NASA CR-66770

AN ANALYTICAL BASIS FOR TIME-MODULATED RANDOM VIBRATION TESTING

BY T. K. HASSELMAN

TRW REPORT NO. 11282-6001-R0-00

JANUARY 1969

Prepared under Contract No. NAS 1-8503

TRW SYSTEMS

Redondo Beach, California

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ABSTRACT

This report considers the time dependent rms response of a base excited single-degree-of-freedom system to time-modulated stationary random vibration. The excitation is characterized by a power spectral density function having an arbitrary bandwidth and center frequency, and by a deterministic modulating function. Closed form solutions are presented for three modulating functions: (1) the step function, (2) the rectangular function, and (3) the decaying exponential function. An approximate solution is provided for arbitrary modulating functions.

A parameter study was made wherein parameters describing system damping, input to system bandwidth and frequency ratios, and modulating function were varied independently. Dimensionless response histories were computed and plotted for these cases. From this study it was concluded that the maximum transient rms response can differ significantly from what would be predicted by enveloping the transients with stationary levels and computing stationary response. Practical examples were suggested in which the difference could be a factor of two in either direction.

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LIST OF SYMBOLS*

E[ ]	Mean value of [ ] averaged over an ensemble of sample records
$f(t)$	Modulating function
$h(t)$	Response of a single-degree-of-freedom system to a unit impulse
i	$\sqrt{-1}$
$K_x(t_1, t_2)$	Autocovariance function of the random process $\underline{\overline{x}}(t)$
$m_x(t)$	Time dependent mean of the random process $\underline{\overline{x}}(t)$
N	Number of steps in a staircase function
p	Center frequency of a narrow-band type power spectral density function
$R_x(t_1, t_2)$	Autocorrelation function of the random process $\underline{\overline{x}}(t)$ (Similar definitions apply for $\underline{\overline{y}}(t)$ and $\underline{\overline{z}}(t)$.)
$R_z^*(t_1-\xi_1, t_2-\xi_2)$	Cross correlation function of two processes turned on at $t = \xi_1$ and $t = \xi_2$, respectively
$S_x(\Omega)$	The two sided power spectral density function of the stationary random process $\underline{\overline{x}}(t)$
t	Independent time parameter
t_o	Cut-off time of the rectangular modulating function
$u(t)$	The unit step function
v_i, $i = 1, 2$	Dummy variables

*Other symbols representing combined constants and time functions appearing in Equations (2) - (6) are defined on pages 9 to 12.

$\overline{\underline{x}}(t)$	A stationary random process
$\overline{\underline{y}}(t)$	A time modulated stationary process
$\overline{\underline{z}}(t)$	The response process for a simple mechanical oscillator
α	The decay rate of the autocorrelation function defined for $\underline{x}(t)$
β	The decay rate of the system impulse response function
ζ	Critical damping ratio for the system
λ	Decay rate of the exponential modulating function
ξ_i	Times at which steps occur in the staircase function
$\overline{\sigma}_x$	The constant rms value of the stationary process $\overline{\underline{x}}(t)$
$\sigma_z(t)$	The time dependent rms response to a time-modulated stationary process
$\overline{\sigma}_z$	The asymptotic value of rms response to a step input of $\overline{\underline{x}}(t)$
τ	Correlation interval, $t_2 - t_1$
τ_1, τ_2	Dummy variables
Ω	Independent frequency parameter
ω	Damped natural frequency of the system
ω_o	Undamped natural frequency of the system

AN ANALYTICAL BASIS FOR TIME-MODULATED RANDOM VIBRATION TESTING

I. INTRODUCTION

Many environments observed in nature are characterized by a limited number of oscillations of a fairly high level. Examples of this class of environments may be measured within a spacecraft during launch, within a transporter driving over rough roads or within a building during an earthquake. The dynamic structure in question -- spacecraft, transporter, or building -- is generally expected to survive a succession of these environments during its service life. The variety of factors influencing each repetition of the environment will create a randomness in the parameters describing the environment, such as frequency content and level.

The design of a laboratory test to adequately simulate these environments is hampered by the limitations of present test equipment and test philosophy. Of choices currently available, a sum of modulated sinusoids appears most appropriate from the standpoint of simulating the desired waveform. Successive application of such a test, however, to simulate the structure's service life has one obvious shortcoming. It fails to account for the anticipated randomness in the service environment.

Time-modulated or shaped random excitation has been considered for the simulation of some of these environments [1,2,3,4].* Application has been rather limited however and no attempt has been made to systematically

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*Numbers in square brackets refer to references at end of the report.

investigate the isolated effects which clearly defined system and input parameters have on the fundamental characteristics of transient rms response. Such an investigation is an important prerequisite to further consideration of this simulation for testing purposes. The primary objective of this study is therefore to provide an analytical basis for further investigation. In so doing, emphasis has been placed on a formulation compatible with present methods of data analysis and laboratory test capabilities.

The analytical procedure discussed herein is based on the transient rms* response of a linear damped mechanical oscillator to a suddenly applied stationary process. This problem was considered by Caughey and Stumpf [5] for a stationary process having an arbitrary power spectrum, but only results for a white process were presented. The notion of response to a suddenly applied stationary input provides a conceptual tie between stationary and nonstationary response in that the transient rms response asymptotically approaches the stationary value after the initial event.

The next part of the analysis concerns the sum of two such step inputs of a stationary process to create a rectangular modulation. This concept is then generalized to include the sum of N step inputs. In this way arbitrary modulating functions are approximated by staircase functions. Reasonably good approximations of transient rms response result from rather crude approximations to the modulating function since response is derived by integrating over the modulating function.

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*All statistical averages discussed in this report are ensemble averages rather than time averages since the operations apply to nonstationary processes.

The analysis is developed in Section II. Idealizations of both structure and environment are first discussed. The integral expressions for the rms response are then presented for each of three different modulating functions. The closed form solutions to these integrals follow. Section III contains a discussion of the parameter study made to investigate the behavior of rms response for each of the three modulating functions. An approximate method to compute rms response for arbitrary modulating functions is developed and discussed in Section IV. A comparison is made with the exact solution derived for the exponential modulating function.

II. ANALYSIS

1. Idealizations

The stated objectives are best served by postulating suitable idealizations for the system and its dynamic environment. Many approaches to structural dynamics analysis rely on the notions of the single-degree-of-freedom system. Among these are analysis by response spectra and the normal mode method. Because of its simplicity and fundamental importance, the single-degree-of-freedom structural model is adopted here. Such a system is specified by its Green's function or impulse response function

$$h(t) = u(t)\left(\frac{1}{\omega} e^{-\beta t} \sin \omega t\right),$$

where

$$\beta = \zeta\omega_o$$

$$\omega = \omega_o\sqrt{1-\zeta^2} \text{ , the damped natural frequency}$$

$$\zeta = \text{fraction of critical damping}$$

$$\omega_o = \text{undamped natural frequency}$$

$$u(t) = \begin{Bmatrix} 0 : & t < 0 \\ 1 : & t \geq 0 \end{Bmatrix} \text{ the unit step function.}$$

A suitable idealization is also sought for the input. It is desirable to optimize the trade-off between simplicity and flexibility. A number of models for nonstationary processes have been investigated. Among these are (1) a finite sum of time-modulated harmonics with random phasing [6], (2) filtered shot noise with time dependent intensity functions [1,3,4] and (3) time-modulated filtered white processes [2,3]. The relationship between

(2) and (3) has been discussed in Reference [3].

A somewhat different technique suggests itself for the simulation of transient random vibration environments in the laboratory. Since the random vibration consoles now used for stationary tests are capable of producing virtually any power spectral density or p.s.d. shape by summing a number filtered white noise processes, it is of interest to consider that class of nonstationary process which can be generated from a stationary process of arbitrary p.s.d. and modulating function. In practice, therefore, one is not restricted to only those p.s.d.'s which correspond to a single filter output. For the purpose of analysis one is not restricted to the exact frequency response characteristics of a single common filter if a simpler expression can be found to specify the second order stochastics of a stationary process.

A stochastic process will be denoted by $\overline{\underline{x}}(t)$, after Barnes*, to distinguish it from deterministic functions written without the double bars. The mean of $\overline{\underline{x}}(t)$ is given by

$$m_x(t) = E[\overline{\underline{x}}(t)] \quad ,$$

and its autocorrelation function by

$$R_x(t_1, t_2) = E[\overline{\underline{x}}(t_1)\, \overline{\underline{x}}(t_2)].$$

If $m_x(t) = 0$ then $R_x(t_1,t_2) = K_x(t_1,t_2)$, the autocovariance function of $\overline{\underline{x}}(t)$. If $\overline{\underline{x}}(t)$ is stationary and ergodic then $R_x(t_1,t_2) = R_x(\tau)$ where

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$\tau = t_2 - t_1$, the correlation interval.

A time-modulated process $\underline{\bar{y}}(t)$ may be specified where

$$\underline{\bar{y}}(t) = f(t)\ \underline{\bar{x}}(t).$$

The modulating function f(t) is deterministic and $\underline{\bar{x}}(t)$ is a stationary ergodic process with zero mean. The process $\underline{\bar{y}}(t)$ is nonstationary. Then

$$m_y(t) = E[f(t)\ \underline{\bar{x}}(t)] = f(t)\ E[\underline{\bar{x}}(t)] = f(t)\ m_x(t) = 0.$$

Similarly

$$R_y(t_1,t_2) = f(t_1)\ f(t_2)\ R_x(t_2 - t_1).$$

The stationary autocorrelation function for the response of a second order filter to white noise excitation is

$$R(\tau) = e^{-\alpha|\tau|} (\cos p\tau - \frac{\alpha}{p} \sin p\ |\tau|)\ ,$$

where

$\tau = t_2 - t_1$, the correlation interval

$\alpha = \xi\ p_o$

$p = p_o \sqrt{1-\xi^2}$

ξ = fraction of critical damping

p_o = undamped filter frequency

Considerable simplification is achieved by dropping the second term. In fact, when $\alpha << p$ the second term may be neglected. This is the case of a lightly damped second order filter. As α/p grows large the importance of the oscillatory term diminishes and the second order filter tends to look more like a first order filter.

The stationary autocorrelation function

$$R_x(\tau) = e^{-\alpha|\tau|} \cos p\tau$$

describes a non-white process of arbitrary bandwidth and center frequency. The Fourier transform of $R_x(\tau)$ yields the corresponding p.s.d. function $S(\Omega)$ which is

$$S_x(\Omega) = \frac{\alpha}{\pi} \left\{ \frac{\alpha^2 + p^2 + \Omega^2}{[\alpha^2 + (p-\Omega)^2]\ [\alpha^2 + (p+\Omega)^2]} \right\}.$$

This function has the same basic characteristics as that for the response of the second order filter to white noise described above. That is for $\alpha << p$, the half power point bandwidth is approximately 2α, and the bandwidth and center frequency can be varied independently permitting the evaluation of a continuous range of individual cases ranging from a pure sinusoid to pure white noise and any center frequency of interest.

Various modulating functions have also been considered. They include the step, rectangular and half sine functions which have been applied to white noise processes [2,5]. An increasing-decreasing exponential function has been applied to the filtered white noise process [3]. The latter was adopted for earthquake simulation problems where α was of the same order as p and where the modulating function varried rather slowly in time.

Three modulating functions f(t) are considered in this section,

$$f(t) = u(t) \quad , \text{ the step function}$$

$$f(t) = u(t) - u\,(t-t_o), \quad \text{the rectangular function}$$

$$f(t) = u(t)e^{-\lambda t} \quad ; \text{ the decaying exponential}$$

2. The Variance of Structural Response

The equation of motion for the displacement response z(t) of a base excited single-degree-of-freedom system with viscous damping is

$$\ddot{z}(t) + 2\beta\,\dot{z}(t) + \omega_o^2\;z(t) = -y(t)$$

where y(t) denotes base acceleration. The solution of the equation for z(t) given zero initial conditions is

$$z(t) = -\int_{-\infty}^{\infty} h(t-\tau)\;y(\tau)\;d\tau$$

where τ is used here as a dummy variable and does not denote correlation interval. The autocorrelation function of the response is given by

$$R_z(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1-\tau_1)\;h(t_2-\tau_2)\;f(\tau_1)\;f(\tau_2)\;R_x(\tau_2-\tau_1)\;d\tau_1 d\tau_2 \qquad (1)$$

The variance of response is obtained by setting $t_1 = t_2 = t$ and then evaluating the integral. These integrals and their solutions are presented for the three modulating functions. The results of the integrations are in dimensionless form where the following constants and functions are defined to simplify the notation:

$$a = \frac{\beta-\alpha}{\omega} \qquad A_{11} = a^2 + p_1^{\,2}$$

$$b = \frac{\beta+\alpha}{\omega} \qquad A_{12} = a^2 + p_2^{\,2}$$

$$p_1 = \frac{\omega+p}{\omega} \qquad A_{21} = b^2 + p_1^{\,2}$$

$$p_2 = \frac{\omega-p}{\omega} \qquad A_{22} = b^2 + p_2^{\,2}$$

$$\overline{A} = -\frac{2}{(a+b)^2+4}\left[\frac{1}{2}\left(\frac{p_1}{A_{11}} + \frac{p_2}{A_{12}}\right) + \frac{a}{2(a+b)}\left(\frac{1}{A_{11}} + \frac{1}{A_{12}}\right)\right] - \overline{D}_3 - \overline{D}_4$$

$$\overline{B} = \frac{a}{2(a+b)}\left(\frac{1}{A_{11}} + \frac{1}{A_{12}}\right)$$

$$\overline{C}_1 = \frac{1}{(a+b)^2+4}\left[\frac{a+b}{2}\left(\frac{p_1}{A_{11}} + \frac{p_2}{A_{12}}\right) + a\left(\frac{1}{A_{11}} + \frac{1}{A_{12}}\right)\right]$$

$$\overline{C}_2 = \frac{1}{(a+b)^2+4}\left[\left(\frac{p_1}{A_{11}} + \frac{p_2}{A_{12}}\right) - a(a+b)\left(\frac{1}{A_{11}} + \frac{1}{A_{12}}\right)\right]$$

$$\overline{D}_1 = -\frac{1}{2}\left(\frac{p_1\ b}{A_{11}\ A_{21}} + \frac{p_2\ b}{A_{12}\ A_{21}} + \frac{a\ p_1}{A_{12}\ A_{21}} - \frac{a\ p_1}{A_{11}\ A_{21}}\right)$$

$$\overline{D}_2 = -\frac{1}{2}\left(\frac{p_1\ b}{A_{11}\ A_{22}} + \frac{p_2\ b}{A_{12}\ A_{22}} - \frac{a\ p_2}{A_{12}\ A_{22}} + \frac{a\ p_2}{A_{11}\ A_{22}}\right)$$

$$\overline{D}_3 = -\frac{1}{2}\left(\frac{{p_1}^2}{A_{11}\ A_{21}} + \frac{p_1\ p_2}{A_{12}\ A_{21}} - \frac{ab}{A_{12}\ A_{21}} + \frac{ab}{A_{11}\ A_{21}}\right)$$

$$\overline{D}_4 = -\frac{1}{2}\left(\frac{p_1\ p_2}{A_{11}\ A_{22}} + \frac{{p_2}^2}{A_{12}\ A_{22}} + \frac{ab}{A_{12}\ A_{22}} - \frac{ab}{A_{11}\ A_{22}}\right)$$

$$\overline{E}_{11} = \frac{ab - {p_1}^2}{2A_{11}A_{21}} \qquad \overline{E}_{21} = \frac{ab + p_2p_1}{2A_{12}A_{21}}$$

$$\overline{E}_{12} = \frac{ab + p_1p_2}{2A_{11}\ A_{22}} \qquad \overline{E}_{22} = \frac{ab - {p_2}^2}{2A_{12}\ A_{22}}$$

$$\overline{E}_{13} = \frac{ap_1 + p_1b}{2A_{11}\ A_{21}} \qquad \overline{E}_{23} = \frac{ap_1 - p_2b}{2A_{12}\ A_{21}}$$

$$\overline{E}_{14} = \frac{ap_2 - p_1 b}{2A_{11}\ A_{22}} \qquad \overline{E}_{24} = \frac{ap_2 + p_2 b}{2A_{12}\ A_{22}}$$

$$\overline{E}_{31} = \frac{p_1 b + ap_1}{2A_{11}\ A_{21}} \qquad \overline{E}_{41} = \frac{p_2 b - bp_1}{2A_{12}\ A_{22}}$$

$$\overline{E}_{32} = \frac{p_1 b - ap_2}{2A_{11}\ A_{22}} \qquad \overline{E}_{42} = \frac{p_2 b + bp_2}{2A_{12}\ A_{22}}$$

$$\overline{E}_{33} = \frac{p_1^{\,2} - ab}{2A_{11}\ A_{21}} \qquad \overline{E}_{43} = \frac{p_1 p_2 + b^2}{2A_{12}\ A_{21}}$$

$$\overline{E}_{34} = \frac{p_1 p_2 + ab}{2A_{11}\ A_{22}} \qquad \overline{E}_{44} = \frac{p_2^{\,2} - b^2}{2A_{12}\ A_{22}}$$

$$B(t) = e^{-(a+b)\omega t}$$

$$C_1(t) = e^{-(a+b)\omega t} \sin 2\omega t$$

$$C_2(t) = e^{-(a+b)\omega t} \cos 2\omega t$$

$$D_1(t) = e^{-b\omega t} \sin p_1\ \omega t$$

$$D_2(t) = e^{-b\omega t} \sin p_2\ \omega t$$

$$D_3(t) = e^{-b\omega t} \cos p_1\ \omega t$$

$$D_4(t) = e^{-b\omega t} \cos p_2 \omega t$$

$$E_1(t) = e^{-a\omega t} \sin p_1 \omega t$$

$$E_2(t) = e^{-a\omega t} \sin p_2 \omega t$$

$$E_3(t) = e^{-a\omega t} \cos p_1 \omega t$$

$$E_4(t) = e^{-a\omega t} \cos p_2 \omega t$$

a. <u>The Step Function: $f(t) = u(t)$</u>

The variance of response to a step modulated input is derived from the autocorrelation function as follows:

$$R_z(t_1, t_2)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t_1-\tau_1)\, h(t_2-\tau_2)\, u(\tau_1)\, u(\tau_2)\, R_x(\tau_2-\tau_1)\, d\tau_1\, d\tau_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(v_1)\, h(v_2)\, u(t_1-v_1)\, u(t_2-v_2)\, R_x\, [(t_2-v_2)-(t_1-v_1)]\, dv_1 dv_2$$

where the change of variables $v_1 = t_1 - \tau_1$ and $v_2 = t_2 - \tau_2$ has been made. Upon letting $t_1 = t_2 = t$,

$$\sigma_z^{\,2}(t) = R_z(t,t) = \int_o^t \int_o^t h(v_1)\, h(v_2)\, R_x(v_1 - v_2)\, dv_1\, dv_2 .$$

In dimensionless form the solution is

$$\omega^4\, \sigma_z^{\,2}(t) = \overline{A} + \overline{B}\, B(t) + \sum_{n=1}^{2} \overline{C}_n\, C_n(t) + \sum_{n=1}^{4} \overline{D}_n\, D_n(t) \qquad (2)$$

b. <u>The Rectangular Function: $f(t) = u(t) - u(t-t_o)$</u>

It is useful to approach the derivation of $\sigma^2_{\,z}(t)$ for this modulating function from a somewhat more general point of view. In so doing, two processes are considered, one applied to the system at $t = \xi_1$ and the other at $t = \xi_2$. The cross correlation of response for $t > \xi_1$, $t > \xi_2$ will be denoted by $R_z^*(t_1-\xi_1,\ t_2-\xi_2)$ where

$$R_z^*(t_1-\xi_1,\ t_2-\xi_2)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t_1-\tau_1)\, h(t_2-\tau_2)\, u(\tau_1-\xi_1)\, u(\tau_2-\xi_2)\, R_x(\tau_2-\tau_1)\, d\tau_1\, d\tau_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(v_1)\, h(v_2)\, u(t_1-\xi_1-v_1)\, u(t_2-\xi_2-v_2)\, R_x[(t_2-v_2)-(t_1-v_1)]\,dv_1 dv_2$$

$$= \int_o^{t-\xi_2} \int_o^{t-\xi_1} h(v_1)\, h(v_2)\, R_x(v_1 - v_2)\, dv_1\, dv_2$$

when $t_1 = t_2 = t$. The response $\sigma_z^2(t)$ for the rectangular input is identical to that for the step input prior to $t = t_o$. For $t>t_o$ it is easily shown that

$$\sigma_z^2(t) = R_z(t,t)$$

$$= R_z^*(t,t) - 2R_z^*(t,\ t-t_o) + R_z^*(t-t_o,\ t-t_o).$$

The first and last terms are known from (a). The general solution for $R_z^*(t-\xi_1,\ t-\xi_2)$ is

$$R_z^*(t-\xi_1,\ t-\xi_2)$$

$$= R_z^*(t-\xi_2,\ t-\xi_2) + \frac{1}{2} \sum_{n=1}^{4} \overline{D}_n\, [D_n(t-\xi_1) - D_n(t-\xi_2)]$$

$$+ \frac{1}{2} \sum_{n=1}^{4} \sum_{m=1}^{4} \overline{E}_{nm}\, E_n(t-\xi_2)\, [D_m(t-\xi_1) - D_m(t-\xi_2)] \qquad (3)$$

After replacing ξ_1 and ξ_2 by 0 and t_o, respectively and cancelling terms it is found that for $t>t_o$,

$$\omega^4 \sigma_z^2(t) = \bar{B} [B(t) - B(t-t_o)] + \sum_{n=1}^{2} \bar{C}_n [C_n (t) - C_n (t-t_o)]$$

$$- \sum_{n=1}^{4} \sum_{m=1}^{4} \bar{E}_{nm} E_n(t-t_o) [D_m(t) - D_m(t-t_o)] \quad (4)$$

c. <u>The Exponential Function: $f(t) = u(t)e^{-\lambda t}$</u>

In this case, noting the similarity between the step input and the decaying exponential, one can write

$$\sigma_z^2(t) = \int_o^t \int_o^t h(v_1) h(v_2) e^{-\lambda(t-v_1)} e^{-\lambda(t-v_2)} R_x(v_1-v_2) dv_1 dv_2$$

But,

$$h(v_1) h(v_2) = \frac{1}{\omega^2} e^{-\beta(v_1+v_2)} \sin \omega v_1 \sin \omega v_2 .$$

Since $e^{-2\lambda t}$ can be taken out of the integral,

$$\sigma_z^2(t) = e^{-2\lambda t} \int_o^t \int_o^t \frac{1}{\omega^2} e^{-(\beta-\lambda)(v_1+v_2)} \sin \omega v_1 \sin \omega v_2 R_x(v_1-v_2) dv_1 dv_2.$$

The solution follows immediately from (a) where β is replaced by $\beta-\lambda$.

$$\omega^4 \; \sigma_z^2(t) \;=\; e^{-2\lambda t} \left\{ \overline{A} + \overline{B}\; B(t) + \sum_{n=1}^{2} \overline{C}_n \; C_n(t) + \sum_{n=1}^{4} \overline{D}_n \; D_n(t) \right\} \tag{5}$$

where in this case

$$a \;=\; \frac{\beta-\lambda-\alpha}{\omega} \quad , \quad b \;=\; \frac{\beta-\lambda-\alpha}{\omega}$$

III. PARAMETER STUDY

For the purpose of this study key parameters were chosen which combine the basic properties of the system and the excitation. They are ζ, α/β, p/ω, $\omega t_o/2\pi$ and λ/β. The first three correspond to the fraction of critical damping in the system, the input to system bandwidth ratio, and the input to system center frequency ratio respectively. The last two terms express the cut-off time of the rectangular modulating function in number of natural periods of system oscillation, and the ratio of input to system decay rates in the case of the exponential modulating function.

The ranges for these parameters were chosen so as to bracket most cases of interest while at the same time providing an indication of trends in the response characteristics as a function of parameter variation. Thus three parameter matrices were created, one for each of the modulating functions. All possible combinations of these choices were considered. The rms response was computed and plotted for each case. These plots are presented in the appendix.

1. Parameter Matrices

The Step Function

Both light and heavy system damping were considered. Narrow, intermediate and broad-band excitation were considered as well as frequency ratios of less than, equal to and greater than one.

$$\zeta = \begin{cases} .01 \\ .1 \end{cases}$$

$$\alpha/\beta = \begin{cases} \left.\begin{matrix} 1 \\ 10 \\ 100 \end{matrix}\right\} & \text{when } \zeta = .01 \\ \left.\begin{matrix} .1 \\ 1 \\ 10 \end{matrix}\right\} & \text{when } \zeta = .1 \end{cases}$$

$$p/\omega = \begin{cases} .5 \\ 1 \\ 2 \end{cases}$$

This matrix contains a total of 18 cases.

The Rectangular Function

The same choices for ζ, α/β and p/ω were retained for five values of cut-off time. The first four illustrate the dependence of residual response on cut-off time within a particular fundamental response period. A period early in the forced response was chosen simply to demonstrate the worst case effects. After a sufficiently long time the residual response becomes independent of cut-off time. The fifth cut-off time was chosen such that the response is stationary prior to cut-off. The parameter $\omega t_o/2\pi$ was therefore assigned the values

$$\omega t_o/2\pi = \left\{ \begin{array}{lll} 1.25 & & \\ 1.5 & & \\ 1.75 & & \\ 2.0 & & \\ 100 & \text{when } \zeta & = .01 \\ 10 & \text{when } \zeta & = .1 \end{array} \right.$$

This matrix contains a total of 90 cases.

The Exponential Function

Again keeping the same choices for ζ, α/β and p/ω, two values were considered for λ/β.

$$\lambda/\beta = \left\{ \begin{array}{ll} \left.\begin{array}{r} 1 \\ 10 \end{array}\right\} & \text{when } \zeta = .01 \\ \left.\begin{array}{r} .1 \\ 1 \end{array}\right\} & \text{when } \zeta = .1 \end{array} \right.$$

These correspond to decaying exponentials which drop from 1.0 to less than .05 in 50 and 5 response periods, respectively. This matrix contains a total of 36 cases.

The plots from these cases were used to infer trends in the behavior of transient response. Additional cases were then considered to confirm and clarify these trends. The results of the parameter study are discussed in the following section.

2. Discussion of Results

Either one of two normalizing factors have been used in plotting. They are $\bar{\sigma}_x$, the rms value of $\bar{x}(t)$, and $\bar{\sigma}_z$, the asymptotic rms value of $\overline{\underline{z}}(t)$ due to a step input of $\overline{\underline{x}}(t)$. The former was used for all cases in the basic parameter study and therefore applies consistently throughout the appendix. The latter is used in this section to help distinguish between the stationary and nonstationary response characteristics. By making the stationary response always unity, emphasis is placed on the transient response characteristics.

The Step Function

Figures 1, 2 and 3 typify the response of a mechanical oscillator to step inputs of the stationary process under consideration. The rms response is, of course, non-negative at all times. Its time history begins at the origin and approaches a constant value asymptotically. This value must be the stationary rms response to the stationary part of the input and is in agreement with the results obtained by evaluating the integral

$$\bar{\sigma}_z^{\,2} = \int_{-\infty}^{\infty} |H(i\Omega)|^2 \, S_x(\Omega) \, d\Omega$$

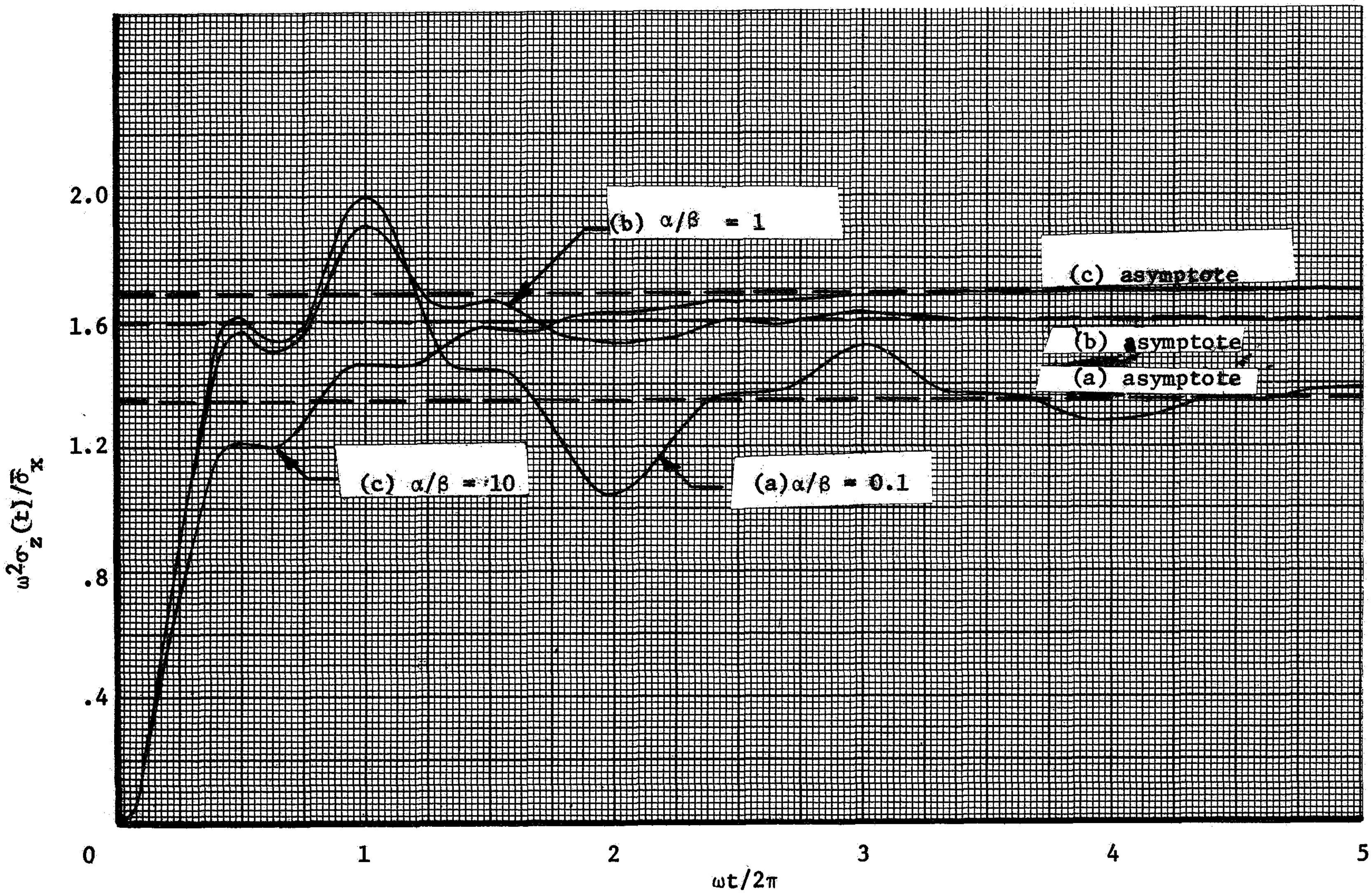

Figure 1 RMS Response for $f(t) = u(t), \zeta = 0.1, p/\omega = 0.5$

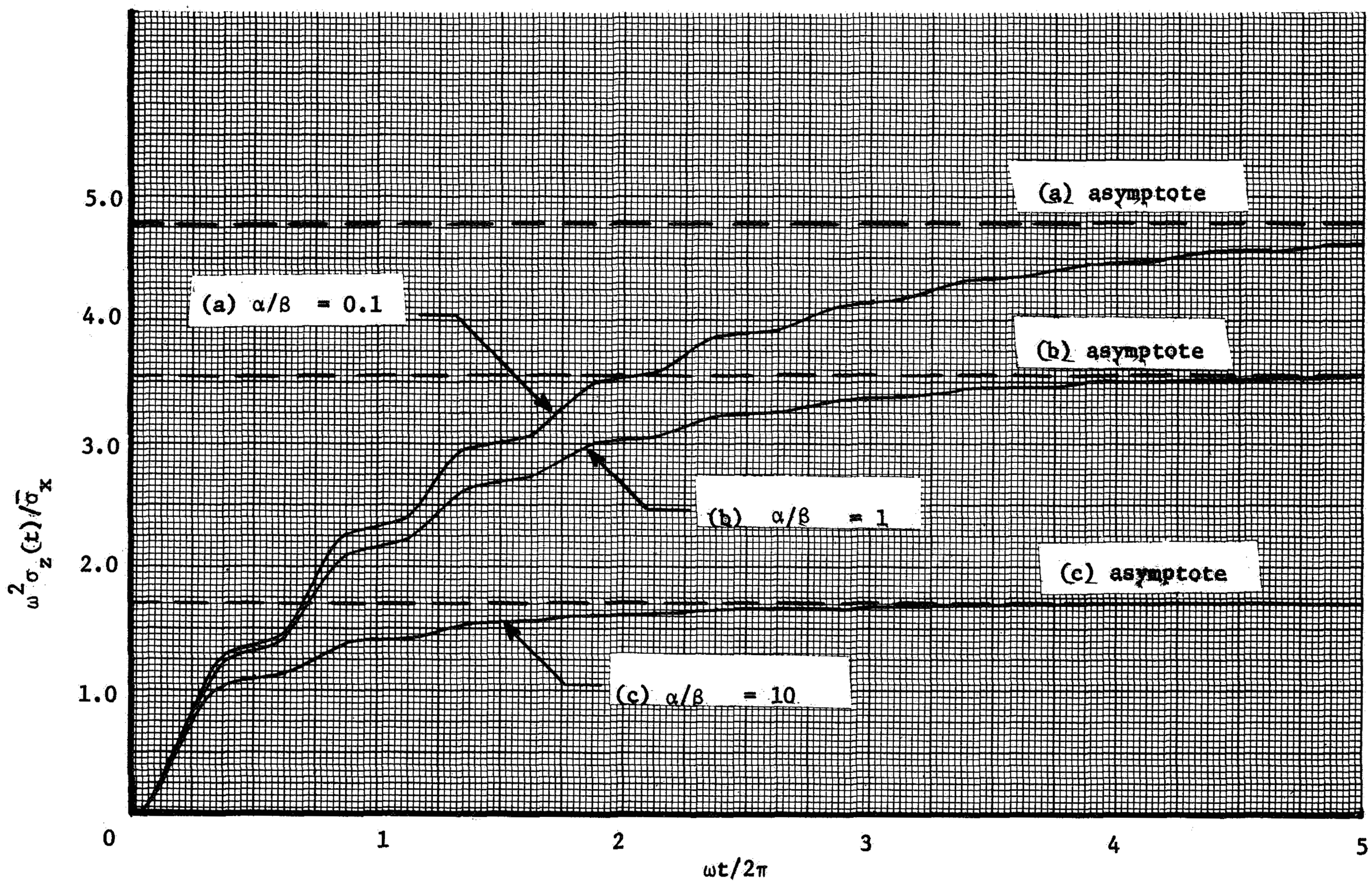

Figure 2 RMS Response for $f(t) = u(t)$, $\zeta = 0.1$, $p/\omega = 1$

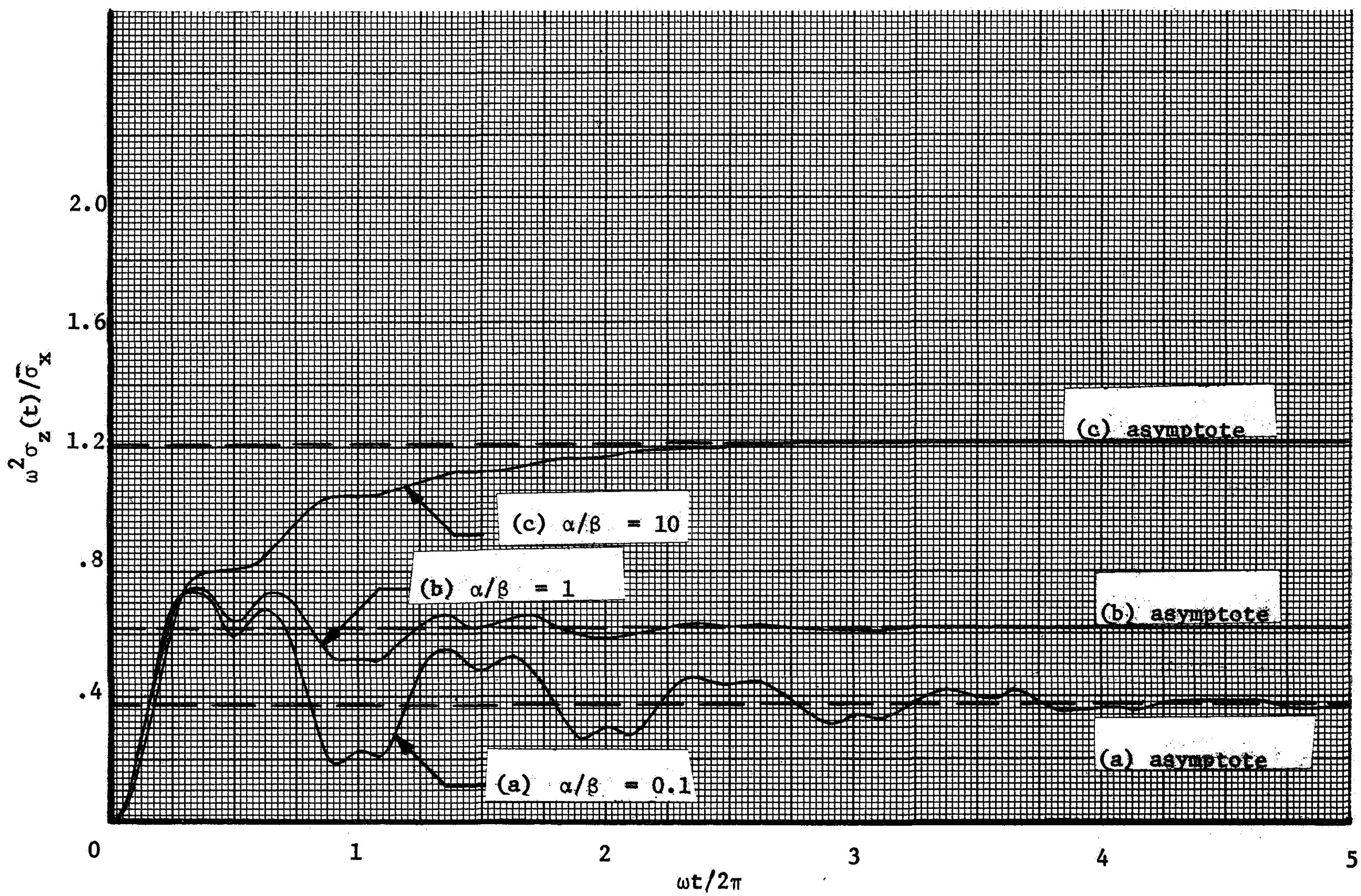

Figure 3 RMS Response for $f(t) = u(t)$, $\zeta = 0.1$, $p/\omega = 2$

commonly used to determine stationary response. The single bar notation $\bar{\sigma}$ is used to denote a constant or stationary value. It is noted in passing that these levels are affected by the input parameters as expected. That is, the area under the product of the two functions $|H(i\Omega)|^2$ and $S_x(\Omega)$ (examples of which are sketched in Figure 4) increases as $p/\omega \to 1$. It increases (up to a point) as α increases for $p/\omega \neq 1$, and decreases as α increases for $p/\omega = 1$. Since $\int_{-\infty}^{\infty} S_x(\Omega)\,d\Omega = 1$, the asymptotic value depends on how that unit area is distributed with respect to the frequency response function of the system.

