) + \frac{\mu_r}{n^2}\,T''\left(\frac{\zeta_x}{J_n}\right)\left(\frac{\zeta_y}{J_n}\right)$$

$$t_{21} = t_{12}$$

$$t_{22} = \frac{T^{+}}{n^2}\left(\frac{k_{2x}+k_{2z}\zeta_x}{J_t}\right)^2 + \frac{\mu_r}{n^2}\,T''\left(\frac{-k_{2x}\zeta_x - k_{2y}\zeta_y + k_{2z}}{J_n\sqrt{k_1^2-J_t^2}}\right)$$

$$\cdot\left(\frac{k_{2y}+k_{2z}\zeta_y - k_{2x}\zeta_x\zeta_y + k_{2y}\zeta_x^2}{J_p}\right)^2 + \frac{\mu_r}{n^2}\,T''\left(\frac{\zeta_y}{J_n}\right)^2$$

$$t_{13} = -\frac{T^+}{n^2}\left(\frac{k_{2y}+k_{2z}\zeta_y}{J_t}\right)\left(\frac{-k_{2y}\zeta_x+k_{2x}\zeta_y}{J_t}\right)$$

$$+\frac{\mu_r}{n^2}T''\left(\frac{-k_{2x}\zeta_x-k_{2y}\zeta_y+k_{2z}}{J_n\sqrt{k_1^2-J_t^2}}\right)\left(\frac{k_{2x}+k_{2z}\zeta_x-k_{2y}\zeta_x\zeta_y+k_{2x}\zeta_y^2}{J_p}\right)$$

$$\cdot\left(\frac{k_{2x}\zeta_x+k_{2y}\zeta_y+k_{2z}\zeta_x^2+k_{2z}\zeta_y^2}{J_p}\right)-\frac{\mu_r}{n^2}T''\left(\frac{\zeta_x}{J_n}\right)\left(\frac{1}{J_n}\right)$$

$$t_{31} = t_{13}$$

$$t_{23} = \frac{T^+}{n^2}\left(\frac{k_{2x}+k_{2z}\zeta_x}{J_t}\right)\left(\frac{-k_{2y}\zeta_x+k_{2x}\zeta_y}{J_t}\right)+\frac{\mu_r}{n^2}T''\left(\frac{-k_{2x}\zeta_x-k_{2y}\zeta_y+k_{2z}}{J_n\sqrt{k_1^2-J_t^2}}\right)$$

$$\cdot\left(\frac{k_{2y}+k_{2z}\zeta_y-k_{2x}\zeta_x\zeta_y+k_{2y}\zeta_x^2}{J_p}\right)\left(\frac{k_{2x}\zeta_x+k_{2y}\zeta_y+k_{2z}\zeta_x^2+k_{2z}\zeta_y^2}{J_p}\right)$$

$$-\frac{\mu_r}{n^2}T''\left(\frac{\zeta_y}{J_n}\right)\left(\frac{1}{J_n}\right)$$

$$t_{32} = t_{23}$$

$$t_{33} = \frac{T^+}{n^2}\left(\frac{-k_{2y}\zeta_x+k_{2x}\zeta_y}{J_t}\right)^2+\frac{\mu_r}{n^2}T''\left(\frac{-k_{2x}\zeta_x-k_{2y}\zeta_y+k_{2z}}{J_n\sqrt{k_1^2-J_t^2}}\right)$$

$$\cdot\left(\frac{k_{2x}\zeta_x+k_{2y}\zeta_y+k_{2z}\zeta_x^2+k_{2z}\zeta_y^2}{J_p}\right)^2+\frac{\mu_r}{n^2}T''\left(\frac{1}{J_n}\right)^2 \quad . \qquad (3\text{-}46)$$

Aside from the tangent plane approximation, expressions for each component of (R) and (T) given in (3-41) and (3-46) are exact, and are seen to be extremely complicated functions of ζ_x and ζ_y, since the Fresnel coefficients are also dependent on these slopes. In order that the integrals, in which ζ_x and ζ_y are variables, be manageable, it is necessary to simplify these expressions with respect to ζ_x and ζ_y. To do so, we assume that the interface is slightly rough, and slopes everywhere are so gentle that we may neglect all the high order terms and retain only up to the first order terms in ζ_x and/or ζ_y. A rough computation shows that slopes of up to 20 degrees from the horizontal plane can be considered for this class of interfaces. Then we can set

$$J_n \approx 1 \quad ,$$

$$J_t \approx k_{2r} + \frac{k_{2z}}{k_{2r}}(k_{2x}\zeta_x + k_{2y}\zeta_y) \quad \text{where } k_{2r}^2 = k_{2x}^2 + k_{2y}^2 \quad ,$$

$$J_p \approx J_t \quad . \tag{3-47}$$

Next we expand the Fresnel coefficients up to the first order using (3-47). Since the Fresnel coefficients with respect to the tangent plane can be written as

$$R^+ = \frac{\mu_r k_{2n} - k_{1n}}{\mu_r k_{2n} + k_{1n}} \quad ,$$

$$T^+ = \frac{2\mu_r k_{2n}}{\mu_r k_{2n} + k_{1n}} \quad ,$$

$$R'' = \frac{n^2 k_{1n} - \mu_r k_{2n}}{n^2 k_{1n} + \mu_r k_{2n}} \quad ,$$

$$T'' = \frac{2n^2 k_{1n}}{n^2 k_{1n} + \mu_r k_{2n}} \quad , \tag{3-48}$$

on inserting k_{1n} and k_{2n} given by (3-45) and expressing in the power series up to the first order (the details are given in Appendix B), we obtain

$$R^+ \approx \frac{\mu_r k_{2z}-k_a}{\mu_r k_{2z}+k_a} - \frac{2\mu_r k_2^2(n^2-1)}{k_a(\mu_r k_{2z}+k_a)^2}(k_{2x}\zeta_x+k_{2y}\zeta_y) \quad ,$$

$$T^+ \approx \frac{2\mu_r k_{2z}}{\mu_r k_{2z}+k_a} - \frac{2\mu_r k_2^2(n^2-1)}{k_a(\mu_r k_{2z}+k_a)^2}(k_{2x}\zeta_x+k_{2y}\zeta_y) \quad ,$$

$$R'' \approx \frac{n^2k_a-\mu_r k_{2z}}{n^2k_a+\mu_r k_{2z}} + \frac{2\mu_r k_1^2(n^2-1)}{k_a(n^2k_a+\mu_r k_{2z})^2}(k_{2x}\zeta_x+k_{2y}\zeta_y) \quad ,$$

$$T'' \approx \frac{2n^2k_a}{n^2k_a+\mu_r k_{2z}} + \frac{2\mu_r k_1^2(n^2-1)}{k_a(n^2k_a+\mu_r k_{2z})^2}(k_{2x}\zeta_x+k_{2y}\zeta_y) \quad , \qquad (3\text{-}49)$$

where we have denoted $\sqrt{k_1^2-k_{2r}^2} = k_a$. For further convenience, we write each of the above expressions in the form

$$R^+ \approx R_o^+ + \frac{R_1^+}{k_a}(k_{2x}\zeta_x+k_{2y}\zeta_y) \quad , \text{ etc.} \quad , \qquad (3\text{-}50)$$

where

$$R_o^+ = \frac{\mu_r k_{2z}-k_a}{\mu_r k_{2z}+k_a} \quad , \quad R_1^+ = -\frac{2\mu_r k_2^2(n^2-1)}{(\mu_r k_{2z}+k_a)^2} \quad , \text{ etc.} \qquad (3\text{-}51)$$

Incorporating (3-50) and continuing application of the first order approximation then give (Appendix C)

$$r_{11} \approx \frac{1}{k_{2r}^2}\{R_o^+k_{2y}^2-R_o''k_{2x}^2+\frac{1}{k_a}(k_{2y}^2R_1^+-k_{2x}^2R_1'')(k_{2x}\zeta_x+k_{2y}\zeta_y)$$

$$-\frac{2k_{2z}}{k_{2r}^2}(k_{2y}^2R_o^+-k_{2x}^2R_o'')(k_{2x}\zeta_x+k_{2y}\zeta_y)+2k_{2z}(R_o^+k_{2y}\zeta_y-R_o''k_{2x}\zeta_x)\}$$

$$r_{12} \approx \frac{-1}{k_{2r}^2}\{k_{2x}k_{2y}(R_o^{\perp}+R_o'') + \frac{k_{2x}k_{2y}}{k_a}(R_1^{\perp}+R_1'')(k_{2x}\zeta_x+k_{2y}\zeta_y)$$

$$- \frac{2k_{2x}k_{2y}k_{2z}}{k_{2r}^2}(R_o^{\perp}+R_o'')(k_{2x}\zeta_x+k_{2y}\zeta_y)+k_{2z}(R_o^{\perp}+R_o'')(k_{2y}\zeta_x+k_{2x}\zeta_y)\}$$

$$r_{21} = r_{12}$$

$$r_{22} \approx \frac{1}{k_{2r}^2}\{R_o^{\perp}k_{2x}^2-R_o''k_{2y}^2+\frac{1}{k_a}(k_{2x}^2R_1^{\perp}-k_{2y}^2R_1'')(k_{2x}\zeta_x+k_{2y}\zeta_y)$$

$$- \frac{2k_{2z}}{k_{2r}^2}(k_{2x}^2R_o^{\perp}-k_{2y}^2R_o'')(k_{2x}\zeta_x+k_{2y}\zeta_y)+2k_{2z}(R_o^{\perp}k_{2x}\zeta_x-R_o''k_{2y}\zeta_y)\}$$

$$r_{13} \approx \frac{1}{k_{2r}^2}\{-R_o^{\perp}k_{2y}(-k_{2y}\zeta_x+k_{2x}\zeta_y)-R_o''k_{2x}(k_{2x}\zeta_x+k_{2y}\zeta_y)\}-R_o''\zeta_x$$

$$r_{31} = r_{13}$$

$$r_{23} \approx \frac{1}{k_{2r}^2}\{R_o^{\perp}k_{2x}(-k_{2y}\zeta_x+k_{2x}\zeta_y)-R_o''k_{2y}(k_{2x}\zeta_x+k_{2y}\zeta_y)\}-R_o''\zeta_y$$

$$r_{32} = r_{23}$$

$$r_{33} \approx R_o''+\frac{R_1''}{k_a}(k_{2x}\zeta_x+k_{2y}\zeta_y) \qquad (3\text{-}52)$$

and

$$t_{11} \approx \frac{1}{n^2k_{2r}^2}[T_o''k_{2y}^2+T_o''\mu_r\frac{k_{2z}}{k_a}k_{2x}^2+\{T_1^{\perp}\frac{k_{2y}^2}{k_a}-2T_o^{\perp}\frac{k_{2y}^2k_{2z}}{k_{2r}^2}$$

$$-T_o''\frac{\mu_r k_{2x}^2}{k_a}(1-\frac{k_{2z}^2}{k_a^2})+\mu_r\frac{k_{2x}^2k_{2z}}{k_a}(\frac{T_1''}{k_a}-2T_o''\frac{k_{2z}}{k_{2r}^2})\}(k_{2x}\zeta_x+k_{2y}\zeta_y)$$

$$+2T_o''\mu_r k_{2x}\frac{k_{2z}^2}{k_a}\zeta_x+2T_o^{\perp}k_{2y}k_{2z}\zeta_y]$$

$$t_{12} \approx \frac{1}{n^2k_{2r}^2}[k_{2x}k_{2y}(-T_o^{+}+T_o''\mu_r \frac{k_{2z}}{k_a})-k_{2x}k_{2y}\{\frac{T_1^{+}}{k_a} - \frac{2k_{2z}}{k_{2r}^2} T_o^{+}$$

$$+T_o'' \frac{\mu_r}{k_a}(1- \frac{k_{2z}^2}{k_a^2})-\mu_r \frac{k_{2z}}{k_a}(\frac{T_1''}{k_a} - \frac{2k_{2z}}{k_{2r}^2})\}(k_{2x}\zeta_x+k_{2y}\zeta_y)$$

$$-k_{2z}(T_o^{+}-\mu_r \frac{k_{2z}}{k_a} T_o'')(k_{2x}\zeta_y+k_{2y}\zeta_x)]$$

$$t_{21} = t_{12}$$

$$t_{22} \approx \frac{1}{n^2k_{2r}^2}[T_o^{+}k_{2x}^2+T_o''\mu_r \frac{k_{2z}}{k_a} k_{2y}^2+\{T_1^{+} \frac{k_{2x}^2}{k_a} -2T_o^{+} \frac{k_{2x}^2k_{2z}}{k_{2r}^2}$$

$$-T_o''\mu_r \frac{k_{2y}^2}{k_a}(1- \frac{k_{2z}^2}{k_a^2})+\mu_r \frac{k_{2y}^2k_{2z}}{k_a}(\frac{T_1''}{k_a} -2T_o'' \frac{k_{2z}}{k_{2r}^2})\}(k_{2x}\zeta_x+k_{2y}\zeta_y)$$

$$+2T_o''\mu_r k_{2y} \frac{k_{2z}^2}{k_a} \zeta_y+2T_o^{+}k_{2x}k_{2z}\zeta_x]$$

$$t_{13} \approx \frac{1}{n^2k_{2r}^2} \{-T_o^{+}k_{2y}(-k_{2y}\zeta_x+k_{2x}\zeta_y)+T_o''\mu_r \frac{k_{2x}k_{2z}}{k_a}(k_{2x}\zeta_x+k_{2y}\zeta_y)\}$$

$$-T_o'' \frac{\mu_r}{n^2} \zeta_x$$

$$t_{31} = t_{13}$$

$$t_{23} \approx \frac{1}{n^2k_{2r}^2} \{T_o^{+}k_{2x}(-k_{2y}\zeta_x+k_{2x}\zeta_y)+T_o''\mu_r \frac{k_{2y}k_{2z}}{k_a}(k_{2x}\zeta_x+k_{2y}\zeta_y)\}$$

$$-T_o'' \frac{\mu_r}{n^2} \zeta_y$$

$$t_{32} = t_{23}$$

$$t_{33} \approx \frac{\mu_r}{n^2}\{T''_o + \frac{T''_1}{k_a}(k_{2x}\zeta_x + k_{2y}\zeta_y)\} \quad . \tag{3-53}$$

According to (3-25), we are actually interested in finding (R) and (T) for a particular incident wave vector, whose components are given by (3-18), namely,

$$\tilde{k}_{2x} = k_2 \frac{x}{R_i} \quad , \quad \tilde{k}_{2y} = k_2 \frac{y}{R_i} \quad , \quad \tilde{k}_{2z} = k_2 \frac{z'}{R_i} \quad , \tag{3-54}$$

where

$$z' = h - \zeta \quad , \quad R_i = \sqrt{r^2 + z'^2} \quad , \quad r = \sqrt{x^2 + y^2} \quad . \tag{3-55}$$

We denote the components of (R) and (T) and the Fresnel coefficients evaluated for (3-54) by tilding. Then from (3-52) and (3-53) we get

$$\tilde{r}_{11} \approx \frac{1}{r^2}\{\tilde{R}^+_o y^2 - \tilde{R}''_o x^2 + \frac{\tilde{R}^+_1 y^2 - \tilde{R}''_1 x^2}{\sqrt{n^2 R_i^2 - r^2}}(x\zeta_x + y\zeta_y) - \frac{2z'}{r^2}(\tilde{R}^+_o y^2$$

$$- \tilde{R}''_o x^2)(x\zeta_x + y\zeta_y) + 2z'(\tilde{R}^+_o y\zeta_y - \tilde{R}''_o x\zeta_x)\}$$

$$\tilde{r}_{12} \approx -\frac{1}{r^2}\{xy(\tilde{R}^+_o + \tilde{R}''_o) + \frac{xy}{\sqrt{n^2 R_i^2 - r^2}}(\tilde{R}^+_1 + \tilde{R}''_1)(x\zeta_x + y\zeta_y)$$

$$- \frac{2xyz'}{r^2}(\tilde{R}^+_o + \tilde{R}''_o)(x\zeta_x + y\zeta_y) + z'(\tilde{R}^+_o + \tilde{R}''_o)(x\zeta_x + y\zeta_y)\}$$

$$\tilde{r}_{21} = \tilde{r}_{12}$$

$$\tilde{r}_{22} \approx \frac{1}{r^2}\{\tilde{R}_o^+x^2-\tilde{R}_o''y^2+\frac{\tilde{R}_1^+x^2-\tilde{R}_1''y^2}{\sqrt{n^2R_i^2-r^2}}(x\zeta_x+y\zeta_y)-\frac{2z'}{r^2}(\tilde{R}_o^+x^2$$

$$-\tilde{R}_o''y^2)(x\zeta_x+y\zeta_y)+2z'(\tilde{R}_o^+x\zeta_x-R_o''y\zeta_y)\}$$

$$\tilde{r}_{13} \approx \frac{1}{r^2}\{-\tilde{R}_o^+y(-y\zeta_x+x\zeta_y)-\tilde{R}_o''x(x\zeta_x+y\zeta_y)\}-\tilde{R}_o''\zeta_x$$

$$\tilde{r}_{31} = \tilde{r}_{13}$$

$$\tilde{r}_{23} \approx \frac{1}{r^2}\{\tilde{R}_o^+x(-y\zeta_x+x\zeta_y)-\tilde{R}_o''y(x\zeta_x+y\zeta_y)\}-\tilde{R}_o''\zeta_y$$

$$\tilde{r}_{32} = \tilde{r}_{23}$$

$$\tilde{r}_{33} \approx \tilde{R}_o''+\frac{\tilde{R}_1''}{\sqrt{n^2R_i^2-r^2}}(x\zeta_x+y\zeta_y) \qquad (3\text{-}56)$$

and

$$\tilde{t}_{11} \approx \frac{1}{n^2r^2}[\tilde{T}_o^+y^2+\frac{\tilde{T}_o''\mu_r z'x^2}{\sqrt{n^2R_i^2-r^2}}+\{\frac{\tilde{T}_1^+y^2}{\sqrt{n^2R_i^2-r^2}}-\frac{2\tilde{T}_o^+y^2z'}{r^2}$$

$$-\frac{\tilde{T}_o''\mu_r x^2(n^2-1)}{\sqrt{n^2R_i^2-r^2}(n^2R_i^2-r^2)}+\frac{\mu_r x^2z'}{\sqrt{n^2R_i^2-r^2}}(\frac{\tilde{T}_1''}{\sqrt{n^2R_i^2-r^2}}$$

$$-\frac{2\tilde{T}_o''z'}{r^2})\}(x\zeta_x+y\zeta_y)+\frac{2\tilde{T}_o''\mu_r xz'^2}{\sqrt{n^2R_i^2-r^2}}\zeta_x+2\tilde{T}_o^+yz'\zeta_y]$$

$$\tilde{t}_{12} \approx \frac{1}{n^2r^2}[xy(-\tilde{T}_o^+ + \frac{\tilde{T}_o''\mu_r z'}{\sqrt{n^2R_i^2-r^2}}) - xy\{\frac{\tilde{T}_1^+}{\sqrt{n^2R_i^2-r^2}} - \frac{2\tilde{T}_o^+ z'}{r^2}$$

$$+ \frac{\tilde{T}_o''\mu_r(n^2-1)}{\sqrt{n^2R_i^2-r^2}\,(n^2R_i^2-r^2)} - \frac{\mu_r z'}{\sqrt{n^2R_i^2-r^2}}\;(\frac{\tilde{T}_1''}{\sqrt{n^2R_i^2-r^2}}$$

$$- \frac{2\tilde{T}_o'' z'}{r^2}\;))\}(x\zeta_x+y\zeta_y) - z'(\tilde{T}_o^+ - \frac{\mu_r\tilde{T}_o'' z'}{\sqrt{n^2R_i^2-r^2}})(x\zeta_y+y\zeta_x)]$$

$$\tilde{t}_{21} = \tilde{t}_{12}$$

$$\tilde{t}_{22} \approx \frac{1}{n^2r^2}[\tilde{T}_o^+x^2 + \frac{\tilde{T}_o''\mu_r z' y^2}{\sqrt{n^2R_i^2-r^2}} + \{\frac{\tilde{T}_1^+x^2}{\sqrt{n^2R_i^2-r^2}} - \frac{2\tilde{T}_o^+x^2 z'}{r^2}$$

$$- \frac{\tilde{T}_o''\mu_r y^2(n^2-1)}{\sqrt{n^2R_i^2-r^2}\,(n^2R_i^2-r^2)} + \frac{\mu_r y^2 z'}{\sqrt{n^2R_i^2-r^2}}\;(\frac{\tilde{T}_1''}{\sqrt{n^2R_i^2-r^2}}$$

$$- \frac{2\tilde{T}_o'' z'}{r^2}\;))\}(x\zeta_x+y\zeta_y) + \frac{2\tilde{T}_o''\mu_r yz'^2}{\sqrt{n^2R_i^2-r^2}}\;\zeta_y + 2\tilde{T}_o^+xz'\zeta_x]$$

$$\tilde{t}_{13} \approx \frac{1}{n^2r^2}\;\{-\tilde{T}_o^+y(-y\zeta_x+x\zeta_y) + \frac{\tilde{T}_o''\mu_r xz'}{\sqrt{n^2R_i^2-r^2}}(x\zeta_x+y\zeta_y)\} - \frac{\tilde{T}_o''\mu_r}{n^2}\;\zeta_x$$

$$\tilde{t}_{31} = \tilde{t}_{13}$$

$$\tilde{t}_{23} \approx \frac{1}{n^2r^2}\;\{\tilde{T}_o^+x(-y\zeta_x+x\zeta_y) + \frac{\tilde{T}_o''\mu_r yz'}{\sqrt{n^2R_i^2-r^2}}(x\zeta_x+y\zeta_y)\} - \frac{\tilde{T}_o''\mu_r}{n^2}\;\zeta_y$$

$$\tilde{t}_{32} = \tilde{t}_{23}$$

$$\tilde{t}_{33} \approx \frac{\mu_r}{n^2}\;\{\tilde{T}_o'' + \frac{\tilde{T}_1''}{\sqrt{n^2R_i^2-r^2}}(x\zeta_x+y\zeta_y)\}\quad . \tag{3-57}$$

