AN INVESTIGATION OF THE OPTIMAL CONDITIONS OF ROCKET MOTION IN THE VICINITY OF A PLANET

ИССЛЕДОВАНИЕ ОПТИМАЛЬНЫХ РЕЖИМОВ ДВИЖЕНИЯ В ОКРЕСТНОСТИ ПЛАНЕТЫ

(Issledovaniye Optimal'nykh Rezhimov Dvizheniya v Okrestnosti planety)

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Annotation

Several problems of the optimal control of the magnitude and direction of thrust during a rocket's maneuver in a vacuum in the vicinity of a spherical planet are examined.

The work is divided into three parts.

In Part I, the problem of optimal programming of the magnitude and direction of rocket thrust in a central field is examined. It was shown that when optimal motion is in a sufficiently close proximity to a planet, there exists a so-called plane of control; i.e., a vector of the reactive forces is transfered in a plane connected with the points of an alternating mass and forward motion in an inertial (Galilean) coordinate system. In other words, the existence of a constant direction with a zero projection of the vector of optimal thrust was demonstrated.

In Part II, there is an examination of a series of approximated analytical solutions to the problem of optimal flight of a craft with low-thrust engines with ideal control of the exhaust velocity. A recommendation is made for the selection of the parameters of a translational (circular) coordinate system, and for the investigation of analytic solutions leading to a synthesis of the control systems of flight (from the overall information). In Part III, a numerical analysis is conducted of the solution of problems on the optimal (by fuel consumption) landing of a spacecraft at a given point on the surface of the moon. An examination is made of the trajectory plane for the descent of a craft in a central gravitational field from an elliptic* selenocentric orbit. An optimal orientation program was found as well as the magnitude limited by the thrust modulus. An examination was made of the influence of various parameters (thrust unit, altitude of the initial orbit of the AES**, the angular distance of the descent sector) on the magnitude of the discharge mass. An analogous problem was solved for the stage of injecting a craft from the moon's surface into the orbit of an ASM***.

Parts I and IIIwere written by V.K. Isayev and B.Kh. Davidson; Part II was by V.K. Isayev and V.V. Sonin.

*Translator's Note: the word elliptic was not used in the Russian text; the word high-low was, however.
** TN: AES, artificial earth satellite
***TN: ASM, artificial satellite of the moon (lunar orbiter)

INTRODUCTION

The purpose of this work is to study the nature of optimal control of the points of motion of the transfer of mass in a vacuum in a central gravitational field.

The solution of the problem was obtained by means of the maximum principle; however, these problems differ by a degree of generality and the peculiarities of the procedures used. The first was devoted to an investigation of the structure of optimal control of the orientation of the reactive forces in a central gravitational field. The existence of analytical solutions for the second problem (transfer between near-circular orbits by means of engines with a regulated exhaust velocity) makes it possible to accomplish a synthesis of optimal control, and a comparison with a precise solution makes it possible to explain the range of its applicability. The third part is devoted to an investigation of the basic peculiarities of a numerical solution of the variation broblem of the soft landing from an ASM orbit of a reactive craft limited by the magnitude of the thrust.

These, at first glance, are unlike problems joined together by one characteristic feature: the trajectory of optimal motion lies at some fairly close proximity to a spherical planet (in the third problem, this assumption has an affect only on the range of the examined orbit).

PART I. One Characteristic of an Optimal Program of Orientation of a Reactive Force During the Dimensional Motion of a Point of Mass Transfer in a Central Gravitational Field.