One of the most outstanding features of these plots is that for narrow-band excitation at a frequency different from system resonance the transient rms response overshoots its asymptotic value. The amount of overshoot can be quite large as evidenced by Figure 3, where for $\zeta = .1$, $p/\omega = 2$ and $\alpha/\beta = .1$, the transient response exceeds the stationary by a factor of 2. The practical significance of this feature depends on the practical significance of the input parameters, particularly on the bandwidth of the input. It is certainly conceivable that such a narrow-band excitation could appear, for example, on the bed of a truck driving over a washboard road. The resonant frequency of a properly isolated payload would be well below the expected frequency of that environment. Consequently, transient displacements might easily exceed those of steady state. Furthermore, these transients are usually of short duration lasting for perhaps only a few periods of payload response so that stationarity is never reached.

Decreasing system damping tends to increase system response. The asymptotic value goes up, the amplitudes of the transient oscillations

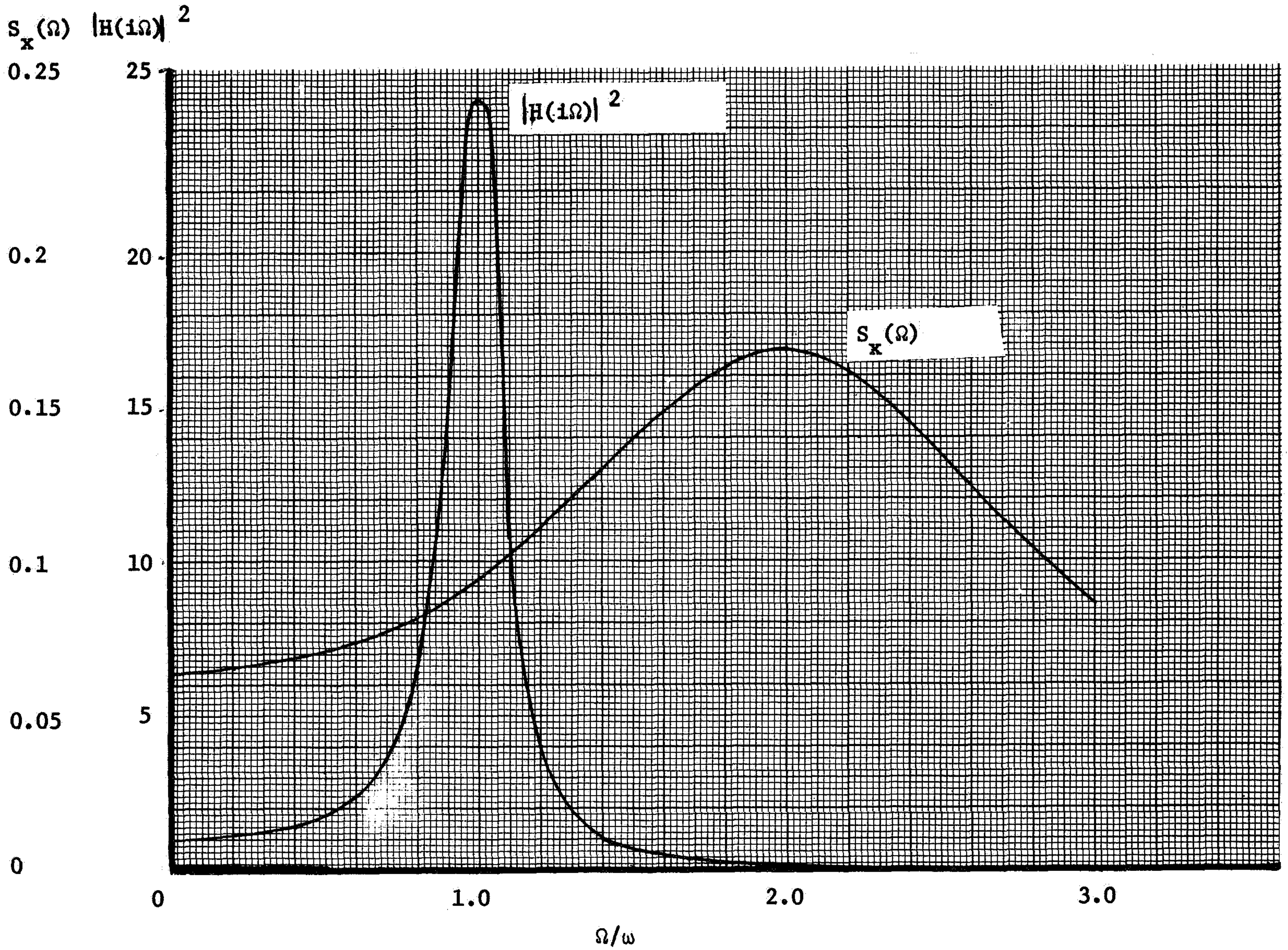

Figure 4 Relationship of Power Spectral Density to Frequency Response Functions

increase and they decay less rapidly. These characteristics are shown in Figure 5.

Variations of the input parameters α and p not only affect the asymptotic response but the initial transients as well. Generally speaking, an increase in α causes stationarity to be reached more quickly as shown in Figure 6 where $\overline{\sigma}_z$ is used as the normalizing factor. Mathematically, this is explained by the presence of the decay term $e^{-(\alpha+\beta)t}$ as a multiplier in Equation (2). This fact is of practical importance when the duration of the input is limited as in the case of a rectangular or an exponential modulating function for example. For then, peak rms response may be affected by the rate of transient decay.

Variations of the parameter p cause another interesting effect. In Figure 2 it is seen that when $p/\omega = 1$ the only oscillatory transient has a frequency of 2ω and that transient overshoot never occurs even for very narrow-band excitation. It can be seen in Equation (2) that in general three frequency components are present: 2ω, $\omega - p$ and $\omega + p$. For example, all three can be seen in Figure 7. As $p/\omega \to 1$, $\omega - p$ becomes small which tends to set up a beating effect as shown in Figure 8. When $p/\omega = 1$ this frequency becomes zero so that the oscillation disappears.

The Rectangular Function

The most important conclusion regarding the response of an oscillator to a rectangular modulation of a stationary process is that the residual response may exceed the forced response, even when the forced response exceeds its asymptotic level. This feature is evidenced in Figure 9 and

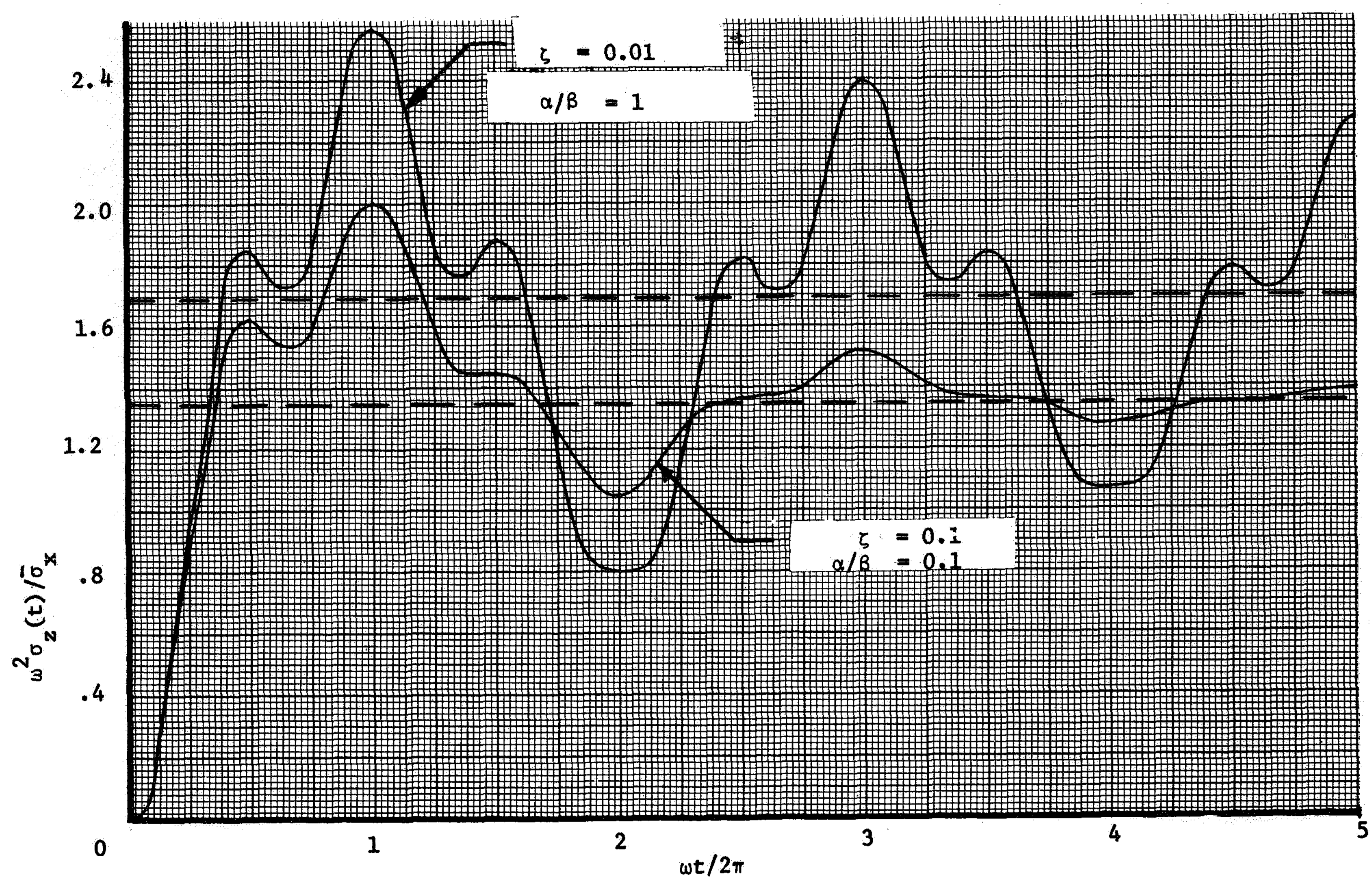

Figure 5 RMS Response for $f(t) = u(t)$, $p/\omega = 0.5$

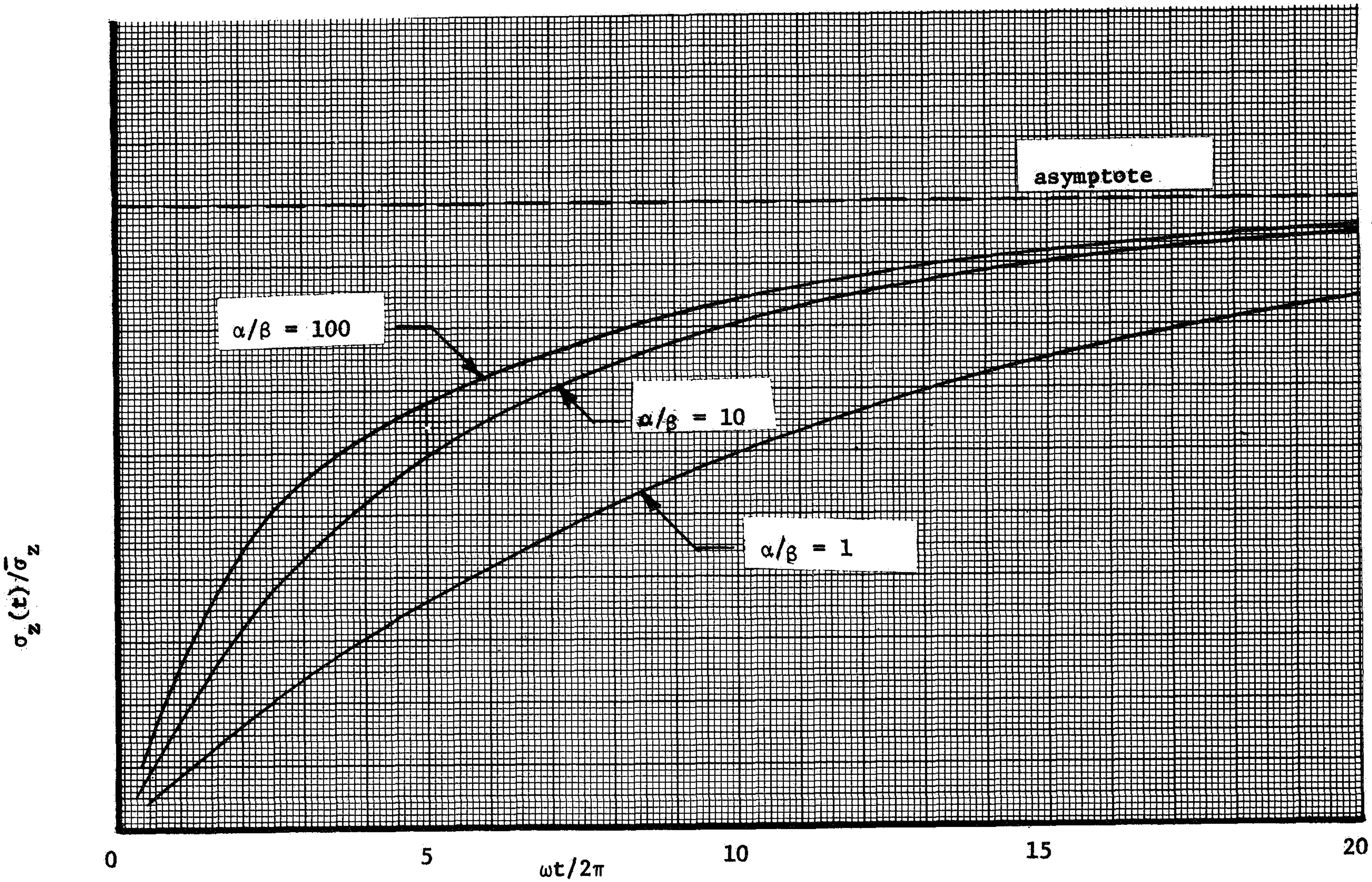

Figure 6 RMS Response for $f(t) = u(t)$, $\zeta = 0.01$, $p/\omega = 1$

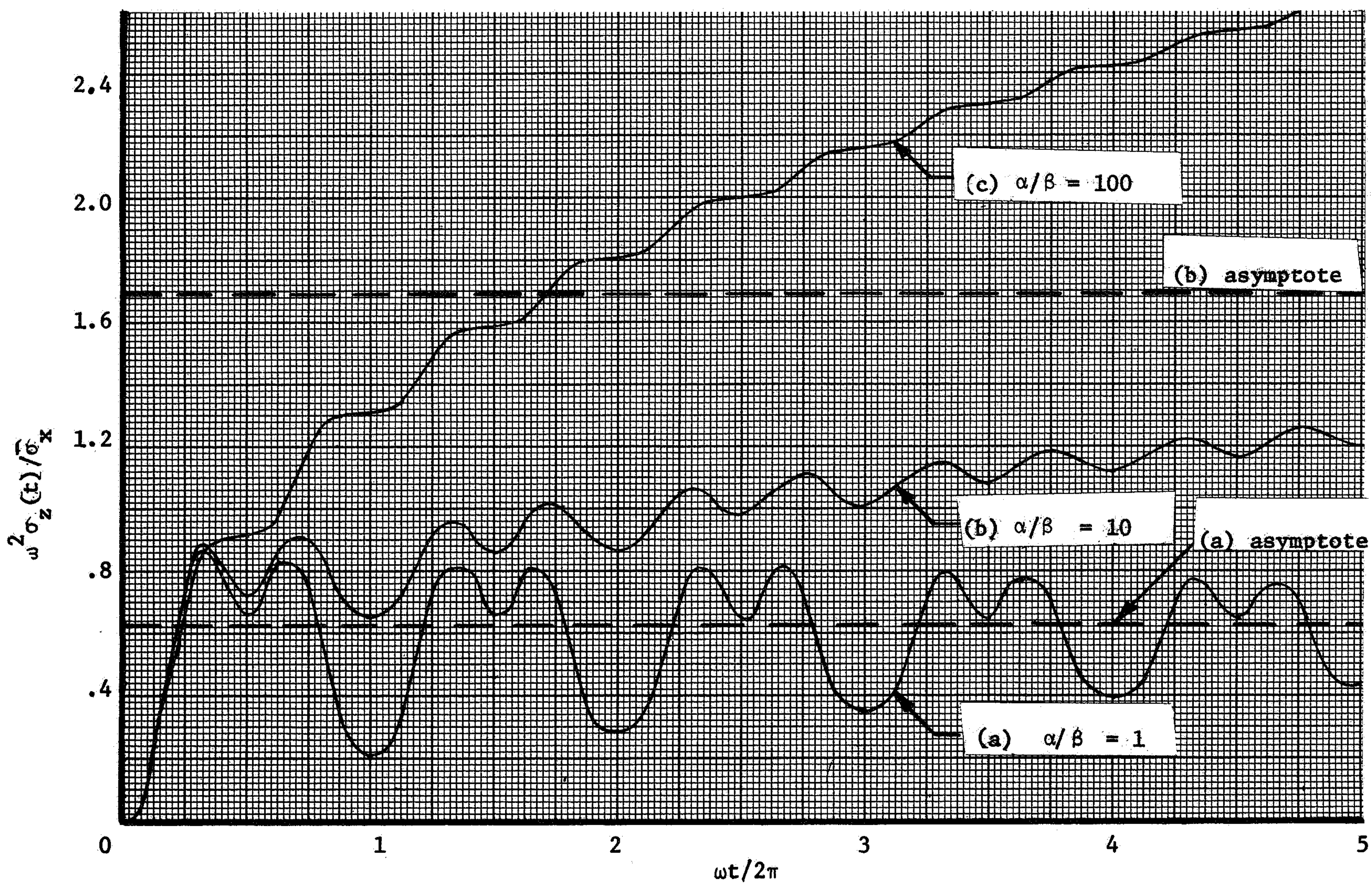

Figure 7 RMS Response for $f(t) = u(t)$, $\zeta = 0.01$, $p/\omega = 2$

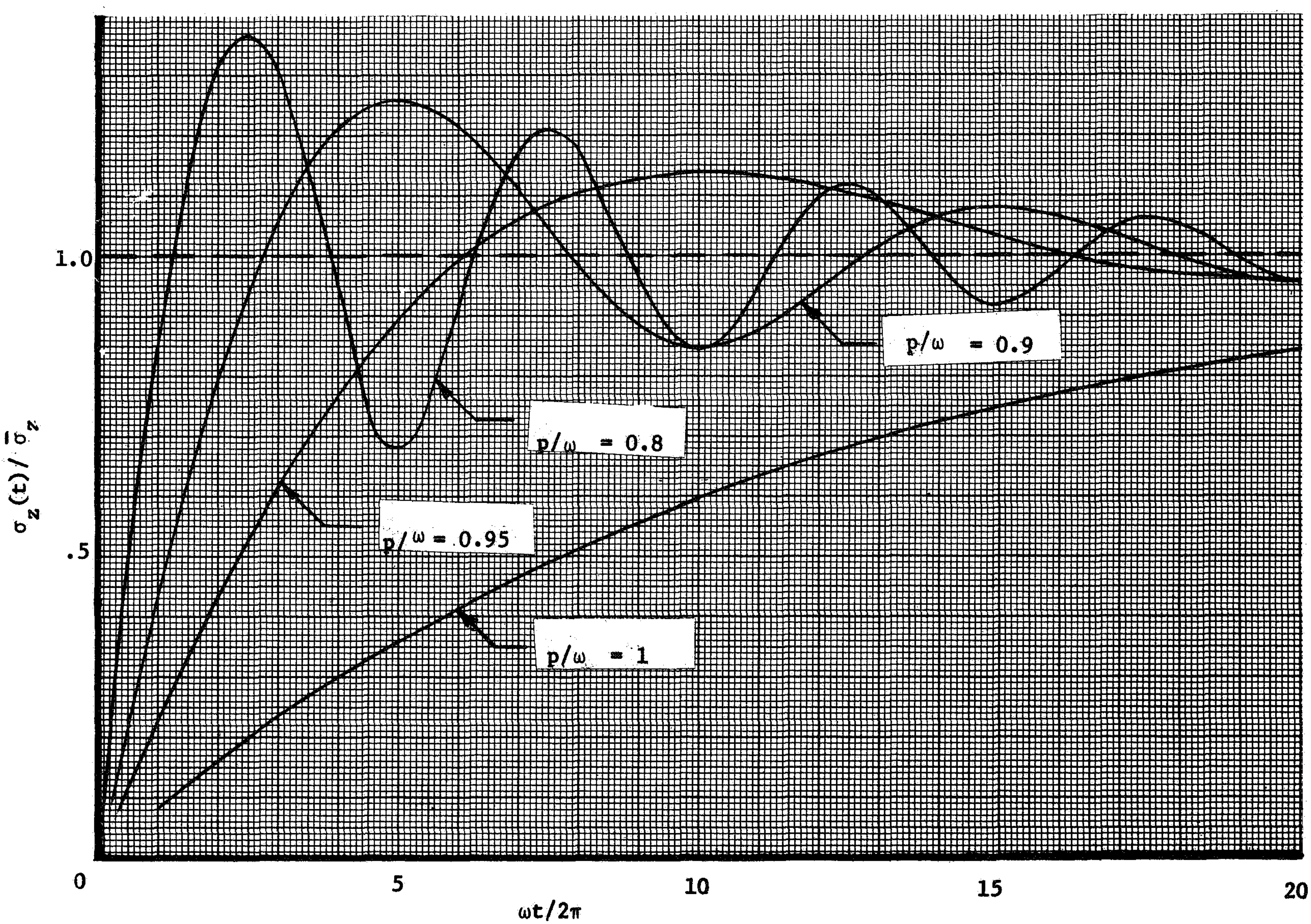

Figure 8 RMS Response for $f(t) = u(t)$, $\zeta = 0.01$, $\alpha/\beta = 1$

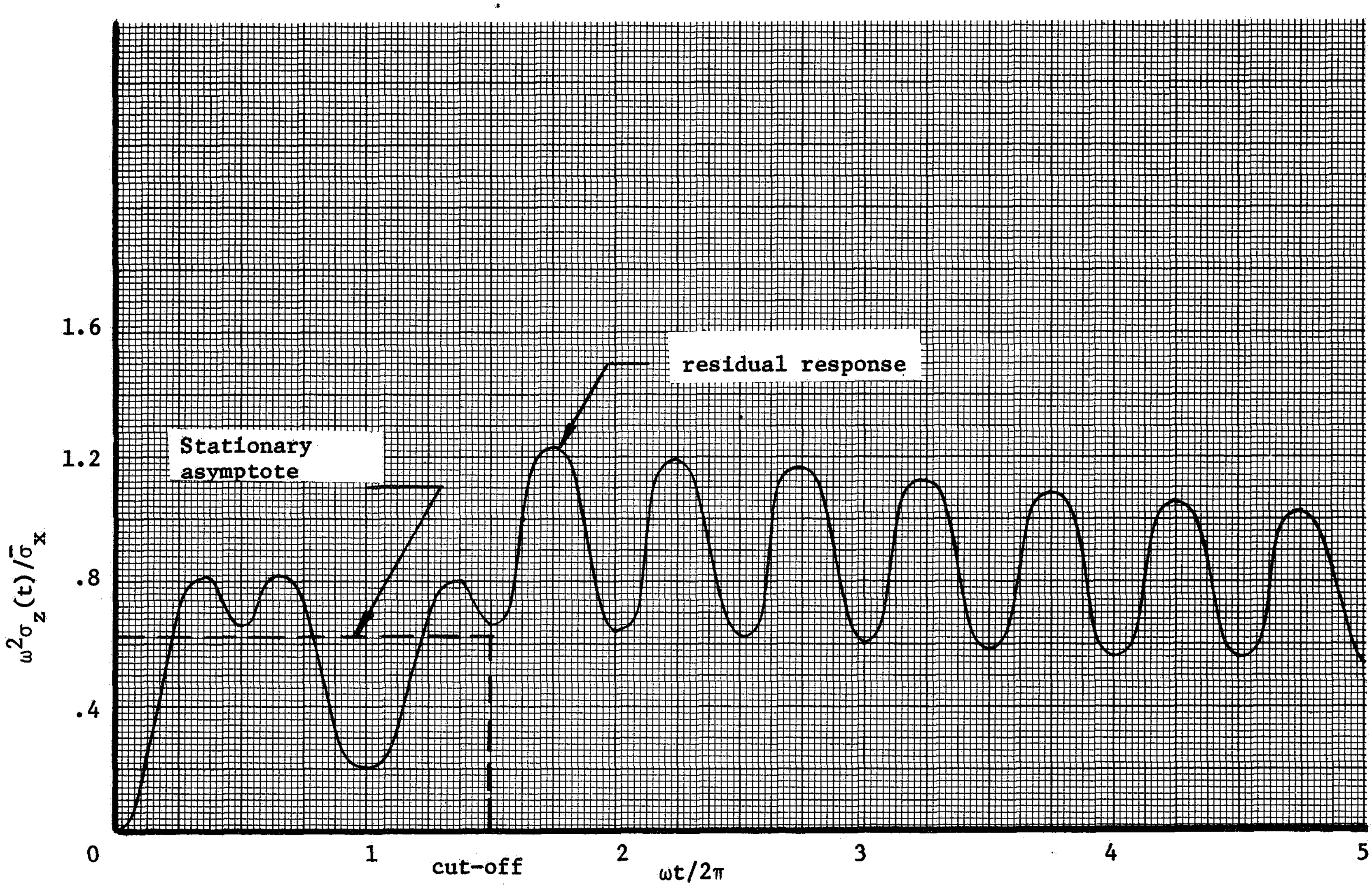

Figure 9 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.01$, $p/\omega = 2$, $\alpha/\beta = 1$, $\omega t_o/2\pi = 1.5$

seems to occur only for narrow-band excitation when $p/\omega > 1$. It is not surprising if one considers the residual response to a sinusoidal input suddenly removed. The residual response depends on the energy stored in the system at the time the input is turned off. This energy is the sum of the kinetic and the strain energies T + U. If steady state is reached prior to cut-off, then for small damping $\dot{z}_{max} \approx p\, z_{max}$.

$$T_{max} = \frac{1}{2} m \dot{z}^2_{max} \approx \frac{1}{2} k \frac{p^2}{\omega^2} z^2_{max} ,$$

and

$$U_{max} = \frac{1}{2} k z^2_{max} .$$

For $p/\omega > 1$, $T_{max} > U_{max}$. Then it is possible for $T + U > U_{max}$ at the time of cut-off in which case the residual displacement response will be greater than the forced response.

In general, if t_o occurs prior to reaching stationarity, the residual response depends upon the time within a particular fundamental period at which cut-off occurs as well as on the total number of periods of forced response. Otherwise it is completely independent of cut-off time. It may still exceed the stationary level, however.

Because of the initial and final high level transients which a system may experience when subjected to a rectangular burst of stationary, narrow-band random, this modulating function is obviously inappropriate for practical use unless such an environment actually exists for the item being tested. This example illustrates the importance of using the proper modulating function rather than one which only simulates duration.

The foregoing conclusions may be of practical importance in another physical situation. Analog power spectral density analyzers are often used to analyze transient data. These devices are basically simple systems having tuneable bandwidths. Some transient data may excite the analyzer in the manner previously described. Depending on the holding circuit used, it is conceivable that during periods of transient behavior erroneous conclusions could be drawn from the data.

The Exponential Function

The rms response to an exponentially modulated input demonstrates some of the more practical aspects of the behavior observed from the step modulation. In general, it can be said that a sharply decaying modulating function tends to eliminate those components of the rms response associated with the instantaneous input after a short time, leaving only a residual type response similar to that observed after cut-off for the rectangular modulating function. This can be seen in Figure 10. On the other hand, the rms response to a slowly decaying modulating function follows the pattern of local stationarity [7] after the initial transients have damped out. That is, it tends to follow the stationary level associated with the instantaneous level of the modulating function as in Figure 11.

It was earlier pointed out that increasing the input bandwidth parameter α could result in higher peak response for an attenuated input if stationary response is held constant. Three response histories for

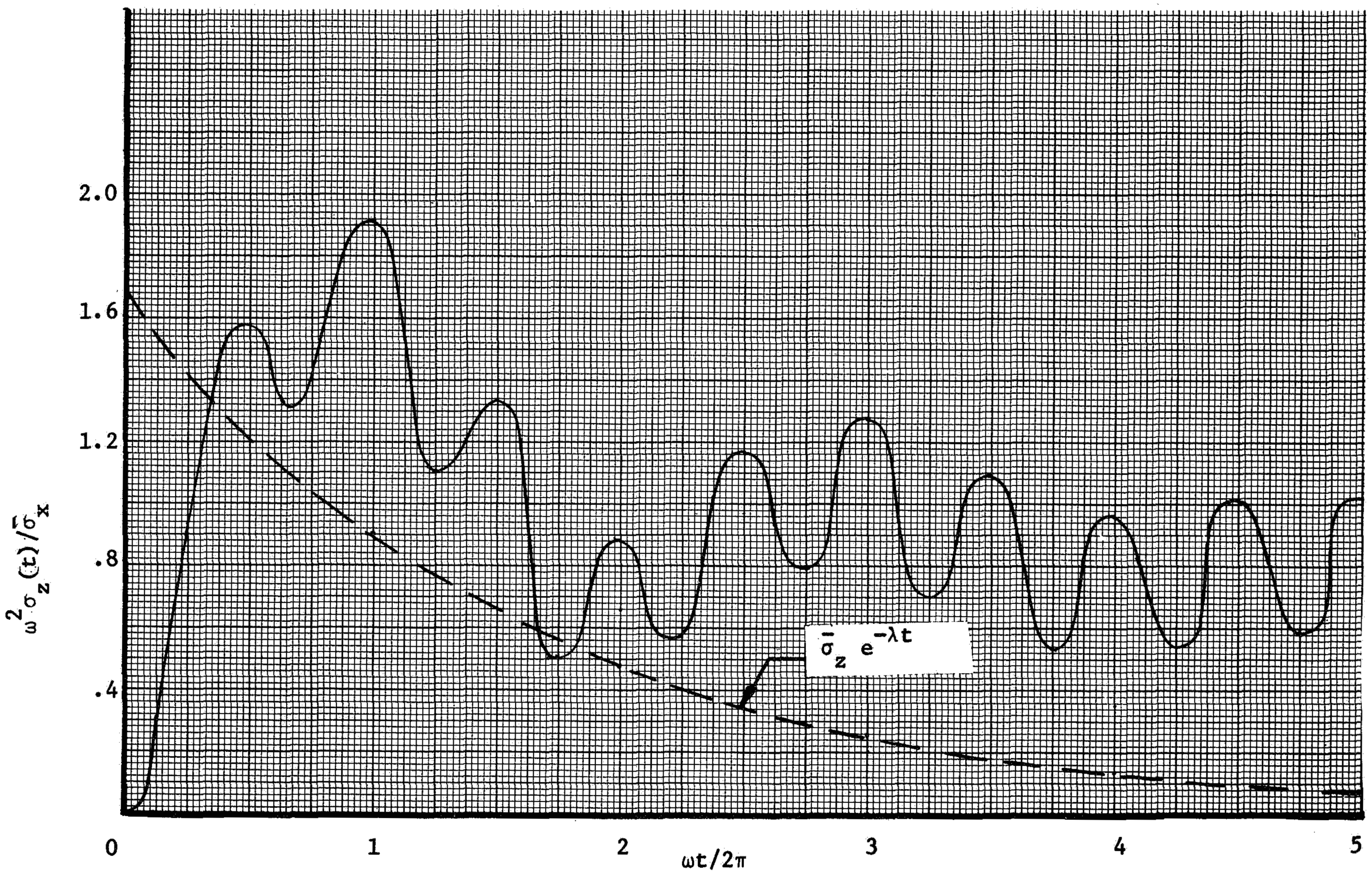

Figure 10 RMS Response for $f(t) = u(t)e^{-\lambda t}, \zeta = 0.01$, $p/\omega = 0.5$, $\alpha/\beta = 1$, $\lambda/\beta = 10$

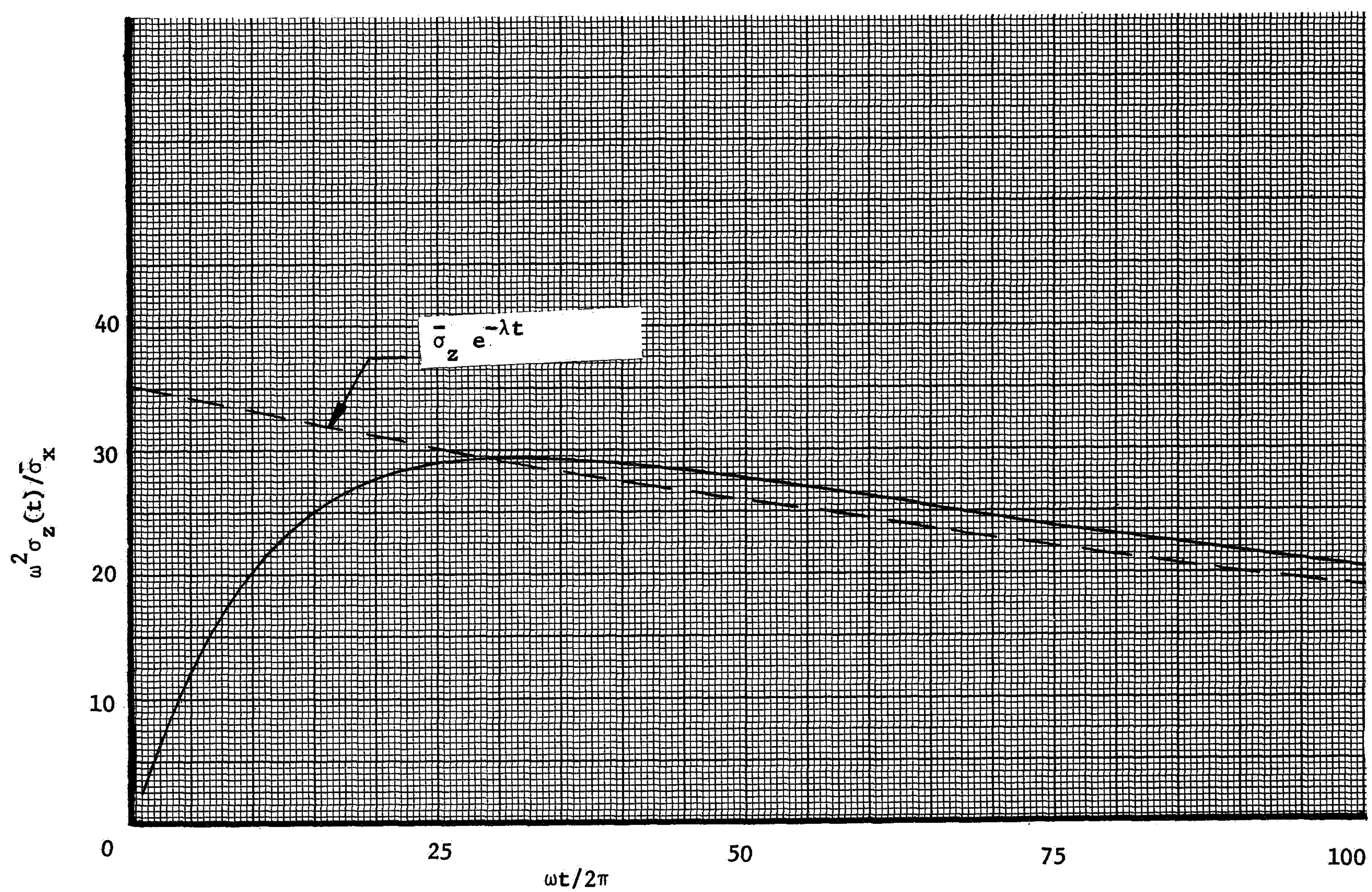

Figure 11 RMS Response for $f(t) = u(t)e^{-\lambda t}$, $= 0.01$, $p/\omega = 1$, $\alpha/\beta = 1$, $\lambda/\beta = 0.1$

different values of α are plotted in Figure 12 where each is normalized to its asymptotic level. These can be compared with Figure 6.