These components of $(\tilde{R})$ and $(\tilde{T})$ given in the above are of course also the components of $[\hat{R}]_{ev}$ and $[\hat{T}]_{ev}$ as used in our integral formulas (3-25) when written in the dyadic form:

$$[\hat{R}]_{ev} = \tilde{r}_{11}\vec{a}_x\vec{a}_x+\tilde{r}_{12}\vec{a}_x\vec{a}_y+\tilde{r}_{13}\vec{a}_x\vec{a}_z+\tilde{r}_{21}\vec{a}_y\vec{a}_x+\tilde{r}_{22}\vec{a}_y\vec{a}_y+\tilde{r}_{23}\vec{a}_y\vec{a}_z$$
$$+\tilde{r}_{31}\vec{a}_z\vec{a}_x+\tilde{r}_{32}\vec{a}_z\vec{a}_y+\tilde{r}_{33}\vec{a}_z\vec{a}_z \quad ,$$

$$[\hat{T}]_{ev} = \tilde{t}_{11}\vec{a}_x\vec{a}_x+\tilde{t}_{12}\vec{a}_x\vec{a}_y+\tilde{t}_{13}\vec{a}_x\vec{a}_z+\tilde{t}_{21}\vec{a}_y\vec{a}_x+\tilde{t}_{22}\vec{a}_y\vec{a}_y+\tilde{t}_{23}\vec{a}_y\vec{a}_z$$
$$+\tilde{t}_{31}\vec{a}_z\vec{a}_x+\tilde{t}_{32}\vec{a}_z\vec{a}_y+\tilde{t}_{33}\vec{a}_z\vec{a}_z \quad .$$

CHAPTER 4

EXPECTED VALUES OF REFLECTED AND TRANSMITTED HERTZ POTENTIALS AND POWER FOR A GAUSSIAN INTERFACE

4.1 Gaussian Random Interface

Natural interfaces such as terrain and sea are always rough. Moreover, the profile of the roughness is never periodic, nor is each element of the profile of simple shape. Thus, in order to deal with boundary value problems involving natural rough interfaces, it appears best to use a statistical approach in which the entire surface is described by a random process and a statistical distribution is prescribed to some random parameter of the interface. With almost no exception, the height coordinate ζ of the interface is chosen for such random parameter. However, a random interface cannot be completely represented by the statistical distribution of ζ alone, since this tells us nothing about the distances between the hills and valleys, that is, about the density of the irregularities. This information is supplied by the autocorrelation function, or its normalized equivalent, the autocorrelation coefficient.

In what follows, we will assume a stationary gaussian random process to describe the rough interface. The use of a gaussian process for a natural interface is by now widely accepted and probably correct considering the fact that the roughness of such an interface is caused by many contributory factors (central limit theorem). This is a fortunate situation for us since manipulations involving gaussian processes are especially simple and convenient. Thus ζ is assumed a gaussian with mean value and variance, both of which are independent of the horizontal coordinates, given by

$$\left\langle \zeta \right\rangle = 0 \quad , \quad \left\langle \zeta^2 \right\rangle = \sigma^2 \quad , \tag{4-1}$$

The probability density function of ζ then becomes

$$f(\zeta) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\zeta^2}{2\sigma^2}} \quad . \tag{4-2}$$

Thus $\zeta(x,y)$ is the random height of the interface from the mean plane $z \equiv 0$. By the same reason as for choosing a gaussian height distribution, one of the most commonly used autocorrelation functions for terrain or sea type interfaces is also of the gaussian form, namely,

$$\rho(r) = \sigma^2 e^{-\frac{r^2}{d^2}} \quad , \tag{4-3}$$

where r is the distance between two points and d is the correlation distance for which $\rho(r)$ falls off to $\sigma^2 e^{-1}$. We have assumed that the roughness of the interface is isotropic (the random process is stationary), so that $\rho(r)$ depends only on the distance between two points, and not on the coordinates of the individual points separately.

Slopes of the interface, ζ_x and ζ_y, are also random processes. By the isotropy of the roughness, all three random variables ζ, ζ_x and ζ_y are mutually independent at a given point and the probability density functions of the two slopes, $f(\zeta_x)$ and $f(\zeta_y)$, must be the same. The form of $f(\zeta_x)$ or $f(\zeta_y)$, however, is not arbitrary with respect to $f(\zeta)$. It turns out that it also becomes gaussian with mean value zero and its variance completely given by the variance and the correlation distance of ζ. Explicitly they are

$$f(\zeta_x) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{\zeta_x^2}{2\sigma_1^2}} ,$$

$$f(\zeta_y) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{\zeta_y^2}{2\sigma_1^2}} , \qquad (4-4)$$

where $\sigma_1^2 = \langle \zeta_x^2 \rangle = \langle \zeta_y^2 \rangle$ is given by

$$\sigma_1 = \sqrt{-[\rho_{xx}]_{r=o}} = \sqrt{-[\rho_{yy}]_{r=o}} \ . \qquad (4-5)$$

σ_1 is related to the r.m.s. slope, σ_s, by [Barrick, 1965, p. 194]

$$\sigma_s = \sqrt{2}\ \sigma_1 \qquad (4-6)$$

which, for a gaussian autocorrelation function (4-3), becomes

$$\sigma_1 = \frac{2\sigma}{d} \qquad (4-7)$$

The above information on our random rough interface should suffice as far as determination of the mean field or the mean Hertz potential is concerned. For calculation of mean pointing vector or mean power, we need to specify a two dimensional statistical distribution of ζ. The gaussian density function of ζ in two dimensions is

$$f(\zeta_1,\zeta_2) = \frac{1}{2\pi\sigma^2\sqrt{1-\gamma^2}} e^{-\frac{\zeta_1^2-2\gamma\zeta_1\zeta_2+\zeta_2^2}{2\sigma^2(1-\gamma^2)}} \tag{4-8}$$

where ζ_1 and ζ_2 are two random variables and γ is the auto-correlation coefficient, $\gamma = \frac{\rho}{\sigma^2}$, which in the gaussian case is simply

$$\gamma = e^{-\frac{r^2}{d^2}} \tag{4-9}$$

4.2 Calculation of Expected Hertz Potentials and Electromagnetic Fields

As the interface function $\zeta(x,y)$ is assumed to be a gaussian random process described in the preceding section, we first evaluate the expected values of the Hertz potentials. By taking the expectation of both hand sides, (3-25) becomes

$$\left\langle \vec{\Pi}_r(Q_2) \right\rangle = \left\langle \frac{k_2}{4\pi i} \int \vec{a}_n \cdot (\vec{a}_{R_i} + \vec{a}_{R_r}) [\hat{R}]_{ev} \cdot \vec{a}_\pi \frac{e^{ik_2R_i}}{R_i} \frac{e^{ik_2R_r}}{R_r} ds \right\rangle ,$$

$$\left\langle \vec{\Pi}_t(Q_1) \right\rangle = \left\langle -\frac{k_1}{4\pi i} \int \vec{a}_n \cdot (\vec{a}_{\tilde{k}_t} + \vec{a}_{R_t}) [\hat{T}]_{ev} \cdot \vec{a}_\pi \frac{e^{ik_2R_i}}{R_i} \frac{e^{ik_1R_t}}{R_t} ds \right\rangle , \tag{4-10}$$

which on interchanging the order of the two operations--the expectation and the area integration--can be written

$$\left\langle \vec{\Pi}_r(Q_2) \right\rangle = \frac{k_2}{4\pi i} \int \left\langle \vec{a}_n \cdot (\vec{a}_{R_i} + \vec{a}_{R_r}) [\hat{R}]_{ev} \cdot \vec{a}_\pi \frac{e^{ik_2 R_i}}{R_i} \frac{e^{ik_2 R_r}}{R_r} \right\rangle ds \quad ,$$

$$\left\langle \vec{\Pi}_t(Q_1) \right\rangle = - \frac{k_1}{4\pi i} \int \left\langle \vec{a}_n \cdot (\vec{a}_{\tilde{k}_t} + \vec{a}_{R_t}) [\hat{T}]_{ev} \cdot \vec{a}_\pi \frac{e^{ik_2 R_i}}{R_i} \frac{e^{ik_1 R_t}}{R_t} \right\rangle ds \quad . \tag{4-11}$$

We now evaluate the expectation of the integrands first, where each integrand is a function of random variables ζ as well as ζ_x and ζ_y.

Specifically, ζ is involved in the integrands through the unit vectors, $\vec{a}_{R_i}$, $\vec{a}_{R_r}$, $\vec{a}_{\tilde{k}_t}$ and $\vec{a}_{R_t}$, and through the distances, R_i, R_r and R_t. ζ also shows in $[\hat{R}]_{ev}$ and $[\hat{T}]_{ev}$ through z', as can be seen from (3-56) and (3-57). In all these expressions, ζ appears always in the form of either $h-\zeta$, $z_2-\zeta$ or $z_1-\zeta$. But, according to our earlier assumptions (Chapter 3), both the source and the observation points are far away from the interface, i.e.,

$$h \gg |\zeta| \quad , \quad z_2 \gg |\zeta| \quad , \quad |z_1| \gg |\zeta| \quad , \tag{4-12}$$

so that except in the principal exponentials we can approximately set

$$h - \zeta \approx h \quad , \quad z_2 - \zeta \approx z_2 \quad , \quad z_1 - \zeta \approx z_1 \quad . \tag{4-13}$$

Therefore

$$\vec{a}_{R_i} \approx \vec{a}_{R_{io}} \quad , \quad \vec{a}_{R_r} \approx \vec{a}_{R_{ro}} \quad , \quad \vec{a}_{R_t} \approx \vec{a}_{R_{to}} \ ;$$

$$R_i \approx R_{io} = \sqrt{x^2 + y^2 + h^2}$$

$$R_r \approx R_{ro} = \sqrt{(x_2 - x)^2 + (y_2 - y)^2 + z_2^2}$$

$$R_t \approx R_{to} = \sqrt{(x_1 - x)^2 + (y_1 - y)^2 + z_1^2} \quad . \qquad (4\text{-}14)$$

Also z' in each component of $[\hat{R}]_{ev}$ and $[\hat{T}]_{ev}$ is replaced by ζ-h. Symbols and geometry are illustrated in Figures 4-1 and 4-2. The expression and its derivation of the unit vector $\vec{a}_{\tilde{k}_{to}}$ are more involved than others because of its dependence on slopes at each point. First we can express

$$\vec{\tilde{k}}_t = \{k_1^2 - (\vec{\tilde{k}}_2 \cdot \vec{a}_p)^2\}^{1/2} \, \vec{a}_n + (\vec{\tilde{k}}_2 \cdot \vec{a}_p)\vec{a}_p \quad , \qquad (4\text{-}15)$$

where expression for $\vec{a}_p$ has been given in (3-40). Reduction to the final expression for $\vec{a}_{\tilde{k}_{to}}$ is given in Appendix D. The result is in the first order (in slopes)

$$\vec{a}_{\tilde{k}_{to}} \approx \frac{1}{nR_{io}}[x+h\zeta_x-(n^2R_{io}^2-r^2)^{1/2}\,\zeta_x \,,\; y+h\zeta_y-(n^2R_{io}^2-r^2)^{1/2}\,\zeta_y \,,$$

$$(n^2R_{io}^2-r^2)^{1/2}+\{1-\frac{h}{\sqrt{n^2R_{io}^2-r^2}}\}(x\zeta_x+y\zeta_y)] \quad , \qquad (4\text{-}16)$$

where the first of the approximation (4-13) has been incorporated.

In the exponentials, we expand as usual the phase function up to the first order and put

$$R_i \approx R_{io} - \vec{a}_{R_{io}} \cdot \vec{\zeta} \,,$$

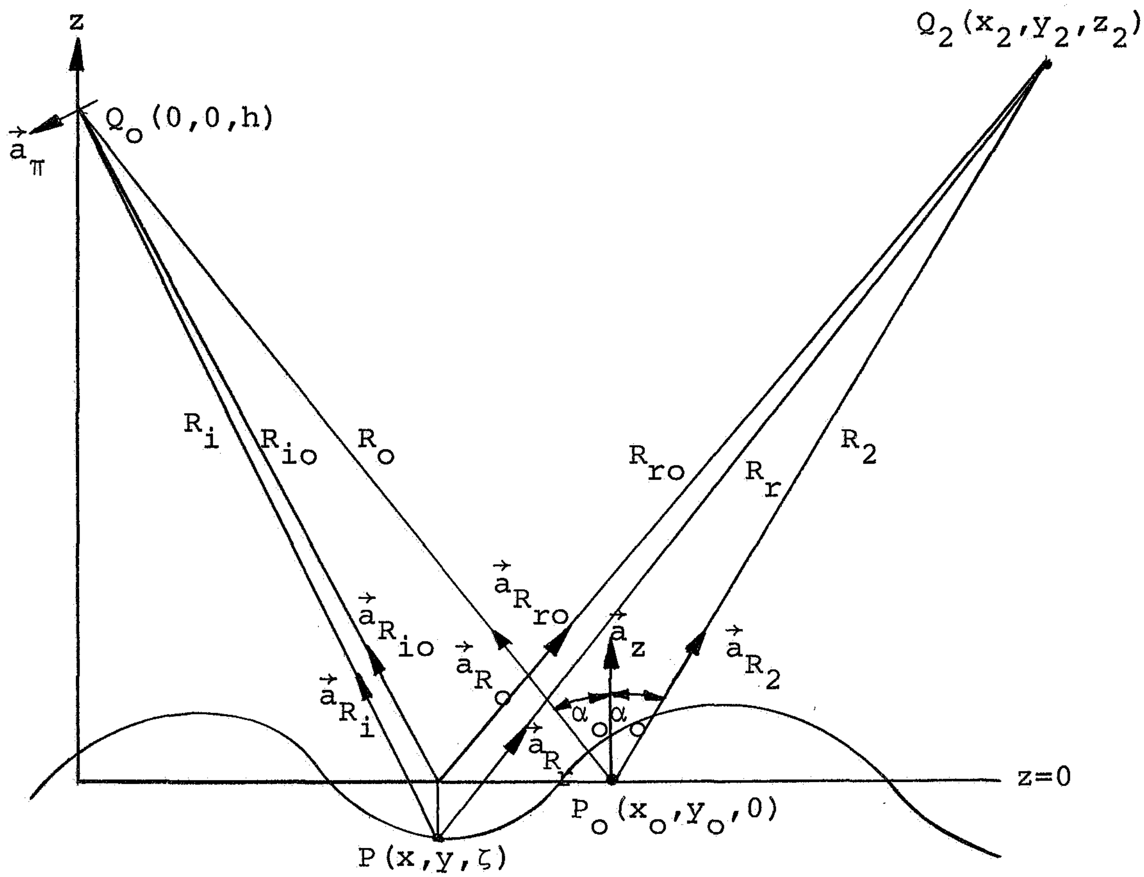

Figure 4-1. More Symbols and Geometry for Reflection

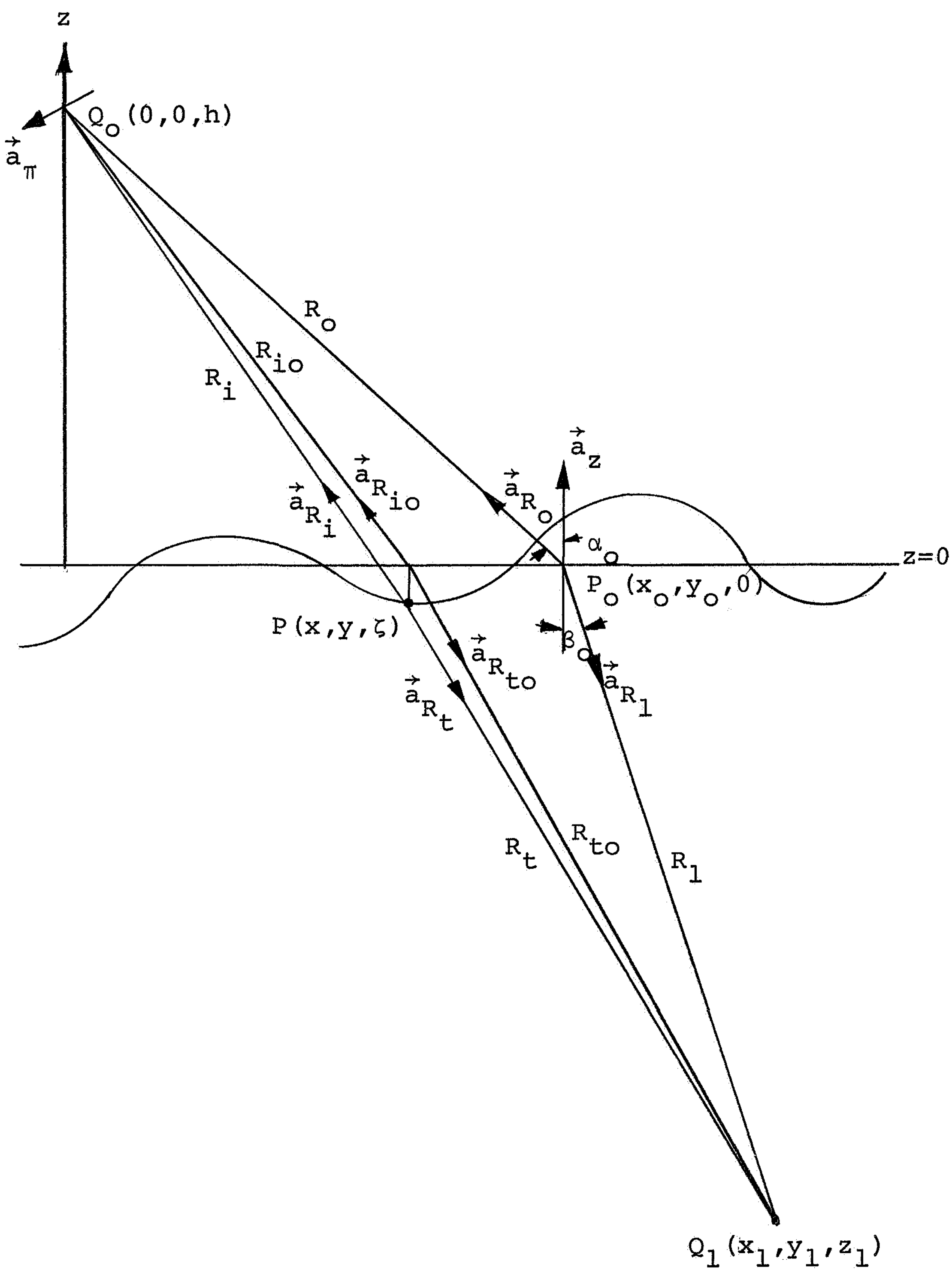

Figure 4-2. More Symbols and Geometry for Transmission

$$R_r \approx R_{ro} - \vec{a}_{R_{ro}} \cdot \vec{\zeta} \quad ,$$

$$R_t \approx R_{to} - \vec{a}_{R_{to}} \cdot \vec{\zeta} \quad , \tag{4-17}$$

where $\vec{\zeta}=\zeta\vec{a}_z$ and

$$\vec{a}_{R_{io}} \cdot \vec{\zeta} = \frac{h}{R_{io}}\zeta \ , \ \vec{a}_{R_{ro}} \cdot \vec{\zeta} = \frac{z_2}{R_{ro}}\zeta \ , \ \vec{a}_{R_{to}} \cdot \vec{\zeta} = \frac{z_1}{R_{to}}\zeta \quad . \tag{4-18}$$

The criteria for the validity of the approximations (4-17) can be given by requiring the second order terms of the Taylor expansion of R_i, R_r or R_t about the mean plane to be much smaller than unity. Thus we have for all x and y

$$\frac{k_2\zeta^2}{2R_{io}} << 1 \quad , \quad \frac{k_2\zeta^2}{2R_{ro}} << 1 \quad , \quad \frac{k_1\zeta^2}{2R_{to}} << 1 \tag{4-19}$$

for each corresponding approximation in (4-17). On incorporating the above approximations, we can now write

$$\left\langle \vec{\Pi}_r(Q_2) \right\rangle = \frac{k_2}{4\pi i} \int \left\langle \vec{a}_n \cdot (\vec{a}_{R_{io}} + \vec{a}_{R_{ro}}) [\hat{R}]_{ev} \cdot \vec{a}_\pi \frac{e^{ik_2(R_{io}+R_{ro})}}{R_{io}R_{ro}} \right.$$

$$\left. \cdot e^{-ik_2\zeta\left(\frac{h}{R_{io}} + \frac{z_2}{R_{ro}}\right)} \right\rangle ds \quad ,$$

$$\left\langle \vec{\Pi}_t(Q_1) \right\rangle = \frac{-k_1}{4\pi i} \int \left\langle \vec{a}_n \cdot (\vec{a}_{\tilde{k}_{to}} + \vec{a}_{R_{to}}) [\hat{T}]_{ev} \cdot \vec{a}_\pi \frac{e^{ik_2(R_{io}+nR_{to})}}{R_{io}R_{to}} \right.$$

$$\left. \cdot e^{-ik_2\zeta\left(\frac{h}{R_{io}} + n\frac{z_1}{R_{to}}\right)} \right\rangle ds \quad . \tag{4-20}$$

Since, by (4-13), $[\hat{R}]_{ev}$ and $[\hat{T}]_{ev}$ are now functions of ζ_x and ζ_y only and do not involve ζ, and since ζ and both ζ_x and ζ_y are independent random variables, we can split the expectation in (4-20) into products of two expectations and write

$$\left\langle \vec{\Pi}_r(Q_2) \right\rangle = \frac{k_2}{4\pi i} \int \left\langle \vec{a}_n \cdot (\vec{a}_{R_{io}} + \vec{a}_{R_{ro}}) [\hat{R}]_{ev} \cdot \vec{a}_\pi \right\rangle \left\langle e^{-ik_2\zeta\left(\frac{h}{R_{io}} + \frac{z_2}{R_{ro}}\right)} \right\rangle \cdot \frac{e^{ik_2(R_{io}+R_{ro})}}{R_{io}R_{ro}} \, ds \quad ,$$

$$\left\langle \vec{\Pi}_t(Q_1) \right\rangle = -\frac{k_1}{4\pi i} \int \left\langle \vec{a}_n \cdot (\vec{a}_{\tilde{k}_t} + \vec{a}_{R_{to}}) [\hat{T}]_{ev} \cdot \vec{a}_\pi \right\rangle \left\langle e^{-ik_2\zeta\left(\frac{h}{R_{io}} + n\frac{z_1}{R_{ro}}\right)} \right\rangle \cdot \frac{e^{ik_2(R_{io}+nR_{to})}}{R_{io}R_{to}} \, ds \quad , \qquad (4\text{-}21)$$

where we have factored some deterministic parts outside the expectation.

The latter expectation in each integral is just the well known characteristic function of ζ. On using the density function given in (4-2), they are found as

$$\left\langle e^{-ik_2\zeta\left(\frac{h}{R_{io}} + \frac{z_2}{R_{ro}}\right)} \right\rangle = e^{-\frac{1}{2}\sigma^2 k_2^2 \left(\frac{h}{R_{io}} + \frac{z_2}{R_{ro}}\right)^2} \quad ,$$

$$\left\langle e^{-ik_2\zeta\left(\frac{h}{R_{io}} + n\frac{z_1}{R_{to}}\right)} \right\rangle = e^{-\frac{1}{2}\sigma^2 k_2^2 \left(\frac{h}{R_{io}} + n\frac{z_1}{R_{to}}\right)^2} \quad . \qquad (4\text{-}22)$$

The arguments of each remaining expectation is up to the second order in ζ_x and ζ_y. We again neglect the second order terms and the expected value of the first order terms vanish since $\langle\zeta_x\rangle=\langle\zeta_y\rangle=0$, thus leaving only the zero order terms

$$\left\langle\vec{a}_n\cdot(\vec{a}_{R_{io}}+\vec{a}_{R_{ro}})[\hat{R}]_{ev}\cdot\vec{a}_\pi\right\rangle \approx \vec{a}_z\cdot(\vec{a}_{R_{io}}+\vec{a}_{R_{ro}})\left\langle[\hat{R}]_{ev}\right\rangle\cdot\vec{a}_\pi \ ,$$

$$\left\langle\vec{a}_n\cdot(\vec{a}_{\tilde{k}_{to}}+\vec{a}_{R_{to}})[\hat{T}]_{ev}\cdot\vec{a}_\pi\right\rangle \approx \vec{a}_z\cdot(\vec{a}_{\tilde{k}_{to}}+\vec{a}_{R_{to}})\left\langle[\hat{T}]_{ev}\right\rangle\cdot\vec{a}_\pi \ , \qquad (4\text{-}23)$$

where now

$$\vec{a}_{\tilde{k}_{to}} = \frac{1}{nR_{io}}\left\{x,y,\left(n^2R_{io}-r^2\right)^{1/2}\right\} \ , \qquad (4\text{-}24)$$

and the components of $\left\langle[\hat{R}]_{ev}\right\rangle$ and $\left\langle[\hat{T}]_{ev}\right\rangle$ are the same as those given in (3-56) and (3-57), but with only the zero order terms:

$$\left\langle\tilde{r}_{11}\right\rangle \approx \frac{1}{r^2}(\tilde{R}_o^{\perp}y^2 - \tilde{R}_o''x^2)$$

$$\left\langle\tilde{r}_{12}\right\rangle \approx -\frac{xy}{r^2}(\tilde{R}_o^{\perp} + \tilde{R}_o'')$$

$$\left\langle\tilde{r}_{21}\right\rangle = \left\langle\tilde{r}_{12}\right\rangle$$

$$\left\langle\tilde{r}_{22}\right\rangle \approx \frac{1}{r^2}(\tilde{R}_o^{\perp}x^2 - \tilde{R}_o''y^2)$$

$$\left\langle\tilde{r}_{13}\right\rangle \approx 0$$

$$\left\langle\tilde{r}_{31}\right\rangle = \left\langle\tilde{r}_{13}\right\rangle$$

$$\left\langle\tilde{r}_{23}\right\rangle \approx 0$$

$$\left\langle\tilde{r}_{32}\right\rangle = \left\langle\tilde{r}_{23}\right\rangle$$

$$\left\langle\tilde{r}_{33}\right\rangle \approx \tilde{R}_o'' \qquad (4\text{-}25)$$

and

$$\left\langle \tilde{t}_{11} \right\rangle \approx \frac{1}{n^2 r^2}(\tilde{T}_o^+ y^2 + \frac{\tilde{T}_o'' \mu_r h x^2}{\sqrt{n^2 R_{io}^2 - r^2}})$$

$$\left\langle \tilde{t}_{12} \right\rangle \approx -\frac{xy}{n^2 r^2}(\tilde{T}_o^+ - \frac{\tilde{T}_o'' \mu_r h}{\sqrt{n^2 R_{io}^2 - r^2}})$$

$$\left\langle \tilde{t}_{21} \right\rangle = \left\langle \tilde{t}_{12} \right\rangle$$

$$\left\langle \tilde{t}_{22} \right\rangle \approx \frac{1}{n^2 r^2}(\tilde{T}_o^+ x^2 + \frac{\tilde{T}_o'' \mu_r h y^2}{\sqrt{n^2 R_{io}^2 - r^2}})$$

$$\left\langle \tilde{t}_{13} \right\rangle \approx 0$$

$$\left\langle \tilde{t}_{31} \right\rangle = \tilde{t}_{13}$$

$$\left\langle \tilde{t}_{23} \right\rangle \approx 0$$

$$\left\langle \tilde{t}_{32} \right\rangle = \tilde{t}_{23}$$

$$\left\langle \tilde{t}_{33} \right\rangle \approx \frac{\mu_r \tilde{T}_o''}{n^2} \quad . \tag{4-26}$$

Since in the first order

$$ds = J_n dxdy \approx dxdy \quad , \tag{4-27}$$

we can now write after inserting (4-23) and (4-24)

$$\left\langle \vec{\Pi}_r(Q_2) \right\rangle = \frac{k_2}{4\pi i} \iint_{-\infty}^{\infty} \vec{a}_z \cdot (\vec{a}_{R_{io}} + \vec{a}_{R_{ro}}) \left\langle [\hat{R}]_{ev} \right\rangle$$

$$\cdot \vec{a}_\pi e^{-\frac{1}{2}\sigma^2 k_2^2 (\frac{h}{R_{io}} + \frac{z_2}{R_{ro}})^2} \frac{e^{ik_2(R_{io}+R_{ro})}}{R_{io} R_{ro}} \, dxdy \quad ,$$

$$\left\langle \vec{\Pi}_t(Q_1) \right\rangle = \frac{-k_1}{4\pi i} \iint_{-\infty}^{\infty} \vec{a}_z \cdot (\vec{a}_{\tilde{k}_{to}} + \vec{a}_{R_{to}}) \left\langle [\hat{T}]_{ev} \right\rangle$$

$$\cdot \vec{a}_\pi e^{-\frac{1}{2}\sigma^2 k_2^2 \left(\frac{h}{R_{io}} + n\frac{z_1}{R_{to}}\right)^2} \frac{e^{ik_2(R_{io}+nR_{to})}}{R_{io}R_{to}} \, dxdy \qquad (4-28)$$

To obtain results in the geometrical optics approximation, we again apply the stationary phase method to (4-28). The general procedure is identical to those used for the integrals in the case of the smooth interface (Chapter 2), except that here we have spatial integrals and thus the stationary point is given in spatial coordinates rather than in wave number coordinates as before. In physical terms, this means that we now neglect those reflected (or transmitted) waves from all parts of the mean plane, except from the small neighborhood of a point at which the phase of the exponential is stationary. With regard to the actual rough interface, this indicates that local orientations of the interface containing the stationary points are so much denser near the stationary point of the mean plane that contributions from other favorably oriented elements of the interface can be neglected. A good example of this can be found in the fact that the image of an electric bulb above a slightly rough desk appears as a blurred reflection in the surface of the desk.

The stationary point is of course the specular point in the case of reflection and the point at which the law of refraction is satisfied (between the rays) in the case of transmission. At this point

$$R_{io} = R_o \quad , \quad R_{ro} = R_2 \quad , \quad R_{to} = R_1 \quad ;$$

$$\vec{a}_z \cdot \vec{a}_{R_{io}} = \vec{a}_z \cdot \vec{a}_{R_{ro}} = \cos\alpha_o \quad , \quad \vec{a}_z \cdot \vec{a}_{\tilde{k}_{to}} = \vec{a}_z \cdot \vec{a}_{R_{to}} = -\cos\beta_o \quad ;$$

$$\left\langle [\hat{R}]_{ev} \right\rangle = \hat{M} \quad , \quad \left\langle [\hat{T}]_{ev} \right\rangle = \hat{N} \quad . \qquad (4\text{-}29)$$

where components of $\hat{M}$ and $\hat{N}$, which we denote m_{ij} and n_{ij}, where $i,j=1,2,3$, are given by

$$m_{11} \approx R^+(\alpha_o)\sin^2\phi_o - R''(\alpha_o)\cos^2\phi_o$$

$$m_{12} \approx -\{R^+(\alpha_o) + R''(\alpha_o)\}\cos\phi_o\sin\phi_o$$

$$m_{21} = m_{12}$$

$$m_{22} \approx R^+(\alpha_o)\cos^2\phi_o - R''(\alpha_o)\sin^2\phi_o$$

$$m_{13} \approx 0$$

$$m_{31} = m_{13}$$

$$m_{23} \approx 0$$

$$m_{32} = m_{23}$$

$$m_{33} \approx R''(\alpha_o) \qquad (4\text{-}30)$$

and

$$n_{11} \approx \frac{1}{n^2} T^+(\alpha_o)\sin^2\phi_o + \frac{T''(\alpha_o)\mu_r\cos\alpha_o\cos^2\phi_o}{n\cos\beta_o}$$

$$n_{12} \approx -\frac{1}{n^2} T^+(\alpha_o) - \frac{T''(\alpha_o)\mu_r\cos\alpha_o}{n\cos\beta_o} \sin\phi_o\cos\phi_o$$

$$n_{21} = n_{12}$$

$$n_{22} \approx \frac{1}{n^2} T^+(\alpha_o)\cos^2\phi_o + \frac{T''(\alpha_o)\mu_r\cos\alpha_o\sin^2\phi_o}{n\cos\beta_o}$$

$$n_{13} \approx 0$$

$$n_{31} = n_{13}$$

$$n_{23} \approx 0$$

$$n_{32} = n_{23}$$

$$n_{33} \approx \frac{\mu_r T''(\alpha_o)}{n^2} \tag{4-31}$$

The explicit forms of the Fresnel coefficients are given by (2-15) and (2-19).

We can now write in place of the integrals (4-28) that

$$\left\langle \vec{\Pi}_r(Q_2) \right\rangle \approx \frac{k_2}{2\pi i} \cos\alpha_o \hat{M}\cdot\vec{a}_\pi e^{-2\sigma^2 k_2^2 \cos^2\alpha_o} \frac{e^{ik_2 R_T}}{R_o R_2} I_r \quad ,$$

$$\left\langle \vec{\Pi}_t(Q_1) \right\rangle \approx \frac{k_1}{2\pi i} \cos\beta_o \hat{N}\cdot\vec{a}_\pi e^{-\frac{1}{2}\sigma^2 k_2^2 (\cos\alpha_o - n\cos\beta_o)^2} \frac{e^{ik_2(R_o+nR_1)}}{R_o R_1} I_t \quad , \tag{4-32}$$

where $R_T=R_o+R_2$, and I_r and I_t are

$$I_r = \iint_{-\infty}^{\infty} e^{\frac{ik_2}{2}\left(x^2\frac{\partial^2}{\partial x^2} + y^2\frac{\partial^2}{\partial y^2} + 2xy\frac{\partial^2}{\partial x\partial y}\right) R_T} dxdy \quad ,$$

$$I_t = \iint_{-\infty}^{\infty} e^{\frac{ik_2}{2}\left(x^2\frac{\partial^2}{\partial x^2} + y^2\frac{\partial^2}{\partial y^2} + 2xy\frac{\partial^2}{\partial x\partial y}\right)(R_o+nR_1)} dxdy, \tag{4-33}$$

in which we mean $\frac{\partial^2}{\partial x^2} R_T = [\frac{\partial^2}{\partial x^2}(R_{io}+R_{ro}]_{ev}$, etc. Carrying out the required differentiations and using the formula (2-39), we can show that

$$I_r = \frac{2\pi i R_o R_2}{k_2 R_T \cos\alpha_o} \quad , \quad I_t = \frac{2\pi i R_o R_1}{k_2 P \cos\beta_o} \quad , \tag{4-34}$$

where P is defined in (2-72), namely,

$$\frac{1}{P} = \frac{\cos\beta_o}{\sqrt{(nR_o+R_1)(nR_o\cos^2\beta_o+R_1\cos^2\alpha_o)}} \quad . \tag{4-35}$$

Thus we finally obtain the results

$$\left\langle \vec{\Pi}_r(Q_2) \right\rangle \approx \hat{M}\cdot\vec{a}_\pi e^{-2\sigma^2 k_2^2 \cos^2\alpha_o} \frac{e^{ik_2 R_T}}{R_T} \quad ,$$

$$\left\langle \vec{\Pi}_t(Q_1) \right\rangle \approx n\hat{N}\cdot\vec{a}_\pi e^{-\frac{1}{2}\sigma^2 k_2^2 (\cos\alpha_o - n\cos\beta_o)^2} \frac{e^{ik_2(R_o+nR_1)}}{P} \quad . \tag{4-36}$$

We again choose without loss of generality $\vec{a}_\pi$ to lie in the xz-plane, i.e., $\vec{a}_\pi = \sin\theta_o \vec{a}_x + \cos\theta_o \vec{a}_z$ and write (4-36) with $\hat{M}\cdot\vec{a}_\pi$ and $\hat{N}\cdot\vec{a}_\pi$ shown explicitly. Then using the following new symbols

$$\delta_r = e^{-2\sigma^2 k_2^2 \cos^2\alpha_o} \quad ,$$

$$\delta_t = e^{-\frac{1}{2}\sigma^2 k_2^2 (\cos\alpha_o - n\cos\beta_o)^2} \quad , \tag{4-37}$$

we have

$$\left\langle \vec{\Pi}_r(Q_2) \right\rangle = [\{R^+(\alpha_o)\sin^2\phi_o - R''(\alpha_o)\cos^2\phi_o\}\sin\theta_o \vec{a}_x$$
$$-\{R^+(\alpha_o)+R''(\alpha_o)\}\sin\theta_o\cos\phi_o\sin\phi_o\vec{a}_y$$
$$+R''(\alpha_o)\cos\theta_o\vec{a}_z]\delta_r \frac{e^{ik_2R_T}}{R_T} \quad ,$$

$$\left\langle \vec{\Pi}_t(Q_1) \right\rangle = [\{T^+(\alpha_o)\sin^2\phi_o + \frac{T''(\alpha_o)\mu_r\cos\alpha_o\cos^2\phi_o}{n\cos\beta_o}\}\sin\theta_o\vec{a}_x$$
$$-\{T^+(\alpha_o) - \frac{T''(\alpha_o)\mu_r\cos\alpha_o}{n\cos\beta_o}\}\sin\theta_o\cos\phi_o\sin\phi_o\vec{a}_y$$
$$+T''(\alpha_o)\mu_r\cos\phi_o\vec{a}_z]\delta_t \frac{e^{ik_2(R_o+nR_1)}}{nP} \quad . \qquad (4\text{-}38)$$

We note that the results above are identical to those of the smooth plane case, namely, (2-48) and (2-55), except for the factors δ_r and δ_t which account for the effect of the roughness of the interface. The limit $\sigma \to 0$ corresponds to the smooth plane case, and indeed in this limit both $\delta_r \to 1$ and $\delta_t \to 1$ so that two expressions in (4-38) become complete identities to (2-48) and (2-55), respectively. Also we note that in the imaginary interface limit $(n \to 1)$, $\left\langle \vec{\Pi}_r \right\rangle \to 0$ and $\left\langle \vec{\Pi}_t \right\rangle \to \exp[ik_2(R_o+R_1)]/(R_o+R_1)$, both regardless of σ, as expected.