The frequency parameter p/ω was varied in the neighborhood of $p/\omega = 1$ to demonstrate the effect of a limited duration input on peak rms response. It can be seen in Figure 13 that as $p/\omega \to 1$ the first peak of the rms response moves to the right and consequently becomes attenuated to a higher degree by the exponential modulating function. These plots can be compared with those in Figure 8.

A practical application of this analysis is to consider the response of a small component to pyrotechnic shock induced transients. These transients may have a frequency distribution centered at 1500 cps with a bandwidth of 1000 cps and have the approximate shape of a decaying exponential of 50 milliseconds duration. If the component has a resonant frequency of 100 cps and 5% damping, its response to the simulated shock environment is that shown in Figure 14. In this case the maximum transient response is about 57% of the stationary level. Lower system damping would reduce this percentage even further.

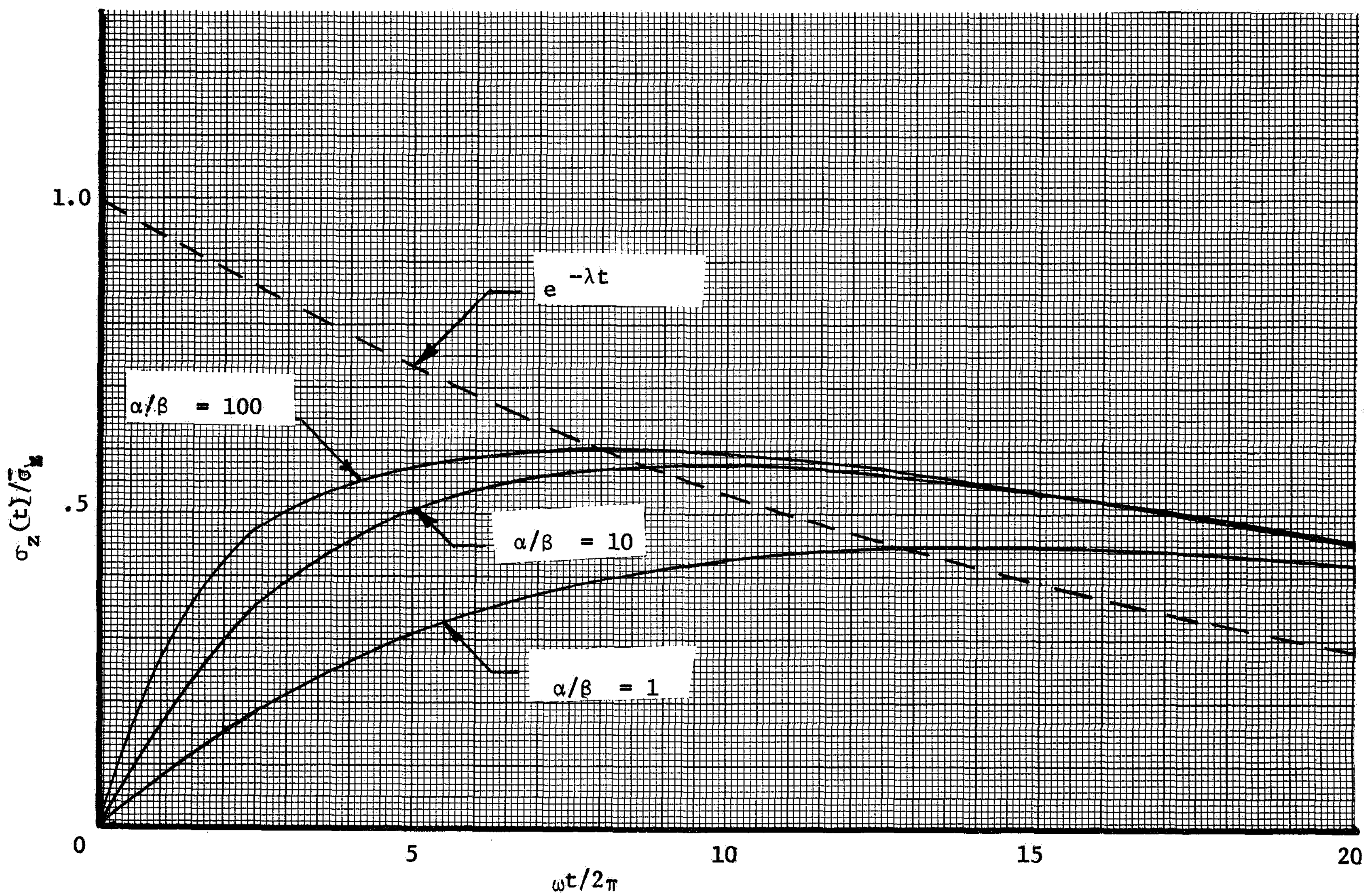

Figure 12 RMS Response for $f(t) = u(t)e^{-\lambda t}$, $\zeta = 0.01$, $p/\omega = 1$, $\lambda/\beta = 1$

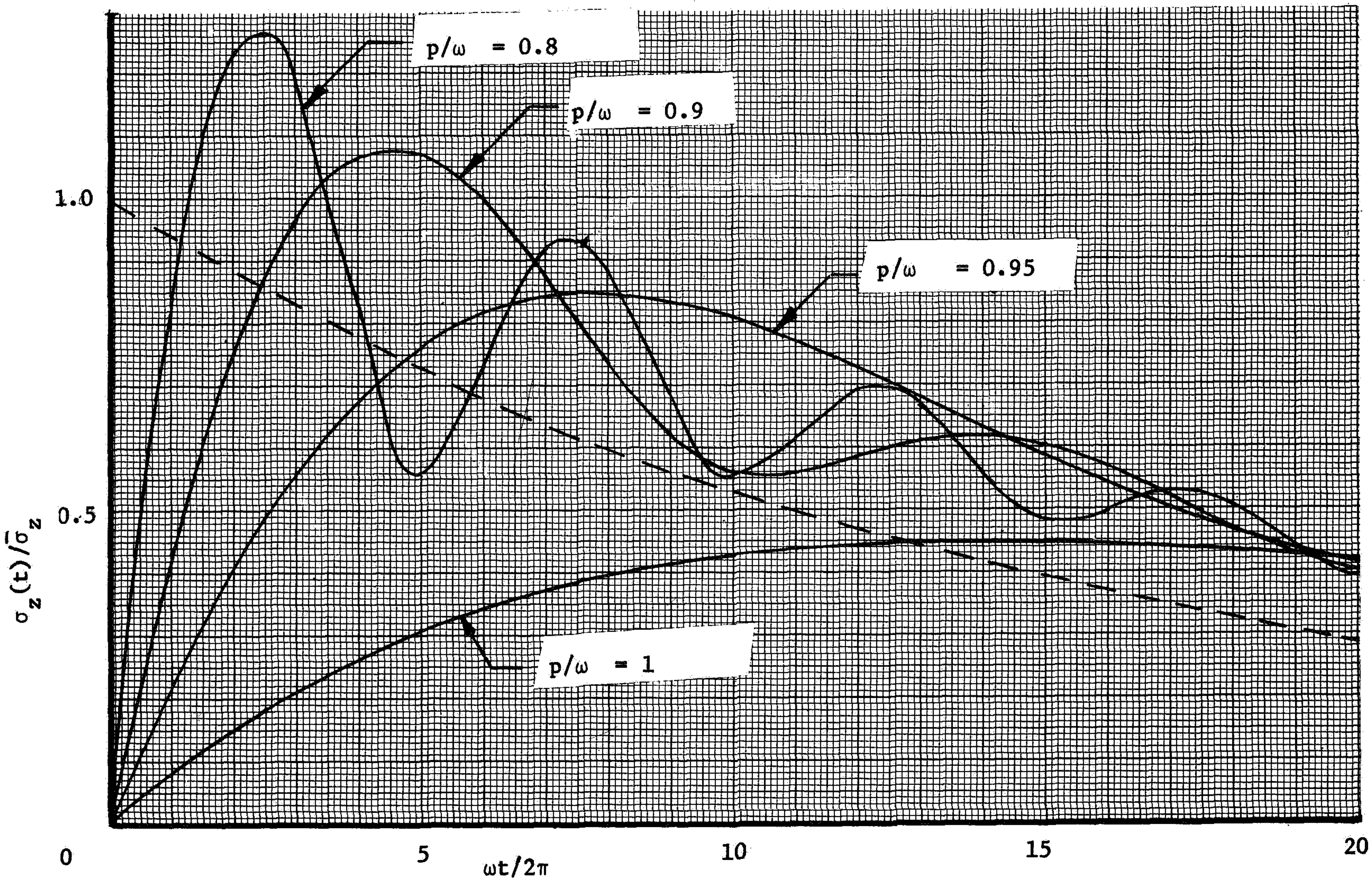

Figure 13 RMS Response for $f(t) = u(t)e^{-\lambda t}$, $\zeta = 0.01$, $\alpha/\beta = 1$, $\lambda/\beta = 1$

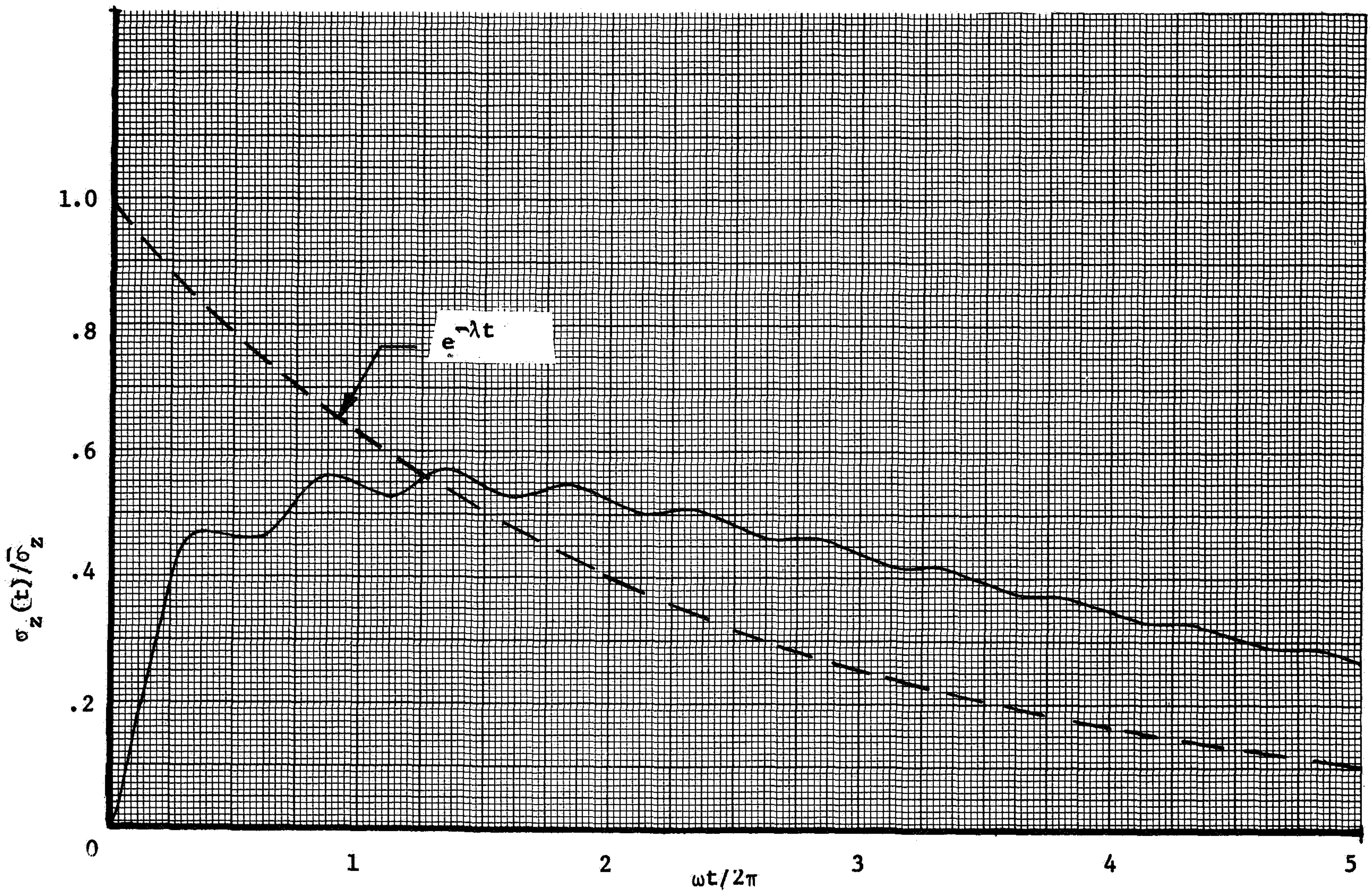

Figure 14 Normalized Component Response to a Simulated Pyrotechnic Shock Environment

IV. SUPERPOSITION

It was suggested earlier that since a function $f(t)$ can be approximated by a sum of step functions, that the rms response to a stationary input modulated by $f(t)$ can be approximated by summing a number of terms $R_z^*(t-\xi_i,\ t-\xi_j)$ similar to that given by Equation (3).

To this end $f(t)$ is replaced by

$$f(t) \approx \sum_{i=1}^{N} a_i u(t-\xi_i) \ .$$

Then according to Equation (1)

$$\sigma_z^2(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t-\tau_1)\ h(t-\tau_2)$$

$$\times \sum_{i=1}^{N} \sum_{j=1}^{N} a_i\ a_j\ u(\tau_1-\xi_i)\ u(\tau_2-\xi_j)\ R_x(\tau_2-\tau_1)\ d\tau_1\ d\tau_2 .$$

If the order of integration and summation is interchanged then

$$\sigma_z^2(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i\ a_j \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t-\tau_1)\ h(t-\tau_2)$$

$$\times \quad u(\tau_1-\xi_i)\ u(\tau_2-\xi_j)\ R_x(\tau_2-\tau_1)\ d\tau_1\ d\tau_2 .$$

A suitable change of variables gives

$$\sigma_z^{2}(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i\, a_j \int_{o}^{t-\xi_2} \int_{o}^{t-\xi_1} h(v_1)\, h(v_2)\, R_x\, (v_1-v_2)\, dv_1\, dv_2$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} a_i\, a_j\, R_z^{*}\, (t-\xi_i,\ t-\xi_j) \quad .$$

The number of computations is cut in half by observing the symmetry of R_z^{*} about the $\xi_1 = \xi_2$ plane. In this case

$$\sigma_z^{2}(t) = \sum_{i=1}^{N} a_i^{2}\, R_z^{*}\, (t-\xi_i,\ t-\xi_i)$$

$$+\ 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} a_i\, a_j\, R_z^{*}\, (t-\xi_i,\ t-\xi_j). \qquad (6)$$

A computer program was written to carry out these computations. To obtain some indication of the accuracy of this approximation, the exponential modulating function was approximated by the staircase function shown in Figure 15. Examples of the results are shown in Figures 16 and 17. The circled dots are points plotted from the approximate solution while the solid lines indicate the exact solution.

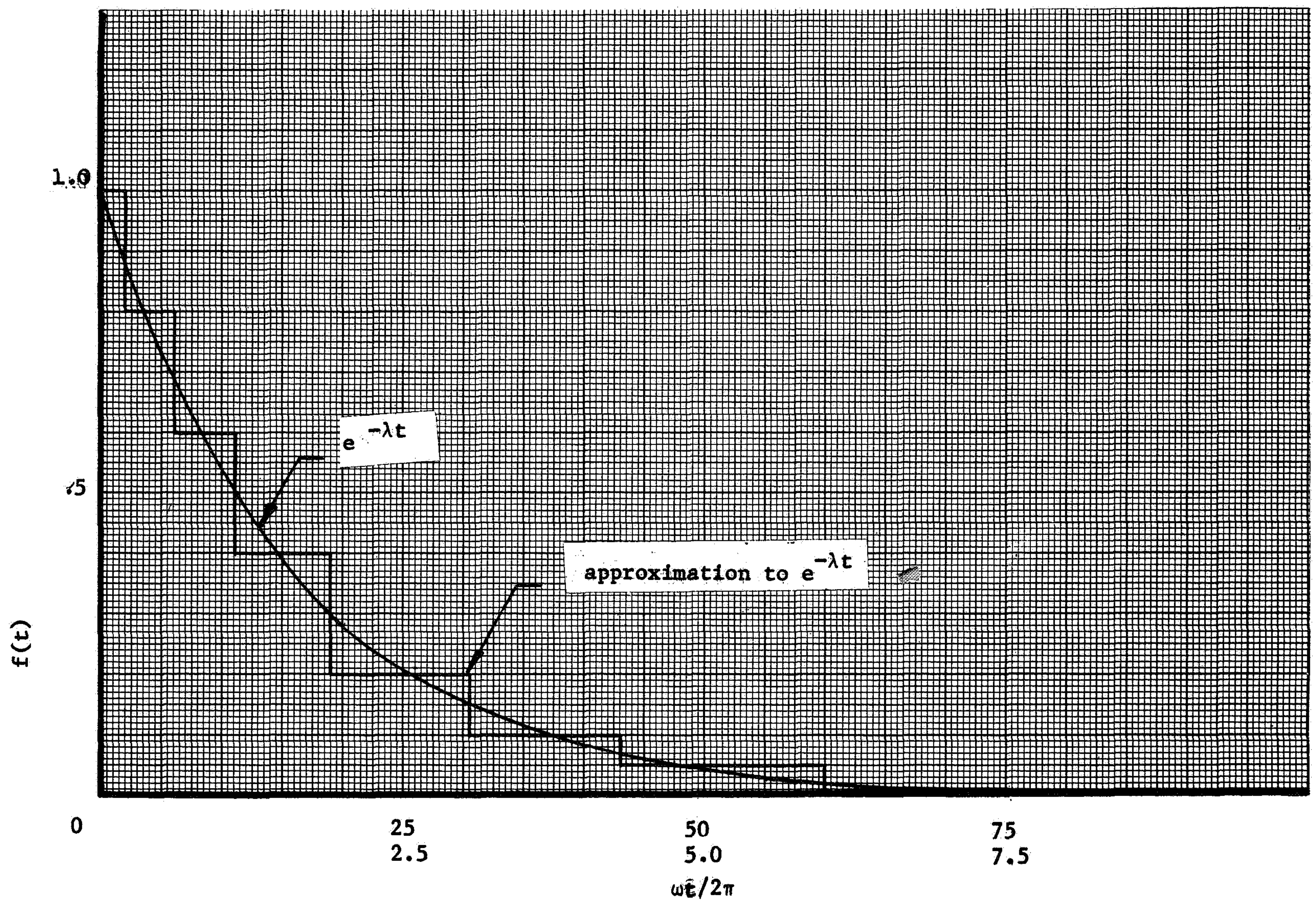

Figure 15 Staircase Function Approximating $f(t) = u(t)e^{-\lambda t}$

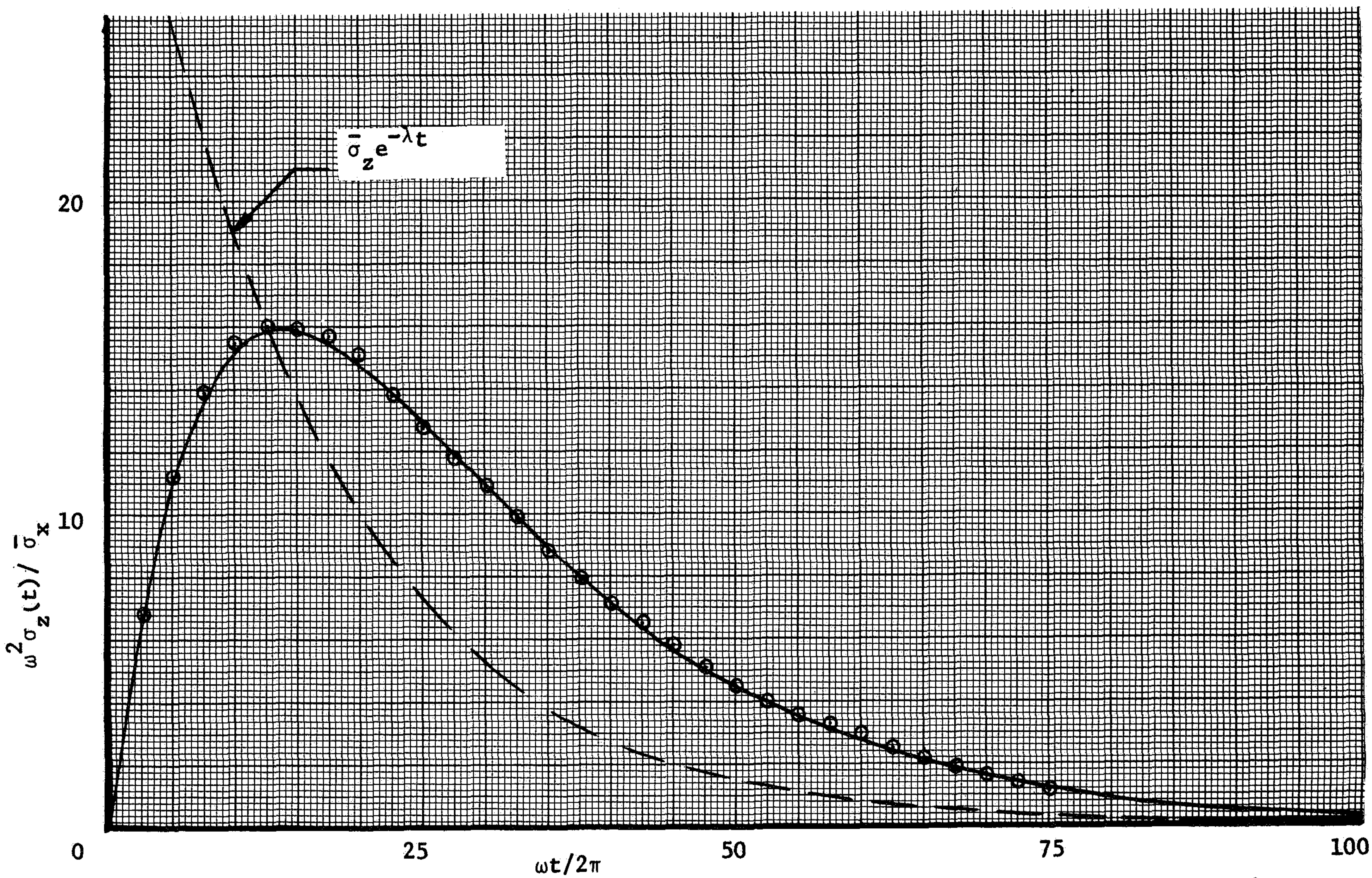

Figure 16 Comparison of Approximate and Exact Solutions for RMS Response $f(t) = u(t)e^{-\lambda t}$, $\zeta = 0.01$, $p/\omega = 1$, $\alpha/\beta = 1$, $\lambda/\beta = 1$

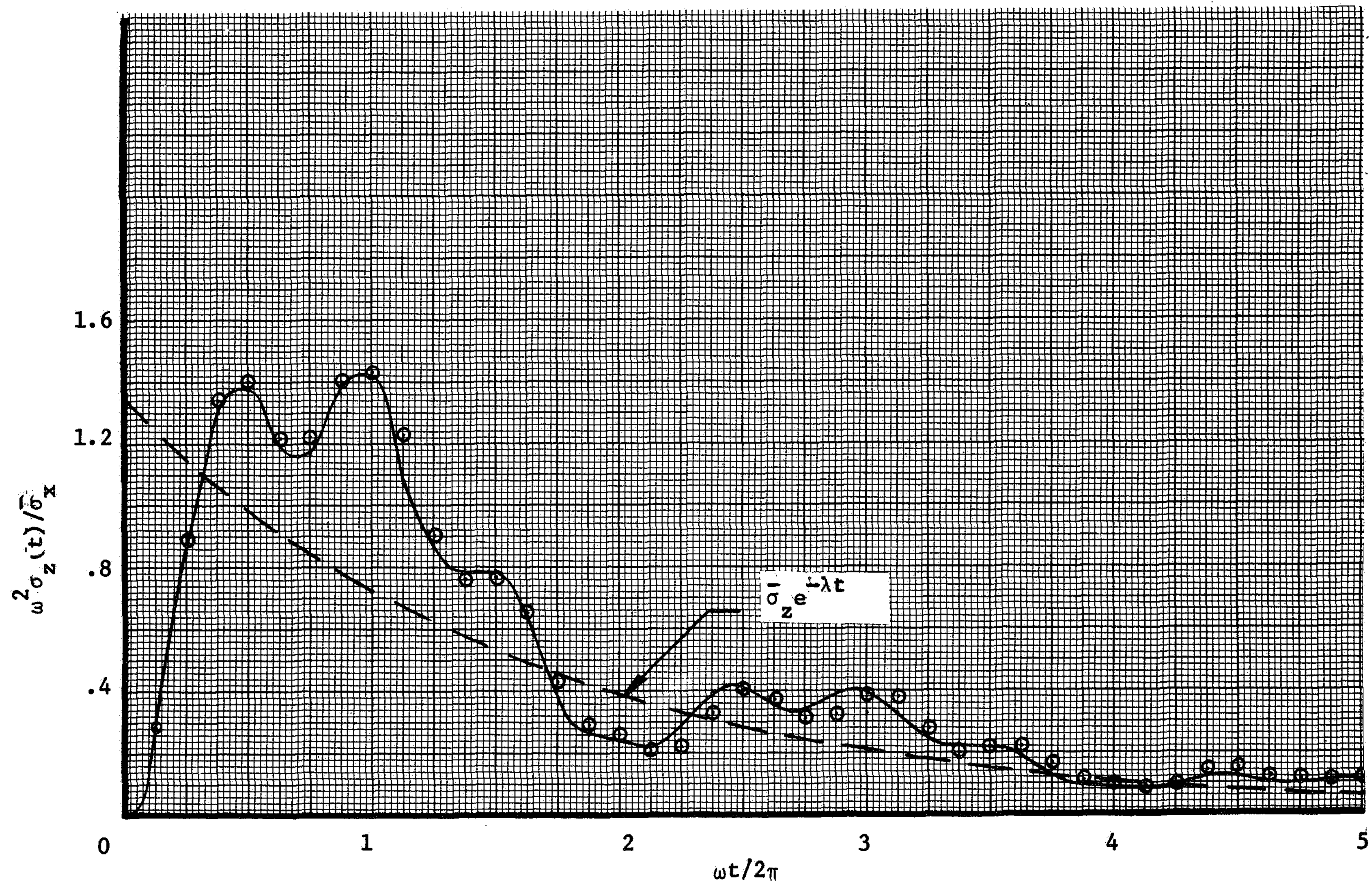

Figure 17 Comparison of Approximate and Exact Solutions for RMS Response $f(t) = u(t)e^{-\lambda t}$

$\zeta = 0.1$, $p/\omega = 0.5$, $\alpha/\beta = 0.1$, $\lambda/\beta = 1$

The agreement has been quite good for those cases considered where the decaying exponential was replaced by an eight step staircase function. The computation of 40 points took about 30 seconds on the TRW SDS-940 time sharing computer.

This technique provides a very useful tool for estimating structural response to nonstationary random environments since the modulating function can be specified pointwise. These points correspond to the instantaneous rms value of the excitation process and can be easily computed from digitized data.

V. CONCLUDING REMARKS

1. Summary

This report has presented the results and conclusions of a parameter study devised to illustrate the behavior of the transient rms response of a simple mechanical oscillator to a time-modulated stationary process of arbitrary bandwidth and center frequency. Closed form solutions were obtained for the step, the rectangular and the decaying exponential modulating functions. A matrix of cases was established in parameters describing system damping, bandwidth ratio, frequency ratio and modulating function. These were varied independently and the resulting rms response histories were computed and plotted. The results were discussed and generalized.

From the step modulation it was learned that transient response levels often exceed their asymptotic or stationary values for a narrow-band input centered at a frequency different from system resonance. Increasing the bandwidth of the input tends to eliminate these transients which are associated with the input center frequency. When the input and system frequencies are close together the response exhibits a beating phenomenon.

From the rectangular modulation it was learned that for narrow-band input the residual response can exceed the forced response if the input frequency is greater than that of the system. The residual response is influenced by cut-off time only if that time occurs before stationarity is reached.

The exponential modulating function has the effect of limiting the residual type response associated with energy build-up in the system as well as attenuating the forced response associated with the instantaneous strength of the input. When rms response was normalized by its stationary level the exponential modulating function attenuated more severely that response associated with a narrower-band input and an input center frequency closer to the resonant frequency of the system.

The results derived from the rectangular modulating function were used to formulate an approximate solution for the response to an arbitrarily modulated stationary input. When the exponential function was replaced by an eight step staircase function, agreement between the approximate and exact solutions was within a few percent for the cases compared.

These results provide an analytical capability for assessing the potential value of a time-modulated random vibration test for specific applications. The parameters chosen to describe the environment are readily observable and permit a quick evaluation of the severity of the environment for the system concerned. Practical examples have been suggested in which structural response to a nonstationary environment can differ by as much as a factor of 2 in either direction from the stationary response resulting from the application of stationary excitation at the same level.

2. Recommendations for Future Work

On the basis of this report, several important topics are recommended for further investigation. They outline the remaining steps leading to the implementation of this method for laboratory testing.

- Demonstrate the validity of the proposed simulation for specific dynamic environments of interest, such as those measured within spacecraft, aircraft, ground vehicles and shock loaded ground structures.

- Investigate the factors relating to test repeatibility as a definitive basis for test specification. A test is conceived to embody a specific number of simulated transients, each random in nature. A proper test would not only simulate the appearance of each transient but the number of them expected in the service life of the test item. For testing purposes, a constraint is placed on the selection of that number. It must be sufficiently large so that repeating the test will yield statistically equivalent results.

- Demonstrate the ability of existing laboratory equipment to provide these excitations.

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APPENDIX: Plots from the Parameter Study

The plots from the basic parameter study for the three modulating functions are presented in this appendix. A summary of cases is contained in the following three tables.

TABLE A I. The Step Function

$f(t) = u(t)$

Figure No.	System damping ζ	Frequency ratio p/ω	Bandwidth ratio α/β
A 1	0.01	0.5	1, 10, 100
A 2	↓	1	↓ ↓ ↓
A 3		2	
A 4	0.1	0.5	0.1, 1, 10
A 5	↓	1	↓ ↓ ↓
A 6		2	

TABLE A II. The Rectangular Function

$$f(t) = u(t) - u(t-t_o)$$

Figure No.	System damping ζ	Frequency ratio p/ω	Bandwidth ratio α/β	Cut-off time $\omega t_o/2\pi$
A 7	0.01	0.5	1, 10, 100	1.25
A 8	↓	↓	↓ ↓ ↓	1.5
A 9				1.75
A 10				2
A 11				100
A 12	↓	1	↓ ↓ ↓	1.25
A 13		↓		1.5
A 14				1.75
A 15				2
A 16				100
A 17	↓	2	↓ ↓ ↓	1.25
A 18		↓		1.5
A 19				1.75
A 20				2
A 21				100

TABLE A II. The Rectangular Function (Cont'd)

$$f(t) = u(t) - u(t-t_o)$$

Figure No.	System damping ζ	Frequency ratio p/ω	Bandwidth ratio α/β	Cut-off time $\omega t_o/2\pi$
A 22	0.1	0.5	0.1, 1, 10	1.25
A 23				1.5
A 24	↓	↓	↓ ↓ ↓	1.75
A 25				2
A 26				10
A 27		1		1.25
A 28				1.5
A 29	↓	↓	↓ ↓ ↓	1.75
A 30				2
A 31				10
A 32		2		1.25
A 33				1.5
A 34	↓	↓	↓ ↓ ↓	1.75
A 35				2
A 36				10

TABLE A III. The Exponential Function

$$f(t) = u(t)e^{-\lambda t}$$

Figure No.	System damping ζ	Frequency ratio p/ω	Bandwidth ratio α/β	Decay ratio λ/β
A 37	0.01	0.5	1, 10, 100	1
A 38	↓	1	↓ ↓ ↓	↓
A 39		2		
A 40	↓	0.5	↓ ↓ ↓	10
A 41		1		↓
A 42		2		
A 43	0.1	0.5	0.1, 1, 10	0.1
A 44	↓	1	↓ ↓ ↓	↓
A 45		2		
A 46	↓	0.5	↓ ↓ ↓	1
A 47		1		↓
A 48		2		

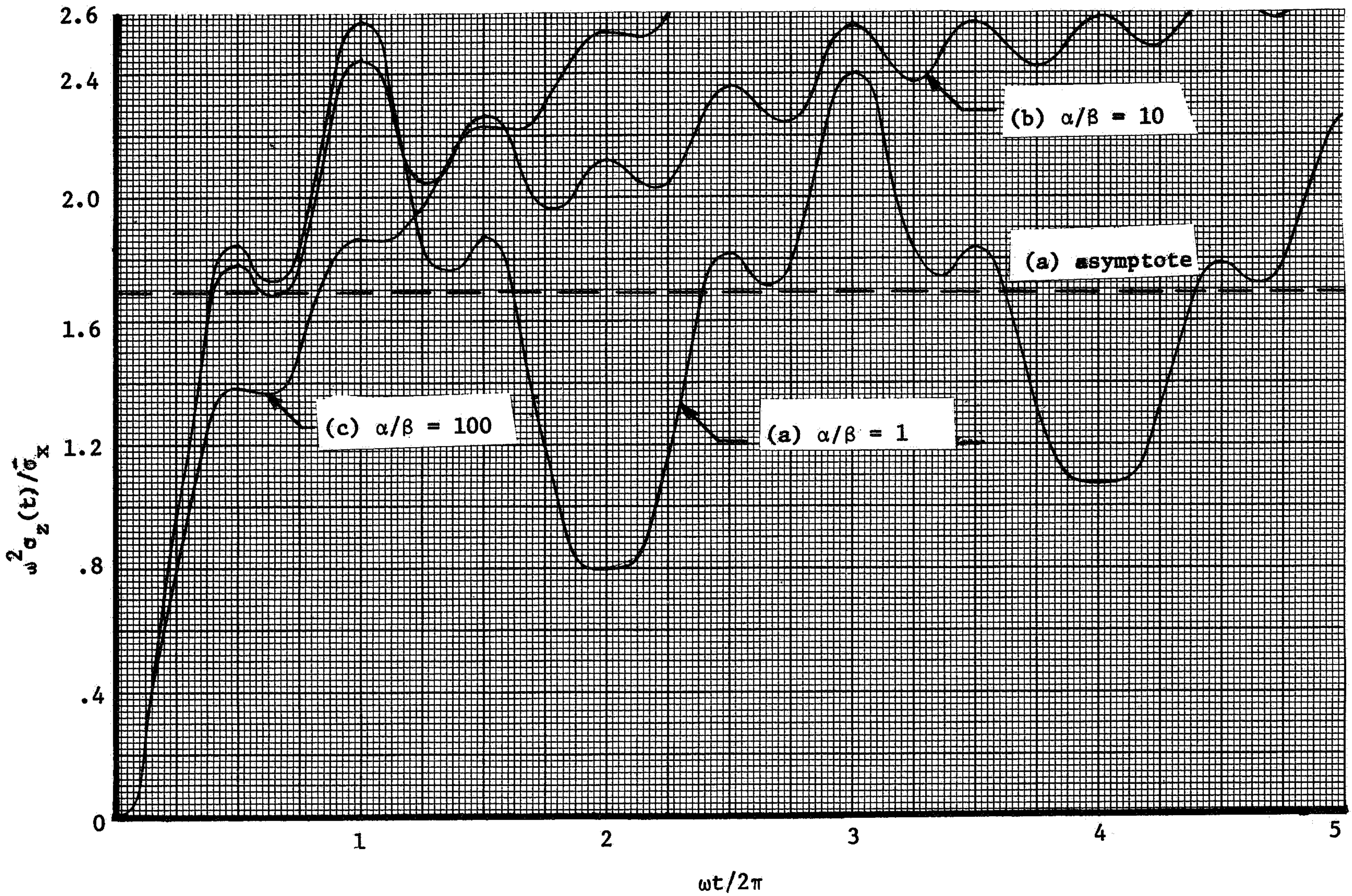

Figure A1 RMS Response for $f(t) = u(t)$, $\zeta = 0.01$, $p/\omega = 0.5$

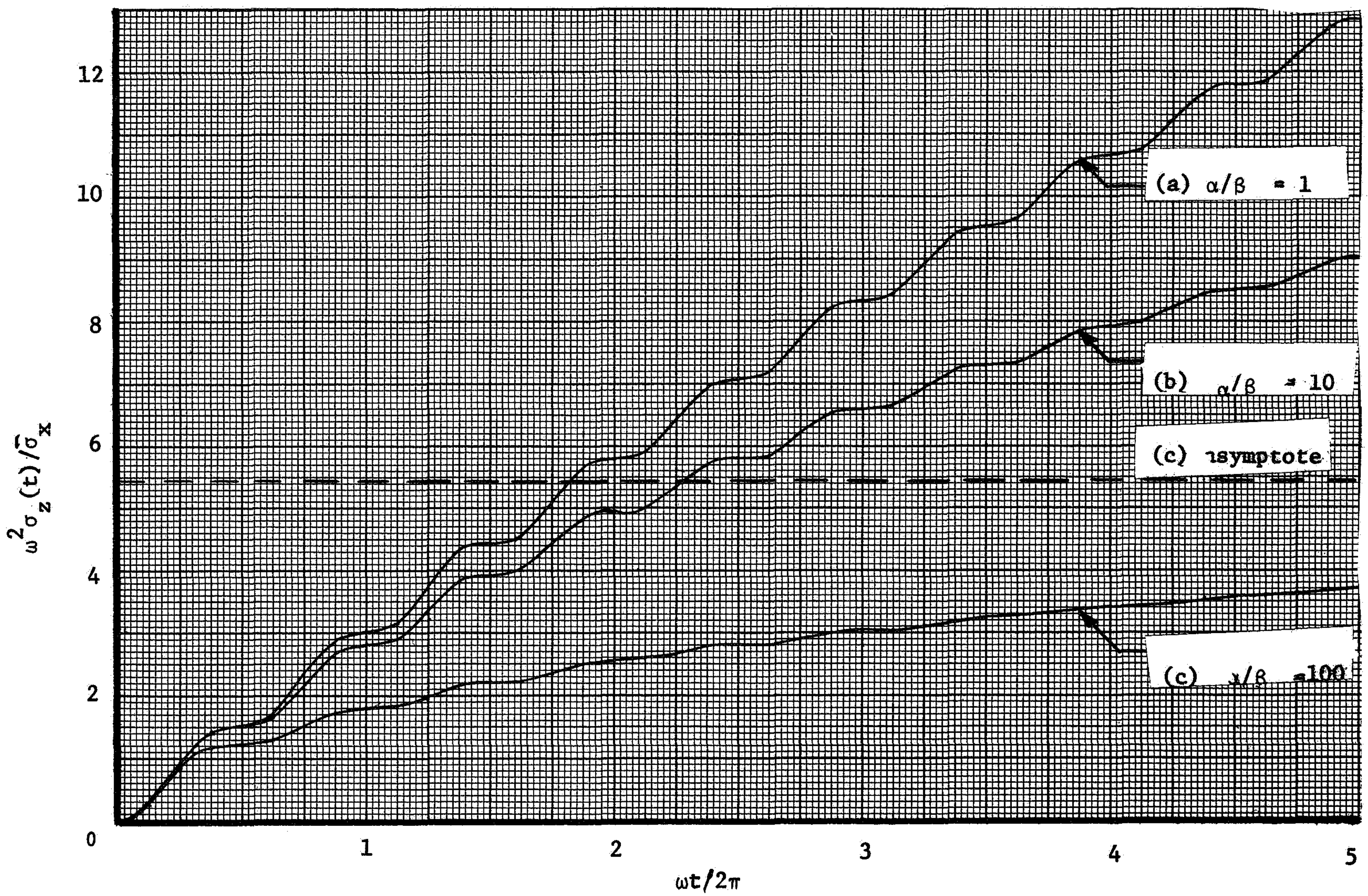

Figure A2 RMS Response for $f(t) = u(t)$, $\zeta = 0.01$, $p/\omega = 1$

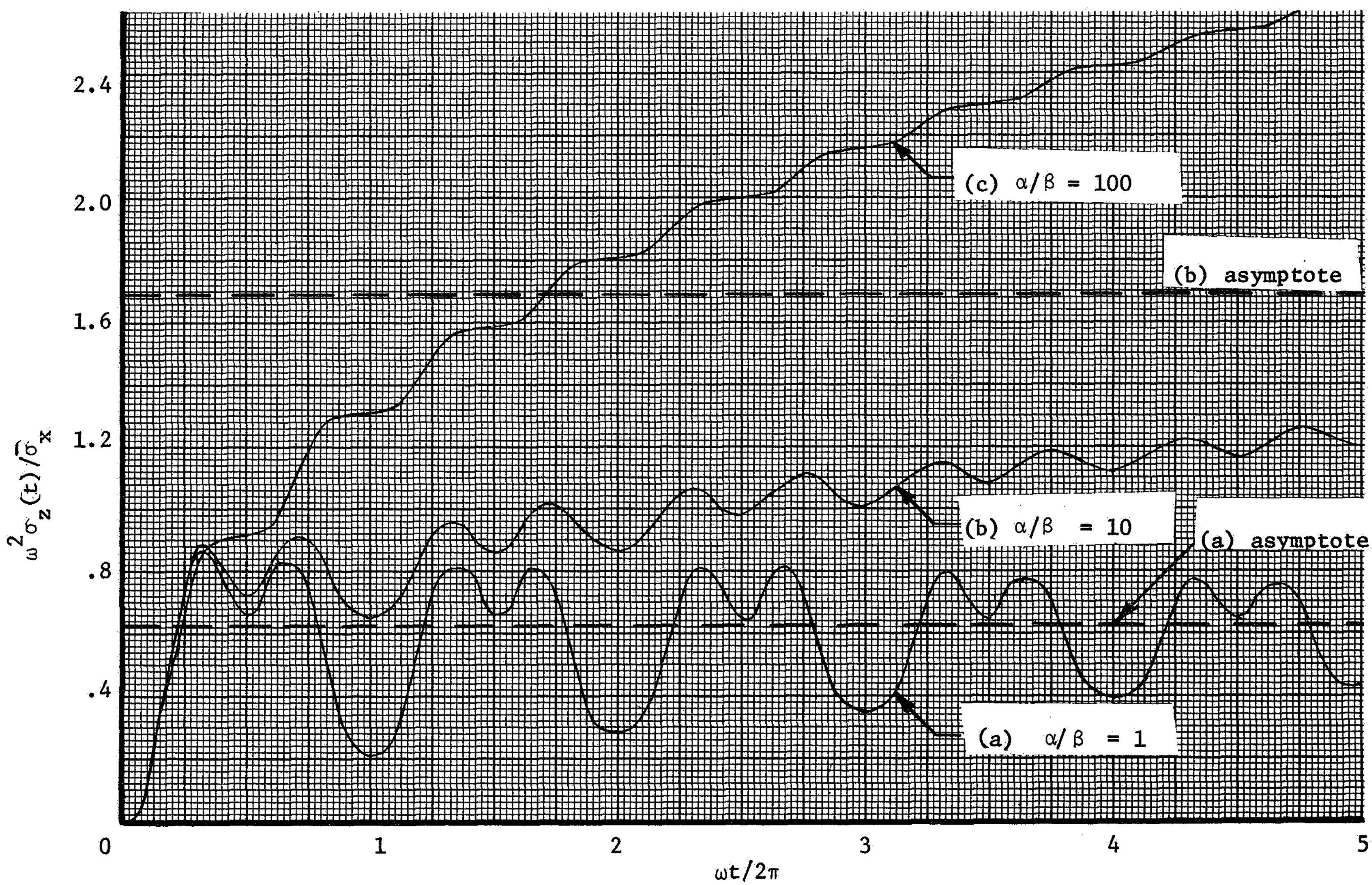

Figure A3 RMS Response for $f(t) = u(t)$, $\zeta = 0.01$, $p/\omega = 2$

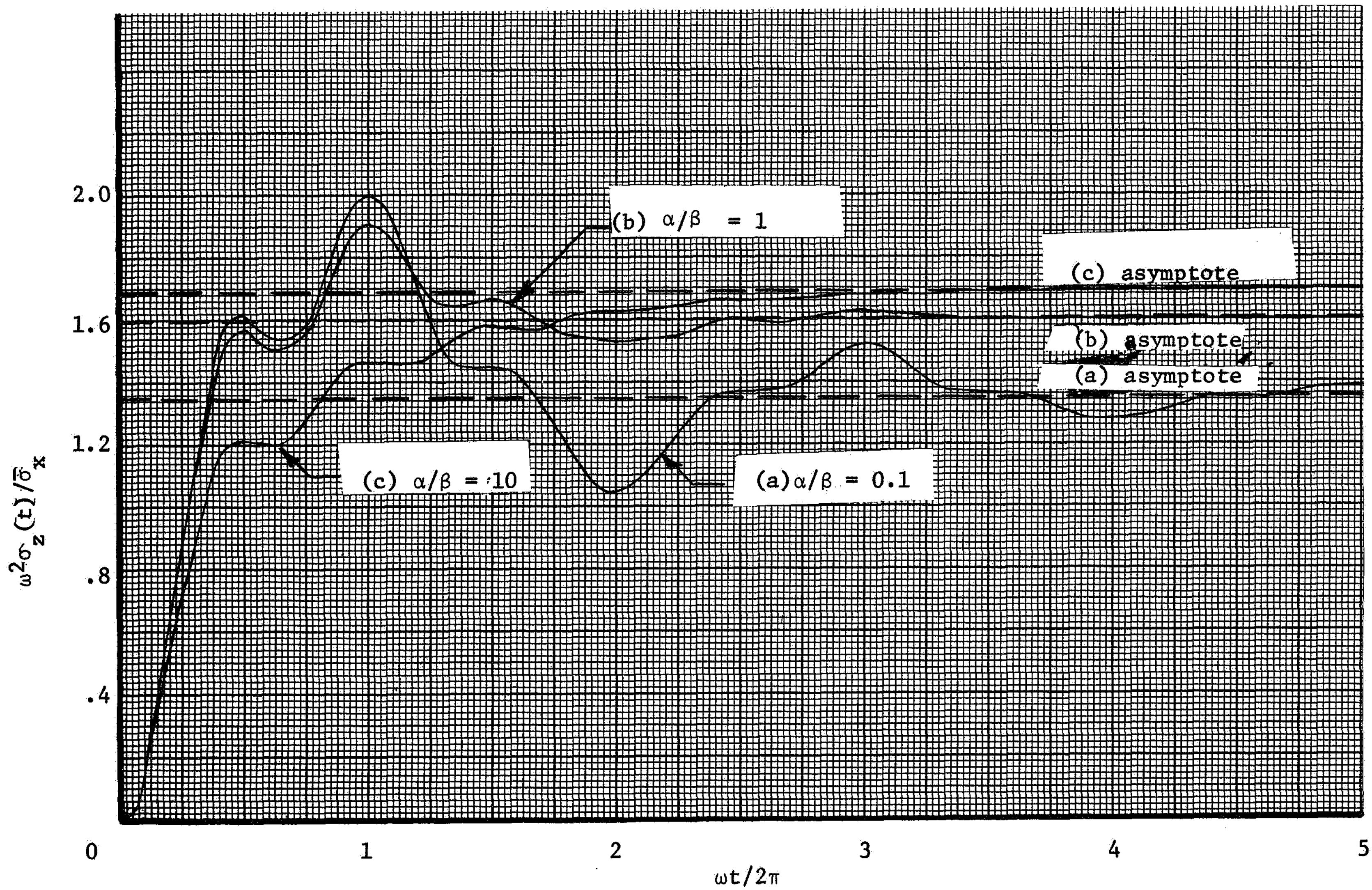

Figure A4 RMS Response for $f(t) = u(t), \zeta = 0.1, p/\omega = 0.5$

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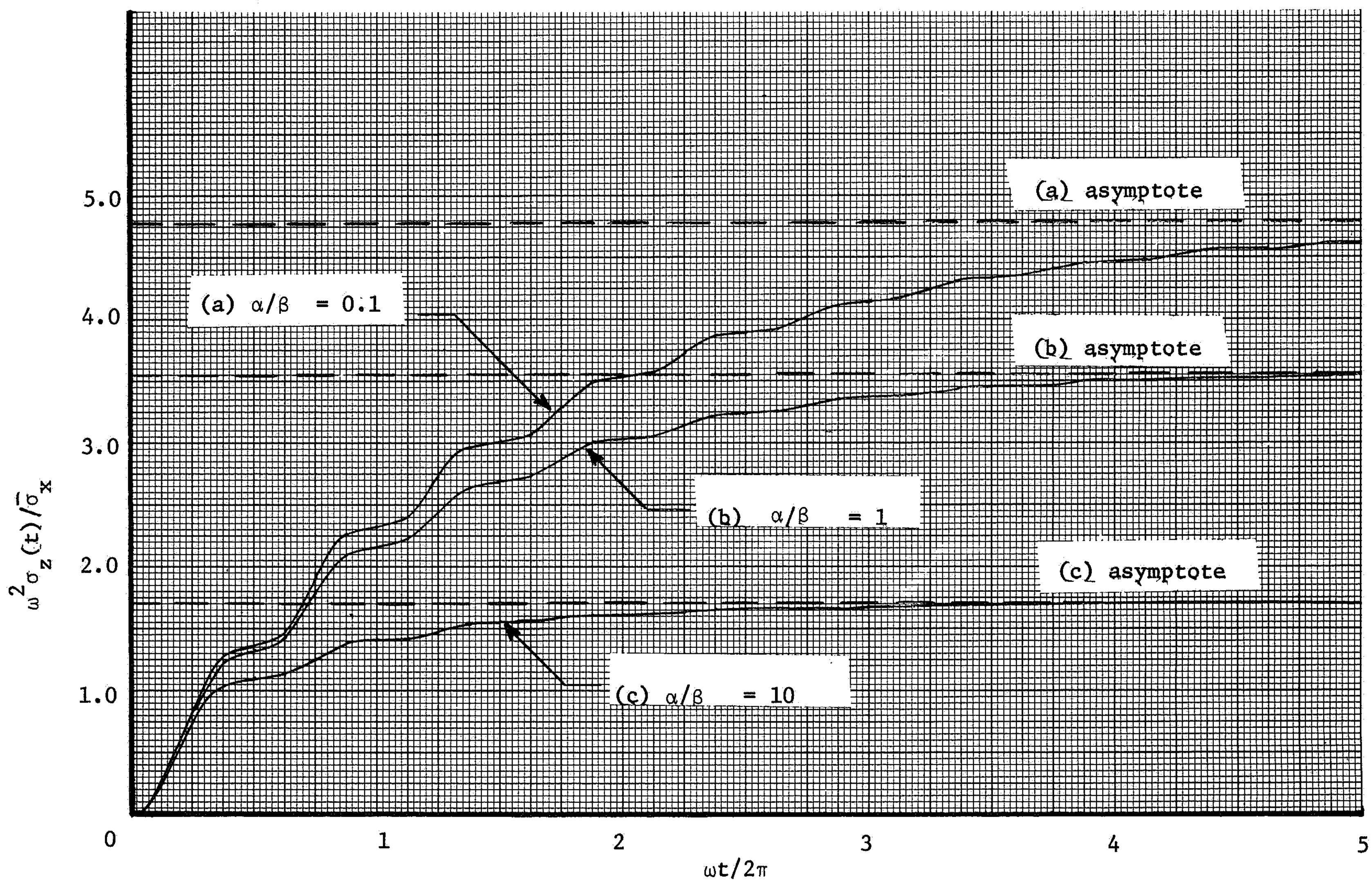

Figure A5 RMS Response for $f(t) = u(t)$, $\zeta = 0.1$, $p/\omega = 1$

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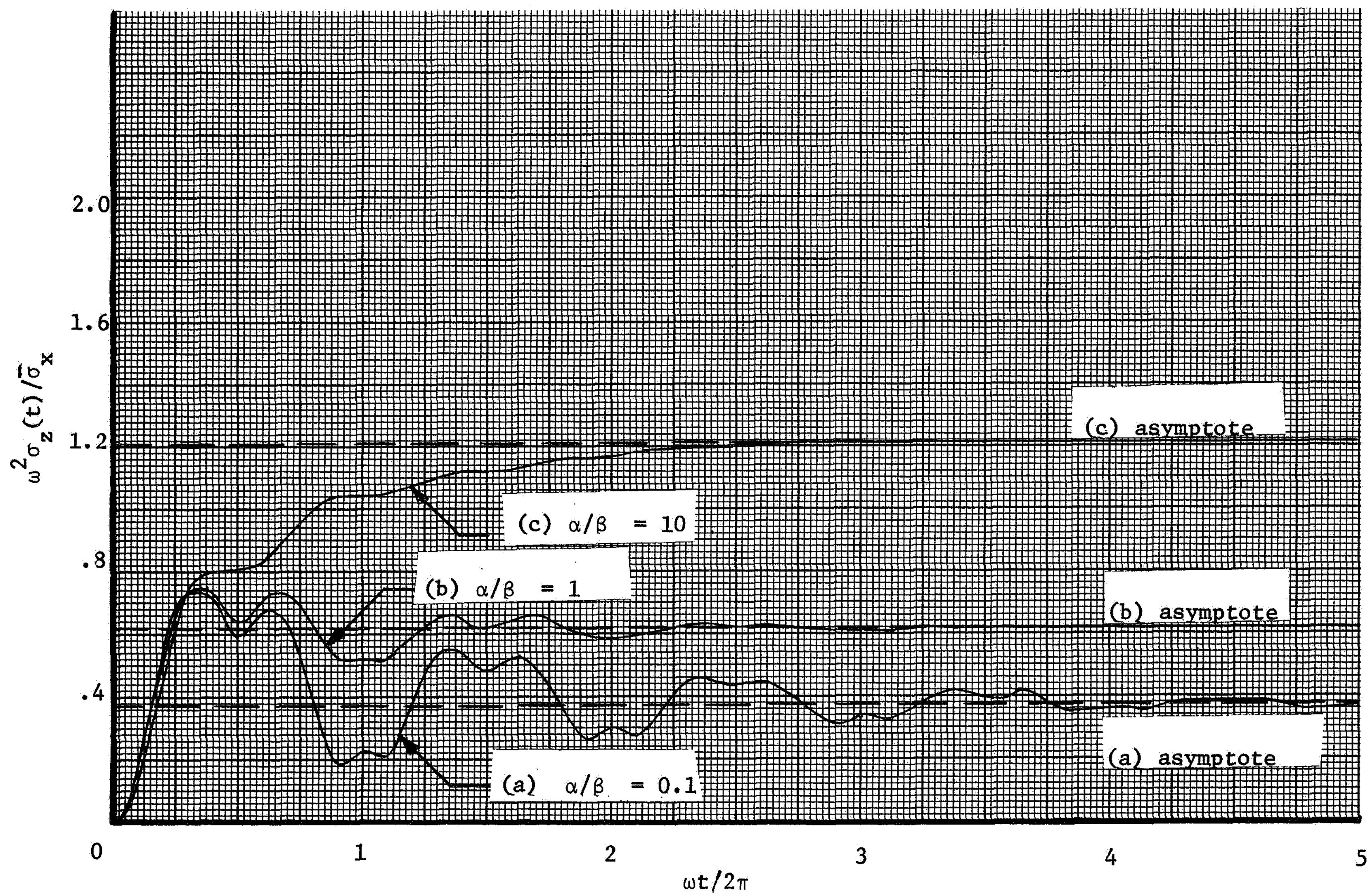

Figure A6 RMS Response for $f(t) = u(t)$, $\zeta = 0.1$, $p/\omega = 2$

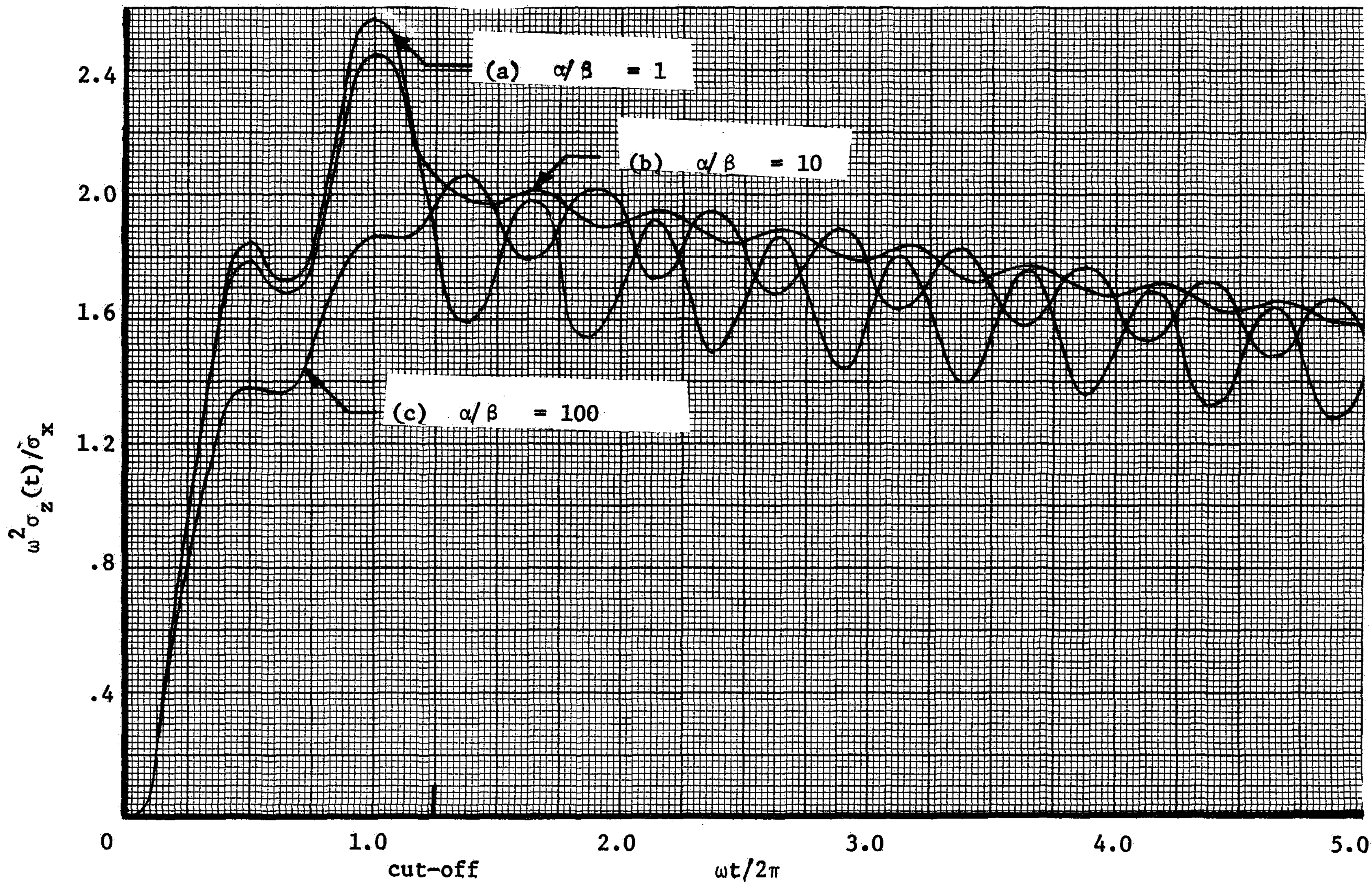

Figure A7 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.01$, $p/\omega = 0.5$, $\omega t_o/2\pi = 1.25$

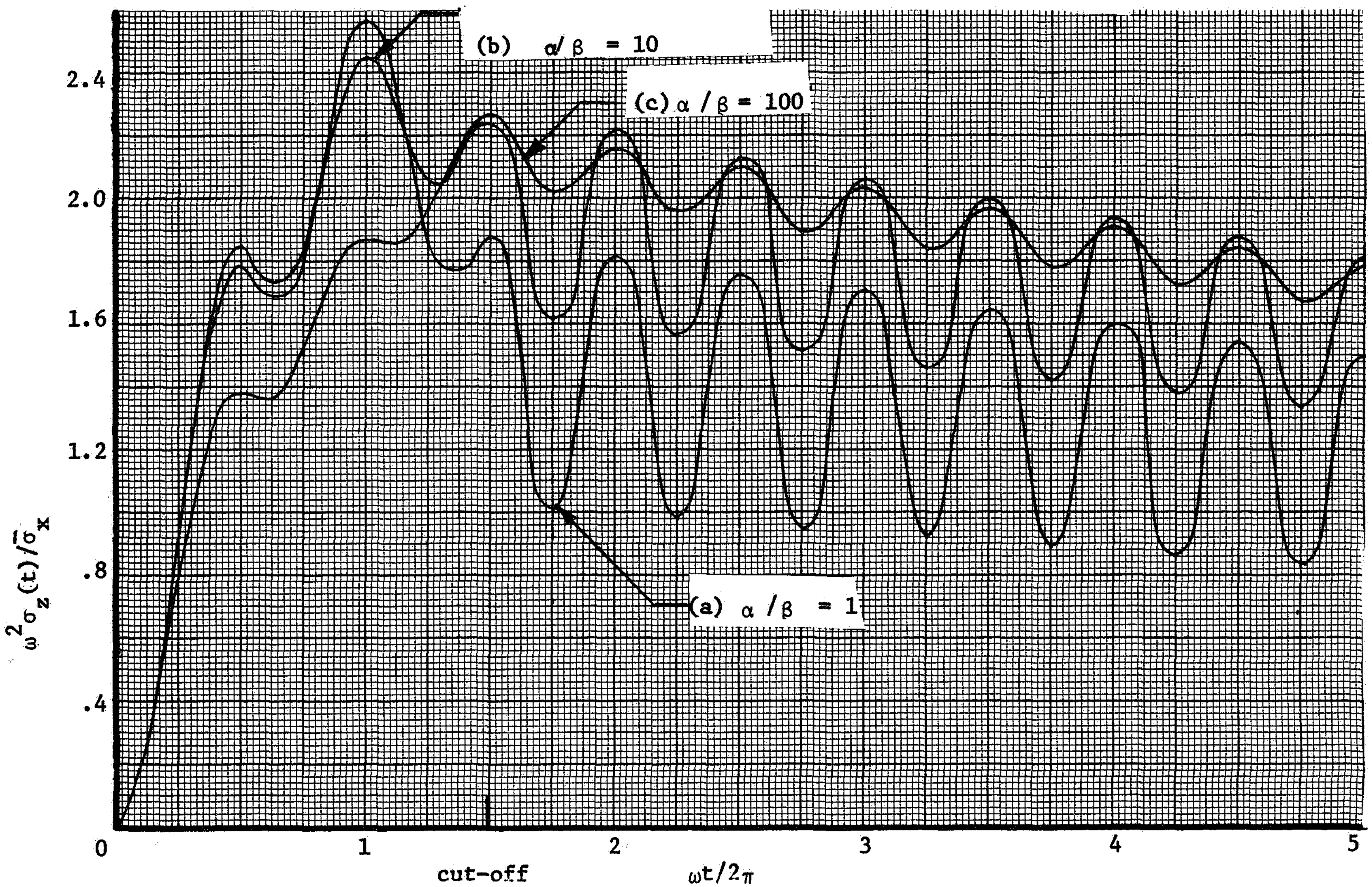

Figure A8 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.01$, $p/\omega = 0.5$, $\omega t_o/2\pi = 1.5$

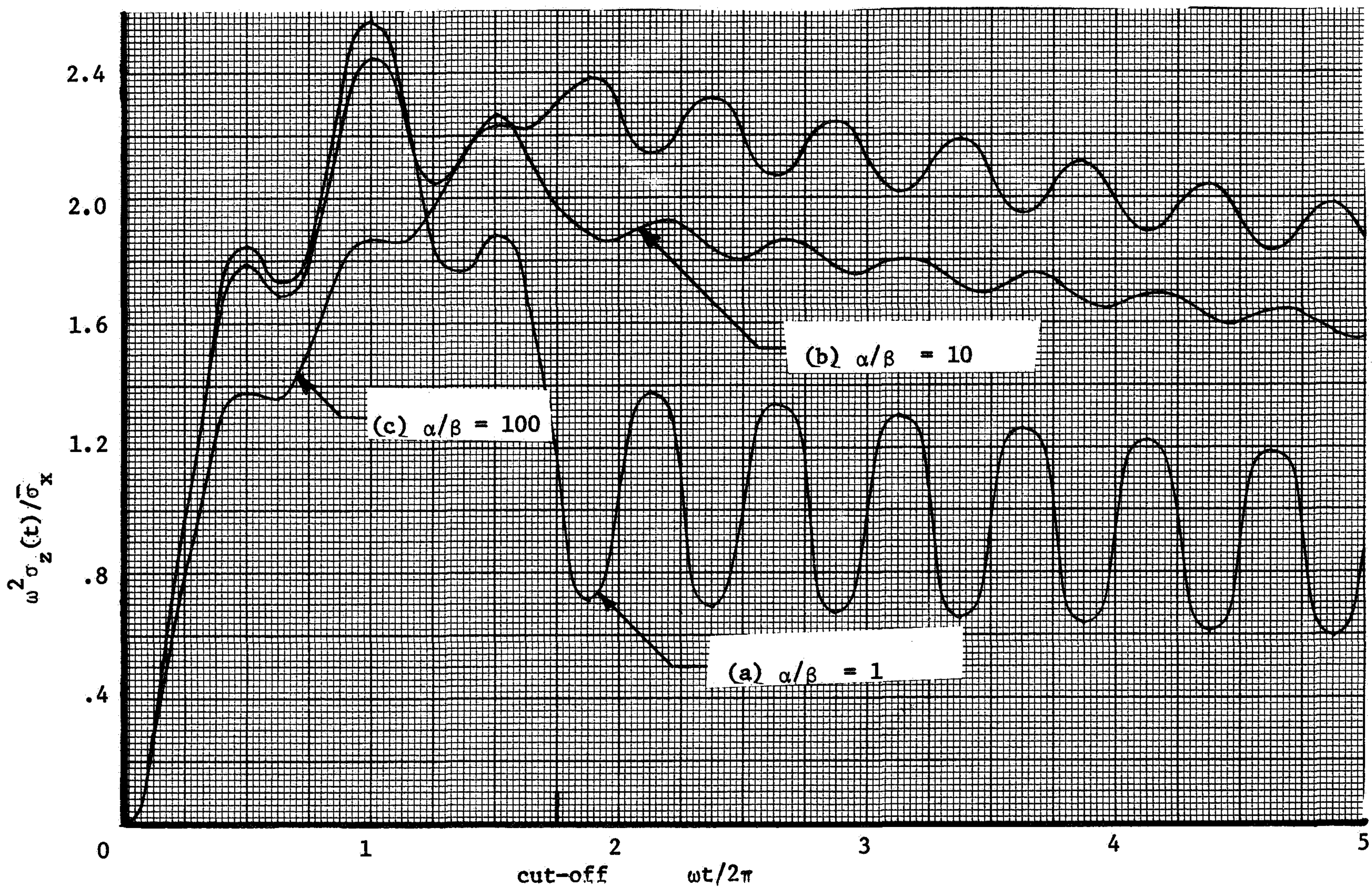

Figure A9 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.01$, p/ω 0.5, $\omega t_o/2\pi = 1.75$

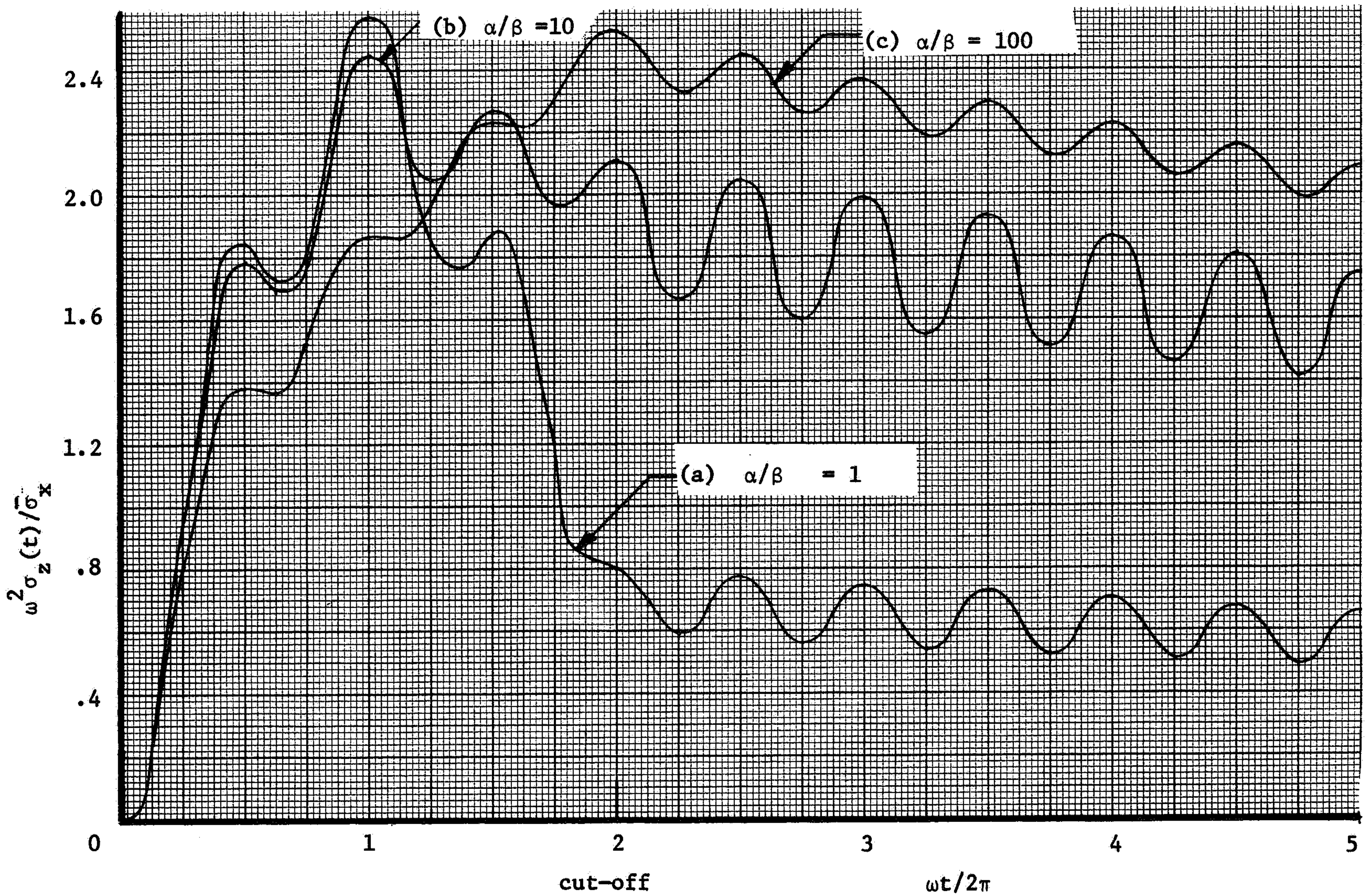

Figure A10 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.01$, $p/\omega = 0.5$, $\omega t_o/2\pi = 2$