The expected values of the reflected and the transmitted electromagnetic fields can be obtained from (4-38) using the equations

$$\left\langle \vec{E}_r(Q_2) \right\rangle = \nabla(\nabla\cdot\left\langle \vec{\Pi}_r(Q_2) \right\rangle) + k_2^2\left\langle \vec{\Pi}_r(Q_2) \right\rangle \quad ,$$

$$\left\langle \vec{H}_r(Q_2) \right\rangle = \frac{k_2^2}{i\omega\mu_2} \nabla\times\left\langle \vec{\Pi}_r(Q_2) \right\rangle \quad ,$$

$$\left\langle \vec{E}_t(Q_1) \right\rangle = \nabla(\nabla\cdot\left\langle \vec{\Pi}_t(Q_1) \right\rangle) + k_1^2\left\langle \vec{\Pi}_t(Q_1) \right\rangle \quad ,$$

$$\left\langle \vec{H}_t(Q_1) \right\rangle = \frac{k_1^2}{i\omega\mu_1} \nabla\times\left\langle \vec{\Pi}_t(Q_1) \right\rangle \quad . \tag{4-39}$$

where the right hand sides of (4-39) are justified by virtue of the linearity of expectation and interchanging the order of expectation and vector (differential) operations. The remaining procedure for carrying out (4-39) is the same as the smooth plane case including the use of the approximations (2-69). The results are simply those given in Section 2.5 modified by the factors δ_r for the reflected fields and δ_t for the transmitted fields.

4.3 Calculation of Expected Power

In the previous section, we have derived expressions for the expected values of the Hertz potentials as well as the electromagnetic fields. However, in practice measurement on an electromagnetic wave is also often made in such a manner that the quantities measured are proportional to the Poynting vector or simply power associated with the electromagnetic wave.

In order to calculate the reflected and the transmitted power, we start from the definition of the Poynting vector. For a steady state sinusoidal radiation, the (time) average Poynting vector is given by

$$\vec{S} = \frac{1}{2} R_e(\vec{E} \times \vec{H}^*) \tag{4-40}$$

where the symbol * means the complex conjugate of the quantity. On writing out $\vec{E}$ and $\vec{H}^*$ into the rectangular components, the reflected and the transmitted Hertz potentials, $\vec{S}_r$ and $\vec{S}_t$, can be given by

$$\vec{S}_r = S_{rx}\vec{a}_x + S_{ry}\vec{a}_y + S_{rz}\vec{a}_z \quad ,$$
$$\vec{S}_t = S_{tx}\vec{a}_x + S_{ty}\vec{a}_y + S_{tz}\vec{a}_z \quad , \tag{4-41}$$

and hence the expected values of $\vec{S}_r$ and $\vec{S}_t$ become

$$\langle\vec{S}_r\rangle = \langle S_{rx}\rangle\vec{a}_x + \langle S_{ry}\rangle\vec{a}_y + \langle S_{rz}\rangle\vec{a}_z \quad ,$$
$$\langle\vec{S}_t\rangle = \langle S_{tx}\rangle\vec{a}_x + \langle S_{ty}\rangle\vec{a}_y + \langle S_{tz}\rangle\vec{a}_z \quad , \tag{4-42}$$

where

$$S_{rx} = \frac{1}{2}R_e\{E_{ry}H^*_{rz} - E_{rz}H^*_{ry}\} \ , \quad S_{ry} = \frac{1}{2}R_e\{E_{rz}H^*_{rx} - E_{rx}H^*_{rz}\} \ ,$$
$$S_{rz} = \frac{1}{2}R_e\{E_{rx}H^*_{ry} - E_{ry}H^*_{rx}\} \ ; \quad S_{tx} = \frac{1}{2}R_e\{E_{ty}H^*_{tz} - E_{tz}H^*_{ty}\} \ ,$$
$$S_{ty} = \frac{1}{2}R_e\{E_{tz}H^*_{tx} - E_{tx}H^*_{tz}\} \ , \quad S_{tz} = \frac{1}{2}R_e\{E_{tx}H^*_{ty} - E_{ty}H^*_{tx}\} \ . \tag{4-43}$$

In general the random functions $\vec{E}$ and $\vec{H}$ are mutually correlated. Thus it is necessary to derive the components of $\vec{E}$ and $\vec{H}$ from $\vec{\Pi}$ in the integral form. This can be done by applying the relations (2-12) to (3-25) under the integral sign. Again performing the differentiations only on the exponentials then yields

$$E_{rx} = \frac{k_2^3}{4\pi i}\int m_r[\{1-\left(\frac{x_2-x}{R_r}\right)^2\}(\tilde{r}_{11}\sin\theta_o+\tilde{r}_{13}\cos\theta_o)$$

$$-\frac{(x_2-x)(y_2-y)}{R_r^2}(\tilde{r}_{21}\sin\theta_o+\tilde{r}_{23}\cos\theta_o)$$

$$-\frac{(x_2-x)(z_2-\zeta)}{R_r^2}(\tilde{r}_{31}\sin\theta_o+\tilde{r}_{33}\cos\theta_o)]\Phi_i\Phi_r ds \quad ,$$

$$E_{ry} = \frac{k_2^3}{4\pi i}\int m_r[-\frac{(x_2-x)(y_2-y)}{R_r^2}(\tilde{r}_{11}\sin\theta_o+\tilde{r}_{13}\cos\theta_o)$$

$$+\{1-\left(\frac{y_2-y}{R_r}\right)^2\}(\tilde{r}_{21}\sin\theta_o+\tilde{r}_{23}\cos\theta_o)$$

$$-\frac{(y_2-y)(z_2-\zeta)}{R_r^2}(\tilde{r}_{31}\sin\theta_o+\tilde{r}_{33}\cos\theta_o)]\Phi_i\Phi_r ds \quad ,$$

$$E_{rz} = \frac{k_2^3}{4\pi i}\int m_r[-\frac{(x_2-x)(z_2-\zeta)}{R_r^2}(\tilde{r}_{11}\sin\theta_o+\tilde{r}_{13}\cos\theta_o)$$

$$-\frac{(y_2-y)(z_2-\zeta)}{R_r^2}(\tilde{r}_{21}\sin\theta_o+\tilde{r}_{23}\cos\theta_o)$$

$$+\{1-\left(\frac{z_2-\zeta}{R_r}\right)^2\}(\tilde{r}_{31}\sin\theta_o+\tilde{r}_{33}\cos\theta_o)]\Phi_i\Phi_r ds \quad ;$$

$$H_{rx} = \frac{k_2^4}{4\pi i\omega\mu_2}\int m_r[\frac{y_2-y}{R_r}(\tilde{r}_{31}\sin\theta_o+\tilde{r}_{33}\cos\theta_o)$$

$$-\frac{z_2-\zeta}{R_r}(\tilde{r}_{21}\sin\theta_o+\tilde{r}_{23}\cos\theta_o)]\Phi_i\Phi_r ds \quad ,$$

$$H_{ry} = \frac{k_2^4}{4\pi i\omega\mu_2}\int m_r[\frac{z_2-\zeta}{R_r}(\tilde{r}_{11}\sin\theta_o+\tilde{r}_{13}\cos\theta_o)$$

$$- \frac{x_2-x}{R_r}(\tilde{r}_{31}\sin\theta_o+\tilde{r}_{33}\cos\theta_o)]\Phi_i\Phi_r ds \quad ,$$

$$H_{rz} = \frac{k_2^4}{4\pi i\omega\mu_2}\int m_r[\frac{x_2-x}{R_r}(\tilde{r}_{21}\sin\theta_o+\tilde{r}_{23}\cos\theta_o)$$

$$- \frac{y_2-y}{R_r}(\tilde{r}_{11}\sin\theta_o+\tilde{r}_{13}\cos\theta_o)]\Phi_i\Phi_r ds \quad , \tag{4-44}$$

and similarly

$$E_{tx} = \frac{k_1^3}{4\pi i}\int m_t[\{1-\left(\frac{x_1-x}{R_t}\right)^2\}(\tilde{t}_{11}\sin\theta_o+\tilde{t}_{13}\cos\theta_o)$$

$$- \frac{(x_1-x)(y_1-y)}{R_t^2}(\tilde{t}_{21}\sin\theta_o+\tilde{t}_{23}\cos\theta_o)$$

$$- \frac{(x_1-x)(z_1-\zeta)}{R_t^2}(\tilde{t}_{31}\sin\theta_o+\tilde{t}_{33}\cos\theta_o)]\Phi_i\Phi_t ds \quad ,$$

$$E_{ty} = \frac{k_1^3}{4\pi i}\int m_t[- \frac{(x_1-x)(y_1-y)}{R_t^2}(\tilde{t}_{11}\sin\theta_o+\tilde{t}_{13}\cos\theta_o)$$

$$+\{1-\left(\frac{y_1-y}{R_t}\right)^2\}(\tilde{t}_{21}\sin\theta_o+\tilde{t}_{23}\cos\theta_o)$$

$$- \frac{(y_1-y)(z_1-\zeta)}{R_t^2}(\tilde{t}_{31}\sin\theta_o+\tilde{t}_{33}\cos\theta_o)]\Phi_i\Phi_t ds \quad ,$$

$$E_{tz} = \frac{k_1^3}{4\pi i}\int m_t[-\frac{(x_1-x)(z_1-\zeta)}{R_t^2}(\tilde{t}_{11}\sin\theta_o+\tilde{t}_{13}\cos\theta_o)$$

$$-\frac{(y_1-y)(z_1-\zeta)}{R_t^2}(\tilde{t}_{21}\sin\theta_o+\tilde{t}_{23}\cos\theta_o)$$

$$+\{1-\left(\frac{z_1-\zeta}{R_t}\right)^2\}(\tilde{t}_{31}\sin\theta_o+\tilde{t}_{33}\cos\theta_o)]\Phi_i\Phi_t ds \quad ;$$

$$H_{tx} = \frac{k_1^4}{4\pi i\omega\mu_1}\int m_t[\frac{y_1-y}{R_t}(\tilde{t}_{31}\sin\theta_o+\tilde{t}_{33}\cos\theta_o)$$

$$-\frac{z_1-\zeta}{R_t}(\tilde{t}_{21}\sin\theta_o+\tilde{t}_{23}\cos\theta_o)]\Phi_i\Phi_t ds \quad ,$$

$$H_{ty} = \frac{k_1^4}{4\pi i\omega\mu_1}\int m_t[\frac{z_1-\zeta}{R_t}(\tilde{t}_{11}\sin\theta_o+\tilde{t}_{13}\cos\theta_o)$$

$$-\frac{x_1-x}{R_t}(\tilde{t}_{31}\sin\theta_o+\tilde{t}_{33}\cos\theta_o)]\Phi_i\Phi_t ds \quad ,$$

$$H_{tz} = \frac{k_1^4}{4\pi i\omega\mu_1}\int m_t[\frac{x_1-x}{R_t}(\tilde{t}_{21}\sin\theta_o+\tilde{t}_{23}\cos\theta_o)$$

$$-\frac{y_1-y}{R_t}(\tilde{t}_{11}\sin\theta_o+\tilde{t}_{13}\cos\theta_o)]\Phi_i\Phi_t ds \quad , \qquad (4\text{-}45)$$

where

$$m_r = \vec{a}_n\cdot(\vec{a}_{R_i}+\vec{a}_{R_r}) \quad , \quad m_t = -\vec{a}_n\cdot(\vec{a}_{\tilde{k}_t}+\vec{a}_{R_t}) \quad ;$$

$$\Phi_i = \frac{e^{ik_2R_i}}{R_i} \quad , \quad \Phi_r = \frac{e^{ik_2R_r}}{R_r} \quad , \quad \Phi_t = \frac{e^{ik_1R_t}}{R_t} \quad ; \qquad (4\text{-}46)$$

and r_{ij}'s and t_{ij}'s are given in (3-56) and (3-57), respectively.

We start calculation of $\vec{S}_r$ given in (4-42) from computation of the first term $\langle E_{ry}H^*_{rz}\rangle$. The primed coordinates and symbols will be used for E_{ry} and the nonprimed for H^*_{rz} for expressing the product of two integrals. On applying the same far-zone approximation, (4-18), to the spherical functions as well as to other factors in the integrand, the expectation of the product of E_{ry} and H^*_{rz} thus becomes

$$\left\langle E_{ry}H^*_{rz}\right\rangle = \frac{k_2^7}{(4\pi)^2\omega\mu_2}\left\langle \iint [m_r' m_r] [- \frac{(x_2-x')(y_2-y')}{(R_{ro}')^2}(\tilde{r}_{11}'\sin\theta_o+\tilde{r}_{13}'\cos\theta_o)$$

$$+\{1-\left(\frac{y_2-y'}{R_{ro}'}\right)^2\}(\tilde{r}_{21}'\sin\theta_o+\tilde{r}_{23}'\cos\theta_o) - \frac{(y_2-y')z_2}{(R_{ro}')^2}(\tilde{r}_{31}'\sin\theta_o$$

$$+\tilde{r}_{33}'\cos\theta_o)] [\frac{x_2-x}{R_{ro}}(\tilde{r}^*_{21}\sin\theta_o+\tilde{r}^*_{22}\cos\theta_o) - \frac{y_2-y}{R_{ro}}(\tilde{r}^*_{11}\sin\theta_o$$

$$+\tilde{r}^*_{13}\cos\theta_o)]\Phi_{io}'\Phi_{ro}'\Phi^*_{io}\Phi^*_{ro}e^{-ik_2\zeta'(\frac{h}{R_{io}'}+\frac{z_2}{R_{ro}'})}$$

$$\cdot e^{ik_2\zeta(\frac{h}{R_{io}}+\frac{z_2}{R_{ro}})} ds'ds\right\rangle \quad . \tag{4-47}$$

First we consider only the zero order slope terms in the brackets, that is, we approximate m_r' and m_r by

$$m_r' \approx \vec{a}_z\cdot(\vec{a}_{R_{io}}'+\vec{a}_{R_{ro}}') \quad , \quad m_r \approx \vec{a}_z\cdot(\vec{a}_{R_{io}}+\vec{a}_{R_{ro}}) \quad , \tag{4-48}$$

and use only zero order r_{ij}'s given in (4-25). The expectation then becomes the joint gaussian characteristic function,

$$\left\langle e^{-k_2 a'\zeta' + ik_2 a\zeta} \right\rangle = e^{-\frac{1}{2} k_2^2 \sigma^2 (a'^2 - 2a'a\gamma + a^2)} \tag{4-49}$$

where

$$a' = \frac{h}{R'_{io}} + \frac{z_2}{R'_{ro}} \quad , \quad a = \frac{h}{R_{io}} + \frac{z_2}{R_{ro}} \quad . \tag{4-50}$$

and the autocorrelation coefficient $\gamma(x',y';x,y)$ has been given in (4-9). We note that if the usual stationary phase method is used to evaluate the integral (4-47) as we have in the previous calculations, then it turns out that the results are identical to those of the smooth plane case and do not depend on the roughness of the interface. This is due to the fact that the exponential (4-47) representing the effect of the interface roughness disappears for $\gamma=1$ and $a=a'$, for, in both primed and nonprimed coodinate systems, the phase is stationary at one and the same point (x_o,y_o). Such results are certainly not very interesting. To avoid this difficulty, we introduce a refinement to the usual stationary phase method by also considering the variation of the correlation coefficient γ in the neighborhood of (x_o,y_o). Thus we expand γ in Taylor's series about (x_o,y_o). To describe this procedure more clearly, we write the integral (4-47) in the following form

$$\left\langle E_{ry} H^*_{rz} \right\rangle = \iiiint f(x,y;x',y') e^{g(x,y;x',y')} e^{i\phi(x',y')} e^{-i\phi(x,y)} dxdydx'dy' \quad , \tag{4-51}$$

where

$$g'(x,y;x',y') = k_2^2\sigma^2 aa'\gamma \quad ,$$

$$\phi(x,y) = k_2(R_{io}+R_{ro}) \quad , \quad \phi(x',y') = k_2(R'_{io}+R'_{ro}) \quad , \tag{4-52}$$

and $f(x,y;x',y')$ is the remainder of the integrand of (4-47) including the constants. We now first evaluate the integral with respect to x and y by writing

$$I(x',y') = [f]_{ev}e^{[g-i\phi]_{ev}}\iint \exp\left[x[g_x]_{ev}+y[g_y]_{ev}+\frac{x^2}{2}[g_{xx}-i\phi_{xx}]_{ev} + \frac{y^2}{2}[g_{yy}-i\phi_{yy}]_{ev}+xy[g_{xy}-i\phi_{xy}]_{ev}\right] dxdy \quad , \tag{4-53}$$

where the indicated evaluation is to be made at the stationary point, (x_o,y_o). (4-53) can be exactly integrated, yielding

$$I(x',y') = -\frac{2}{\sqrt{4AD-B^2}}[f]_{ev}e^{-\frac{1}{4A}\left\{C^2+\frac{2(AF-BC)^2}{4AD-B^2}\right\}}e^{[g-i\phi]_{ev}} \quad . \tag{4-54}$$

The new symbols in (4-54) represent

$$A = \frac{1}{2}[g_{xx} - i\phi_{xx}]_{ev}$$

$$B = [g_{xy} - i\phi_{xy}]_{ev}$$

$$C = [g_x]_{ev}$$

$$D = \frac{1}{2}[g_{yy} - i\phi_{yy}]_{ev}$$

$$F = [g_y]_{ev} \quad . \tag{4-55}$$

Next noting in the right hand side of (4-54) the exponential in front is much more slowly changing than the last one, we expand the exponent of the latter exponential only up to the second order in x' and y' about (x_o, y_o). Carrying out the remaining integration of (4-51) with respect to x' and y' then gives

$$\left\langle E_{ry}H^*_{rz}\right\rangle = \frac{4\pi^2}{|4AD-B^2|}\,[f]_{ev} \quad , \tag{4-56}$$

where the functions, f, A, B, and D are all to be evaluated at (x_o, y_o) in both the primed as well as the nonprimed coordinates. Substituting explicit expressions for the symbols, we obtain

$$\begin{aligned}\left\langle E_{ry}H^*_{rz}\right\rangle = K_2\{&-\sin^2\alpha_o\sin\phi_o\cos\phi_o(m_{11}\sin\theta_o+m_{13}\cos\theta_o)\\&+(1-\sin^2\alpha_o\sin^2\phi_o)(m_{21}\sin\theta_o+m_{23}\cos\theta_o)\\&-\sin\alpha_o\cos\alpha_o\sin\phi_o(m_{31}\sin\theta_o+m_{33}\cos\theta_o)\}\\&\cdot\{\sin\alpha_o\cos\phi_o(m^*_{21}\sin\theta_o+m^*_{22}\cos\theta_o)\\&-\sin\alpha_o\sin\phi_o(m^*_{11}\sin\theta_o+m^*_{13}\cos\theta_o)\} \quad ,\end{aligned} \tag{4-57}$$

where m_{ij}'s are given in (4-30) and

$$K_2 = \frac{k_2^5}{\omega\mu_2 R_T^2\sqrt{1+Q_2}} \quad , \tag{4-58}$$

Q_2 is derived in Appendix E:

$$Q_2 = 4k_2^2\sigma_s^4\left(\frac{1}{R_o}+\frac{1}{R_2}\right)^{-4}\left\{4k_2^2\sigma_s^4\cos^4\alpha_o+\left(\frac{1}{R_o}+\frac{1}{R_2}\right)^2+\left(\frac{1}{R_o}+\frac{1}{R_2}\right)^2\cos^4\alpha_o\right\} \quad , \tag{4-59}$$

where σ_s is the r.m.s. slope.