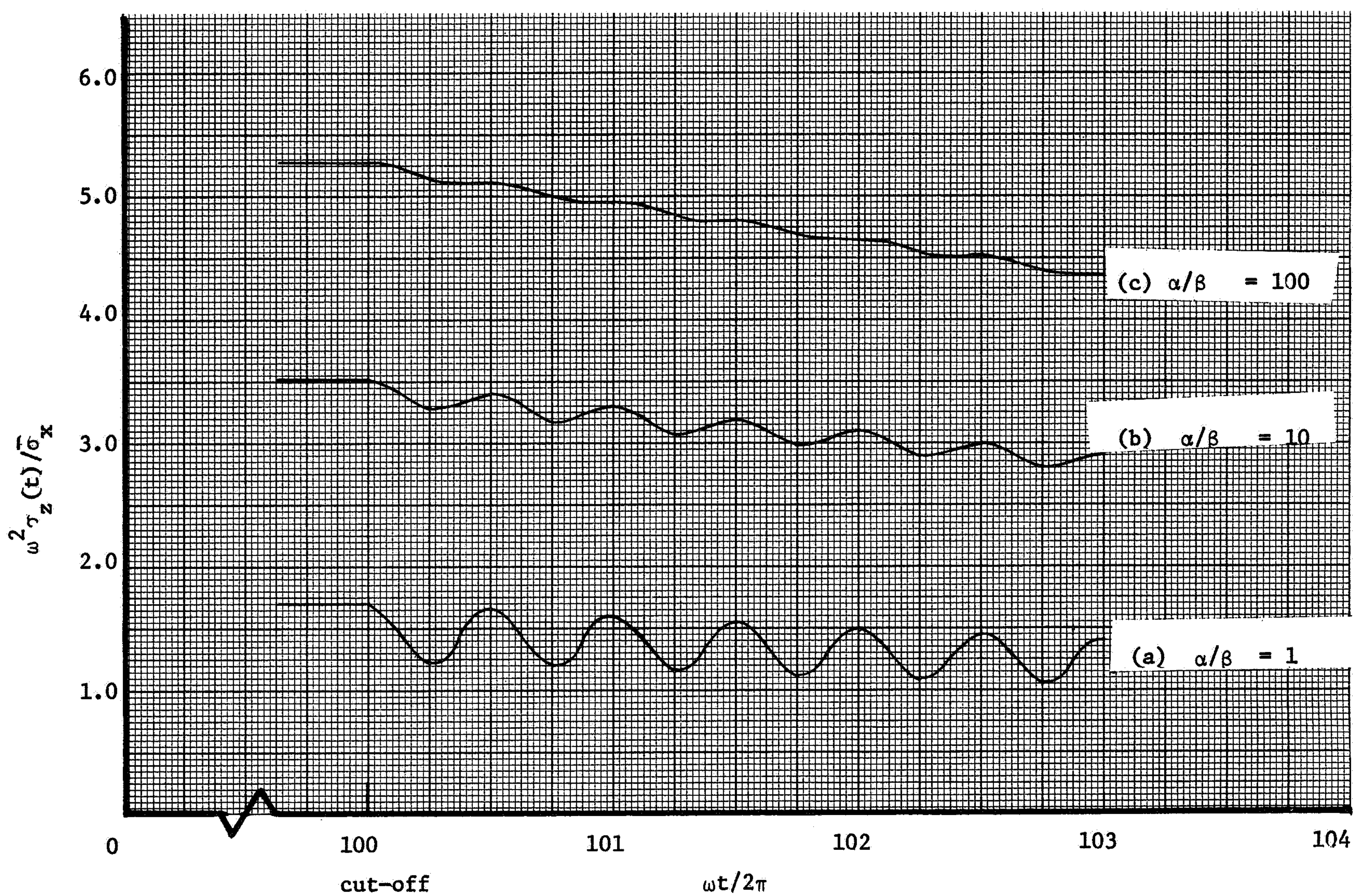

Figure A11 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.01$, $p/\omega = 0.5$, $\omega t_o/2\pi = 100$

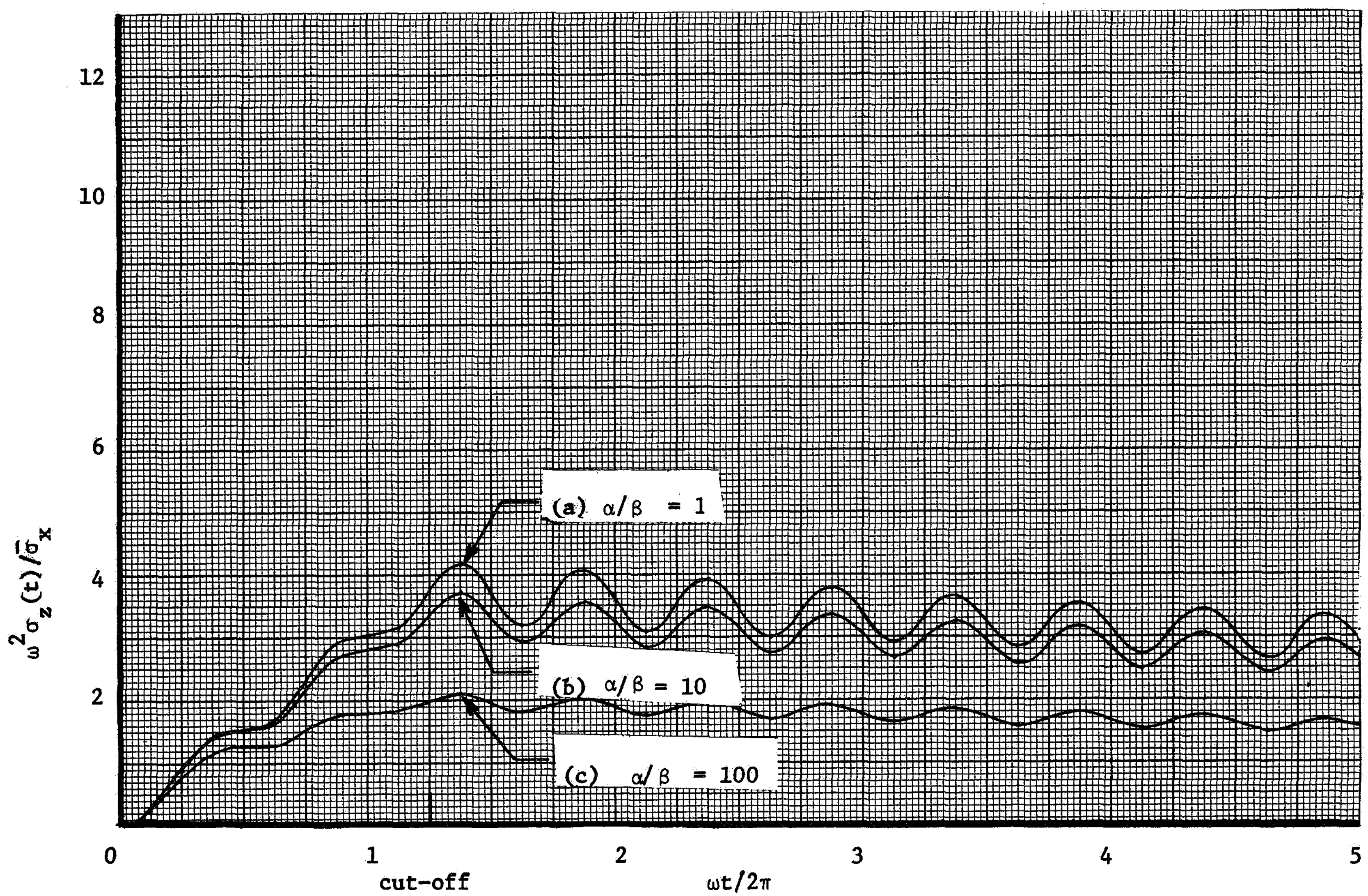

Figure A12 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.01$, $p/\omega = 1$, $\omega t_o/2\pi = 1.25$

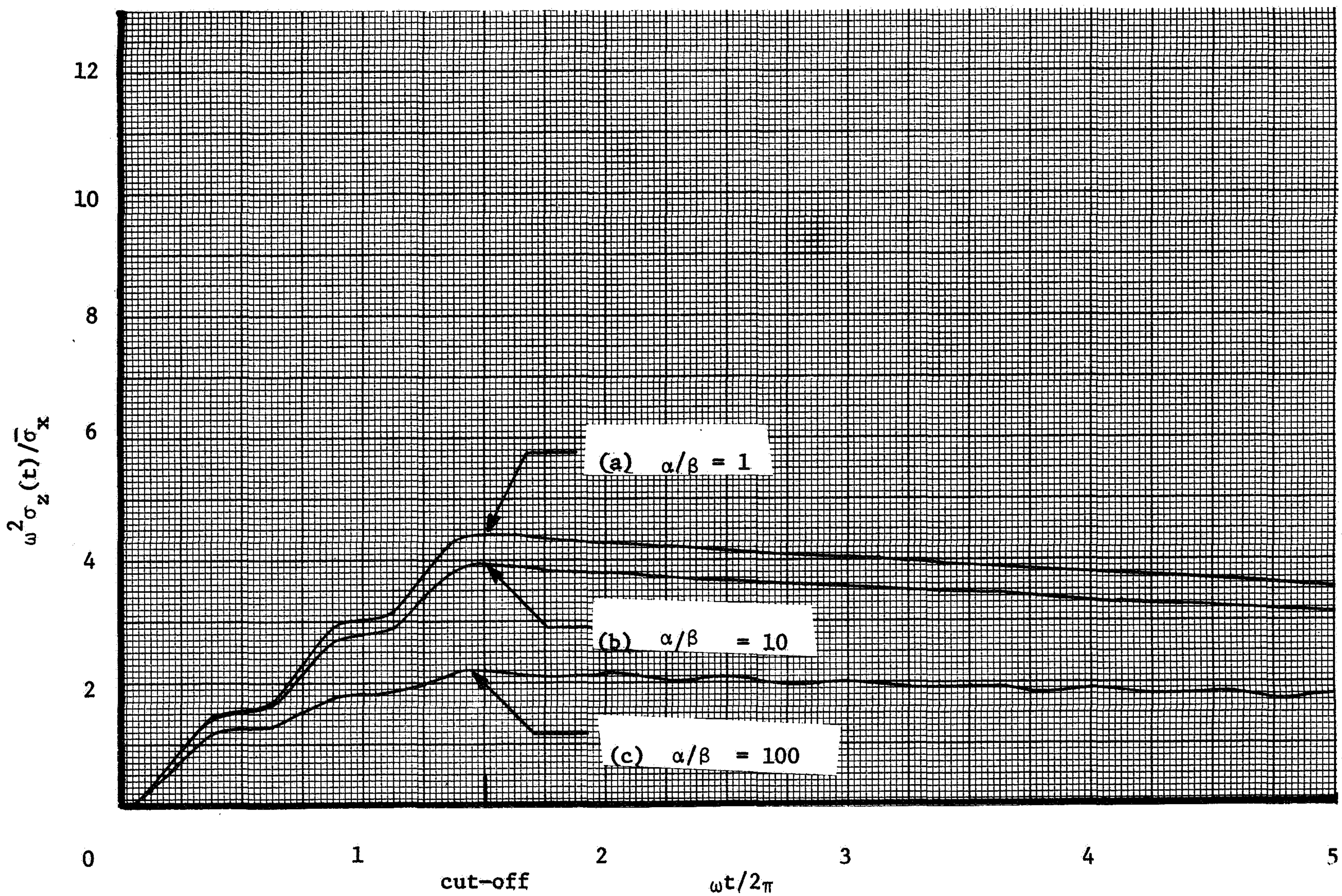

Figure A13 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.01$, $p/\omega = 1$, $\omega t_o/2\pi = 1.5$

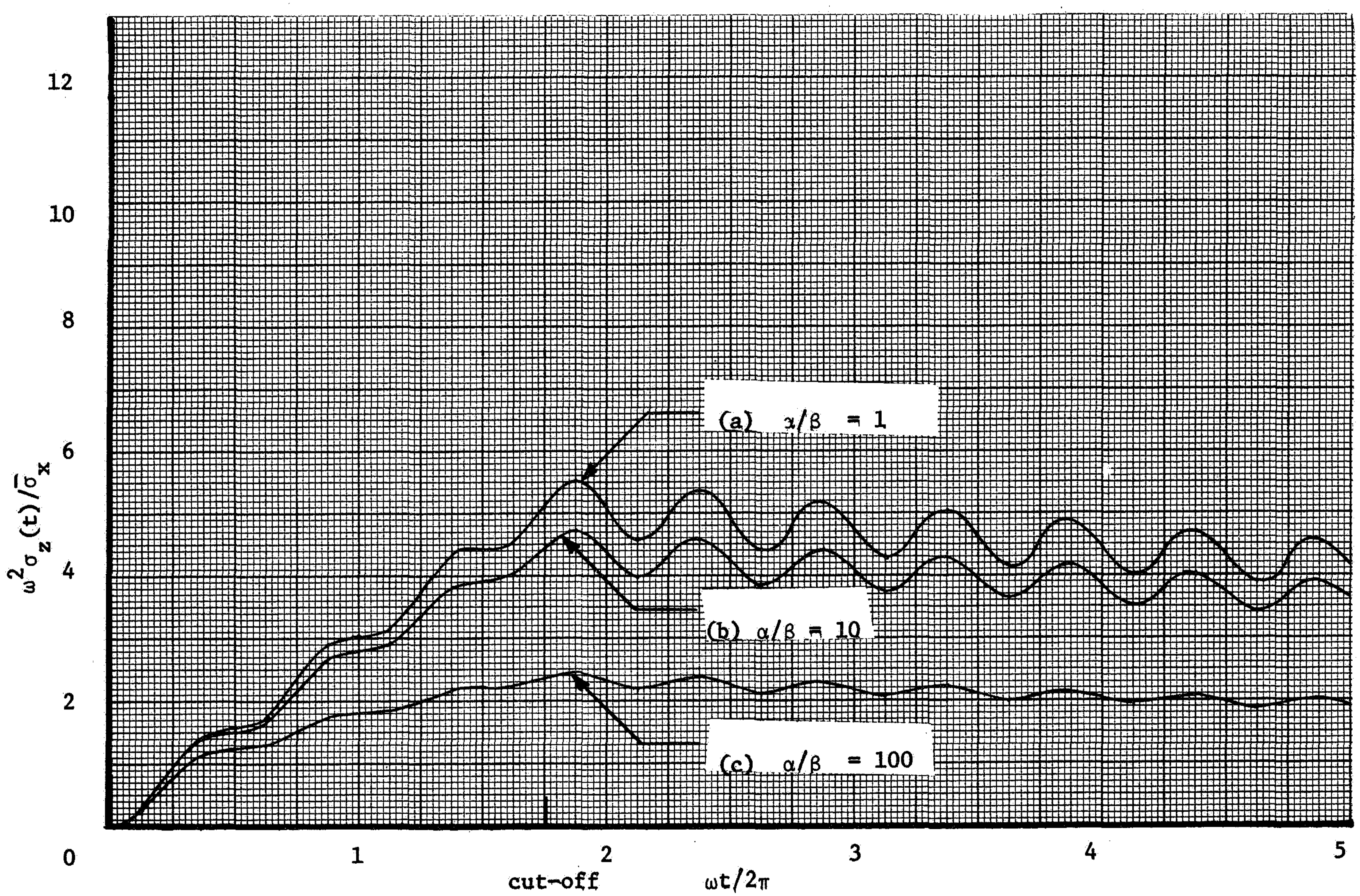

Figure A14 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.01$, $p/\omega = 1$, $\omega t_o/2\pi = 1.75$

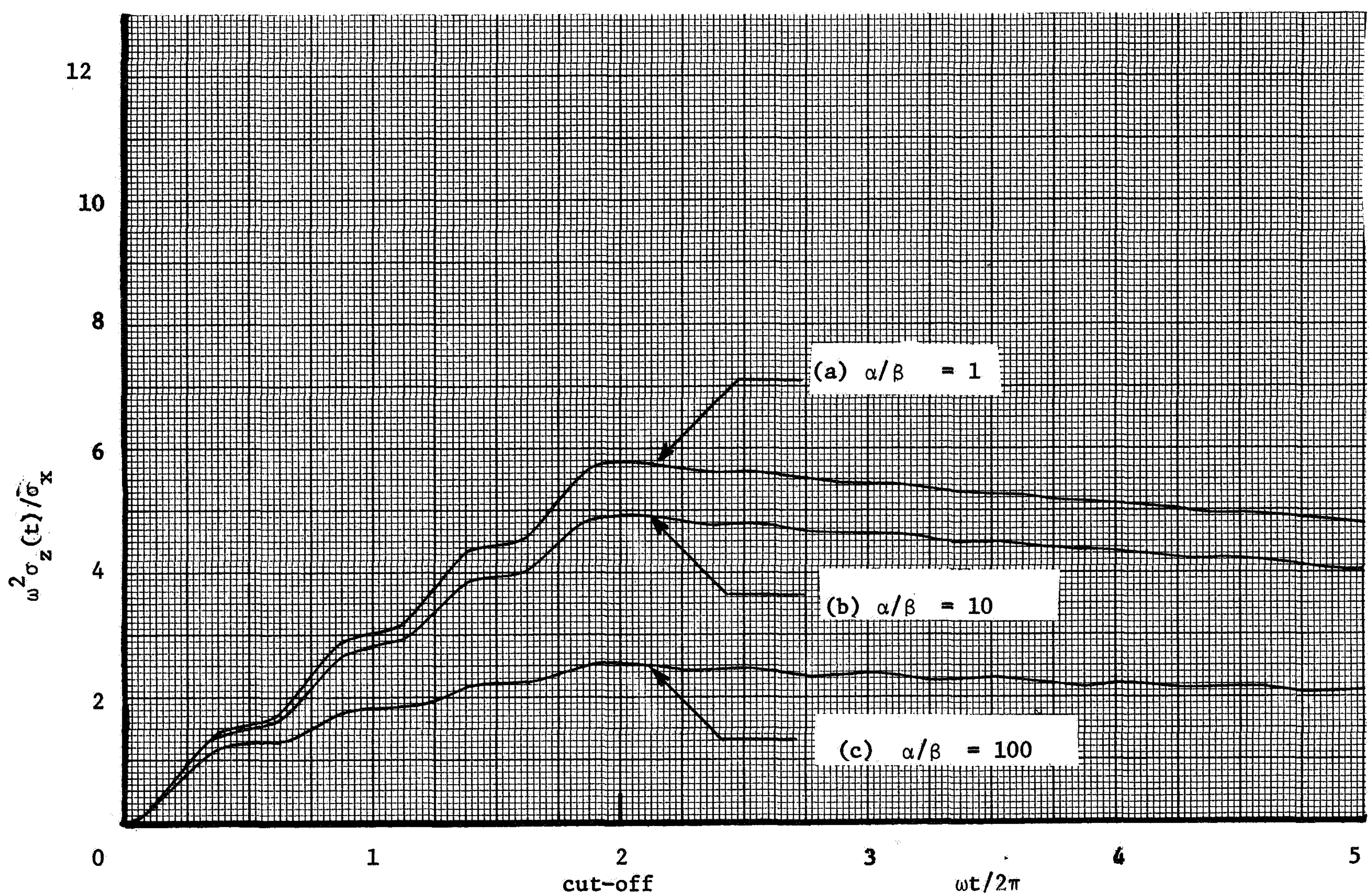

Figure A15 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.01$, $p/\omega = 1$, $\omega t_o/2\pi = 2$

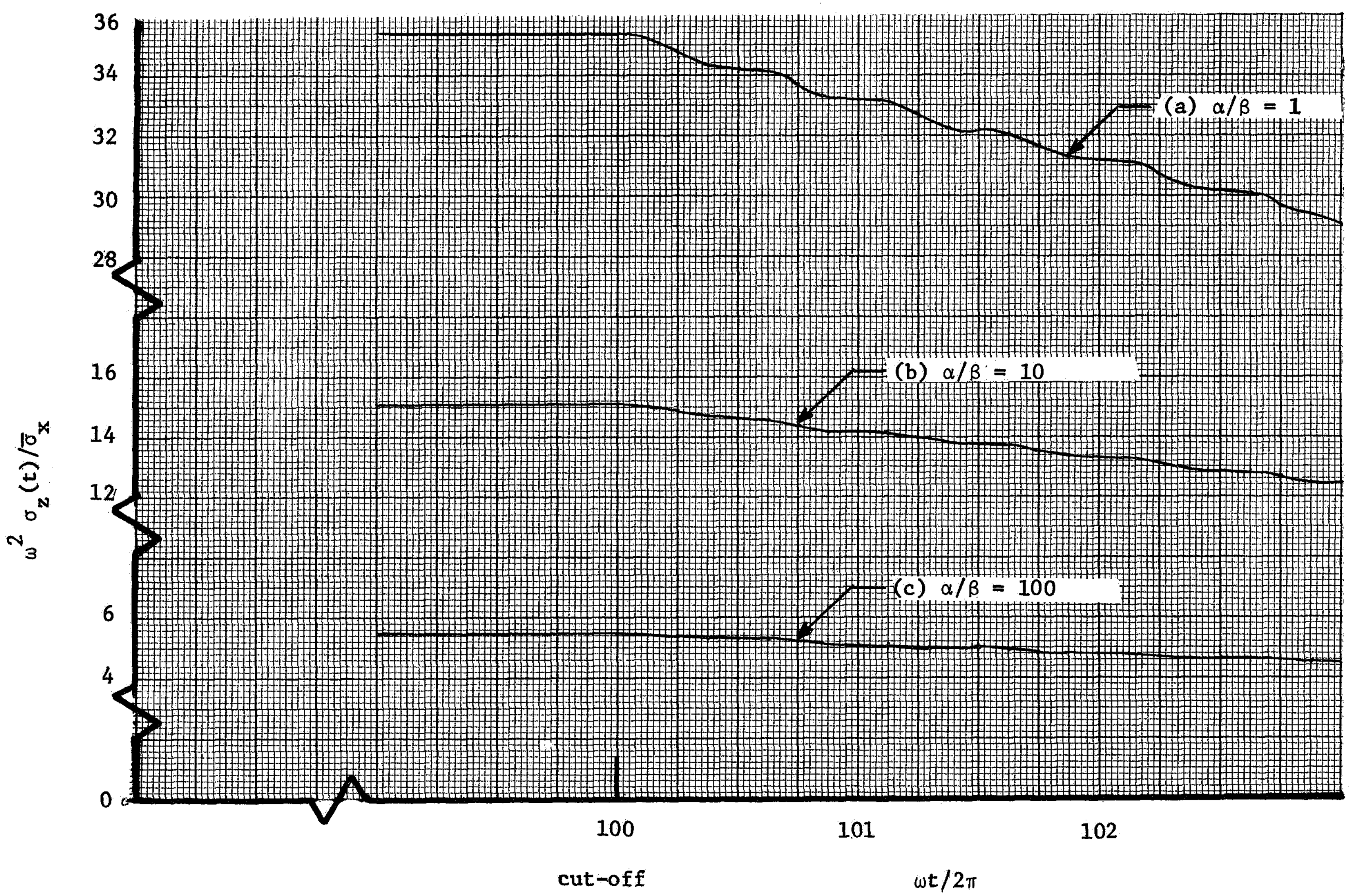

Figure A16 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.01$, $p/\omega = 1$, $\omega t_o/2\pi = 100$

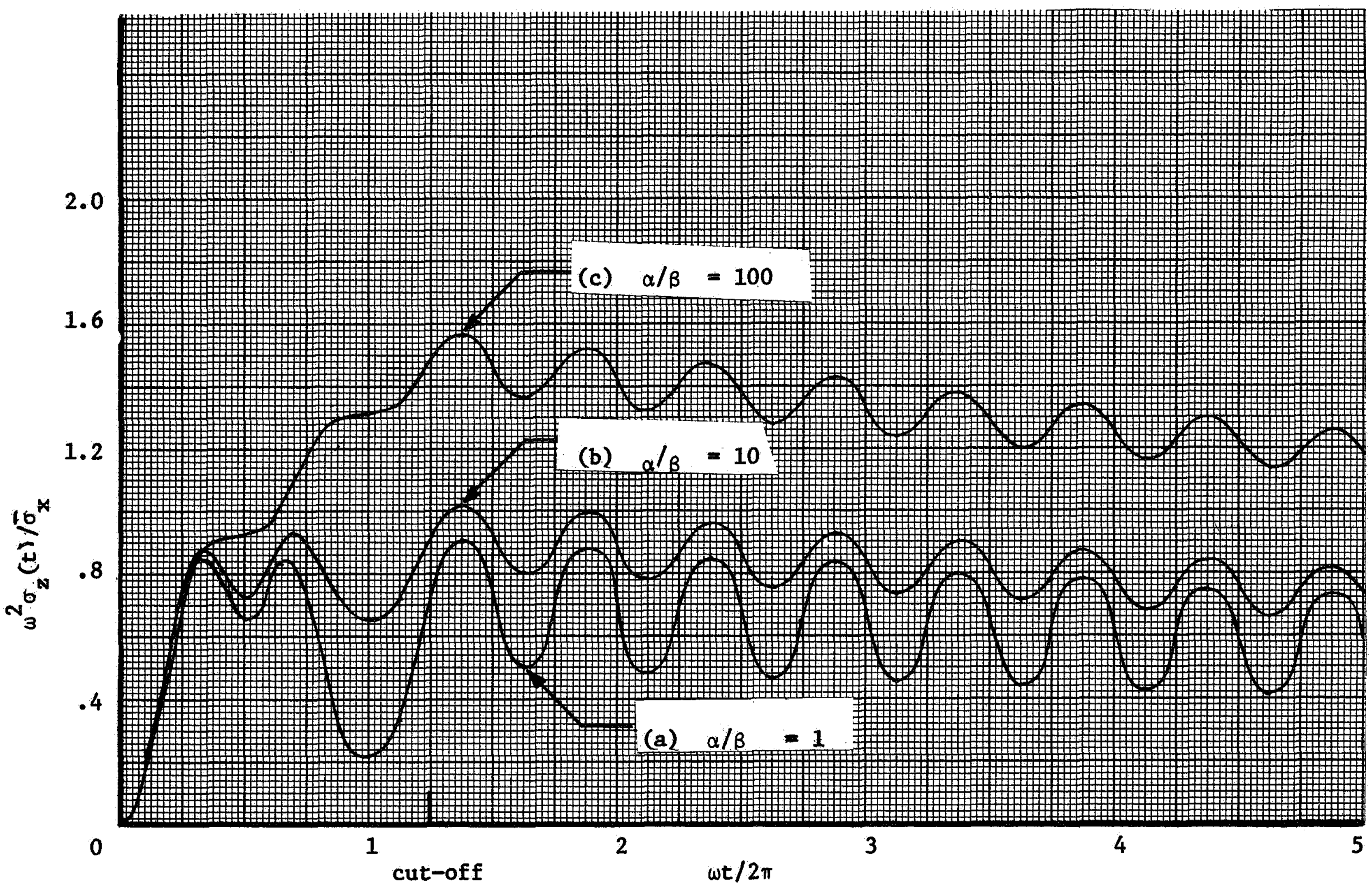

Figure A17 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.01$, $p/\omega = 2$, $\omega t_o/2\pi = 1.25$

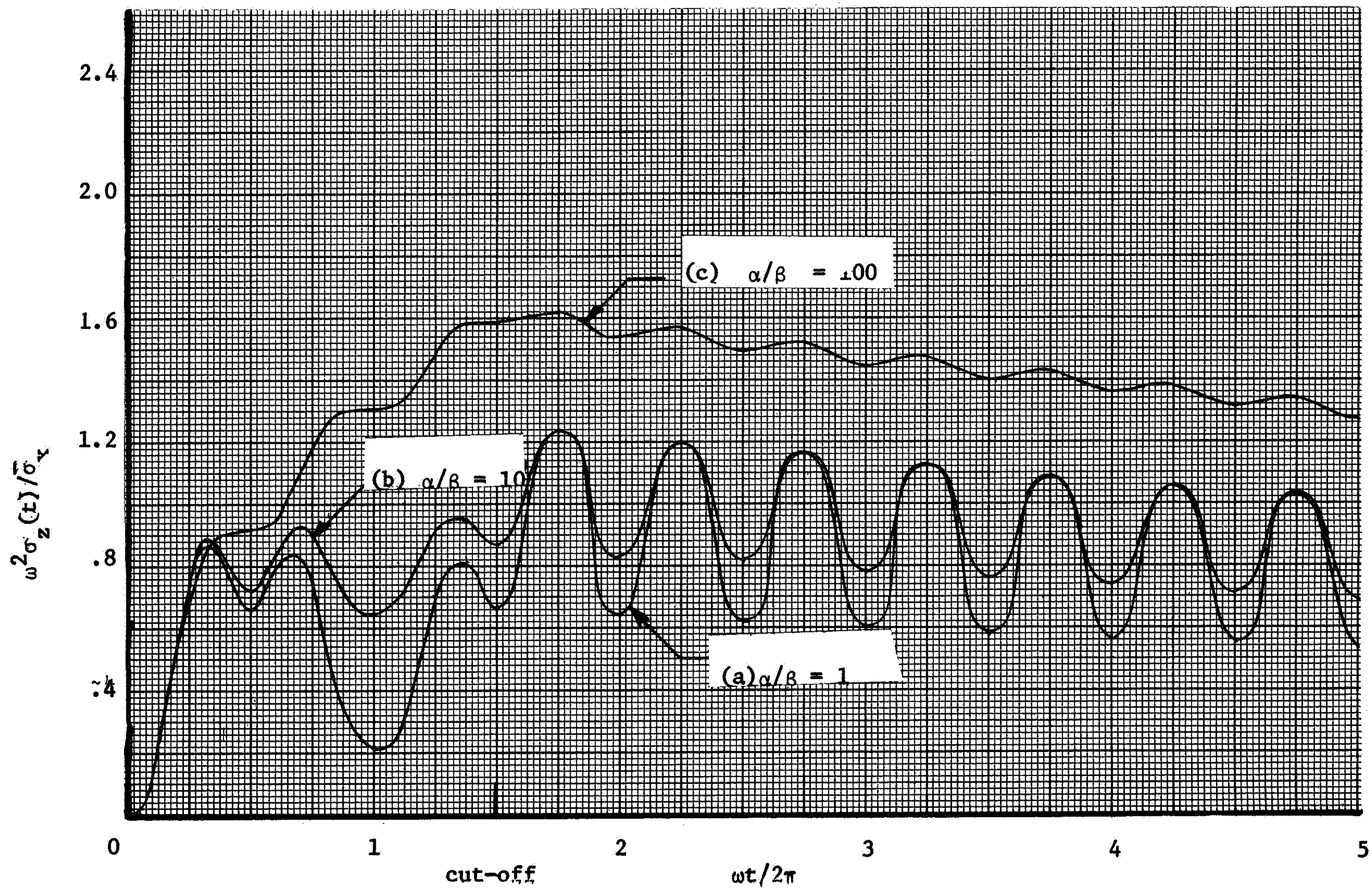

Figure A18 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.01$, $p/\omega = 2$, $\omega t_o/2\pi = 1.5$

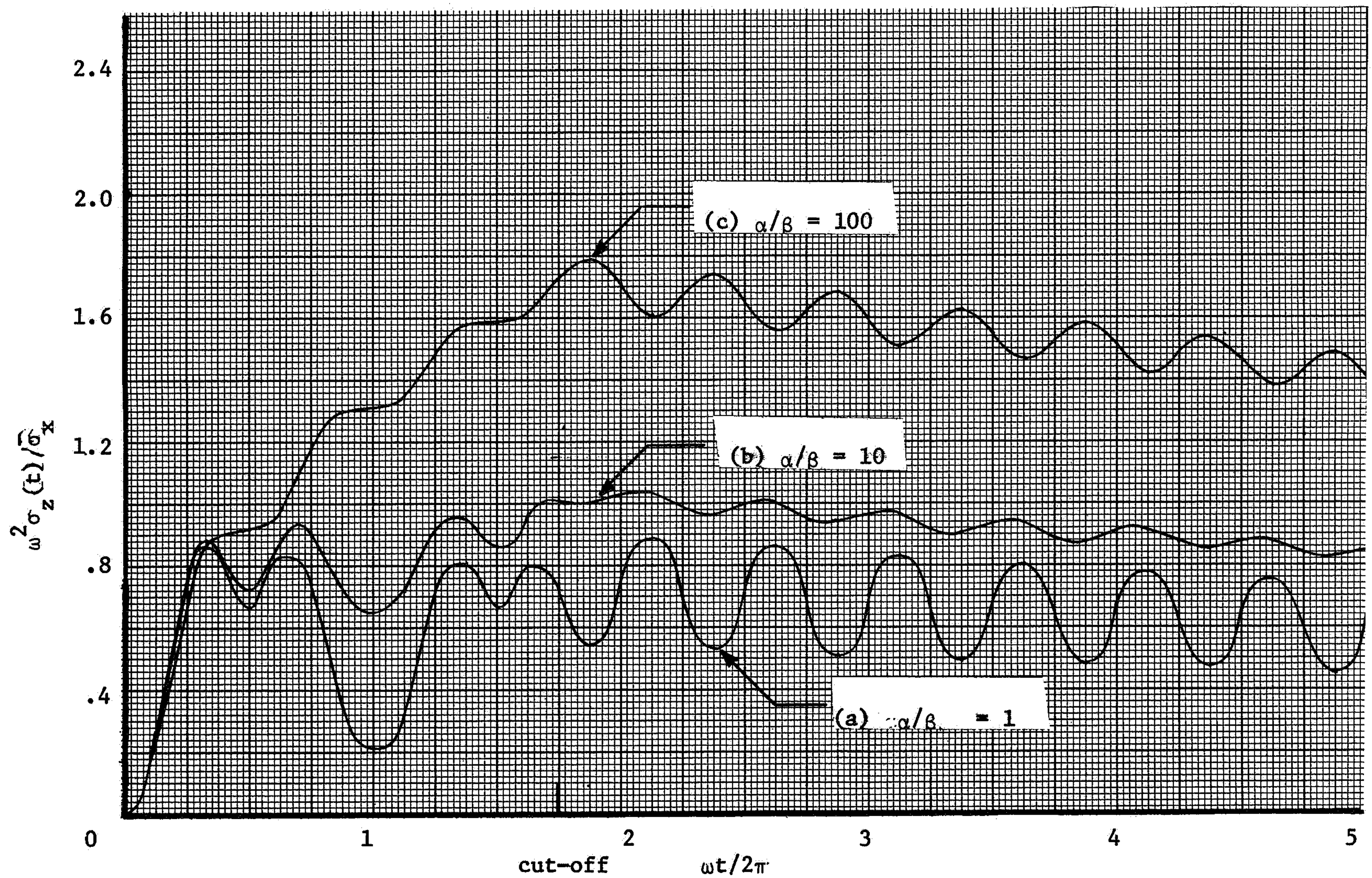

Figure A19 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.01$, $p/\omega = 2$, $\omega t_o/2\pi = 1.75$

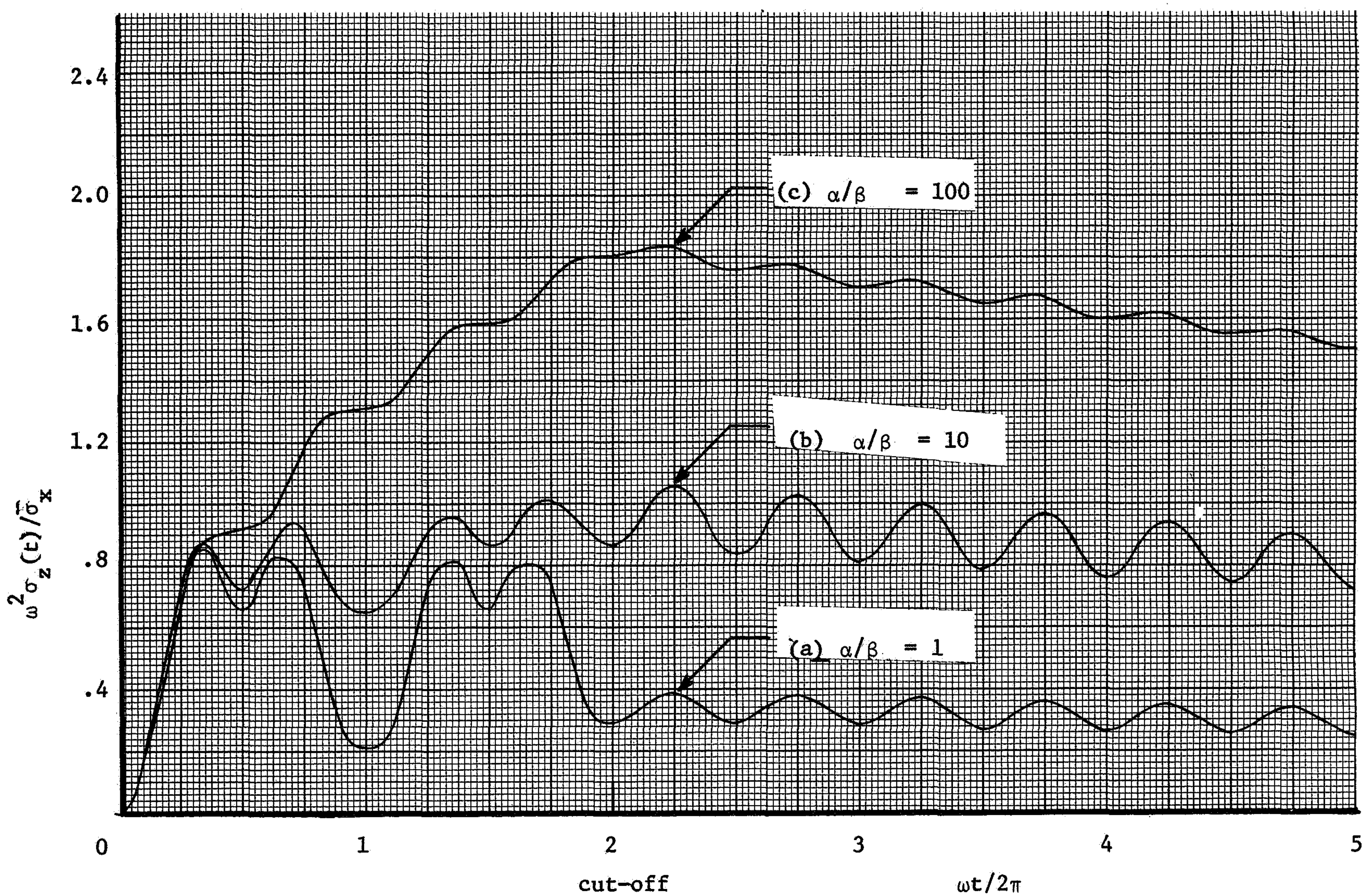

Figure A20 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.01$, $p/\omega = 2$, $\omega t_o/2\pi = 2$