Other expectations of similar products of components of $\vec{E}_r$ and $\vec{H}_r$ can be found in an identical manner. Using the definition (4-43), we thus obtain

$$\left\langle S_{rx}\right\rangle = R_e\Big[\frac{K_2}{2}[\{-\sin^2\alpha_o\sin\phi_o\cos\phi_o(m_{11}\sin\theta_o+m_{13}\cos\theta_o)$$
$$+(1-\sin^2\alpha_o\sin^2\phi_o)(m_{21}\sin\theta_o+m_{23}\cos\theta_o)-\sin\alpha_o\cos\alpha_o\sin\phi_o$$
$$\cdot(m_{31}\sin\theta_o+m_{33}\cos\theta_o)\}\{\sin\alpha_o\cos\phi_o(m_{21}^*\sin\theta_o+m_{23}^*\cos\theta_o)$$
$$-\sin\alpha_o\sin\phi_o(m_{11}^*\sin\theta_o+m_{13}^*\cos\theta_o)\}-\{-\sin\alpha_o\cos\alpha_o\cos\phi_o$$
$$\cdot(m_{11}\sin\theta_o+m_{13}\cos\theta_o)-\sin\alpha_o\cos\alpha_o\sin\phi_o(m_{21}\sin\theta_o+m_{23}\cos\theta_o)$$
$$+\sin^2\alpha_o(m_{31}\sin\theta_o+m_{33}\cos\theta_o)\}\{\cos\alpha_o(m_{11}^*\sin\theta_o+m_{13}^*\cos\theta_o)$$
$$-\sin\alpha_o\cos\phi_o(m_{31}^*\sin\theta_o+m_{33}^*\cos\theta_o)\}]\Big] \quad ,$$

$$\left\langle S_{ry}\right\rangle = R_e\Big[\frac{K_2}{2}[\{-\sin\alpha_o\cos\alpha_o\cos\phi_o(m_{11}\sin\theta_o+m_{13}\cos\theta_o)-\sin\alpha_o\cos\alpha_o\sin\phi_o$$
$$\cdot(m_{21}\sin\theta_o+m_{23}\cos\theta_o)+\sin^2\alpha_o(m_{31}\sin\theta_o+m_{33}\cos\theta_o)\}$$
$$\cdot\{\sin\alpha_o\sin\phi_o(m_{31}^*\sin\theta_o+m_{33}^*\cos\theta_o)-\cos\alpha_o(m_{21}^*\sin\theta_o+m_{23}^*\cos\theta_o)\}$$
$$-\{(1-\sin^2\alpha_o\cos^2\phi_o)(m_{11}\sin\theta_o+m_{13}\cos\theta_o)-\sin^2\alpha_o\sin\phi_o\cos\phi_o$$
$$\cdot(m_{21}\sin\theta_o+m_{23}\cos\theta_o)-\sin\alpha_o\cos\alpha_o\cos\phi_o(m_{31}\sin\theta_o+m_{33}\cos\theta_o)\}$$
$$\cdot\{\sin\alpha_o\cos\phi_o(m_{21}^*\sin\theta_o+m_{23}^*\cos\theta_o)-\sin\alpha_o\sin\phi_o$$
$$\cdot(m_{11}^*\sin\theta_o+m_{13}^*\cos\theta_o)\}]\Big] \quad ,$$

$$\left\langle S_{rz} \right\rangle = R_e \Bigg[\frac{K_2}{2} [\{(1-\sin^2\alpha_o\cos^2\phi_o)(m_{11}\sin\theta_o+m_{13}\cos\theta_o)$$

$$-\sin^2\alpha_o\sin\phi_o\cos\phi_o(m_{31}\sin\theta_o+m_{33}\cos\theta_o)-\sin\alpha_o\cos\alpha_o\cos\phi_o$$

$$\cdot(m_{31}\sin\theta_o+m_{33}\cos\theta_o)\}\{\cos\alpha_o(m^*_{11}\sin\theta_o+m^*_{13}\cos\theta_o)$$

$$-\sin\alpha_o\cos\phi_o(m^*_{31}\sin\theta_o+m^*_{33}\cos\theta_o)\}-\{-\sin^2\alpha_o\sin\phi_o\cos\phi_o$$

$$\cdot(m_{11}\sin\theta_o+m_{13}\cos\theta_o)+(1-\sin^2\alpha_o\sin^2\phi_o)(m_{21}\sin\theta_o+m_{23}\cos\theta_o)$$

$$-\sin\alpha_o\cos\alpha_o\sin\phi_o(m_{31}\sin\theta_o+m_{33}\cos\theta_o)\}\{\sin\alpha_o\sin\phi_o$$

$$\cdot(m^*_{31}\sin\theta_o+m^*_{33}\cos\theta_o)-\cos\alpha_o(m^*_{21}\sin\theta_o+m^*_{23}\cos\theta_o)\}]\Bigg] . \quad (4\text{-}60)$$

In order to find the components of S_t , we let

$$m'_t \approx -\vec{a}_z\cdot(\vec{a}'_{k_t}+\vec{a}'_{R_t}) \quad , \quad m_t = -\vec{a}_z\cdot(\vec{a}_{\tilde{k}_t}+\vec{a}_{R_t}) \quad , \quad (4\text{-}61)$$

and use zero order terms of t_{ij}'s given in (4-26). For this case, we have

$$e^{-ik_2b'\zeta'+ik_2b} = e^{-\frac{1}{2}k_2^2\sigma^2(b'^2-2b'b\gamma+b^2)} \quad , \quad (4\text{-}62)$$

where

$$b' = \frac{h}{R'_{io}} + n\frac{z_1}{R'_{to}} \quad , \quad b = \frac{h}{R_{io}} + n\frac{z_1}{R_{to}} \quad , \quad (4\text{-}63)$$

and obtain

$$\left\langle S_{tx}\right\rangle = R_e\Big[\frac{K_1}{2}[\{-\sin^2\beta_o\sin\phi_o\cos\phi_o(n_{11}\sin\theta_o+n_{13}\cos\theta_o)$$

$$+(1-\sin^2\beta_o\sin^2\phi_o)(n_{21}\sin\theta_o+n_{23}\cos\theta_o)-\sin\beta_o\cos\beta_o\sin\phi_o$$

$$\cdot(n_{31}\sin\theta_o+n_{33}\cos\theta_o)\}\{\sin\beta_o\cos\phi_o(n^*_{21}\sin\theta_o+n^*_{23}\cos\theta_o)$$

$$-\sin\beta_o\sin\phi_o(n^*_{11}\sin\theta_o+n^*_{13}\cos\theta_o)\}-\{-\sin\beta_o\cos\beta_o\cos\phi_o$$

$$\cdot(n_{11}\sin\theta_o+n_{13}\cos\theta_o)-\sin\beta_o\cos\beta_o\sin\phi_o(n_{21}\sin\theta_o+n_{23}\cos\theta_o)$$

$$+\sin^2\beta_o(n_{31}\sin\theta_o+n_{33}\cos\theta_o)\}\{\cos\beta_o(n^*_{11}\sin\theta_o+n^*_{13}\cos\theta_o)$$

$$-\sin\beta_o\cos\phi_o(n^*_{31}\sin\theta_o+n^*_{33}\cos\theta_o)\}]\Big] \quad ,$$

$$\left\langle S_{ty}\right\rangle = R_e\Big[\frac{K_1}{2}[\{-\sin\beta_o\cos\beta_o\cos\phi_o(n_{11}\sin\theta_o+n_{13}\cos\theta_o)-\sin\beta_o\cos\beta_o\sin\phi_o$$

$$\cdot(n_{21}\sin\theta_o+n_{23}\cos\theta_o)+\sin^2\beta_o(n_{31}\sin\theta_o+n_{33}\cos\theta_o)\}$$

$$\cdot\{\sin\beta_o\sin\phi_o(n^*_{31}\sin\theta_o+n^*_{33}\cos\theta_o)-\cos\beta_o(n^*_{21}\sin\theta_o+n^*_{23}\cos\theta_o)\}$$

$$-\{(1-\sin^2\beta_o\cos^2\phi_o)(n_{11}\sin\theta_o+n_{13}\cos\theta_o)-\sin^2\beta_o\sin\phi_o\cos\phi_o$$

$$\cdot(n_{21}\sin\theta_o+n_{23}\cos\theta_o)-\sin\beta_o\cos\beta_o\cos\phi_o(n_{31}\sin\theta_o+n_{33}\cos\theta_o)\}$$

$$\cdot\{\sin\beta_o\cos\phi_o(n^*_{21}\sin\theta_o+n^*_{23}\cos\theta_o)-\sin\beta_o\sin\phi_o$$

$$\cdot(n^*_{11}\sin\theta_o+n^*_{13}\cos\theta_o)\}]\Big] \quad ,$$

$$\left\langle S_{tz}\right\rangle = R_e\Big[\frac{K_1}{2}[\{(1-\sin^2\beta_o\cos^2\phi_o)(n_{11}\sin\theta_o+n_{13}\cos\theta_o)-\sin^2\beta_o\sin\phi_o\cos\phi_o$$

$$\cdot(n_{21}\sin\theta_o+n_{23}\cos\theta_o)-\sin\beta_o\cos\beta_o\cos\phi_o(n_{31}\sin\theta_o+n_{33}\cos\theta_o)\}$$

$$\cdot\{\cos\beta_o(n^*_{11}\sin\theta_o+n^*_{13}\cos\theta_o)-\sin\beta_o\cos\phi_o(n^*_{31}\sin\theta_o+n^*_{33}\cos\theta_o)\}$$

$$-\{-\sin^2\beta_o\sin^2\phi_o\cos\phi_o(n_{11}\sin\theta_o+n_{13}\cos\theta_o)+(1-\sin^2\beta_o\sin^2\phi_o)$$

$$\cdot(n_{21}\sin\theta_o+n_{23}\cos\theta_o)-\sin\beta_o\cos\beta_o\sin\phi_o(n_{31}\sin\theta_o+n_{33}\cos\theta_o)\}$$

$$\cdot\{\sin\beta_o\sin\phi_o(n^*_{31}\sin\theta_o+n^*_{33}\cos\theta_o)-\cos\beta_o(n^*_{21}\sin\theta_o+n^*_{23}\cos\theta_o)\}]\Big] \qquad (4\text{-}64)$$

where

$$K_1 = \frac{n^2 k_1^5}{\omega\mu_1 P^2 \sqrt{1+Q_1}} \quad , \tag{4-65}$$

and P has been given in (4-35). Q_1 is also derived in Appendix E:

$$Q_1 = \frac{k_2^2\sigma_s^4}{2}\left(\frac{1}{R_o} + \frac{n}{R_1}\right)^{-2}\left(\frac{\cos^2\alpha_o}{R_o} + \frac{n\cos^2\beta_o}{R_1}\right)^{-2}(\cos\alpha_o - n\cos\beta_o)^4$$

$$\cdot\left\{-\left(\frac{1}{R_o} + \frac{n}{R_1}\right)\left(\frac{\cos^2\alpha_o}{R_o} + \frac{n\cos^2\beta_o}{R_1}\right) + \frac{k_2^2\sigma_s^4}{8}(\cos\alpha_o - n\cos\beta_o)^4\right.$$

$$+ \frac{1}{2}\left(\frac{1}{R_o} + \frac{n}{R_1}\right)^2 + \left(\frac{1}{R_o} + \frac{n}{R_1}\right)\left(\frac{\cos^2\alpha_o}{R_o} + \frac{n\cos^2\beta_o}{R_1}\right)$$

$$\left. + \frac{1}{2}\left(\frac{\cos^2\alpha_o}{R_o} + \frac{n\cos^2\beta_o}{R_1}\right)^2\right\} \quad . \tag{4-66}$$

$(1+Q_2)^{-1/2}$ and $(1+Q_1)^{-1/2}$ in (4-60) and (4-65) are the correction factors due to the roughness of the interface, which in the smooth plane limit approach unity as expected.

Next we examine the case when for a given σ_s the source and the observation points recede even further away from the interface. The last two terms in (4-59) eventually become negligible and thus

$$Q_2 \approx 16k_2^4\sigma_s^8\left(\frac{1}{R_o} + \frac{1}{R_2}\right)^{-4}\cos^4\alpha_o \quad , \tag{4-67}$$

and

$$1 + Q_2 \approx Q_2 \quad . \tag{4-68}$$

Therefore from (4-58)

$$K_2 \approx \frac{k_2^5}{\omega\mu_2 R_T^2 \sqrt{Q_2}} \quad \frac{k_2^3}{4\omega\mu_2\sigma_s^4 R_o^2 \cos^2\alpha_o} \quad . \tag{4-69}$$

(4-69) indicates that by (4-60) the reflected power asymptotically becomes inversely proportional to the fourth power of r.m.s. slope. Equivalently, by substituting in σ_s the expression (4-7), it can be also said that the reflected power becomes inversely and directly proportional to the fourth power of r.m.s. height and the correlation distance, respectively.

If we do the same for K_1 in (4-65) by increasing R_o and R_1, then

$$Q_1 \approx \frac{k_2^4\sigma_s^8}{16}\left(\frac{1}{R_o} + \frac{n}{R_1}\right)^{-2}\left(\frac{\cos^2\alpha_o}{R_o} + \frac{n\cos^2\beta_o}{R_1}\right)^{-2}(\cos\alpha_o - n\cos\beta_o)^8 \quad , \tag{4-70}$$

and for sufficiently large n

$$1 + Q_1 \approx Q_1 \quad , \tag{4-71}$$

so that

$$K_1 \approx \frac{n^2 k_1^5}{\omega\mu_1 P^2 \sqrt{Q_1}} \approx \frac{4n^4 k_1^3 \cos^2\beta_o}{\omega\mu_1\sigma_s^4 R_o^2 R_1^2 (\cos\alpha_o - n\cos\beta_o)^4} \quad . \tag{4-72}$$

It is seen that the dependence of K_1 of $\langle \vec{S}_t \rangle$ on σ_s or σ/d is similar to that in the previous case. The dependence of K_1 on k_1 is as k_1^3. However, other factors in each component of

$\langle \vec{S}_t \rangle$ are proportional to k_1^{-4} so that when the lower medium is perfectly conducting $\langle \vec{S}_t \rangle$ correctly becomes zero.

Next we consider higher order terms in the integrand of (4-47). We write (4-47) up to the first order terms as

$$\langle E_{ry} H^*_{rz} \rangle = \frac{k_2^7}{(4\pi)^2 \omega\mu_2} \iiiint_{-\infty}^{\infty} \Big\langle (u_o + u_1 \zeta_x + u_2 \zeta_y + u_3 \zeta'_{x'} + u_4 \zeta'_{y'})$$

$$e^{-ik_2 a'\zeta' + ik_2 a\zeta} \Big\rangle \Phi'_{io} \Phi'_{ro} \Phi^*_{io} \Phi^*_{ro} dx'dy'dxdy \qquad (4\text{-}73)$$

where the coefficients u's are independent of random variables and can be computed in a straightforward manner by writing out the integrand in the first order. We can then show that (See Appendix F)

$$\Big\langle \zeta_x e^{-ik_2 a'\zeta' + ik_2 a\zeta} \Big\rangle = -ik_2 a'\sigma^2 \gamma_x e^{-\frac{1}{2} k_2^2 \sigma^2 (a'^2 - 2a'a\gamma + a^2)}$$

$$\Big\langle \zeta_y e^{-ik_2 a'\zeta' + ik_2 a\zeta} \Big\rangle = -ik_2 a'\sigma^2 \gamma_y e^{-\frac{1}{2} k_2^2 \sigma^2 (a'^2 - 2a'a\gamma + a^2)}$$

$$\Big\langle \zeta'_{x'} e^{-ik_2 a'\zeta' + ik_2 a\zeta} \Big\rangle = ik_2 a\sigma^2 \gamma_{x'} e^{-\frac{1}{2} k_2^2 \sigma^2 (a'^2 - 2a'a\gamma + a^2)}$$

$$\Big\langle \zeta'_{y'} e^{-ik_2 a'\zeta' + ik_2 a\zeta} \Big\rangle = ik_2 a\sigma^2 \gamma_{y'} e^{-\frac{1}{2} k_2^2 \sigma^2 (a'^2 - 2a'a\gamma + a^2)} \qquad (4\text{-}74)$$

However, at the origin ($r=0$),

$$\gamma_x = \gamma_y = \gamma_{x'} = \gamma_{y'} = 0 \quad , \qquad (4\text{-}75)$$

leaving only the zero order term in (4-73). The first order terms vanish in all other such expectations and thus the zero order results (4-60) and (4-64) are actually equivalent to at least the first order approximation.

Lastly we consider the second order terms. Those second order terms that show up as we write out the product of three first order brackets in the integrand of (4-47) represent only part of the entire second order terms, since we have neglected such terms all along up to that point. We could, of course, include from the beginning all the second order terms if we had so desired, which obviously would have turned out quite tedious if not difficult. More interesting, however, is to calculate the general form for the second order power. Thus again appropriately denoting each coefficient of such terms by symbols, we write

$$\left\langle E_{ry}H^*_{rz}\right\rangle = \frac{k_2^7}{(4\pi)^2\omega\mu_2}\iiiint_{-\infty}^{\infty}\Big\langle\big(u_o+u_5\zeta_x^2+u_6\zeta_y^2+u_7\zeta_x\zeta_y+u_8\zeta_x\zeta'_{x'}$$

$$+u_9\zeta_x\zeta'_{y'}+u_{10}\zeta_y\zeta'_{x'}+u_{11}\zeta_y\zeta'_{y'}+u_{12}\zeta'^2_{x'}+u_{13}\zeta'^2_{y'}+u_{14}\zeta'_{x'}\zeta'_{y'}\big)$$

$$\cdot e^{-ik_2a'\zeta'+ik_2a\zeta}\Big\rangle\Phi'_{io}\Phi'_{ro}\Phi^*_{io}\Phi^*_{ro}\,dx'dy'dxdy \quad , \qquad (4\text{-}76)$$

and in addition to (4-49) make the following substitutions (see Appendix F):

$$\left\langle\zeta_x^2 e^{-ik_2a'\zeta'+ik_2a\zeta}\right\rangle = (\sigma_1^2-k_2^2\sigma^4a'^2\gamma_x^2)\,e^{-\frac{1}{2}k_2^2\sigma^2(a'^2-2a'a\gamma+a^2)}$$

$$\left\langle \zeta'^{2}_{x'} e^{-ik_2 a'\zeta' + ik_2 a\zeta} \right\rangle = (\sigma_1^2 - k_2^2\sigma^4 a^2 \gamma_{x'}^2)\, e^{-\frac{1}{2} k_2^2 \sigma^2 (a'^2 - 2a'a\gamma + a^2)}$$

$$\left\langle \zeta_x \zeta_y e^{-ik_2 a'\zeta' + ik_2 a\zeta} \right\rangle = -k_2^2\sigma^4 \gamma_x \gamma_y e^{-\frac{1}{2} k_2^2 \sigma^2 (a'^2 - 2a'a\gamma + a^2)}$$

$$\left\langle \zeta_x \zeta'_{x'} e^{-ik_2 a'\zeta' + ik_2 a\zeta} \right\rangle = (\sigma^2 \gamma_{xx'} + k_2^2\sigma^4 aa' \gamma_x \gamma_{x'})\, e^{-\frac{1}{2} k_2^2 \sigma^2 (a'^2 - 2a'a\gamma + a^2)} . \tag{4-77}$$

Other second order expectations can be written on inspection of (4-77):

$$\left\langle \zeta^{2}_{y} e^{-ik_2 a'\zeta' + ik_2 a\zeta} \right\rangle = (\sigma_1^2 - k_2^2\sigma^4 a'^2 \gamma_{y}^2)\, e^{-\frac{1}{2} k_2^2 \sigma^2 (a'^2 - 2a'a\gamma + a^2)}$$

$$\left\langle \zeta'^{2}_{y'} e^{-ik_2 a'\zeta' + ik_2 a\zeta} \right\rangle = (\sigma_1^2 - k_2^2\sigma^4 a^2 \gamma_{y'}^2)\, e^{-\frac{1}{2} k_2^2 \sigma^2 (a'^2 - 2a'a\gamma + a^2)}$$

$$\left\langle \zeta'_{x'} \zeta'_{y'} e^{-ik_2 a'\zeta' + ik_2 a\zeta} \right\rangle = -k_2^2\sigma^4 \gamma_{x'} \gamma_{y'} e^{-\frac{1}{2} k_2^2 \sigma^2 (a'^2 - 2a'a\gamma + a^2)}$$

$$\left\langle \zeta_{x'} \zeta'_{y'} e^{-ik_2 a'\zeta' + ik_2 a\zeta} \right\rangle = (\sigma^2 \gamma_{xy'} + k^2\sigma^4 aa' \gamma_x \gamma_{y'})\, e^{-\frac{1}{2} k_2^2 \sigma^2 (a'^2 - 2a'a\gamma + a^2)}$$

$$\left\langle \zeta'_{x'} \zeta_{y} e^{-ik_2 a'\zeta' + ik_2 a\zeta} \right\rangle = (\sigma^2 \gamma_{x'y} + k_2^2\sigma^2 aa' \gamma_{x'} \gamma_{y})\, e^{-\frac{1}{2} k_2^2 \sigma^2 (a'^2 - 2a'a\gamma + a^2)}$$

$$\left\langle \zeta_{y} \zeta'_{y'} e^{-ik_2 a'\zeta' + ik_2 a\zeta} \right\rangle = (\sigma^2 \gamma_{yy'} + k_2^2\sigma^4 aa' \gamma_{y} \gamma_{y'})\, e^{-\frac{1}{2} k_2^2 \sigma^2 (a'^2 - 2a'a\gamma + a^2)} . \tag{4-78}$$

For r=0, (4-77) and (4-78) become

$$\left\langle \zeta_x^2 e^{-ik_2a'\zeta'+ik_2a\zeta} \right\rangle = \left\langle \zeta_{x'}'^2 e^{-ik_2a'\zeta'+ik_2a\zeta} \right\rangle = \left\langle \zeta_y^2 e^{-ik_2a'\zeta'+ik_2a\zeta} \right\rangle$$

$$= \left\langle \zeta_{y'}'^2 e^{-ik_2a'\zeta'+ik_2a\zeta} \right\rangle = \left\langle \zeta_x \zeta_{x'}' e^{-ik_2a'\zeta'+ik_2a\zeta} \right\rangle$$

$$= \left\langle \zeta_y \zeta_y' \, e^{-ik_2a'\zeta'+ik_2a\zeta} \right\rangle = \sigma_1^2 \quad ,$$

$$\left\langle \zeta_x \zeta_y e^{-ik_2a'\zeta'+ik_2a\zeta} \right\rangle = \left\langle \zeta_{x'}' \zeta_{y'}' e^{-ik_2a'\zeta'+ik_2a\zeta} \right\rangle$$

$$= \left\langle \zeta_x \zeta_{y'}' e^{-ik_2a'\zeta'+ik_2a\zeta} \right\rangle = \left\langle \zeta_{x'}' \zeta_y e^{-ik_2a'\zeta'+ik_2a\zeta} \right\rangle = 0 \quad . \tag{4-79}$$

Thus the modified stationary phase method yields

$$\left\langle E_{ry} H_{rz}^* \right\rangle = \frac{k_2^5}{4\omega\mu_2 R_T^2 \cos^2\alpha_o \sqrt{1+Q_2}} \left([u_o]_{ev} + [u_s]_{ev} \sigma_1^2 \right) \quad , \tag{4-80}$$

where $[u_o]_{ev}$ has been evaluated in (4-57) and

$$u_s = u_5 + u_6 + u_8 + u_{11} + u_{12} + u_{13} \quad . \tag{4-81}$$

Other similar expectations which are required for computing $\langle \vec{S}_r \rangle$ will have the same form of results except for different u_o and u_s. Similarly, a typical expectation in the case of transmitted power becomes

$$\left\langle E_{ty}H_{tz}^{*}\right\rangle = \frac{k_1^5}{4\omega\mu_1 P^2\cos^2\alpha_o\sqrt{1+Q_1}}([v_o]_{ev}+[v_s]_{ev}\sigma_1^2) \qquad (4\text{-}82)$$

where the nature of v_o and v_s are similar to that of u_o and u_s.

For small Q_1 the second order power is seen to be directly proportional to the mean square slope of the interface, or equivalently, to the mean square height of the interface.

4.4 Determination of Mean Square Slope by Experiment

It is evident from (4-69) and (4-72) that the mean square (m.s.) slope for a class of natural rough surfaces can be determined by experiment. For this purpose, it is more convenient to consider the reflected power and the monostatic geometry, namely, the geometry that the source and the observation points are at the same point. Then the usual overflight test can be performed where a single airborne antenna is used for transmitting and receiving. We assume that the velocity of the airplane can be neglected so that the antenna can be considered stationary in order to permit the steady state analysis. The normal or near-normal signal incidence (strictly, with respect to the stationary phase point) further simplifies the analysis.

On putting $\alpha_o = \frac{\pi}{2}$ in (4-60), the horizontal components of $\langle \vec{S}_r \rangle$ vanish and the backscattered component is found as

$$\left\langle S_{rz}\right\rangle = \frac{K_2}{2}\left(|m_{11}|^2 + |m_{21}|^2\right) \quad , \qquad (4\text{-}83)$$

where we have also put $\theta_o = \frac{\pi}{2}$ considering a horizontal dipole so that the maximum radiation is in the vertical direction. Also $\alpha_o = \frac{\pi}{2}$ and (4-69) give

$$K_2 = \frac{k_2^3 C_o^2}{4\omega\mu_2 \sigma_s^4 R_o^2 R_2^2} \quad . \tag{4-84}$$

The extra factor C_o^2 is introduced now to account for the strength of the transmitted field, which we have put as unity so far in our investigation. Thus this corresponds to writing our incident Hertz potential as

$$\vec{\Pi}_i = \vec{a}_\pi C_o \frac{e^{ik_2 R}}{R} \quad . \tag{4-85}$$

On inserting m_{11} and m_{21} given in (4-30) with $\alpha_o = \frac{\pi}{2}$, (4-83) becomes

$$\langle S_{rz} \rangle = \frac{K_2}{2} |m_{11}|^2 = \frac{K_2}{2} \left| \frac{k_1 - k_2}{k_1 + k_2} \right|^2 \quad , \tag{4-86}$$

since

$$R^{+}(\tfrac{\pi}{2}) = \frac{1-n}{1+n} \quad , \quad R''(\tfrac{\pi}{2}) = -\frac{1-n}{1+n} \quad , \tag{4-87}$$

and

$$m_{21} = 0 \quad . \tag{4-88}$$

Using (4-84) and (4-86), we can write for σ_s^2

$$\sigma_s^2 = \frac{1}{2R_o R_2} \sqrt{\frac{k_2^3 C_o^2}{2\omega\mu_2 \langle S_{rz} \rangle}} \left| \frac{k_1 - k_2}{k_1 + k_2} \right| \quad . \tag{4-89}$$

Define the following quantities:

P_T = peak power radiated by the airborne antenna

G_T = gain of the transmitting antenna over an isotropic antenna

G_{SD} = gain of an electric dipole antenna over an isotropic antenna = $\frac{3}{2}$.

Then the power that must be radiated by the dipole to give the same power density in the main lobe is given by [Erteza, et al., 1965]

$$W_{SD} = \frac{P_T G_T}{G_{SD}} = \frac{2P_T G_T}{3} \tag{4-90}$$

C_o can be given in terms of W_{SD} by

$$C_o = \sqrt{\frac{3W_{SD}c^3}{4\pi\varepsilon_o\omega^4}} \quad . \tag{4-91}$$

Since

$$P_r = \frac{\langle S_{rz}\rangle G_R \lambda^2}{4\pi} \quad , \tag{4-92}$$

where G_R is the gain of the receiving antenna over an isotropic antenna, we finally obtain from (4-89), (4-90) and (4-91)

$$\sigma_s^2 = \frac{1}{4k_2^2 R_o R_2} \sqrt{\frac{G_T G_R P_T}{P_r}} \left|\frac{k_1-k_2}{k_1+k_2}\right| \quad . \tag{4-93}$$

Usually the gains of an antenna in transmitting and receiving are identical. Thus

$$R_o = R_2 \quad , \quad G_T = G_R = G \tag{4-94}$$

and

$$\sigma_s^2 = \frac{G}{4k_2^2R^2}\sqrt{\frac{P_T}{P_r}}\left|\frac{k_1-k_2}{k_1+k_2}\right| \quad . \tag{4-95}$$

If the lower medium is highly conducting so that $k_1 >> k_2$, then the last factor in (4-95) can be neglected, further simplifying the results

$$\sigma_s^2 = \frac{G}{4k_2^2R^2}\sqrt{\frac{P_T}{P_r}} \quad . \tag{4-96}$$

Thus by measuring P_r and R, σ_s^2 can be determined. Here it should be remembered that P_r and P_t are steady state powers. However, most radars employ pulse signals rather than CW signals. If a pulse used is of fairly long duration compared to the carrier wavelength, then the wave train in a single pulse can be to a good approximation considered a steady state wave with the carrier frequency ω_o.

It is interesting to see that if we assume a vertical dipole instead of a horizontal one, by letting $\theta_o=0$ in (4-60), we find

$$\left\langle S_{rz}\right\rangle = \frac{K_2}{2}\left(m_{13}{}^2 + m_{23}{}^2\right) \simeq 0 \tag{4-97}$$

since $m_{13} \simeq m_{23} \simeq 0$. This result is not surprising, which simply verifies the well known fact that the radiation field is zero along the dipole axis.

CHAPTER 5

SUMMARY AND CONCLUSIONS

Using the concept of Hertz potentials and a new plane wave approach, integral expressions describing the reflected and the transmitted Hertz potentials everywhere due to an arbitrarily oriented dipole source have been obtained for a smooth and a rough infinite plane interface. It is assumed that the upper medium (k_2) in which the dipole is situated is the air and the lower medium (k_1) has an arbitrary medium property, both media being homogeneous and isotropic.

After an introductory remark and a review on some of the more important work on the classical dipole-earth problems and on the rough surface scattering theories in Chapter 1, Chapter 2 deals with the smooth plane boundary case. The integrals (2-30) are exact in this case and the integrations have been carried out by using the method of stationary phase. Restrictions on the validity of such results have been discussed in great detail. For these results to be applicable, it is found that both the source and the observation points cannot simultaneously approach the interface in either case of reflection and transmission. However, there is one exception. For a horizontal dipole source, the aforementioned restriction holds true only for the reflected field when the observation point is located in or near the normal direction to the dipole axis. On the other hand, if the observation point lies in or near the direction of the dipole axis, then, even for a grazing incidence, the results become valid, provided the source and the observation points are far apart. We believe that this interesting situation has received attention for the first time. Also such a particularly simple form of our results for the reflected and transmitted Hertz potentials is probably new, except perhaps for the form of the reflected Hertz potential in the case of a vertical dipole source. The results corresponding to two distinctive dipole orientations, namely, vertical and horizontal, have been obtained by specializing a parameter in the general results. Specializations have been also made with respect to the medium property of the lower medium. As $k_1 \to k_2$ (no interface) and $k_1 \to \infty$ (perfectly conducting), the results obtained in the geometrical optics approximation gradually improve in their applicability until finally in the limit they become exact and reduce to well known expressions. From the Hertz potentials, the electromagnetic field expressions are also derived.

Another interesting and apparently important observation has been made on integrals (2-30), although the same observation can be made on the final results also. As we specialize the dipole direction to be horizontal, the vertical (z-) component term vanishes and we are left with only the horizontal (x- and y-) components of the Hertz potential. If we choose the x-direction as the dipole direction, the incident Hertz potential can be described by Π_x alone, and hence, consequently, the total Hertz potential everywhere will have only the horizontal components. This contradicts all the previous work on the classical horizontal dipole problem in which the Hertz potential everywhere is shown to always have x- and z-components. The entire section 2.6 has been devoted to clarifying this apparent discrepancy. It has been successfully shown with illustration that the Hertz potential is not unique in the horizontal dipole problem (or in any boundary value problem for that matter) and that as to the resolution of Hertz potentials there are altogether four possible resolutions: They are $\vec{\Pi}=(\Pi_x,\Pi_y,0)$, $\vec{\Pi}=(\Pi_x,0,\Pi_z)$, $\vec{\Pi}=(0,\Pi_y,\Pi_z)$ and $\vec{\Pi}=(\Pi_x,\Pi_y,\Pi_z)$. The second one is the resolution chosen by others. If Sommerfeld himself has not been mistaken on the nonuniqueness of the Hertz potential resolution, his remark in assuming the particular resolution is certainly misleading, which undoubtedly has led others to believing his resolution to be unique.

In Chapter 3, we derive the vector Helmholtz integral using a somewhat more general method. From this integral an integral formulation has been developed for an arbitrary nonplanar interface given by $z=\zeta(x,y)$. This is also believed to be the first time that an electromagnetic scattering from a rough interface has been formulated starting from the vector Helmholtz integral using the Hertz potentials. With respect to the rough interface the following assumptions have been made:

1. The radius of curvature at every point of the rough interface is much greater than the wavelength.
2. Shadowing effects are neglected.

3. Multiple scattering is neglected.

4. Both the source and the observation points are far from the interface.

5. The interface is assumed to be a stationary random process with a gaussian height distribution.

6. The height variation ζ from the mean plane is assumed to be small and the slope everywhere is such that the second order slope terms can be neglected. Statistically this is equivalent to small σ/d.

7. The lower medium is homogeneous, isotropic and otherwise arbitrary.

The single most important approximation that is used in our formulation of the scattering from the rough interface is the tangent plane approximation, which is the reason why the first three assumptions in the above must be made. Integrals (3-25) obtained using this approximation represent the reflected and the transmitted Hertz potentials at points far from the interface. The reflection and the transmission coefficients in the integrands are dyadic quantities so that they account for the different coefficients for each component and also for the change of polarizations (with respect to a fixed coordinate system) on reflection or transmission.

Chapter 4 is concerned with the carrying out of those integrals with the assumption of $\zeta(x,y)$ as a gaussian random process. By approximating the phase variation from the mean plane which is constant for a stationary process, the stationary phase method is applied with respect to the mean plane. Each of the resulting expected Hertz potentials and the electromagnetic fields are identical to the smooth plane interface case except for an exponential factor representing the effect of the roughness.

Expressions for the expected power (poynting vectors) are also derived, using a modified stationary phase method. The modification is made in order to account for the variation of the correlation function in the neighborhood of the stationary point. The results show an interesting dependence on the roughness of the interface. As the source and the observation points recede from the interface, the reflected and the transmitted power eventually become inversely proportional to the fourth power of the r.m.s. slope. It is shown that this asymptotic result for the reflected case can be applied to an experimental determination of the r.m.s. slope using overflight tests. The result is specialized to a monostatic case and the normal incidence is considered. The general expression for the higher order power is also indicated. It is shown that the first order power vanishes and the second order power is virtually directly proportional to the mean square slope.