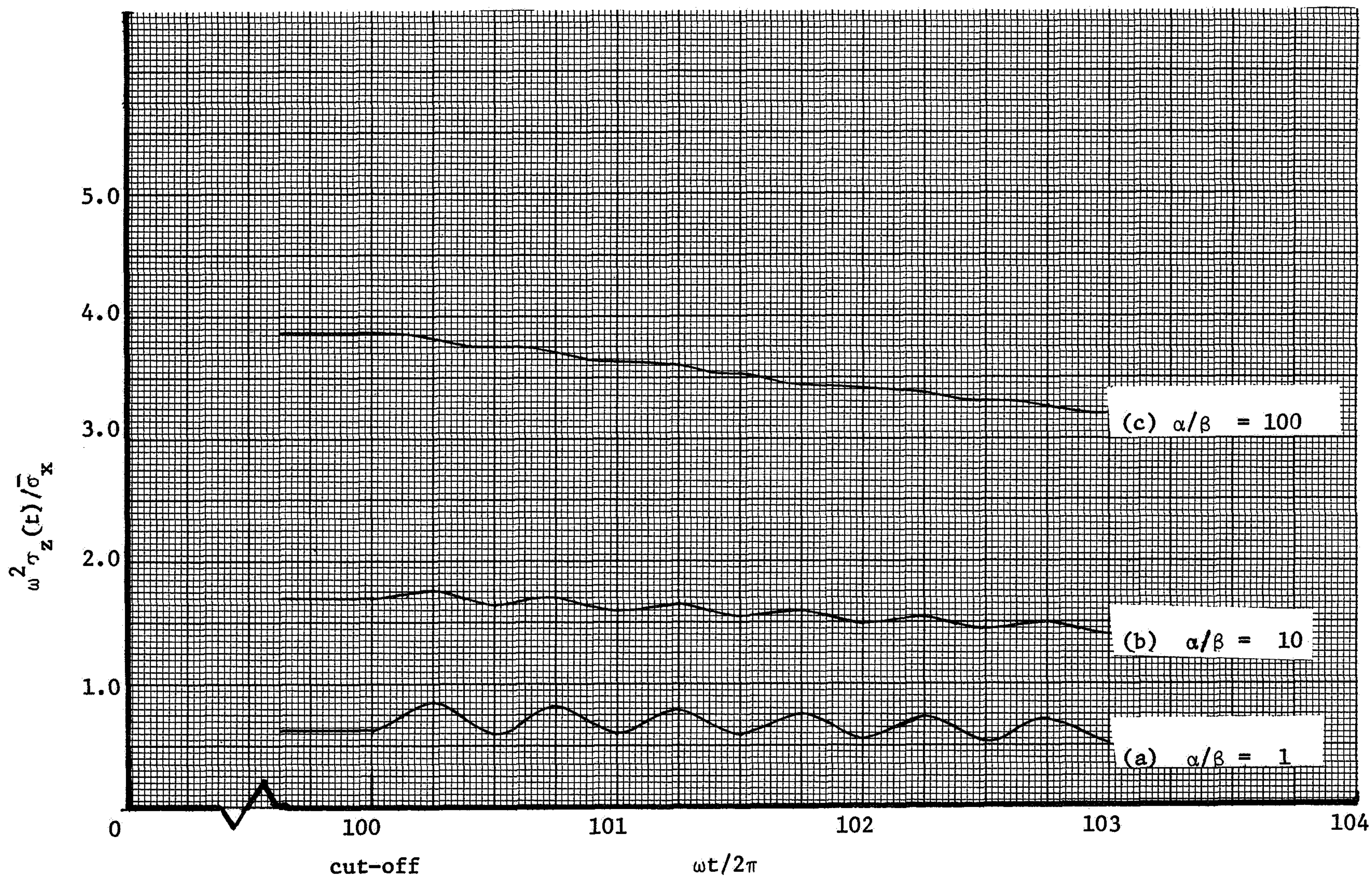

Figure A21 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.01$, $p/\omega = 2$, $\omega t_o/2\pi = 100$

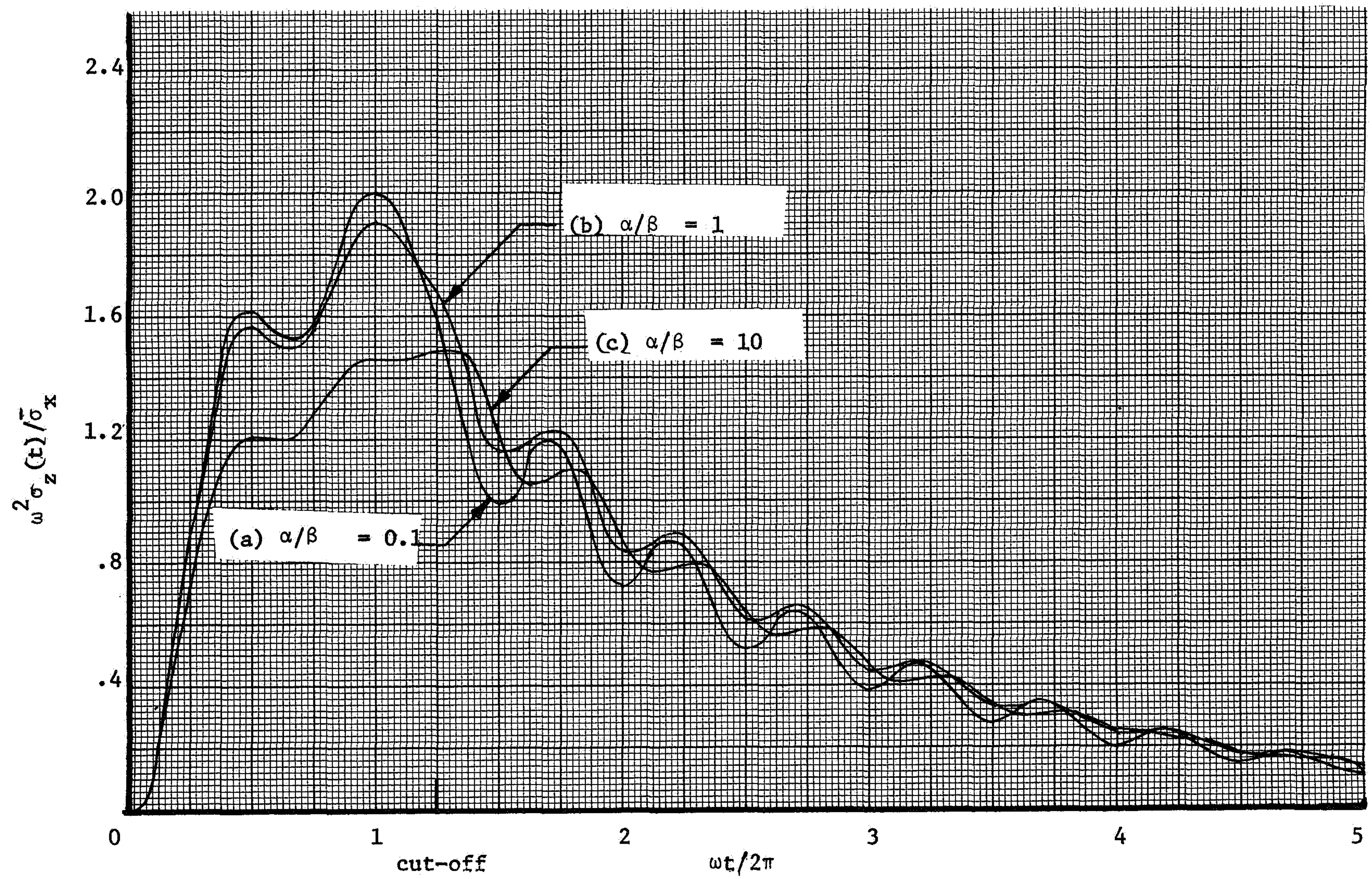

Figure A22 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.1$, $p/\omega = 0.5$, $\omega t_o 2\pi = 1.25$

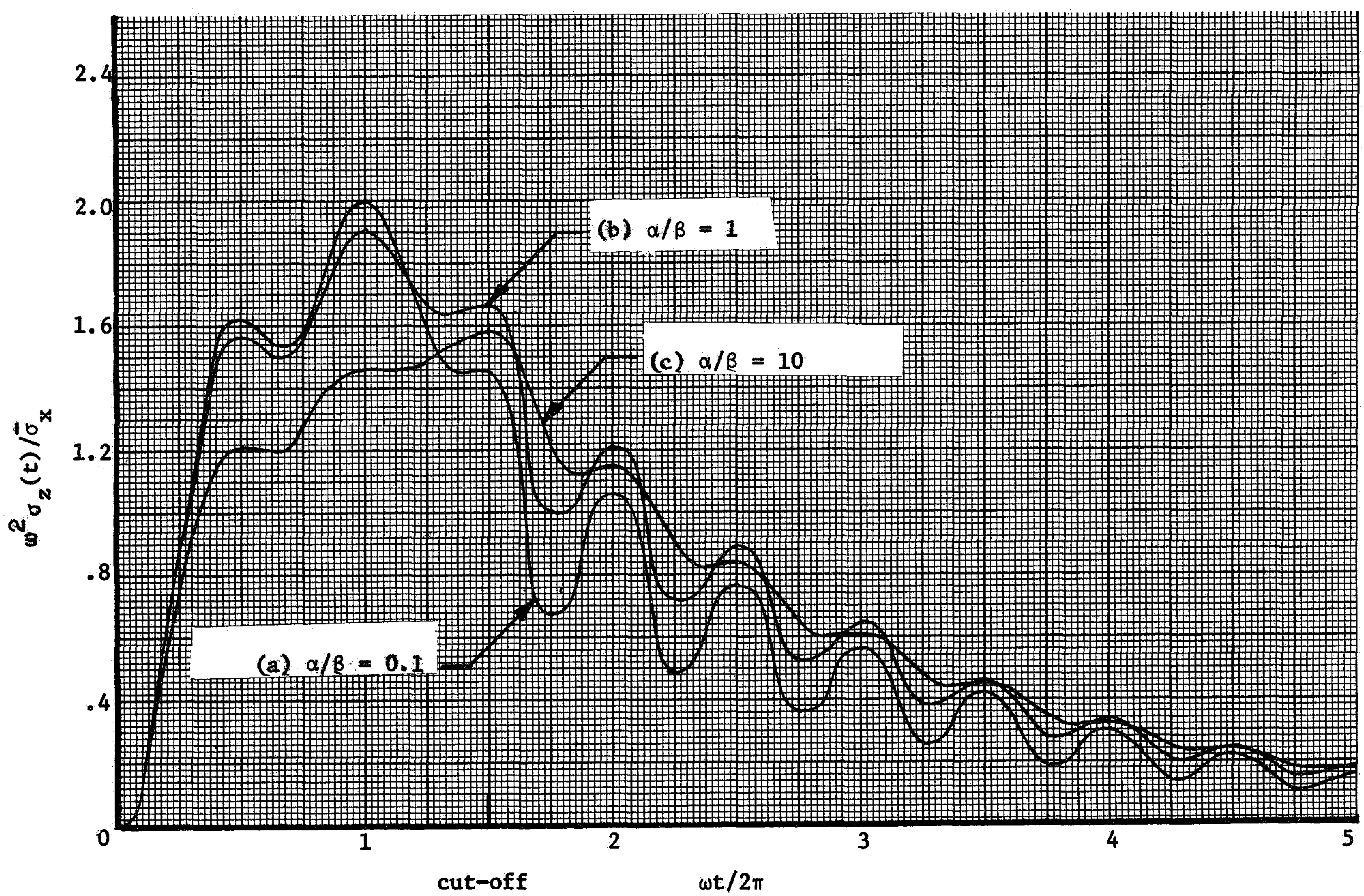

Figure A23 RMS Response for $f(t) = \mathcal{U}(t) - \mathcal{U}(t-t_o)$, $\zeta = 0.1$, $p/\omega = 0.5$, $\omega t_o/2\pi = 1.5$

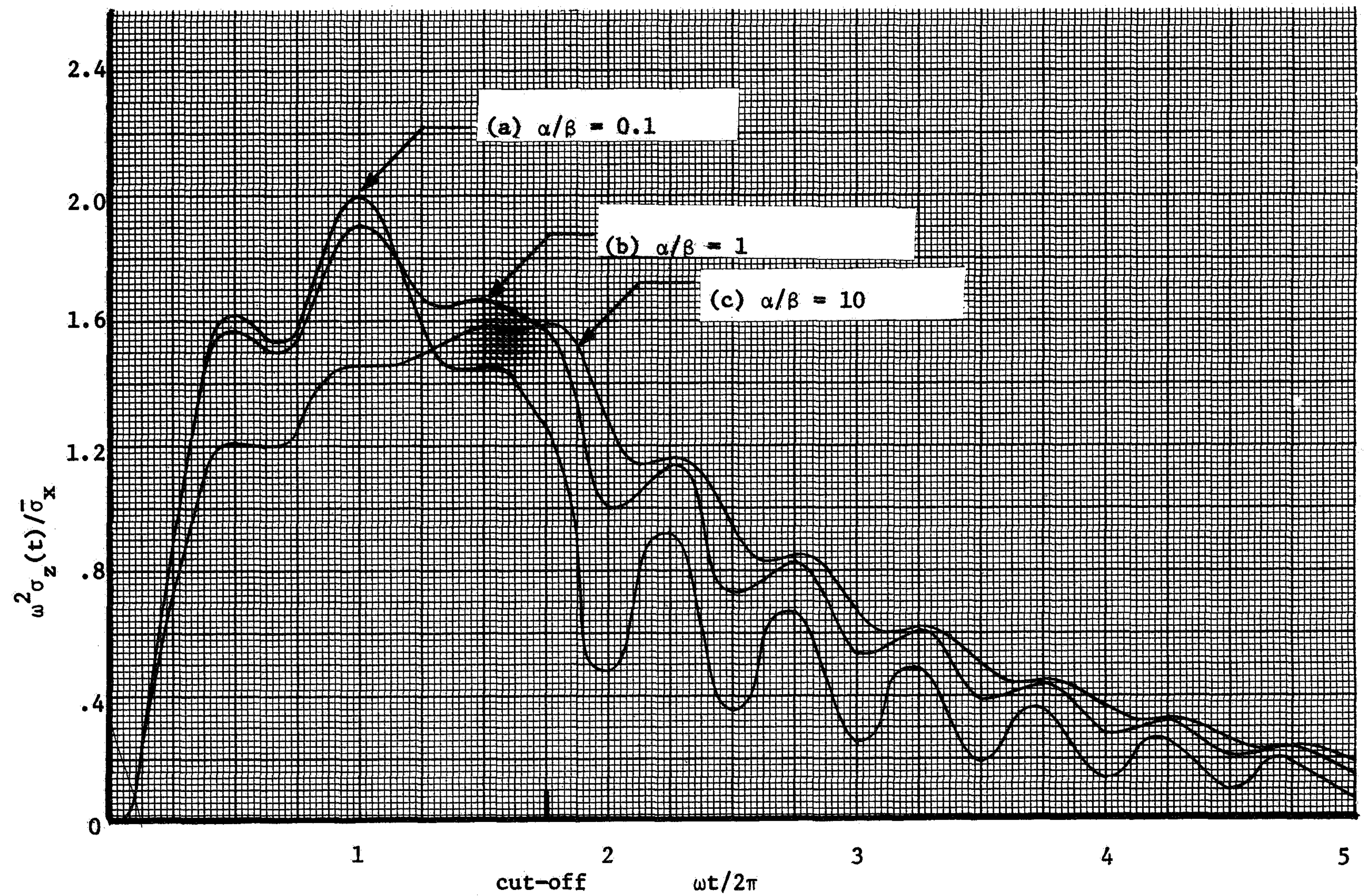

Figure A24 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.1$, $p/\omega = 0.5$, $\omega t_o/2\pi = 1.75$

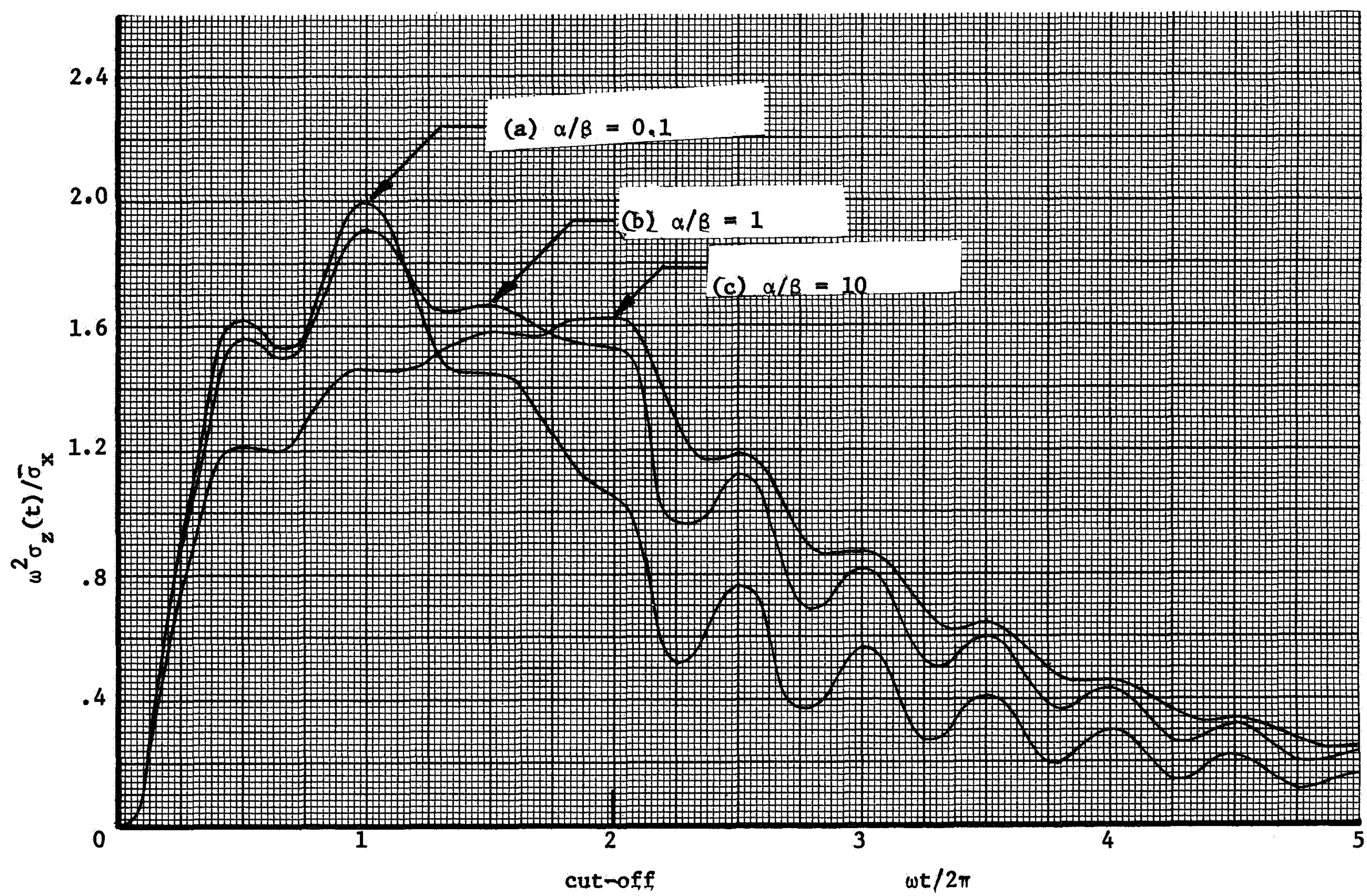

Figure A25 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.1$, $p/\omega = 0.5$, $\omega t_o/2\pi = 2$

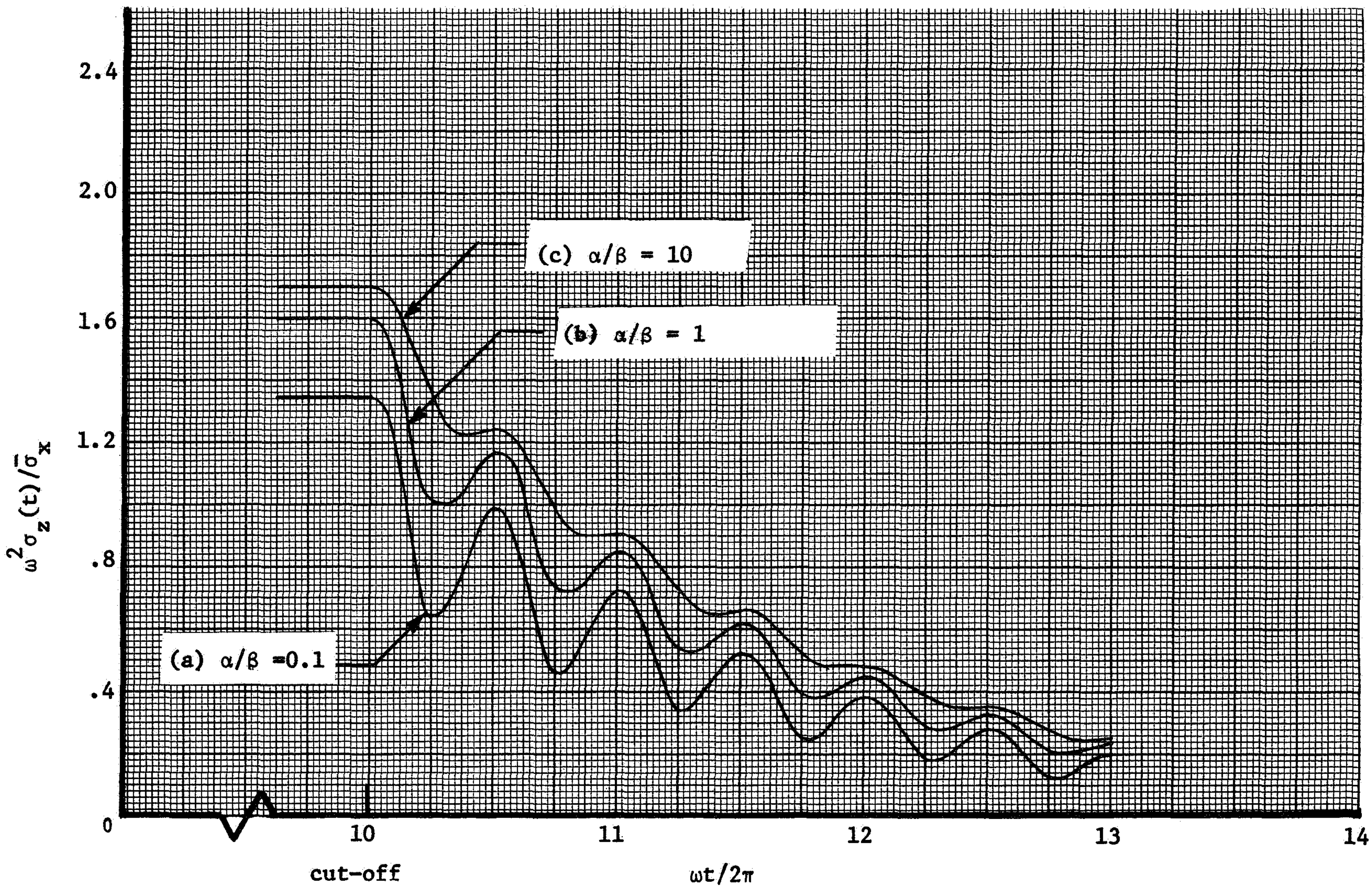

Figure A26 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.1$, $p/\omega = 0.5$, $\omega t_o/2\pi = 10$

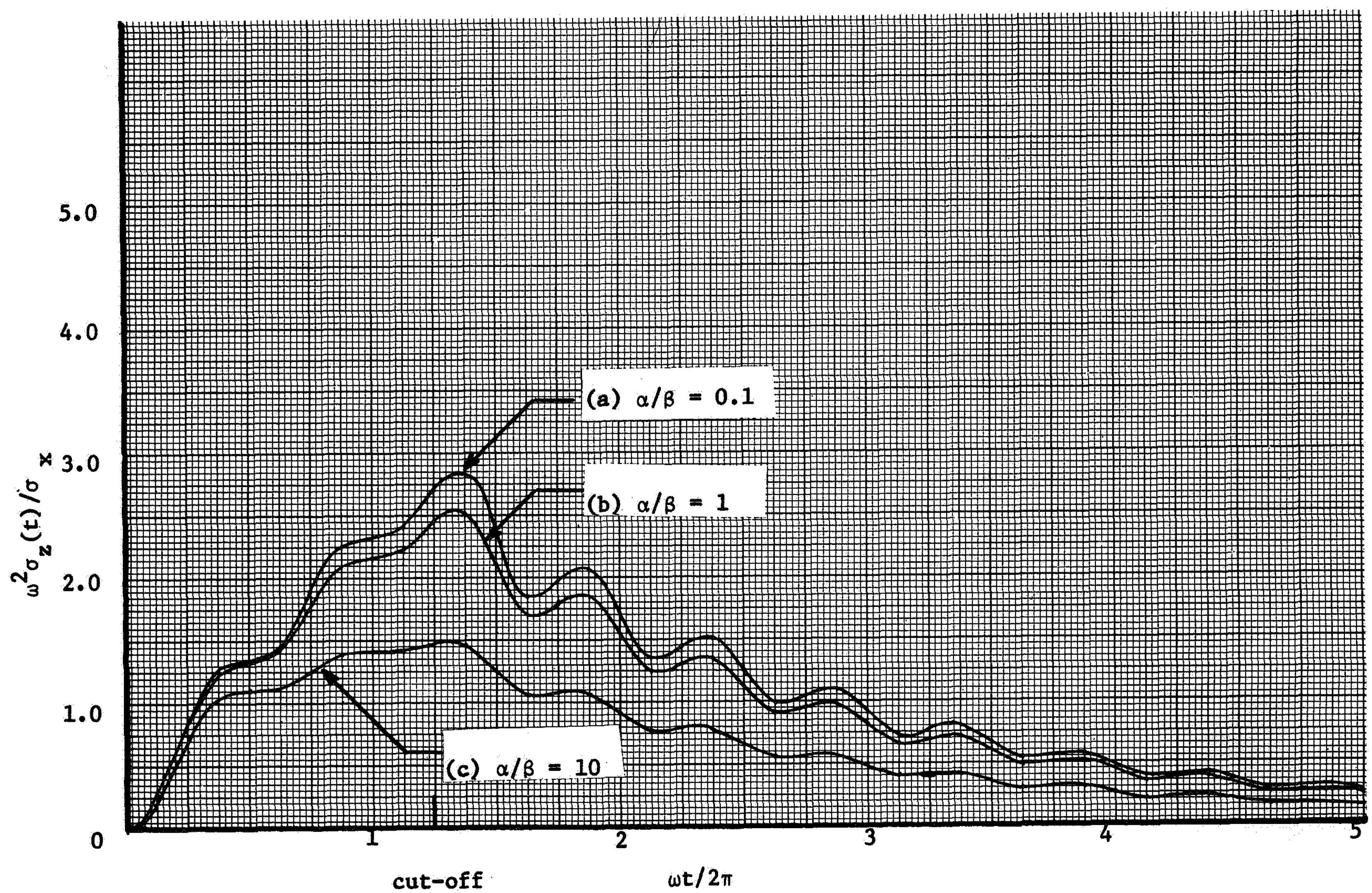

Figure A27 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.1$, $p/\omega = 1$, $\omega t_o/2\pi = 1.25$

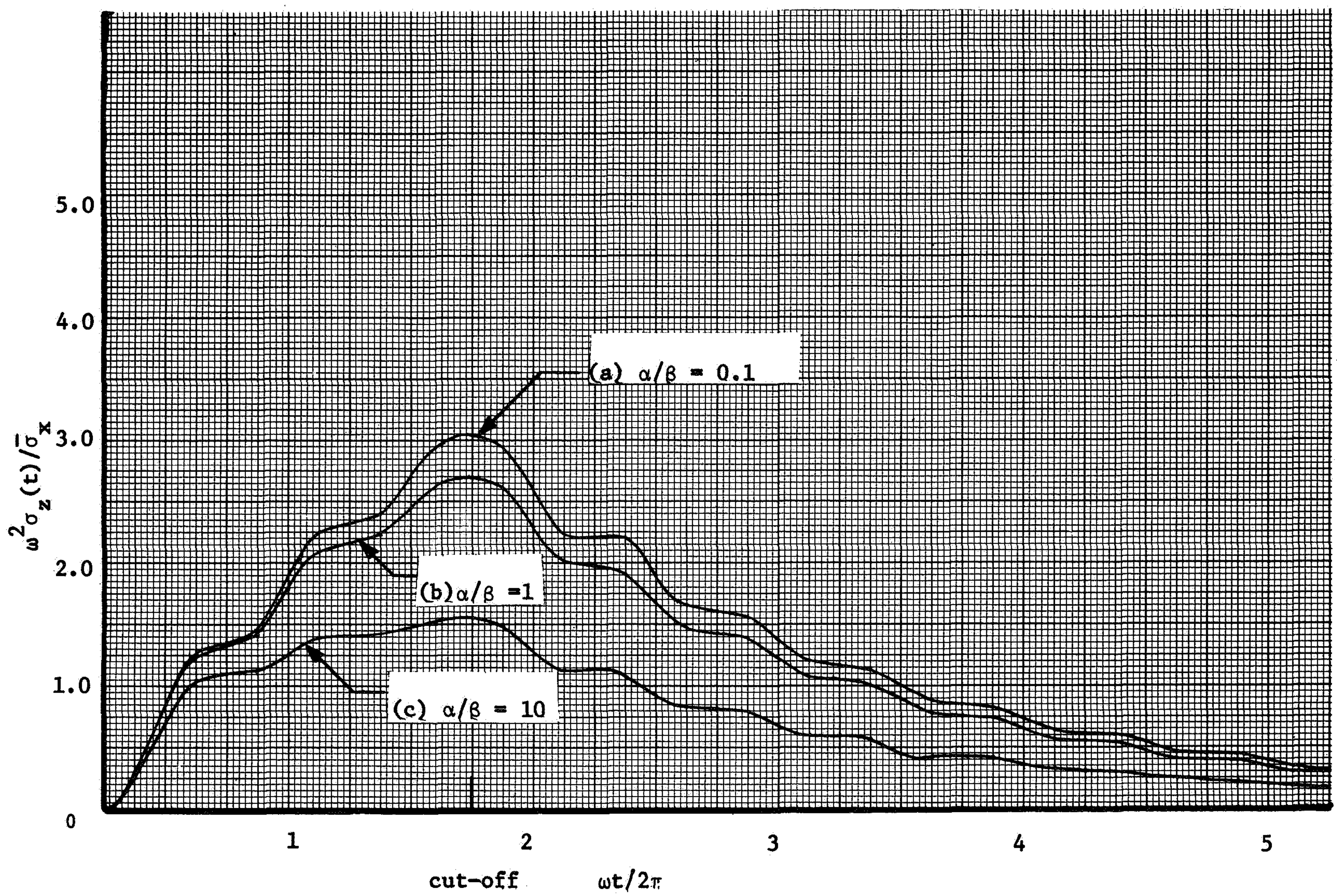

Figure A28 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.1$, $p/\omega = 1$, $\omega t_o/2\pi = 1.5$

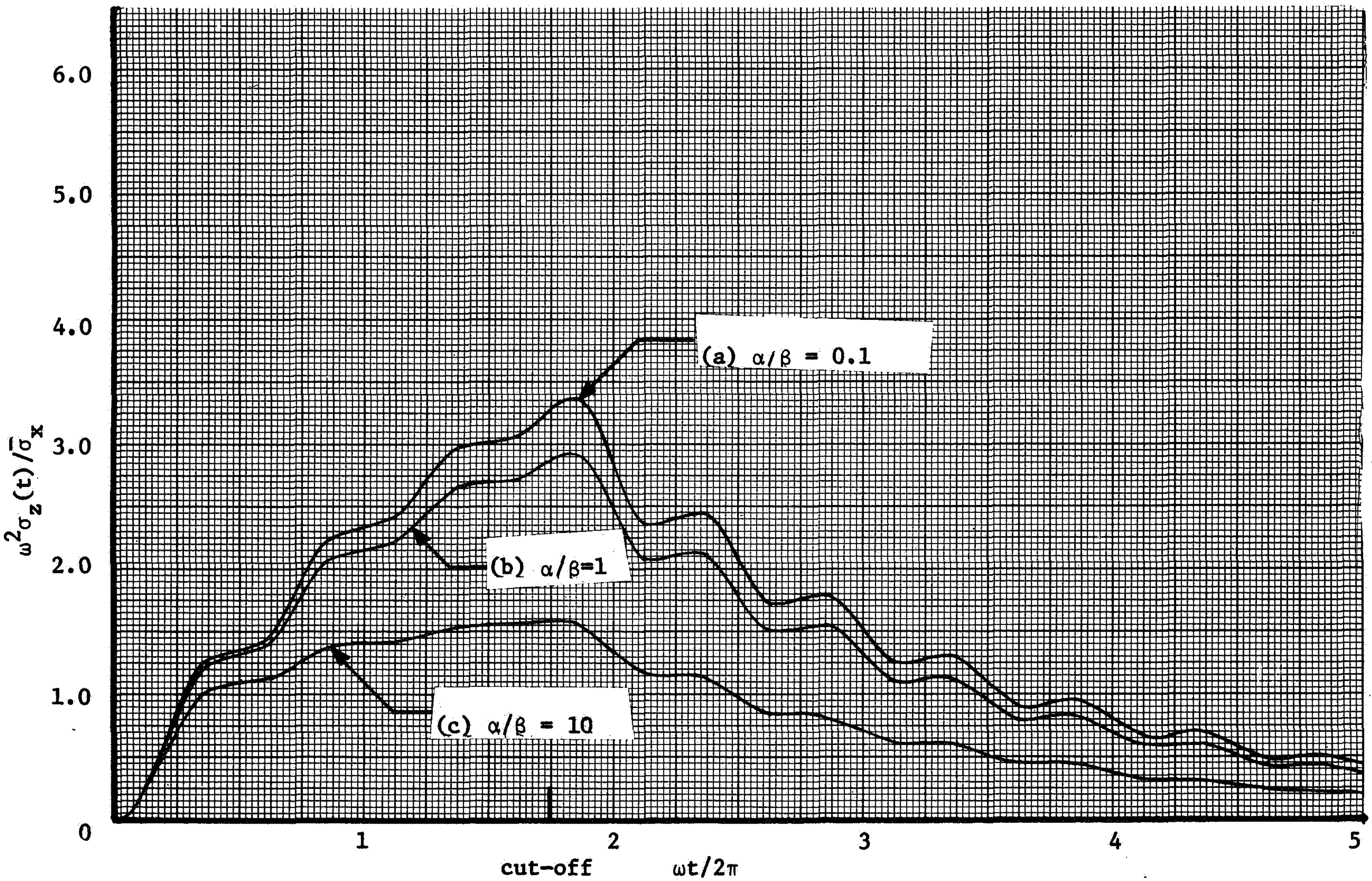

Figure A29 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.1$, $p/\omega = 0.5$, $\omega t_o/2\pi = 1.75$