APPENDIX A

CALCULATION OF ELECTROMAGNETIC FIELDS

For the vertical dipole case, the reflected Hertz potential is given from (2-48) as

$$\Pi_{rz} = R''(\alpha_o)\,\frac{e^{ik_2R_T}}{R_T},\quad \Pi_{rx} \equiv \Pi_{ry} \equiv 0. \qquad \text{(A-1)}$$

$\vec{E}_r$ and $\vec{H}_r$ can thus be derived from

$$E_{rx} = \frac{\partial^2\Pi_{rz}}{\partial x_2 \partial z_2},$$

$$E_{ry} = \frac{\partial^2\Pi_{rz}}{\partial y_2 \partial z_2},$$

$$E_{rz} = \frac{\partial^2\Pi_{rz}}{\partial z_2^{\,2}} + k_2^{\,2}\Pi_{rz};$$

$$H_{rx} = \frac{k_2^{\,2}}{i\omega\mu_2}\,\frac{\partial\Pi_{rz}}{\partial y_2},$$

$$H_{ry} = -\frac{k_2^{\,2}}{i\omega\mu_2}\,\frac{\partial\Pi_{rz}}{\partial x_2},$$

$$H_{rz} = 0. \qquad \text{(A-2)}$$

On using the approximations (2-69), we at once obtain

$$E_{rx} = -k_2^{\,2}\sin\alpha_o\cos\alpha_o\cos\phi_o R''(\alpha_o)\,\frac{e^{ik_2R_T}}{R_T},$$

$$E_{ry} = -k_2^{\,2}\sin\alpha_o\cos\alpha_o\sin\phi_o\, R''(\alpha_o)\,\frac{e^{ik_2R_T}}{R_T},$$

$$E_{rz} = k_2^{\,2}\sin^2\alpha_o\, R''(\alpha_o)\,\frac{e^{ik_2R_T}}{R_T};$$

$$H_{rx} = \frac{k_2^{\,3}}{\omega\mu_2} \sin \alpha_o \sin \phi_o \, R''(\alpha_o) \frac{e^{ik_2R_T}}{R_T},$$

$$H_{ry} = \frac{k_2^{\,3}}{\omega\mu_2} \sin \alpha_o \cos \phi_o \, R''(\alpha_o) \frac{e^{ik_2R_T}}{R_T},$$

$$H_{rz} = 0. \qquad \text{(A-3)}$$

These can be converted to the cylindrical components by using

$$E_{rr} = E_{rx} \cos \phi_o + E_{ry} \sin \phi_o ,$$

$$E_{r\phi} = -E_{rx} \sin \phi_o + E_{ry} \cos \phi_o ,$$

$$E_{rz} = \text{Unchanged} \qquad \text{(A-4)}$$

and similar ones for the magnetic fields.

The components for $\vec{E}_t$ and $\vec{H}_t$ for the vertical case can also be found analogously, which we will neglect to show.

The manipulations for the horizontal dipole case is a little more involved but again straightforward, which will also be neglected of showing.

APPENDIX B

DERIVATION OF (3-49)

Exact expression for the reflection and the transmission coefficients are given in (3-48). It is first necessary to express k_{2n} and k_{1n} in terms of the rectangular components (x,y,z). Since

$$\vec{k}_2 = (k_{2x},\ k_{2y},\ k_{2z}),$$

$$\vec{a}_n = \frac{1}{J_n}(-\zeta_x,\ -\zeta_y,\ 1) \approx (-\zeta_x, -\zeta_y,\ 1), \qquad \text{(B-1)}$$

we have

$$k_{2n} = \vec{k}_2 \cdot \vec{a}_n = -k_{2x}\zeta_x - k_{2x}\zeta_y + k_{2z}. \qquad \text{(B-2)}$$

By (3-43) and (3-47),

$$\begin{aligned} k_{1n} &= \sqrt{k_1^2 - (\vec{k}_2 \cdot \vec{a}_p)^2} \\ &= \sqrt{k_1^2 - J_t^2} \\ &\approx \sqrt{k_1^2 - k_{2r}^2 - 2k_{2z}(k_{2x}\zeta_x + k_{2y}\zeta_y)} \\ &\approx \sqrt{k_1^2 - k_{2r}^2} - \frac{k_{2z}}{\sqrt{k_1^2 - k_{2r}^2}}(k_{2x}\zeta_x + k_{2y}\zeta_y) \\ &= k_a - \frac{k_{2z}}{k_a}(k_{2x}\zeta_x + k_{2y}\zeta_y), \end{aligned} \qquad \text{(B-3)}$$

where $k_a = \sqrt{k_1^2 - k_{2r}^2}$.

Therefore,

$$R^{\perp} = \frac{\mu_r k_{2n} - k_{1n}}{\mu_r k_{2n} + k_{1n}}$$

$$= \frac{\mu_r(k_{2z} - k_{2x}\zeta_x - k_{2y}\zeta_y) - k_a + \frac{k_{2z}}{k_a}(k_{2x}\zeta_x + k_{2y}\zeta_y)}{\mu_r(k_{2z} - k_{2x}\zeta_x - k_{2y}\zeta_y) + k_a - \frac{k_{2z}}{k_a}(k_{2x}\zeta_x + k_{2y}\zeta_y)}$$

$$= \{\mu_r k_{2z} - k_a + (\frac{k_{2z}}{k_a} - \mu_r)(k_{2x}\zeta_x + k_{2y}\zeta_y)\}$$

$$\cdot\{\mu_r k_{2z} + k_a - (\frac{k_{2z}}{k_a} + \mu_r)(k_{2x}\zeta_x + k_{2y}\zeta_y)\}^{-1}$$

$$\approx \{\mu_r k_{2z} - k_a + (\frac{k_{2z}}{k_a} - \mu_r)(k_{2x}\zeta_x + k_{2y}\zeta_y)\}(\mu_r k_{2z} + k_a)^{-1}$$

$$\cdot\{1 + (\mu_r k_{2z} + k_a)^{-1}(\frac{k_{2z}}{k_a} + \mu_r)(k_{2x}\zeta_x + k_{2y}\zeta_y)\}$$

$$= \frac{\mu_r k_{2z} - k_a}{\mu_r k_{2z} + k_a} + \left\{\frac{k_{2z} - \mu_r k_a}{k_a(\mu_r k_{2z} + k_a)} + \frac{(\mu_r k_{2z} - k_a)(k_{2z} + \mu_r k_a)}{k_a(\mu_r k_{2z} + k_a)^2}\right\}$$

$$\cdot(k_{2x}\zeta_x + k_{2y}\zeta_y)$$

$$= \frac{\mu_r k_{2z} - k_a}{\mu_r k_{2z} + k_a} - \frac{2\mu_r k_2^2(n^2 - 1)}{k_a(\mu_r k_{2z} + k_a)^2}(k_{2x}\zeta_x + k_{2y}\zeta_y). \qquad \text{(B-4)}$$

Similar calculations can yield the remaining three equations for $T^{\perp}$, R'' and T''. $T^{\perp}$ and T'' can also be obtained from $R^{\perp}$ and R'' through the relations

$$T^{\perp} = 1 + R^{\perp}, \; T'' = 1 + R'', \qquad \text{(B-5)}$$

which can be easily checked out in (3-49).

APPENDIX C

DERIVATIONS OF (3-52) AND (3-53)

We first show the derivation of (3-52). On using (3-47) and further performing the first order approximations, each term in r_{11} of (3-41) reduces to the following:

$$\text{First Term} \approx R^{\perp}\left\{\frac{k_{2y}^{2}+2k_{2z}k_{2y}\zeta_y}{k_{2r}^{2}+2k_{2z}(k_{2x}\zeta_x+k_{2y}\zeta_y)}\right\}$$

$$\approx R^{\perp}\ (k_{2y}^{2}+k_{2z}k_{2y}\zeta_y)k_{2r}^{-2}\{1-\frac{2k_{2z}}{k_{2r}^{2}}\ (k_{2x}\zeta_x+k_{2y}\zeta_y)\}$$

$$\approx R^{\perp}k_{2r}^{-2}\{k_{2y}^{2}-\frac{2k_{2z}k_{2y}^{2}}{k_{2r}^{2}}$$

$$\cdot\ (k_{2x}\zeta_x+k_{2y}\zeta_y)+2k_{2z}k_{2y}\zeta_y\} \qquad \text{(C-1)}$$

which by (3-50) further becomes

$$\text{First Term} \approx k_{2r}^{-2}\{k_{2y}^{2}R_o^{+}+(\frac{k_{2y}^{2}R_1^{\perp}}{k_a}-\frac{2k_{2z}k_{2y}^{2}}{k_{2r}^{2}}R_o^{\perp})$$

$$\cdot\ (k_{2x}\zeta_x+k_{2y}\zeta_y)+2k_{2z}k_{2y}R_o^{+}\zeta_y\} \qquad \text{(C-2)}$$

Similarly

$$\text{Second Term} \approx -R''(k_{2x}^{2}+2k_{2z}k_{2x}\zeta_x)\ k_{2r}^{-2}$$

$$\cdot\ \{1-\frac{2k_{2z}}{k_{2r}^{2}}\ (k_{2x}\zeta_x+k_{2y}\zeta_y)\}$$

$$\approx k_{2r}^{-2}\{-k_{2x}^{2}R_o''-(\frac{k_{2x}^{2}R_1''}{k_a}-\frac{2k_{2x}k_{2x}^{2}}{k_{2r}^{2}}R_o'')$$

$$\cdot\ (k_{2x}\zeta_x+k_{2y}\zeta_y)\ -\ 2k_{2z}k_{2x}R_o''\zeta_x\} \qquad \text{(C-3)}$$

Third Term ≈ 0. (C-4)

Adding all three terms now gives r_{11}.

Next r_{12} is the sum of the following terms:

$$\text{First Term} \approx -R^{+}k_{2r}^{-2}\{k_{2x}k_{2y}+k_{2z}(k_{2x}\zeta_y+k_{2y}\zeta_x)\}$$

$$\cdot\{1 - \frac{2k_{2z}}{k_{2r}^{2}}(k_{2x}\zeta_x+k_{2y}\zeta_y)\}$$

$$\approx -R^{+}k_{2r}^{-2}\{k_{2x}k_{2y}+k_{2z}(k_{2x}\zeta_y+k_{2y}\zeta_x)$$

$$-\frac{2k_{2z}k_{2x}k_{2y}}{k_{2r}^{2}}(k_{2x}\zeta_x+k_{2y}\zeta_y)\}$$

$$\approx -\frac{1}{k_{2r}^{2}}\{k_{2x}k_{2y}R_o^{+} + k_{2z}R_o^{+}(k_{2x}\zeta_y+k_{2y}\zeta_x)$$

$$-\frac{2k_{2z}k_{2x}k_{2y}}{k_{2r}^{2}}R_o^{+}(k_{2x}\zeta_x+k_{2y}\zeta_y)+\frac{k_{2x}k_{2y}}{k_a}R_1^{+}$$

$$\cdot(k_{2x}\zeta_x+k_{2y}\zeta_y)\} \tag{C-5}$$

$$\text{Second Term} \approx -R''\left(\frac{k_{2z}\zeta_x+k_{2x}}{J_t}\right)\left(\frac{k_{2z}\zeta_y+k_{2y}}{J_t}\right)$$

$$\approx -\frac{1}{k_{2r}^{2}}\{k_{2x}k_{2y}R_o''+k_{2z}R_o''(k_{2x}\zeta_y+k_{2y}\zeta_x) - \frac{2k_{2z}k_{2x}k_{2y}}{k_{2r}^{2}}R_o''$$

$$\cdot(k_{2x}\zeta_x+k_{2y}\zeta_y) + \frac{k_{2x}k_{2y}}{k_a}R_1''$$

$$\cdot(k_{2x}\zeta_x+k_{2y}\zeta_y)\} \tag{C-6}$$

Third Term ≈ 0 (C-7)

r_{22} can be obtained by exchanging x and y in every subscript in the expression of r_{11}.

Next r_{13} is found by adding

$$\text{First Term} \approx -R^{\perp} k_{2r}^{-2} k_{2y} (-k_{2y}\zeta_x + k_{2x}\zeta_y)$$

$$\cdot \{1 - \frac{2k_{2z}}{k_{2r}^2} (k_{2x}\zeta_x + k_{2y}\zeta_y)\}$$

$$\approx -R_o^{\perp} k_{2r}^{-2} k_{2y} \; (-k_{2y}\zeta_x + k_{2x}\zeta_y) \tag{C-8}$$

$$\text{Second Term} \approx -R_o'' k_{2r}^{-2} k_{2x} (k_{2x}\zeta_x + k_{2y}\zeta_y) \tag{C-9}$$

$$\text{Third Term} \approx -R_o''\zeta_x \tag{C-10}$$

r_{23} is also obtained by exchanging x and y in every subscript in r_{13}.

$r_{33} \approx$ Third Term

$$\approx R_o'' + \frac{R_1''}{k_a} (k_{2x}\zeta_x + k_{2y}\zeta_y) \tag{C-11}$$

t_{ij}'s will now be derived. For t_{11} we can write

$$\text{First Term} \approx \frac{1}{n^2 k_{2r}^2} \{k_{2y}^2 T_o^{\perp} + (\frac{k_{2y}^2 T_1^{+}}{k_a} - \frac{2k_{2z}k_{2y}^2}{k_{2r}^2} T_o^{\perp})$$

$$\cdot (k_{2x}\zeta_x + k_{2y}\zeta_y) + 2k_{2z}k_{2y}T_o^{+}\zeta_y\} \tag{C-12}$$

$$\text{Second Term} \approx \frac{\mu_r}{n^2} T'' \{k_{2z} - (k_{2x}\zeta_x + k_{2y}\zeta_y)\}$$

$$\cdot \{k_1^2 - k_{2r}^2 - 2k_{2z}(k_{2x}\zeta_x + k_{2y}\zeta_y)\}^{1/2}$$

$$\cdot\ k_{2r}^{-2}(k_{2x}^2+2k_{2z}k_{2x}\zeta_x)\ \{1-\frac{2k_{2z}}{k_{2r}^2}(k_{2x}\zeta_x+k_{2y}\zeta_y)\}$$

$$\approx \frac{\mu_r}{n^2} T''k_a^{-1}\{k_{2z}-(k_{2x}\zeta_x+k_{2y}\zeta_y)+\frac{k_{2z}^2}{k_a^2}(k_{2x}\zeta_x+k_{2y}\zeta_y)\}$$

$$\cdot\ k_{2r}^{-2}\ \{k_{2x}^2+2k_{2z}k_{2x}\zeta_x-\frac{2k_{2z}k_{2x}^2}{k_{2r}^2}(k_{2x}\zeta_x+k_{2y}\zeta_y)\}$$

$$\approx \frac{\mu_r T''}{n^2k_ak_{2r}^2}\ \{k_{2z}k_{2x}^2+k_{2x}^2(\frac{k_{2z}^2}{k_a^2}-1)(k_{2x}\zeta_x+k_{2y}\zeta_y)$$

$$+\ 2k_{2z}^2k_{2x}\zeta_x-\frac{2k_{2z}^2k_{2x}^2}{k_{2r}^2}(k_{2x}\zeta_x+k_{2y}\zeta_y)\}$$

$$\approx \frac{\mu_r}{n^2k_ak_{2r}^2}\ [k_{2z}k_{2x}^2T_o''+\{k_{2z}k_{2x}^2\frac{T_1''}{k_a}+k_{2x}^2T_o''(\frac{k_{2z}^2}{k_a^2}-1)$$

$$-\frac{2k_{2z}^2k_{2x}^2}{k_{2r}^2}T_o''\}\ (k_{2x}\zeta_x+k_{2y}\zeta_y)+2k_{2z}^2k_{2x}T_o''\zeta_x] \qquad \text{(C-13)}$$

$$\text{Third Term} \approx 0 \qquad \text{(C-14)}$$

Adding the above three terms, we get t_{11}. For t_{12},

$$\text{First Term} \approx -\frac{1}{n^2k_{2r}^2}\ \{k_{2x}k_{2y}T_o^{+}+k_{2z}T_o^{+}(k_{2x}\zeta_y+k_{2y}\zeta_x)$$

$$-\ \frac{2k_{2z}k_{2x}k_{2y}}{k_{2r}^2}T_o^{+}\ (k_{2x}\zeta_x+k_{2y}\zeta_y)$$

$$+\frac{k_{2x}k_{2y}}{k_a}T_1^{+}\ (k_{2x}\zeta_x+k_{2y}\zeta_y)\} \qquad \text{(C-15)}$$

$$\text{Second Term} \approx \frac{\mu_r}{n^2k_{2r}^2k_a}\ \{k_{2z}-(1-\frac{k_{2z}^2}{k_a^2})(k_{2x}\zeta_x+k_{2y}\zeta_y)\}$$

$$\cdot\ \{k_{2x}k_{2y}T_o''+k_{2z}T_o''(k_{2x}\zeta_y+k_{2y}\zeta_x)-\frac{2k_{2z}k_{2x}k_{2y}}{k_{2r}^2}T_o''$$

$$\cdot\ (k_{2x}\zeta_x+k_{2y}\zeta_y)+\frac{k_{2x}k_{2y}}{k_a}T_1''\ (k_{2x}\zeta_x+k_{2y}\zeta_y)\}$$

$$\approx \frac{\mu_r}{n^2 k_{2r}^2 k_a} [k_{2z}k_{2x}k_{2y}T''_o - \{k_{2x}k_{2y}T''_o(1 - \frac{k_{2z}^2}{k_a^2})$$

$$+ \frac{2k_{2z}^2 k_{2x}k_{2y}}{k_{2r}^2} T''_o - \frac{k_{2x}k_{2y}k_{2z}}{k_a} T''_1\}(k_{2x}\zeta_x + k_{2y}\zeta_y)$$

$$+ k_{2z}^2 T''_o (k_{2x}\zeta_y + k_{2y}\zeta_x)] \qquad \text{(C-16)}$$

$$\text{Third Term} \approx 0 \qquad \text{(C-17)}$$

from which t_{12} follows. t_{13} obtains from

$$\text{First Term} \approx - \frac{T_o^{\perp}}{n^2 k_{2r}^2} k_{2y}(-k_{2y}\zeta_x + k_{2x}\zeta_y) \qquad \text{(C-18)}$$

$$\text{Second Term} \approx \frac{\mu_r T''_o k_{2x}}{n^2 k_{2r}^2 k_a} k_{2z}(k_{2x}\zeta_x + k_{2y}\zeta_y) \qquad \text{(C-19)}$$

$$\text{Third Term} \approx - \frac{\mu_r}{n^2} T''_o \zeta_x \qquad \text{(C-20)}$$

t_{22} and t_{23} are found by exchanging x and y in t_{11} and t_{13} respectively.

$$t_{33} \approx \text{Third Term}$$

$$\approx \frac{\mu_r}{n^2} \{T''_o + \frac{T''_1}{k_a} (k_{2x}\zeta_x + k_{2y}\zeta_y)\} \qquad \text{(C-21)}$$

APPENDIX D

DERIVATION OF (4-14)

By decomposing $\vec{\vec{k}}_t$ into $\vec{a}_n$ and $\vec{a}_p$ directions, we have

$$\vec{\vec{k}}_t = (\vec{\vec{k}}_t \cdot \vec{a}_n)\vec{a}_n + (\vec{\vec{k}}_t \cdot \vec{a}_p)\vec{a}_p$$

$$= \{k_1^2 - (\vec{\vec{k}}_t \cdot \vec{a}_p)^2\}^{1/2} \vec{a}_n + (\vec{\vec{k}}_t \cdot \vec{a}_p)\vec{a}_p$$

$$= \{k_1^2 - (\vec{\vec{k}}_2 \cdot \vec{a}_p)^2\}^{1/2} \vec{a}_n + (\vec{\vec{k}}_2 \cdot \vec{a}_p)\vec{a}_p \quad \text{(D-1)}$$

In terms of components

$$\vec{\vec{k}}_2 = \frac{k_2}{R_i}(x, y, h-\zeta) \approx \frac{k_2}{R_i}(x, y, h), \quad \text{(D-2)}$$

$$\vec{a}_n = \frac{1}{J_n}(-\zeta_x, -\zeta_y, 1) \approx (-\zeta_x, -\zeta_y, 1), \quad \text{(D-3)}$$

and hence

$$\vec{\vec{k}}_2 \cdot \vec{a}_n \approx \frac{k_2}{R_i}(-x\zeta_x - y\zeta_y + h). \quad \text{(D-4)}$$