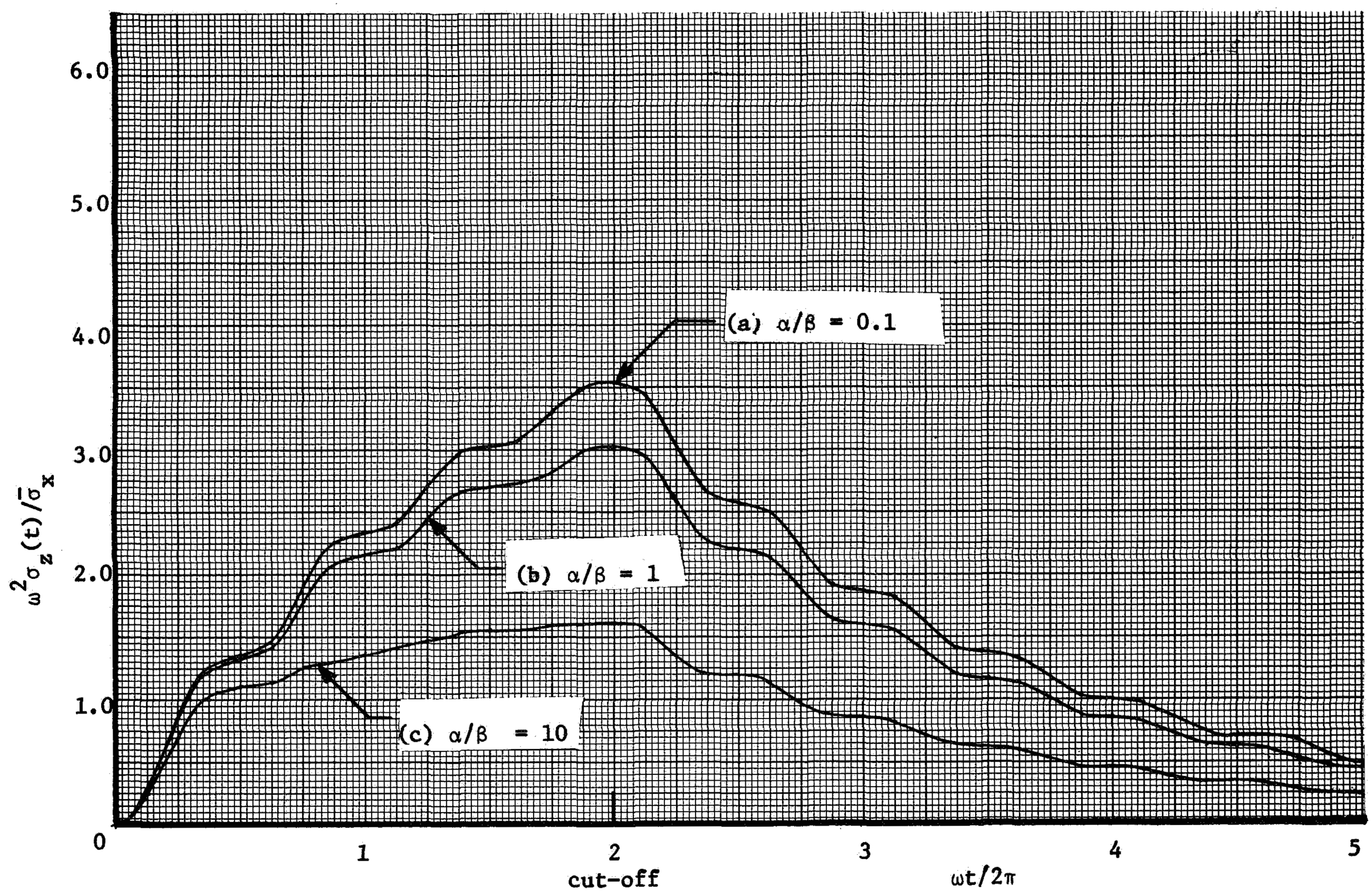

Figure A30 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.1$, $p/\omega = 0.5$, $\omega t_o/2\pi = 2$

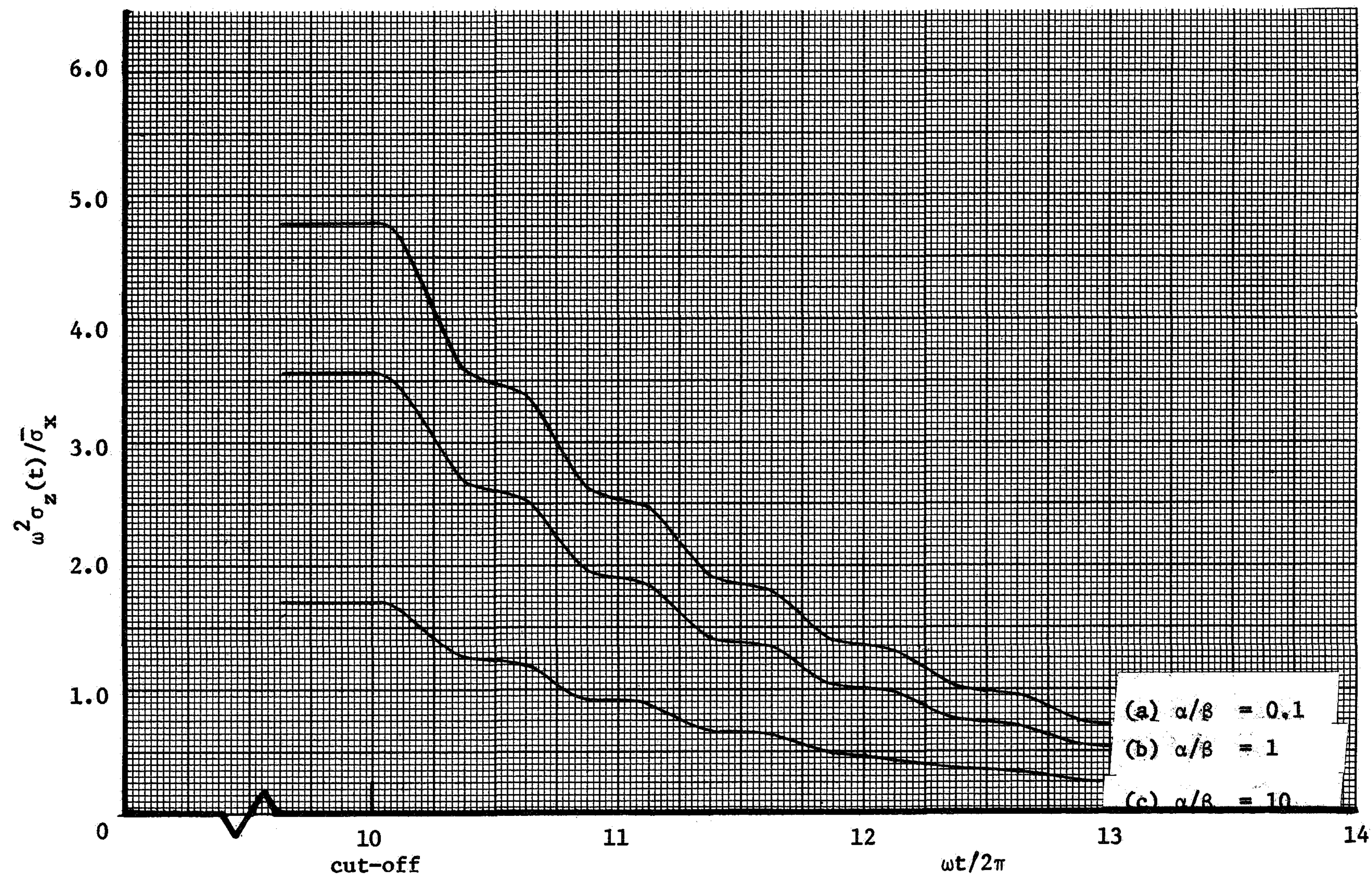

Figure A31 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.1$, $p/\omega = 0.5$, $\omega t_o/2\pi = 10$

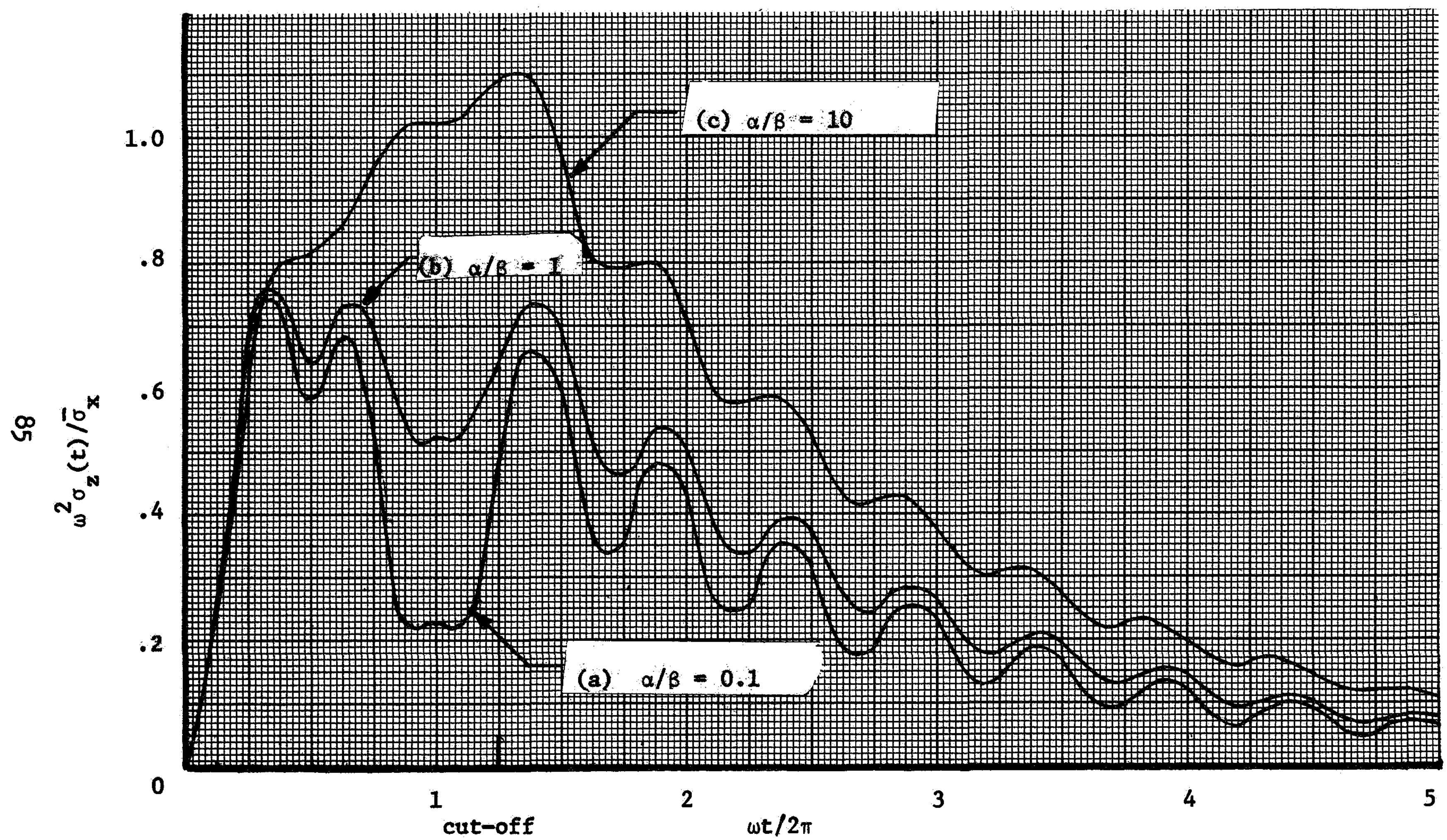

Figure A32 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.1$, $p/\omega = 2$, $\omega t_o/2\pi = 1.25$

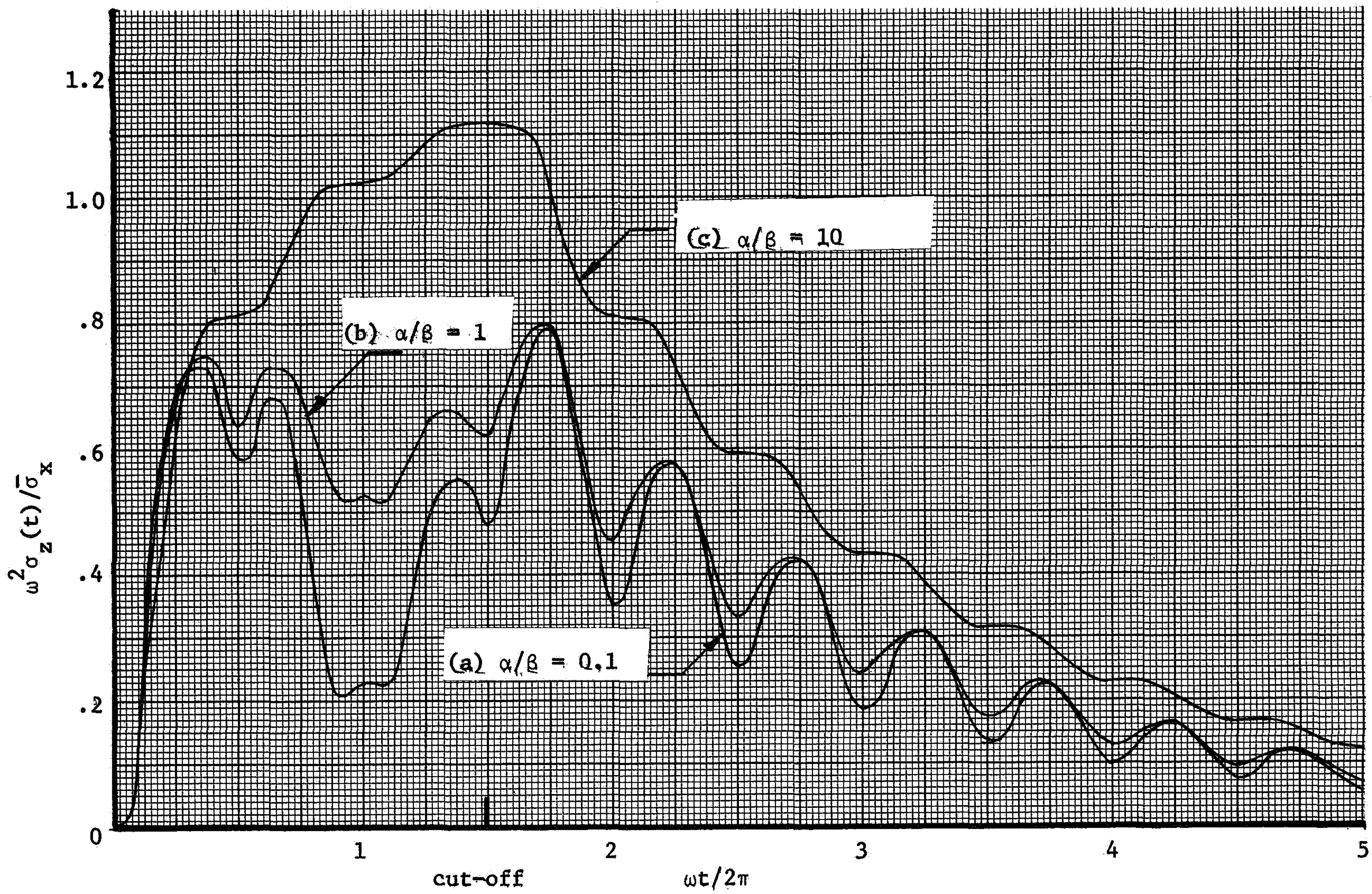

Figure A33 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.1$, $p/\omega = 2$, $\omega t_o/2\pi = 1.5$

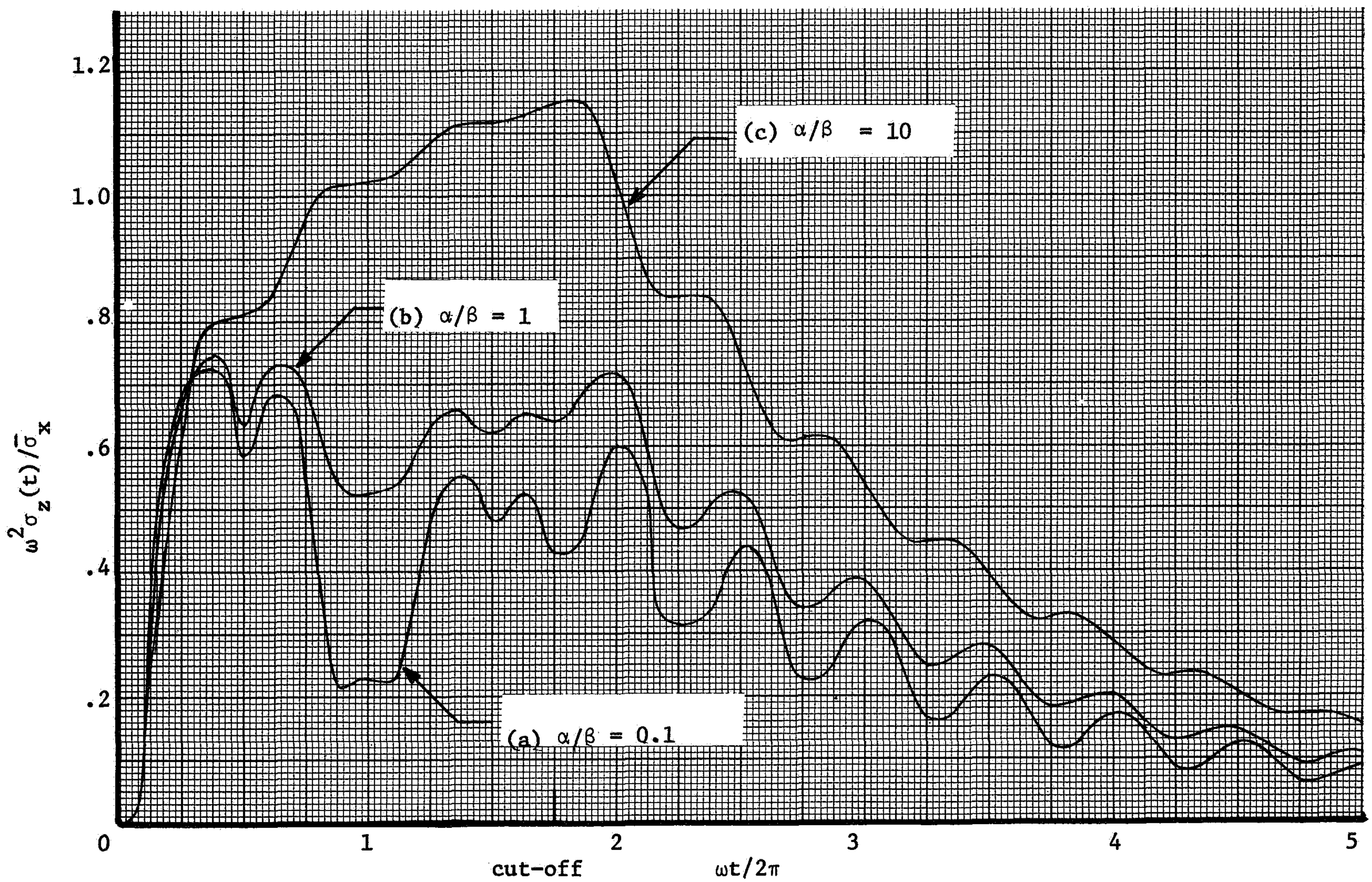

Figure A34 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.1$, $p/\omega = 2$, $\omega t_o/2\pi = 1.75$

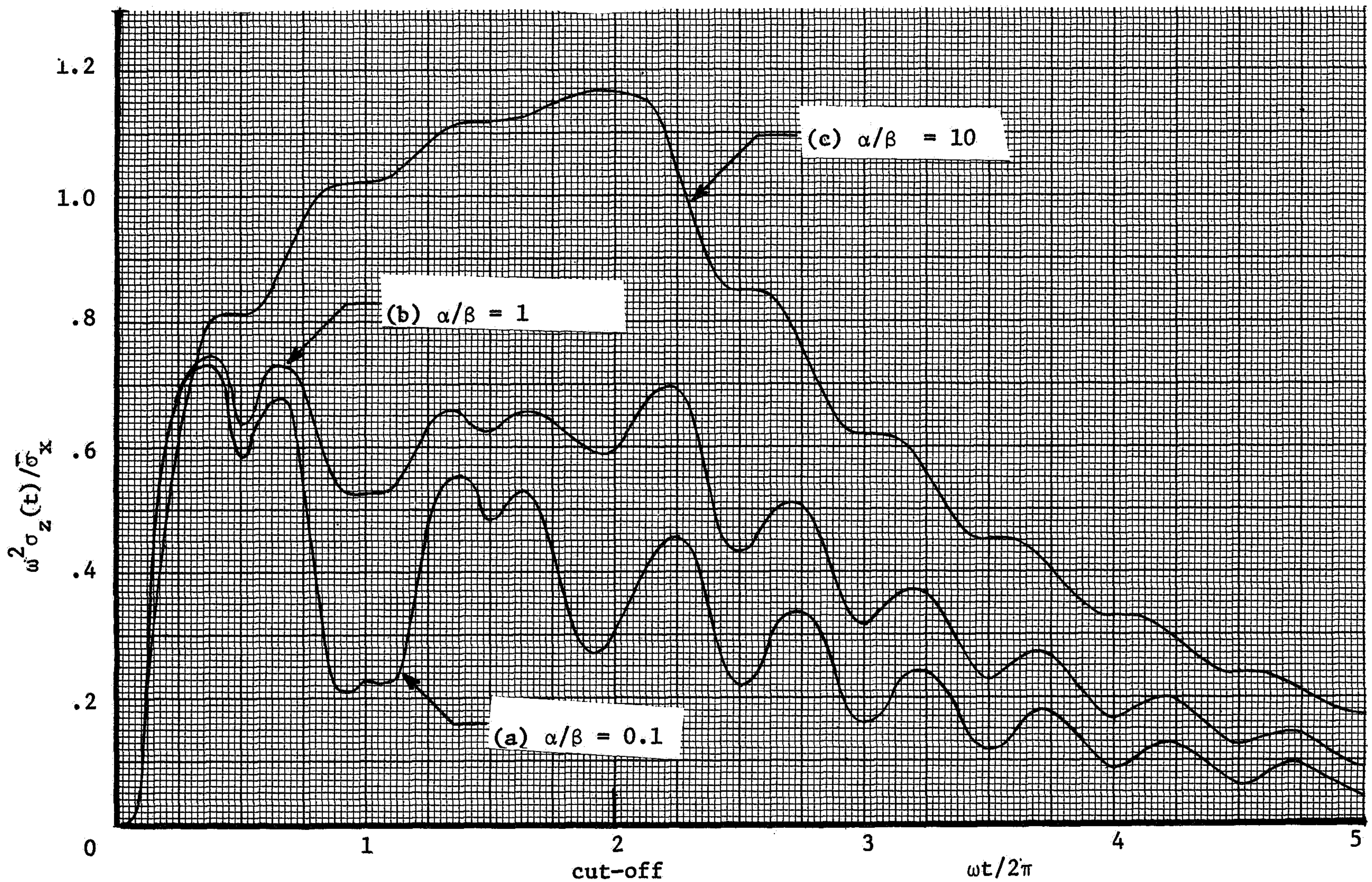

Figure A35 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.1$, $p/\omega = 2$, $\omega t_o/2\pi = 2$

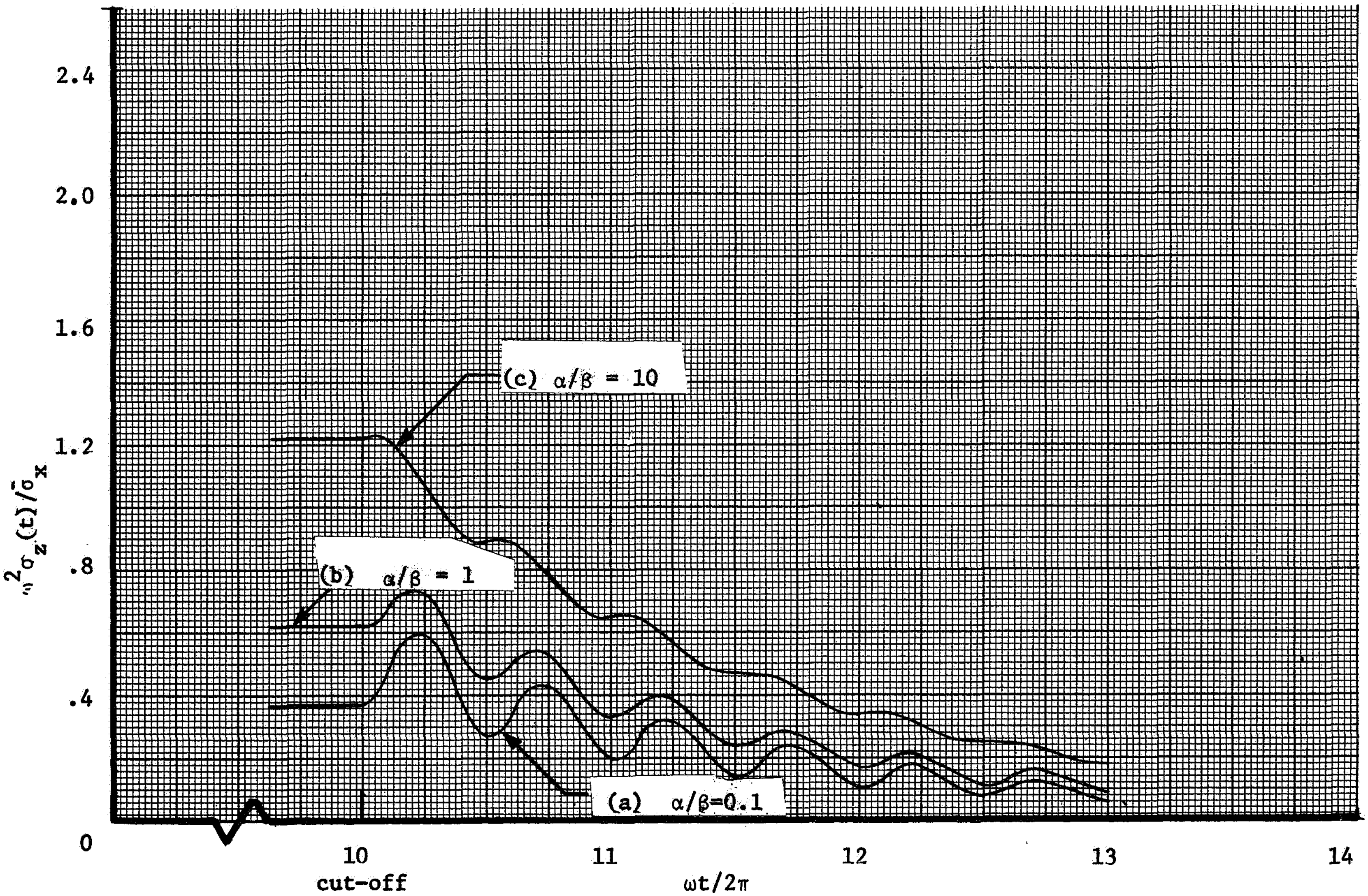

Figure A36 RMS Response for $f(t) = u(t) - u(t-t_o)$, $\zeta = 0.1$, $p/\omega = 2$, $\omega t_o/2\pi = 10$

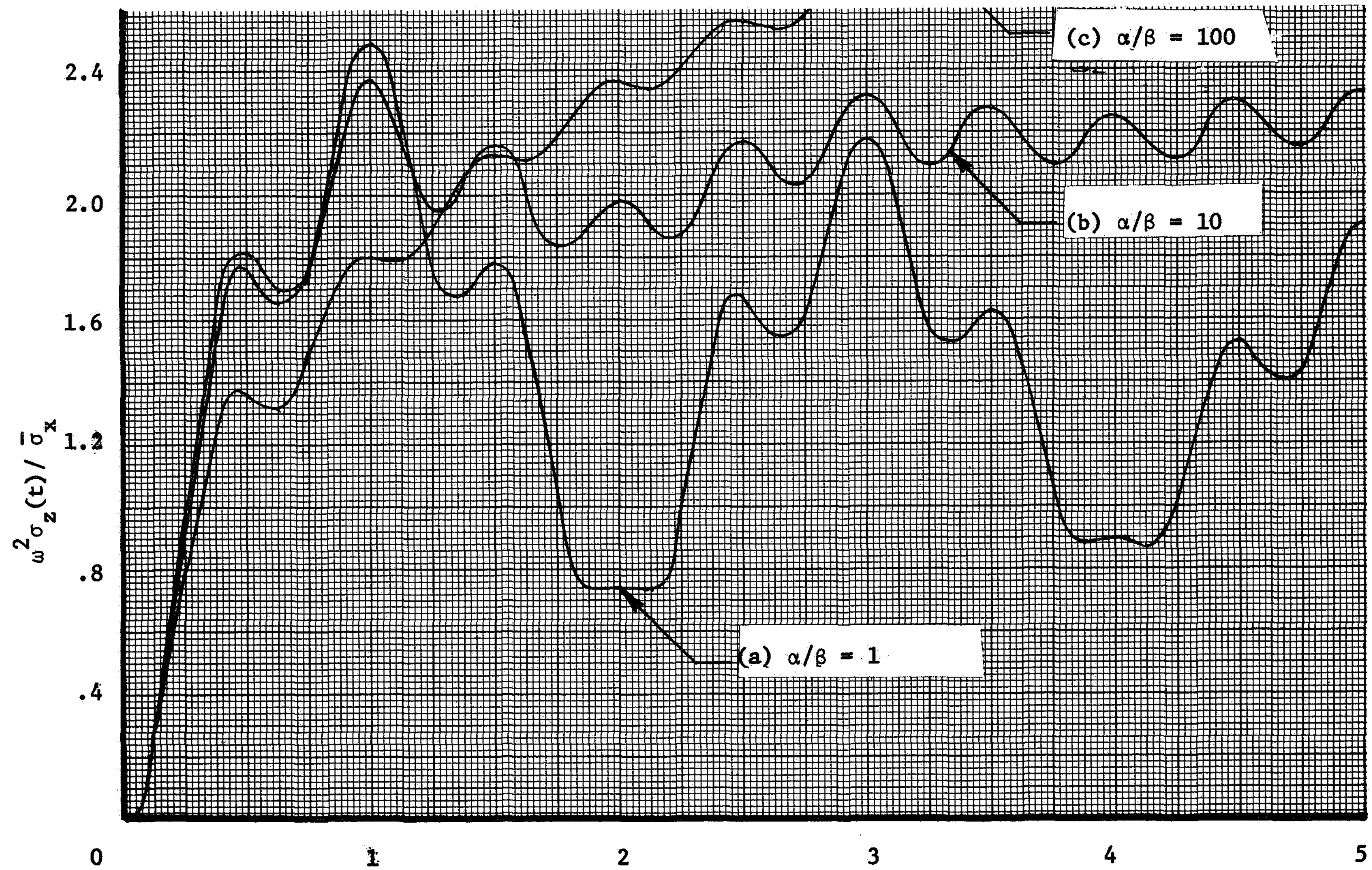

Figure A37 RMS Response for $f(t) = u(t)\, e^{-\lambda t}$, $\zeta = 0.01$, $p/\omega = 0.5$, $\lambda/\beta = 1$

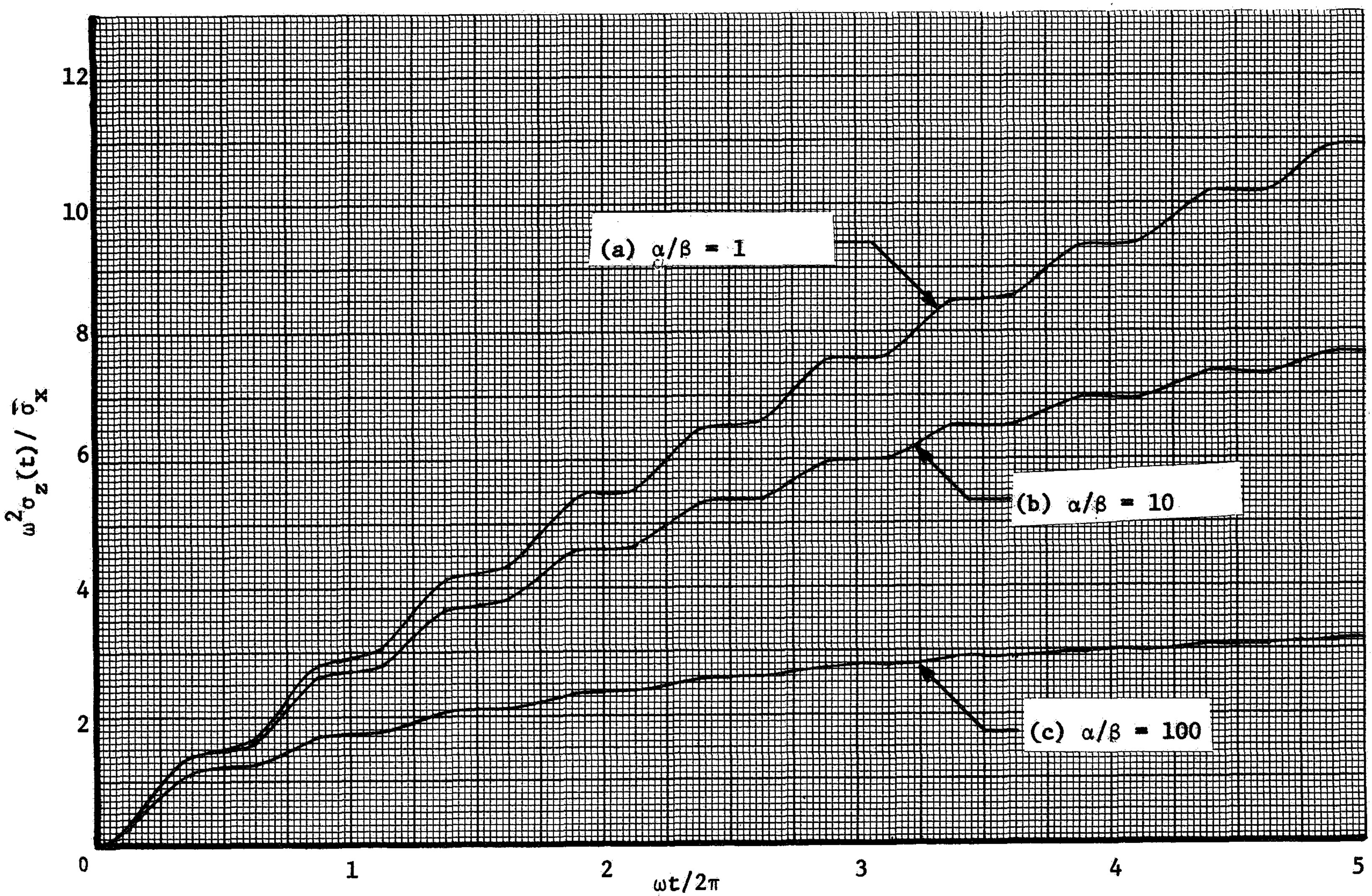

Figure A38 RMS Response for $f(t) = u(t)e^{-\lambda t}$, $\zeta = 0.01$, $p/\omega = 1$, $\lambda/\beta = 1$

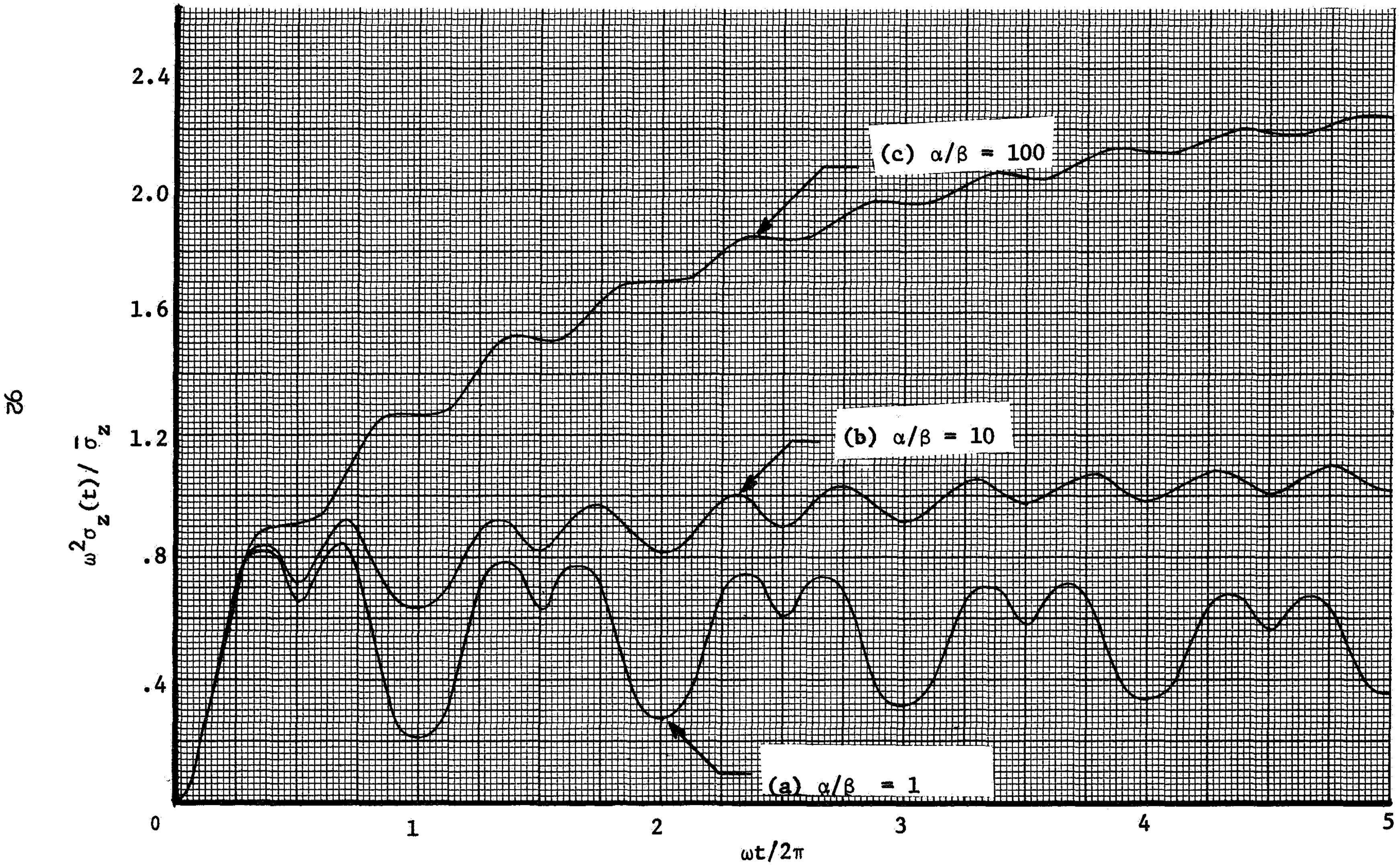

Figure A39 RMS Response for $f(t) = u(t)e^{-\lambda t}$, $\zeta = 0.01$, $p/\omega = 2$, $\lambda/\beta = 1$

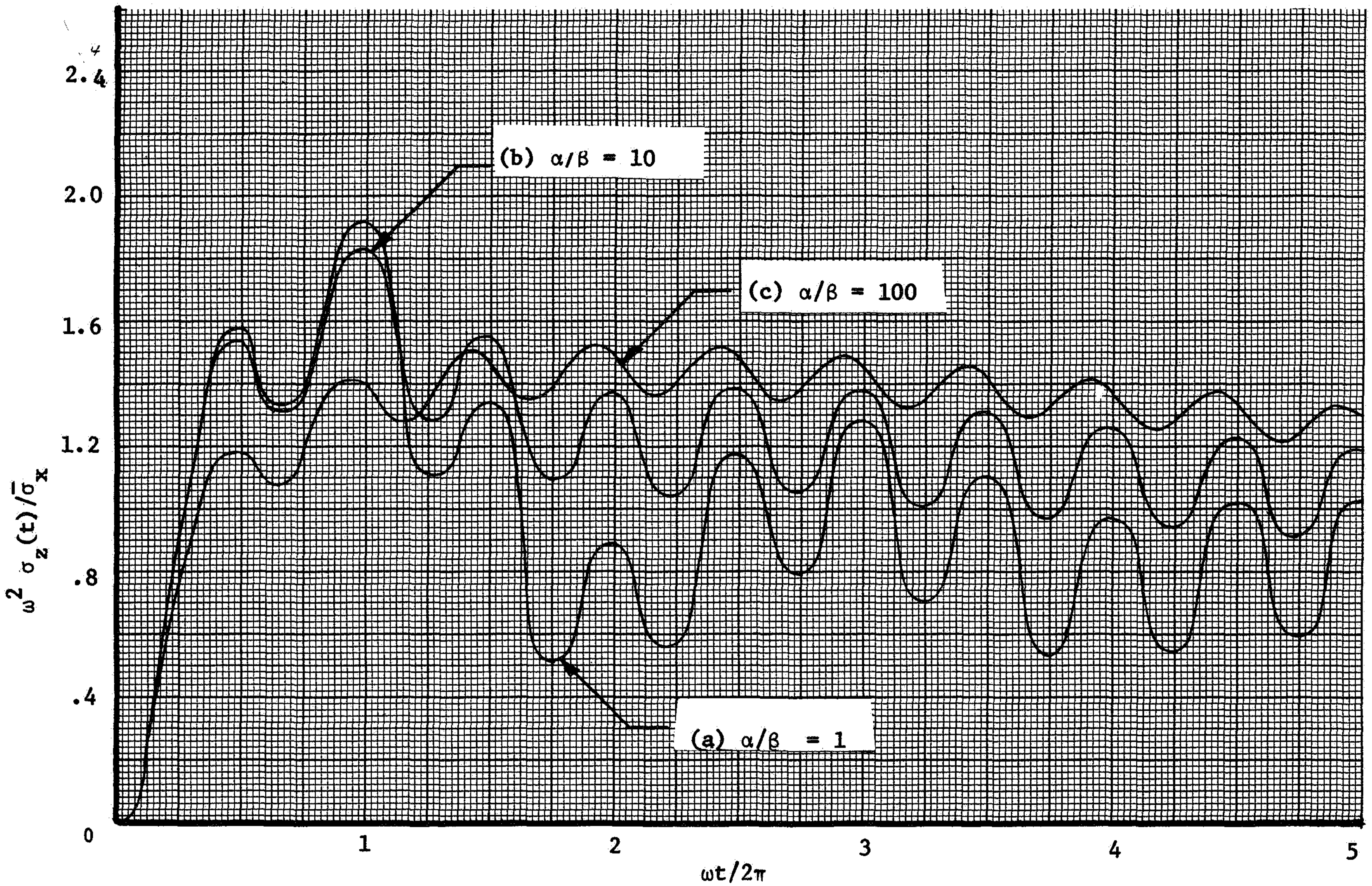

Figure A40 RMS Response for $f(t) = u(t)e^{-\lambda t}$, $\zeta = 0.01$, $p/\omega = 0.5$, $\lambda/\beta = 10$

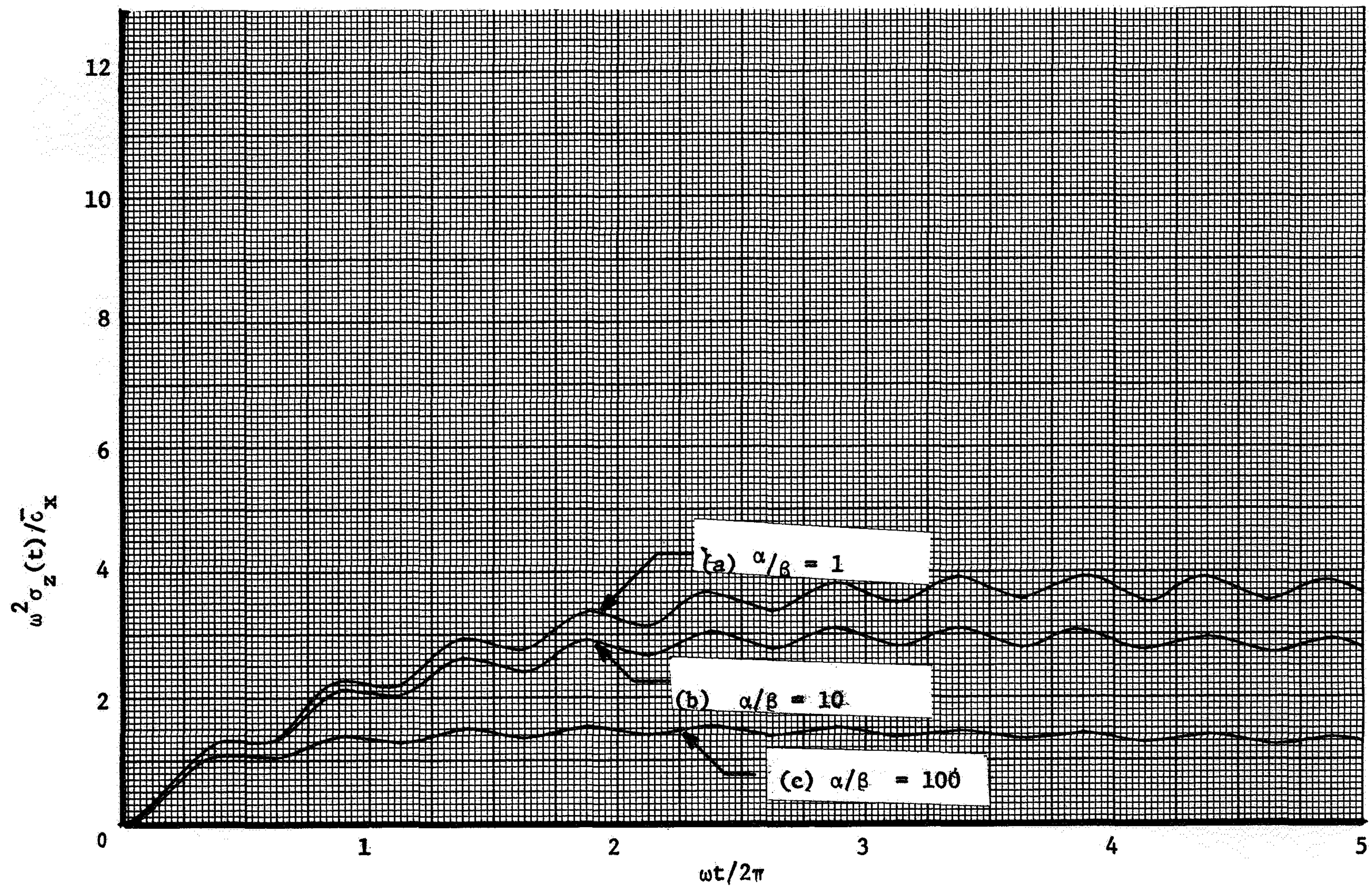

Figure A41 RMS Response for $f(t) = u(t)\,e^{-\beta t}$, $\zeta = 0.01$, $p/\omega = 1$, $\sqrt{\beta} = 10$

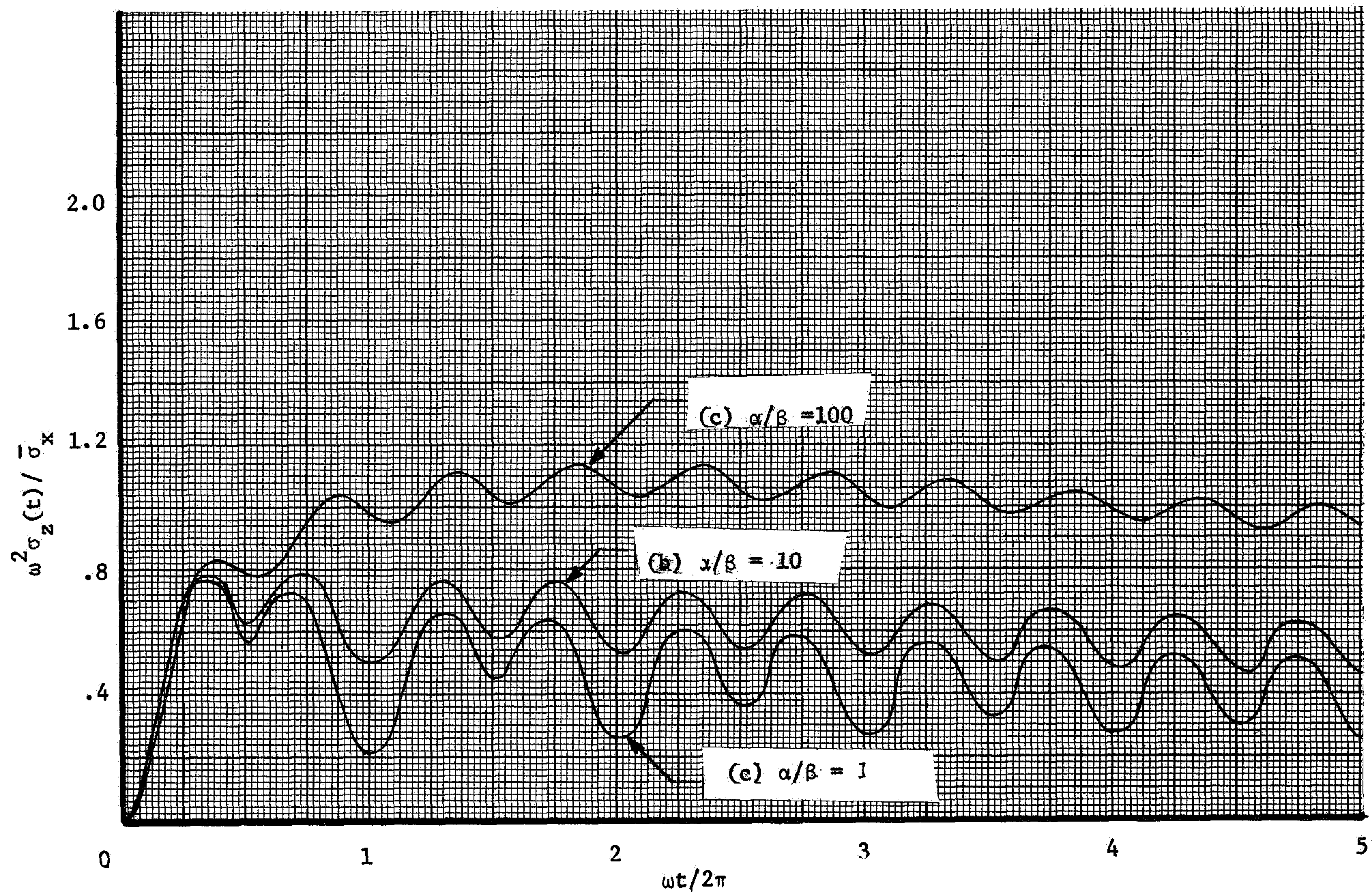

Figure A42 RMS Response for $f(t) = u(t)e^{-\lambda t}$, $\zeta = 0.01$, $p/\omega = 2$, $\lambda/\beta = 10$

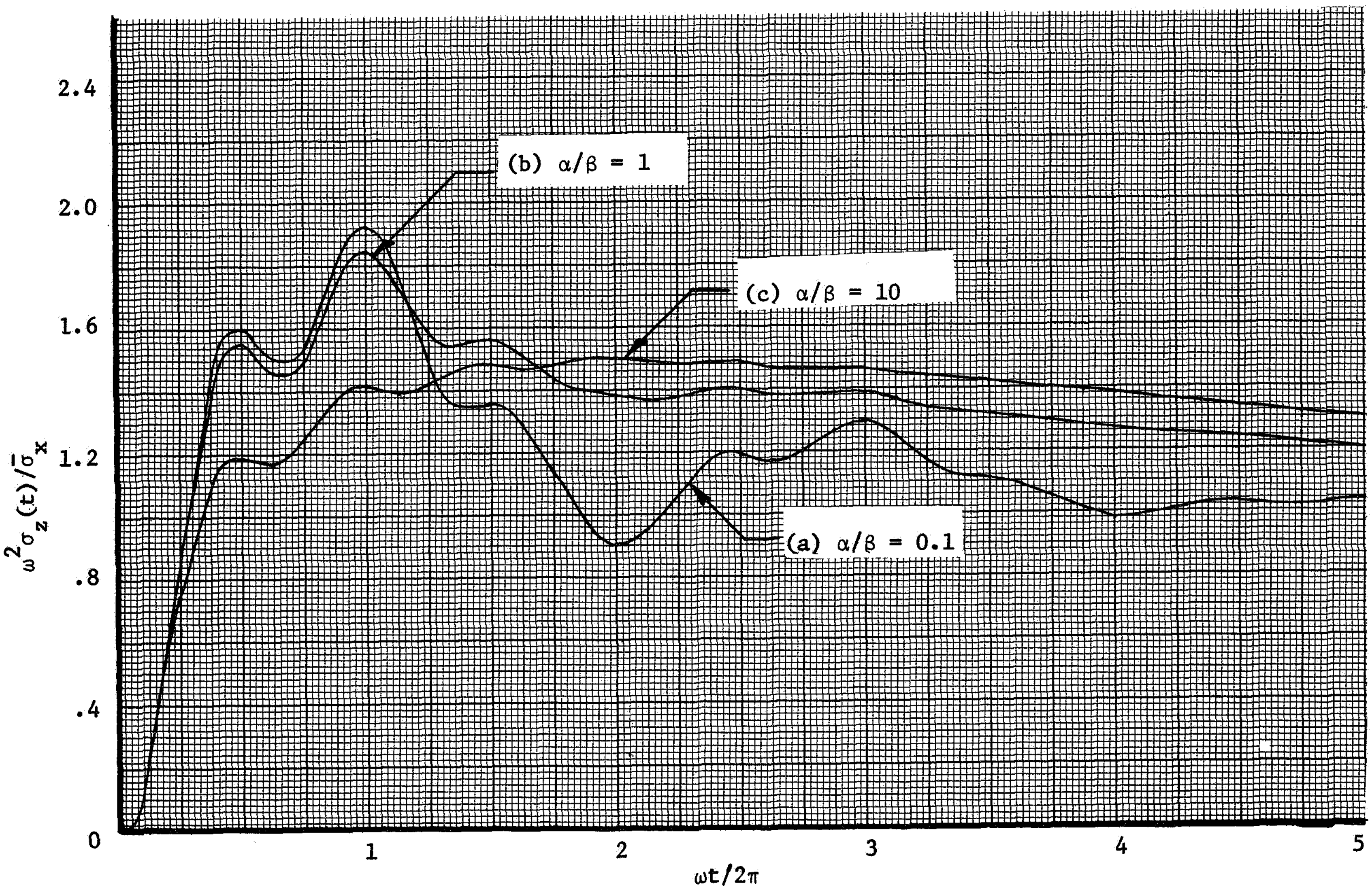

Figure A43 RMS Response for $f(t) = u(t)e^{-\lambda t}$, $\zeta = 0.1$, $p/\omega = 0.5$, $\lambda/\beta = 0.1$

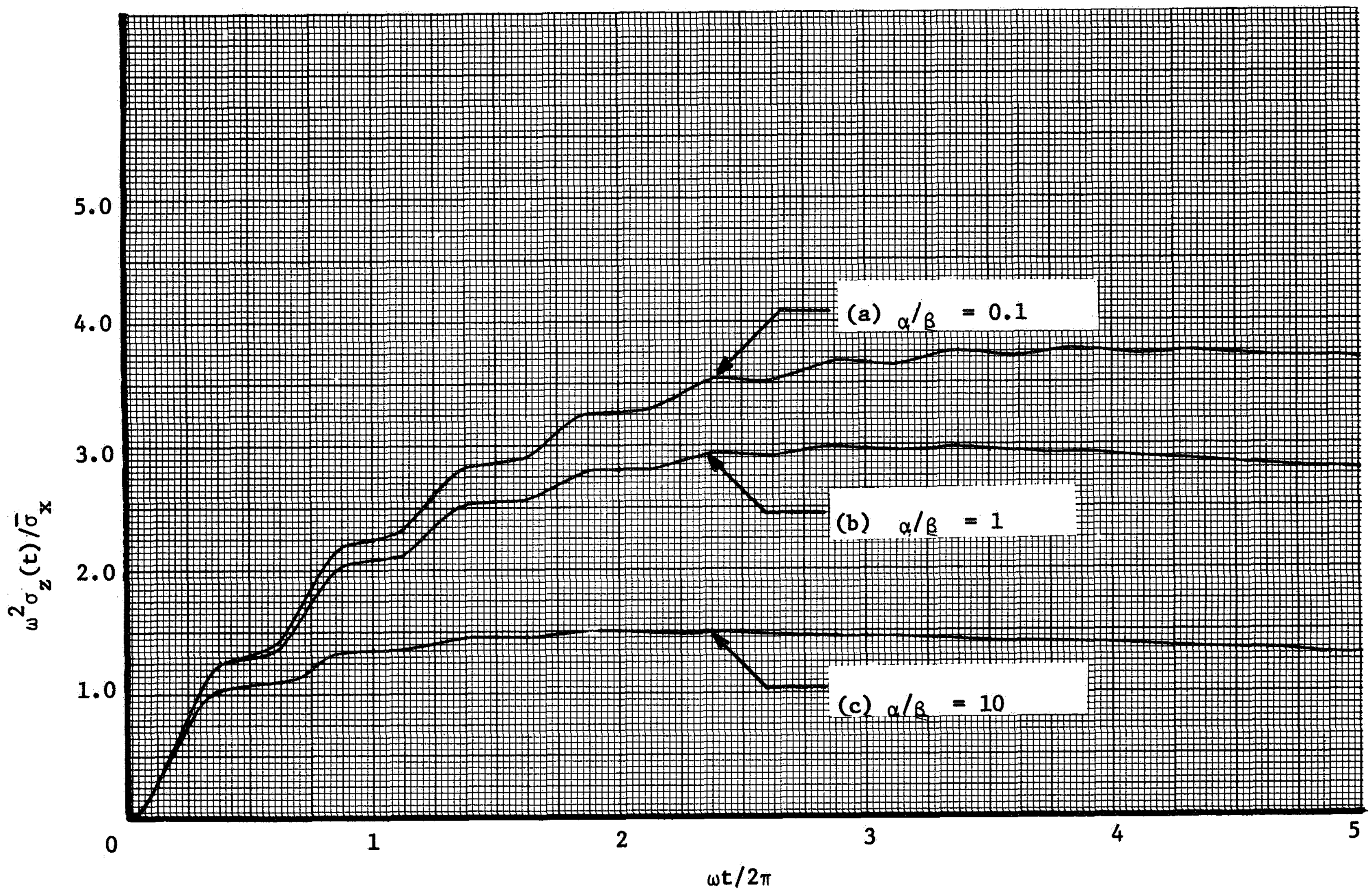

Figure A44 RMS Response for $f(t) = u(t)e^{-\lambda t}, \zeta = 0.1,\ p/\omega = 1,\ \lambda/\beta = 0.1$

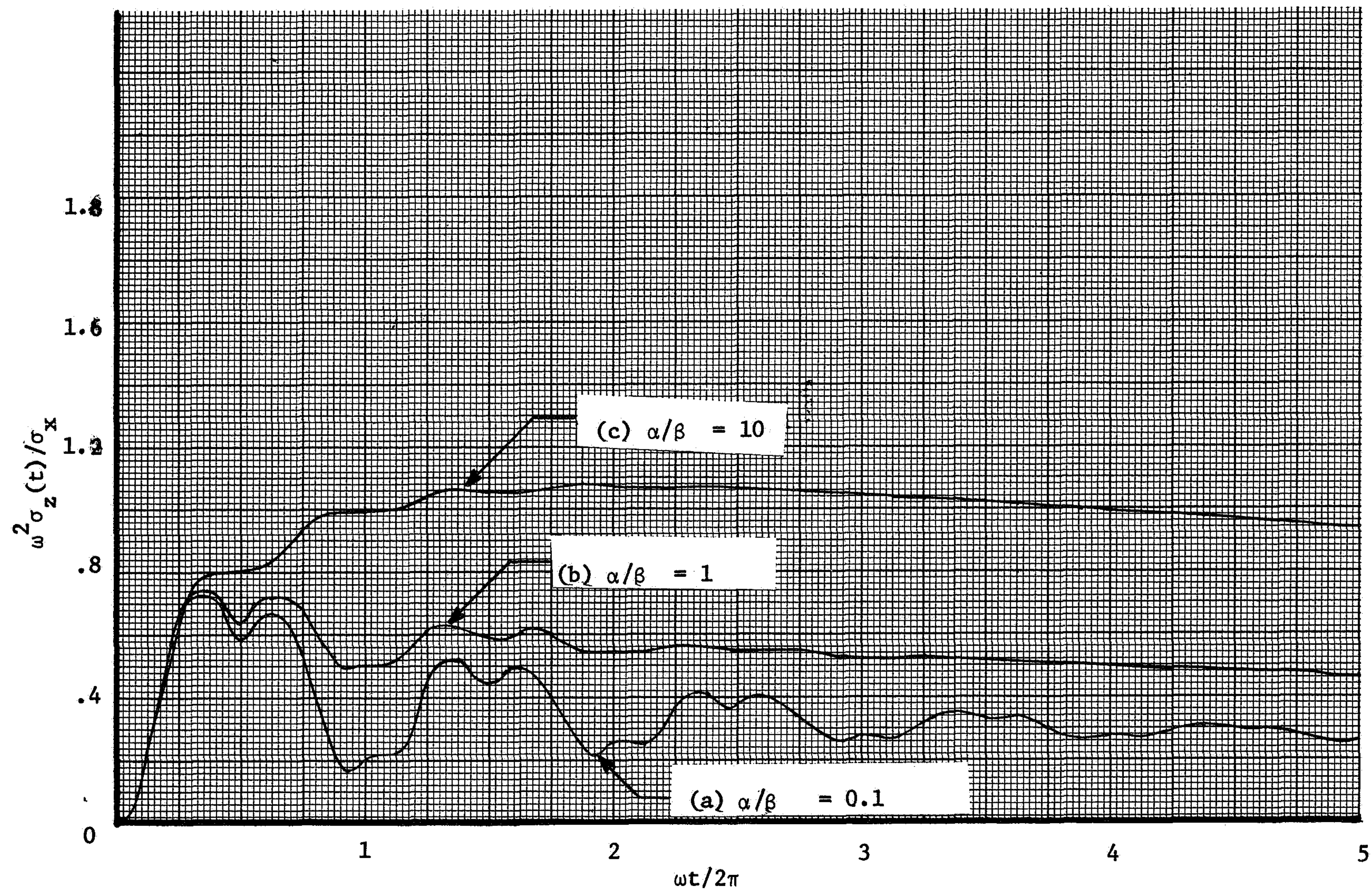

Figure A45 RMS Response for $f(t) = u(t)e^{-\lambda t}$, $\zeta = 0.1$, $p/\omega = 2$ $\lambda/\beta = 0.1$

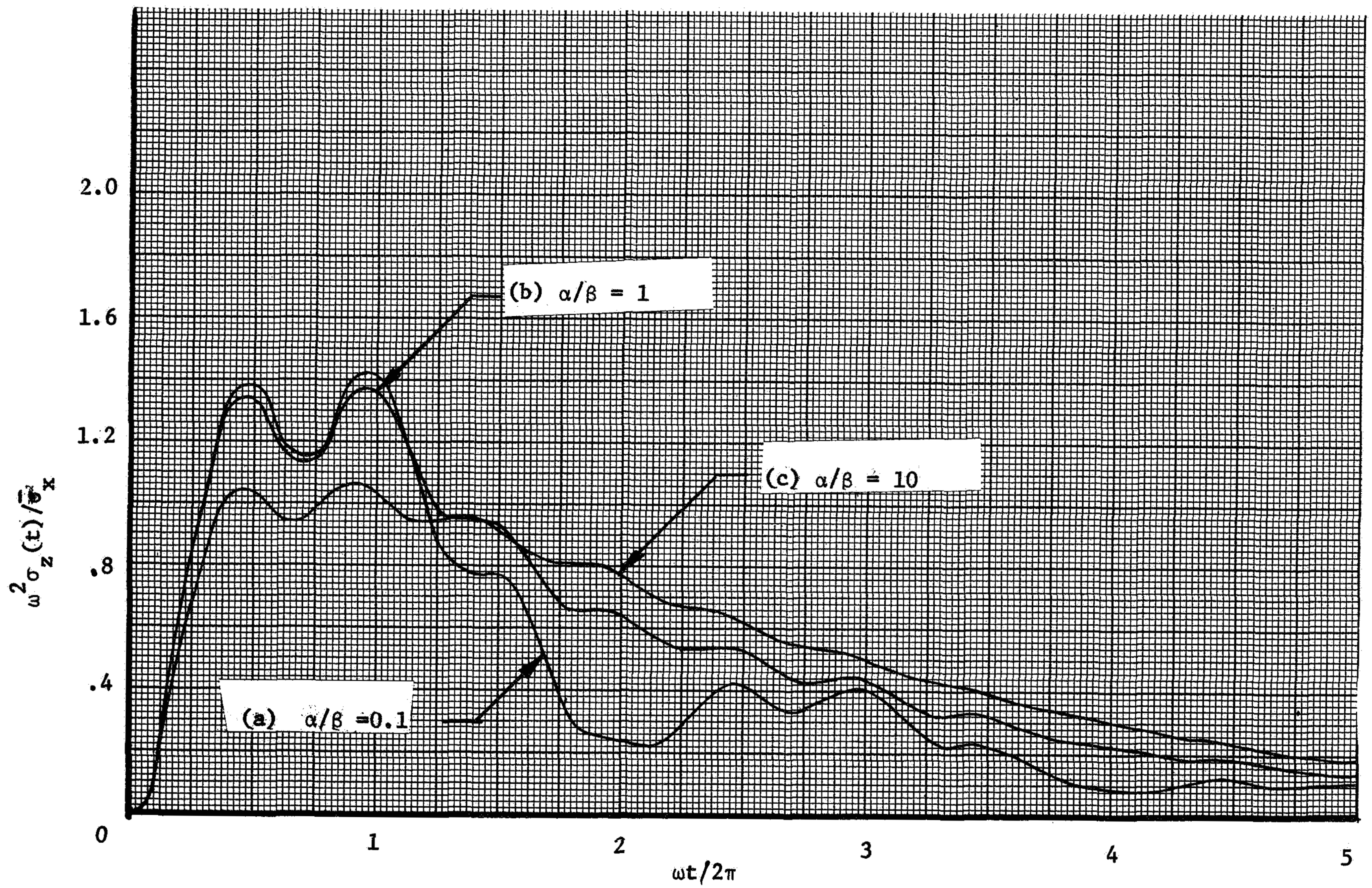

Figure A46 RMS Response for $f(t) = u(t)e^{-\lambda t}$, $\zeta = 0.1$, $p/\omega = 0.5$, $\lambda/\beta = 1$

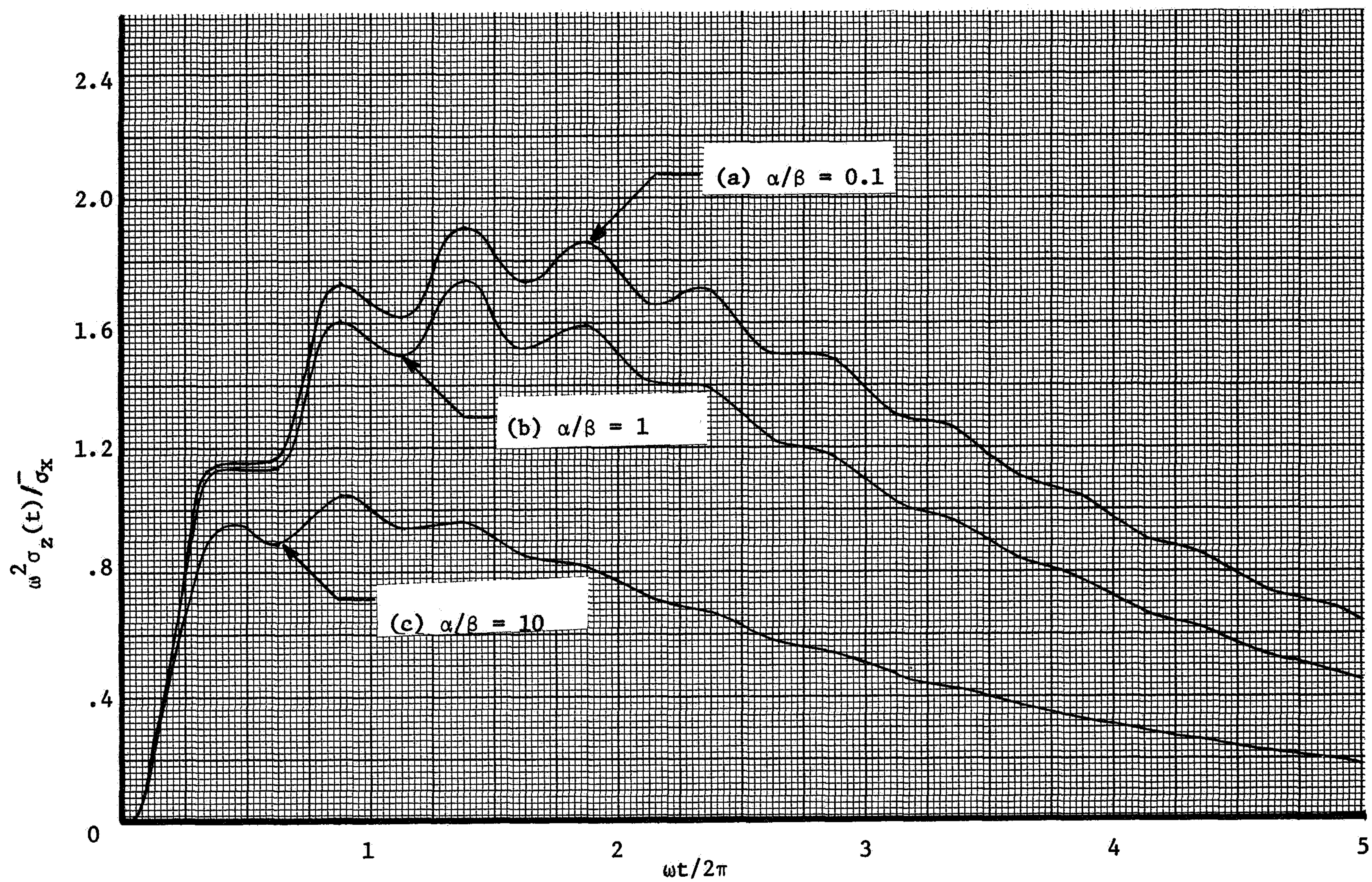

Figure A47 RMS Response for $f(t) = u(t)e^{-\lambda t}$, $\zeta = 0.1$, $p/\omega = 1$, $\lambda/\beta = 1$

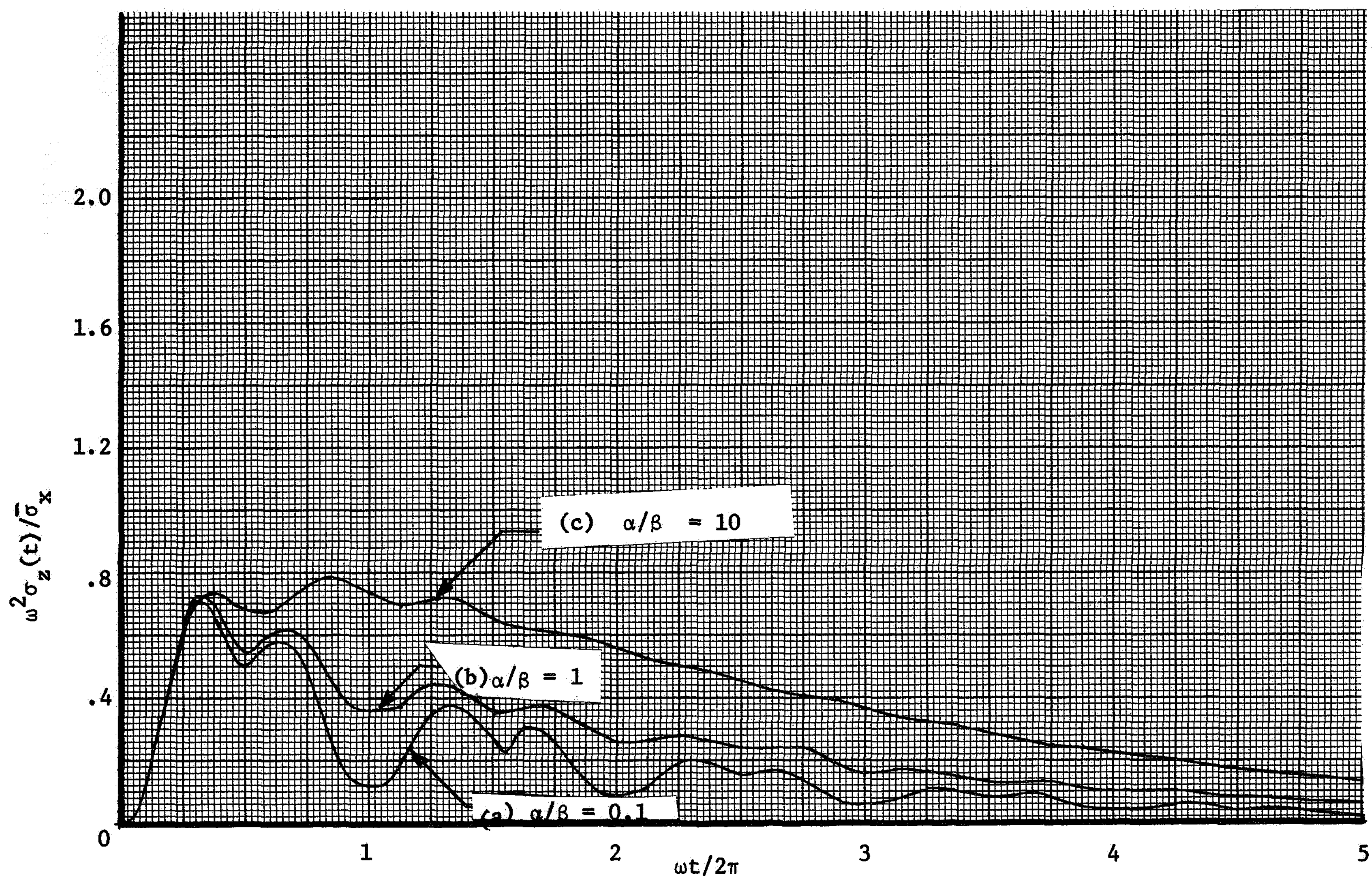

Figure A48 RMS Response for $f(t) = u(t)e^{-\lambda t}$, $\zeta = 0.1$, $p/\omega = 2$, $\lambda/\beta = 1$