From (3-40) $\vec{a}_p$ can be written

$$\vec{a}_p = \frac{k_2}{J_p R_i}(x + h\zeta_x - y\zeta_x\zeta_y + x\zeta_y^2,\ y + h\zeta_y - x\zeta_x\zeta_y + y\zeta_x^2,$$

$$x\zeta_x + y\zeta_y + h\zeta_x^2 + y\zeta_y^2) \quad \text{(D-5)}$$

Since from (3-47)

$$J_p \approx J_t \approx \frac{k_2}{R_i}\{1 + \frac{h}{r^2}(x\zeta_x + y\zeta_y)\} \quad \text{(D-6)}$$

(D-5) in the first order becomes

$$\vec{a}_p \approx \frac{1}{r}\{1 - \frac{h}{r^2}(x\zeta_x + y\zeta_y)\}(x + h\zeta_x, y + h\zeta_y, x\zeta_x + y\zeta_y)$$

$$\approx \frac{1}{r}\{x + h\zeta_x - \frac{hx}{r^2}(x\zeta_x + y\zeta_y),\ y + h\zeta_y - \frac{hx}{r^2}(x\zeta_x + y\zeta_y),$$

$$x\zeta_x + y\zeta_y\} \quad \text{(D-7)}$$

and

$$\vec{\tilde{k}}_2\cdot\vec{a}_p \approx \frac{k_2}{rR_i}\{r^2 + h(x\zeta_x+y\zeta_y)\},$$

$$(\vec{\tilde{k}}_2\cdot\vec{a}_p)^2 \approx \frac{k_2^{\,2}}{R_i^{\,2}}\{r^2+2h(x\zeta_x+y\zeta_y)\}. \qquad \text{(D-8)}$$

Finally

$$\vec{\tilde{k}}_t \approx \left(k_1^{\,2} - \frac{k_2^{\,2}r^2}{R_i^{\,2}}\right)^{1/2}\{1 - \frac{k_2^{\,2}h(x\zeta_x+y\zeta_y)}{k_1^{\,2}R_i^{\,2}-k_2^{\,2}r^2}\}(-\zeta_x,-\zeta_y,1)$$

$$+\frac{k_2}{r^2R_i}\{r^2+h(x\zeta_x+y\zeta_y)\}\{x+h\zeta_x-\frac{hx}{r^2}(x\zeta_x+y\zeta_y),$$

$$y+h\zeta_y-\frac{hy}{r^2}(x\zeta_x+y\zeta_y),\ x\zeta_x+y\zeta_y\}$$

$$\approx \left[-\left(k_1^{\,2}-\frac{k_2^{\,2}r^2}{R_i^{\,2}}\right)^{1/2}\zeta_x,\ -\left(k_1^{\,2}-\frac{k_2^{\,2}r^2}{R_i^{\,2}}\right)^{1/2}\zeta_y,\right.$$

$$\left.\left(k_1^{\,2}-\frac{k_2^{\,2}r^2}{R_i^{\,2}}\right)^{1/2}\left\{1-\frac{k_2^{\,2}h(x\zeta_x+y\zeta_y)}{k_1^{\,2}R_i^{\,2}-k_2^{\,2}r^2}\right\}\right]$$

$$+\frac{k_2}{R_i}\{x+h\zeta_x,\ y+h\zeta_y,\ x\zeta_x+y\zeta_y\}$$

$$=\frac{k_2}{R_i}[x+h\zeta_x-(n^2R_i^{\,2}-r^2)^{1/2}\zeta_x,\ y+h\zeta_y-(n^2R_i^{\,2}-r^2)^{1/2}\zeta_y,$$

$$(n^2R_i^{\,2}-r^2)^{1/2}+\left\{1-\frac{h}{(n^2R_i^{\,2}-r^2)^{1/2}}\right\}(x\zeta_x+y\zeta_y)] \qquad \text{(D-9)}$$

or

$$\vec{a}_{\tilde{k}_t} \approx \frac{1}{nR_i}\left[x+h\zeta_x-(n^2R_i^{\,2}-r^2)^{1/2}\zeta_x,\ y+h\zeta_y-(n^2R_i^{\,2}-r^2)^{1/2}\zeta_y,\right.$$

$$\left.(n^2R_i^{\,2}-r^2)^{1/2}+\{1-\frac{h}{(n^2R_i^{\,2}-r^2)^{1/2}}\}(x\zeta_x+y\zeta_y)\right] \qquad \text{(D-10)}$$

so that

$$\vec{a}_{\tilde{k}_{to}} \approx \frac{1}{nR_{io}} \Big[x+h\zeta_x-(n^2R_{io}{}^2-r^2)^{1/2}\zeta_x ,$$

$$y+h\zeta_y - (n^2R_{io}{}^2-r^2)^{1/2}\zeta_y ,$$

$$(n^2R_{io}{}^2-r^2)^{1/2} + \left\{ 1- \frac{h}{(n^2R_{io}-r^2)^{1/2}} \right\}$$

$$\cdot(x\zeta_x+y\zeta_y)\Big] \tag{D-11}$$

APPENDIX E

DERIVATION OF Q_2 AND Q_1

We will first derive Q_2. Square of the denominator of (4-56) can be given by

$$|4AD-B^2|^2 = [Re\{4AD-B^2\}]^2 + [Im\{4AD-B^2\}]^2 \quad , \tag{E-1}$$

where from (4-55)

$$Re\{4AD-B^2\} = [g_{xx}]_{ev}[g_{yy}]_{ev} - [\phi_{xx}]_{ev}[\phi_{yy}]_{ev} - [g_{xy}]^2_{ev} + [\phi_{xy}]^2_{ev} \quad ,$$

$$Im\{4AD-B^2\} = - [\phi_{xx}]_{ev}[g_{yy}]_{ev} - [\phi_{yy}]_{ev}[g_{xx}]_{ev} +2[g_{xy}]_{ev}[\phi_{xy}]_{ev} \quad . \tag{E-2}$$

Evaluations are made at the stationary point (x_o,y_o) and in both primed and nonprimed coordinate system. g and ϕ are defined in (4-52). If we assume that a and a' are much more slowly varying than γ, we have

$$g_{xx} = k_2^2\sigma^2 aa'\gamma_{xx} \tag{E-3}$$

and hence

$$\begin{aligned} [g_{xx}]_{ev} &= [k_2^2\sigma^2 aa'\gamma_{xx}]_{ev} \\ &= 4k_2^2\cos^2\alpha_o\sigma^2[\gamma_{xx}]_{ev} \\ &= -2k_2^2\sigma_s{}^2\cos^2\alpha_o \quad . \end{aligned} \tag{E-4}$$

Similarly we can find

$$[g_{yy}]_{ev} = -\ 2k_2^2\sigma_s{}^2\cos^2\alpha_o,$$

$$[g_{xy}]_{ev} = 0. \qquad (E-5)$$

Also

$$[\phi_{xx}]_{ev} = k_2(\frac{1}{R_o}+\frac{1}{R_2})(1-\sin^2\alpha_o\cos^2\phi_o),$$

$$[\phi_{yy}]_{ev} = k_2(\frac{1}{R_o}+\frac{1}{R_2})(1-\sin^2\alpha_o\sin^2\phi_o),$$

$$[\phi_{xy}]_{ev} = -\ k_2(\frac{1}{R_o}+\frac{1}{R_2})\sin^2\alpha_o\sin\phi_o\cos\phi_o. \qquad (E-6)$$

(E-2) thus becomes

$$\mathrm{Re}\{4AD-B^2\} = 4k_2^4\sigma_s{}^4\cos^4\alpha_o - k_2{}^2(\frac{1}{R_o}+\frac{1}{R_2})^2\cos^2\alpha_o,$$

$$\mathrm{Im}\{4AD-B^2\} = 2k_2^3\sigma_s{}^2(\frac{1}{R_o}+\frac{1}{R_2})(1+\cos^2\alpha_o)\cos^2\alpha_o. \qquad (E-7)$$

Therefore by squaring and adding the two equations in the above, we get

$$\begin{aligned}|4AD-B^2|^2 &= k_2^4(\frac{1}{R_o}+\frac{1}{R_2})^4\cos^4\alpha_o + 16k_2^8\sigma_s{}^8\cos\alpha_o \\ &+ 4k_2^6\sigma_s{}^4(\frac{1}{R_o}+\frac{1}{R_2})^2\cos^4\alpha_o \\ &+ 4k_2^6\sigma_s{}^4(\frac{1}{R_o}+\frac{1}{R_2})^2\cos^8\alpha_o \\ &= k_2^4(\frac{1}{R_o}+\frac{1}{R_2})^4\cos^4\alpha_o[1 + 4k_2^2\sigma_s{}^4(\frac{1}{R_o}+\frac{1}{R_2})^{-4} \\ &\cdot\{4k_2^2\sigma_s{}^4\cos^4\alpha_o + (\frac{1}{R_o}+\frac{1}{R_2})^2 \\ &+ (\frac{1}{R_o}+\frac{1}{R_2})^2\cos^4\alpha_o\}], \qquad (E-8)\end{aligned}$$

from which (4-59) follows.

Q_1 can be found much the same way as above. Here

$$[g_{xx}]_{ev} = -\frac{1}{2}k_2^2\sigma_s^2(\cos\alpha_o - n\cos\beta_o)^2,$$

$$[g_{yy}]_{ev} = -\frac{1}{2}k_2^2\sigma_s^2(\cos\alpha_o - n\cos\beta_o)^2,$$

$$[g_{xy}]_{ev} = 0,$$

$$[\phi_{xx}]_{ev} = \frac{k_2}{R_o}(1-\sin^2\alpha_o\cos^2\phi_o) + \frac{k_1}{R_o}(1-\sin^2\beta_o\cos^2\phi_o),$$

$$[\phi_{yy}]_{ev} = \frac{k_2}{R_o}(1-\sin^2\alpha_o\sin^2\phi_o) + \frac{k_1}{R_1}(1-\sin^2\beta_o\sin^2\phi_o),$$

$$[\phi_{xy}]_{ev} = -k_2\left(\frac{\sin^2\alpha_o}{R_o} + n\frac{\sin^2\beta_o}{R_1}\right). \tag{E-9}$$

Thus

$$\mathrm{Re}\{4AD-B^2\} = \frac{1}{4}k_2^4\sigma_s^4(\cos\alpha_o - n\cos\beta_o)^4 - \frac{k_2^2}{R_o^2}\cos^2\alpha_o - \frac{k_2k_1}{R_oR_1}(\cos^2\alpha_o + \cos^2\beta_o) - \frac{k_1^2}{R_1^2}\cos^2\beta_o,$$

$$\mathrm{Im}\{4AD-B^2\} = -\frac{k_2^3\sigma_s^2}{2}(\cos\alpha_o - n\cos\beta_o)^2\left\{\frac{1+\cos^2\alpha_o}{R_o} + n\frac{(1+\cos^2\beta_o)}{R_1}\right\}, \tag{E-10}$$

and therefore we finally obtain

$$|4AD-B^2|^2 = k_2^4\left(\frac{1}{R_o}+\frac{n}{R_1}\right)^2\left(\frac{\cos^2\alpha_o}{R_o} + \frac{n\cos^2\beta_o}{R_1}\right)^2 - \frac{k_2^6\sigma_s^4}{2}\left(\frac{1}{R_o}+\frac{n}{R_1}\right)(\cos\alpha_o - n\cos\beta_o)^4 \cdot\left(\frac{\cos^2\alpha_o}{R_o} + \frac{n\cos^2\beta_o}{R_1}\right) + \frac{k_2^8\sigma_s^8}{16}(\cos\alpha_o - n\cos\beta_o)^8$$

$$+ \frac{k_2^6 \sigma_s^4}{4} (\cos\alpha_o - n\cos\beta_o)^4$$

$$\cdot (\frac{1}{R_o} + \frac{n}{R_1})^2 + \frac{k_2^6 \sigma_s^4}{2} (\cos\alpha_o - n\cos\beta_o)^4 (\frac{1}{R_o} + \frac{n}{R_1})$$

$$\cdot (\frac{\cos^2\alpha_o}{R_o} + \frac{n\cos^2\beta_o}{R_1}) + \frac{K_2^6 \sigma_s^4}{4} (\cos\alpha_o - n\cos\beta_o)$$

$$\cdot (\frac{\cos^2\alpha_o}{R_o} + \frac{n\cos^2\beta_o}{R_1})^2$$

$$= k_2^4 (\frac{1}{R_o} + \frac{n}{R_1})^2 (\frac{\cos^2\alpha_o}{R_o} + \frac{n\cos^2\beta_o}{R_1})^2 [1 + \frac{1}{2} k_2^2 \sigma_s^4 (\frac{1}{R_o} + \frac{n}{R_1})^{-2}$$

$$\cdot (\frac{\cos^2\alpha_o}{R_o} + \frac{n\cos^2\beta_o}{R_1})^{-2} (\cos\alpha_o - n\cos\beta_o)^4 \{-(\frac{1}{R_o} + \frac{n}{R_1})$$

$$\cdot (\frac{\cos^2\alpha_o}{R_o} + \frac{n\cos^2\beta_o}{R_1}) + \frac{k_2^2 \alpha_s^4}{8} (\cos\alpha_o - n\cos\beta_o)^4$$

$$+ \frac{1}{2}(\frac{1}{R_o} + \frac{n}{R_1})^2 + (\frac{1}{R_o} + \frac{n}{R_1})(\frac{\cos^2\alpha_o}{R_o} + \frac{n\cos^2\beta_o}{R_1})$$

$$+ \frac{1}{2}(\frac{\cos^2\alpha_o}{R_o} + \frac{n\cos^2\beta_o}{R_1})^2 \}] \quad . \qquad \text{(E-11)}$$

Q_1 is evidently the second term in the bracket.

APPENDIX F

EVALUATION OF ENSEMBLE AVERAGES

Expected value of the following exponential is given (Stogryn, 1967):

$$\langle e^{N_1\zeta+N_2\zeta'+N_3\zeta_x+N_4\zeta'_{x'}}\rangle = e^{(1/2)N^T\Gamma N} \qquad \text{(F-1)}$$

where N is the column matrix and N^T is its transpose:

$$N^T = (N_1\ N_2\ N_3\ N_4), \qquad \text{(F-2)}$$

and Γ is the variance-covariance matrix

$$\Gamma = \begin{pmatrix} \langle\zeta^2\rangle & \langle\zeta\zeta'\rangle & \langle\zeta\zeta_x\rangle & \langle\zeta\zeta'_{x'}\rangle \\ \langle\zeta'\zeta\rangle & \langle\zeta'^2\rangle & \langle\zeta'\zeta_x\rangle & \langle\zeta'\zeta'_{x'}\rangle \\ \langle\zeta_x\zeta\rangle & \langle\zeta_x\zeta'\rangle & \langle\zeta_x^2\rangle & \langle\zeta_x\zeta'_{x'}\rangle \\ \langle\zeta'_{x'}\zeta\rangle & \langle\zeta'_{x'}\zeta'\rangle & \langle\zeta'_{x'}\zeta_x\rangle & \langle\zeta'_{x'}{}^2\rangle \end{pmatrix}. \qquad \text{(F-3)}$$

For a stationary random process, each component of Γ becomes

$$\Gamma = \begin{pmatrix} \sigma^2 & \sigma^2\gamma & 0 & \sigma^2\gamma_{x'} \\ \sigma^2\gamma & \gamma^2 & \sigma^2\gamma_x & 0 \\ 0 & \sigma^2\gamma_x & \sigma_1{}^2 & \sigma^2\gamma_{xx'} \\ \sigma^2\gamma_{x'} & 0 & \sigma^2\gamma_{xx'} & \sigma_1{}^2 \end{pmatrix} \qquad \text{(F-4)}$$

Thus

$$\begin{aligned} N^T\Gamma N = {} & N_1{}^2\sigma^2 + N_2{}^2\sigma^2+N_3{}^2\sigma_1{}^2+N_4{}^2\sigma_1{}^2+2N_1N_2\sigma^2\gamma \\ & + 2N_1N_4\sigma^2\gamma_{x'} + 2N_2N_3\sigma^2\gamma_x \\ & + 2N_3N_4\sigma^2\gamma_{xx'}. \end{aligned} \qquad \text{(F-5)}$$

Thus a typical expectation in the integrand of (4-73) can be written as

$$\langle \zeta_x e^{ik_2(a\zeta - a'\zeta')} \rangle = \frac{\partial}{\partial N_3} \langle e^{N_1\zeta + N_2\zeta' + N_3\zeta_x + N_4\zeta'_{x'}} \rangle \Big|_{\vec{N}=\vec{N}_o}$$

(F-6)

where

$$\vec{N} = (N_1,\ N_2,\ N_3,\ N_4),\ \vec{N}_o = (ik_2 a,\ -ik_2 a',\ 0,\ 0)$$

(F-7)

By (F-1),(F-6) reduces to

$$\langle \zeta_x e^{ik_2(a\zeta - a'\zeta')} \rangle = (N_3\sigma_1{}^2 + N_2\sigma^2\gamma_x + N_4\sigma^2\gamma_{xx'})$$

$$\cdot e^{(1/2)N^T\Gamma N} \Big|_{\vec{N} = \vec{N}_o}$$

$$= -ik_2 a'\sigma^2\gamma_x\ e^{(1/2)k_2{}^2\sigma^2(a'^2 - 2a'a\gamma + a^2)} \qquad \text{(F-8)}$$

Other first order expections can be evaluated similarly.

For the second order expectations we write, for example,

$$\langle \zeta_x{}^2 e^{ik_2(a\zeta - a'\zeta')} \rangle$$

$$= \frac{\partial^2}{\partial N_3{}^2} \langle e^{N_1\zeta + N_2\zeta' + N_3\zeta_x + N_4\zeta'_{x'}} \rangle \Big|_{\vec{N} = \vec{N}_o}$$

$$\langle \zeta_x\zeta'_{x'}\ e^{ik_2(a\zeta - a'\zeta')} \rangle$$

$$= \frac{\partial^2}{\partial N_3\partial N_4} \langle e^{N_1\zeta + N_2\zeta' + N_3\zeta_x + N_4\zeta'_{x'}} \rangle \Big|_{\vec{N} = \vec{N}_o}$$

(F-9)

For mixed (in x and y) second order expectations, we can then write

$$<\zeta_x \zeta_y e^{ik_2(a\zeta - a'\zeta')}>$$

$$= \frac{\partial^2}{\partial N_3 \partial N_4} <e^{N_1\zeta + N_2\zeta' + N_3\zeta_x + N_4\zeta_x}> \Bigg|_{\vec{N} = \vec{N}_o} \qquad \text{(F-10)}$$

where Γ is now given by

$$\Gamma = \begin{pmatrix} <\zeta^2> & <\zeta\zeta'> & <\zeta\zeta_x> & <\zeta\zeta_y> \\ <\zeta'\zeta> & <\zeta'^2> & <\zeta'\zeta_x> & <\zeta'\zeta_y> \\ <\zeta_x\zeta> & <\zeta_x\zeta'> & <\zeta_x^2> & <\zeta_x\zeta_y> \\ <\zeta_y\zeta> & <\zeta_y\zeta'> & <\zeta_y\zeta_x> & <\zeta_y^2> \end{pmatrix}$$

$$= \begin{pmatrix} \sigma^2 & \sigma^2\gamma & 0 & 0 \\ \sigma^2\gamma & \sigma^2 & \sigma^2\gamma_x & \sigma^2\gamma_y \\ 0 & \sigma^2\gamma x & \sigma_1^2 & 0 \\ 0 & \sigma^2\gamma_y & 0 & \sigma_1^2 \end{pmatrix} \qquad \text{(F-11)}$$

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