We examined a process of optimal dimensional motion of the points of mass transfer in a vacuum in a central gravitational field. An equation for motion in a Cartesian inertial coordinate system connected with a center of gravity takes the view that

$$\dot{u} = \frac{P}{m}\cos\vartheta\cos\phi - \frac{\mu}{R^3}x,$$

$$\dot{v} = \frac{P}{m}\sin\vartheta - \frac{\mu}{R^3}y,$$

$$\dot{w} = \frac{P}{m}\cos\vartheta\sin\phi - \frac{\mu}{R^3}z,$$

$$\dot{x} = u, \quad \dot{y} = v, \quad \dot{z} = w,$$

$$\dot{m} = -\frac{P}{c}.$$

Here, u, v, w comprise the velocity vector,
x, y, z are the coordinate points
$R=(x^2 + y^2 + z^2)^{1/2}$
m=the mass points, $m=\frac{M(t)}{M(0)}$
μ=constant gravitation of a planet
P=thrust of the engine relative to
c=exhaust velocity of a jet stream
θ, ϕ=angles of thrust orientation

The operating functions, subject to optimization, may be P, c, θ, and ϕ. The structure of the optimal control θ and ϕ does not depend on a method of selection of the P and c optimal program or assigned *a priori*. Below, for definiteness, we examine motion on the assumption that $c=const$. P may change from null to P_{max}.

By means of the L.S. Pontryagin maximum principle (Refs. 1 and 2), the momentary magnitude of controlling functions are determined from the conditions of a minimum of the function $\mathcal{H}$

$$P = \begin{cases} P_{MAX}, & \Theta > 0, \\ 0, & \Theta < 0, \end{cases} \qquad /I.2/$$

$$\Theta = \rho + \frac{m p_m}{c},$$

$$\rho = (p_u^2 + p_v^2 + p_w^2)^{1/2}.$$

$$\sin\vartheta = -\frac{p_v}{\rho}, \quad \cos\vartheta = \frac{z}{\rho},$$

$$\sin\phi = -\frac{p_w}{z}, \quad \cos\phi = \frac{p_u}{z},$$

$$z = (p_u^2 + p_w^2)^{1/2}. \qquad /I.3/$$

From Eq. (1.3), it is seen that the thrust vector $\vec{P}$ is a collinear vector with components Pu, Pv, and Pw (Fig. 1); i.e., the radius vector of the p-trajectory (Ref. 3). A type of p-trajectory completely determines the nature of the optimal orientation program. Conjugate variables of p_i are determined by the following equation system:

$$\dot{p}_u = -p_x, \quad \dot{p}_v = -p_y, \quad \dot{p}_w = -p_z,$$

$$\dot{p}_x = \frac{\mu}{R^3}\left[p_u - \frac{3x}{R^2}(xp_u + yp_v + zp_w)\right],$$

$$\dot{p}_y = \frac{\mu}{R^3}\left[p_v - \frac{3y}{R^2}(xp_u + yp_v + zp_w)\right], \qquad /I.4/$$

$$\dot{p}_z = \frac{\mu}{R^3}\left[p_w - \frac{3z}{R^2}(xp_u + yp_v + zp_w)\right].$$

In addition, in Eqs. (1.1) and (1.4) there follows a consideration of the necessary conditions of control optimality of Eqs. (1.2) and (1.3).

The system in Eq. (1.4) clearly depends on the coordinate and may not be separately integrated. This circumstance does not make it possible to determine the structure of optimal control, in general, in the case of motion in a central field.

In Ref. 4 there was presented a uniform spherical model of a gravitational field which approximately described a field of gravitational force in some spherical layer the thickness of which is sufficiently small compared with an average distance of R_0 points from the center of attraction. In a uniformly-spherical field there is assumed a linear relation of the component gravitational acceleration to the coordinate

$$g_x = -\nu^2 x, \quad g_y = -\nu^2 y, \quad g_z = -\nu^2 z. \qquad /I.5/$$

R_0 is the radius of a circular orbit in the vicinity of which a motion $\nu^2 = g_0/R_0$ results.

The conjugate system, in this case, is especially simple:

$$\dot{P}_u = -P_x, \quad \dot{P}_v = -P_y, \quad \dot{P}_w = -P_z,$$

$$\dot{P}_x = \nu^2 P_u, \quad \dot{P}_y = \nu^2 P_v, \quad \dot{P}_z = \nu^2 P_w, \qquad /I.6/$$

$$\dot{P}_m = -\frac{P}{m^2}\rho.$$

The first six equations of this system, which completely determine the nature of the optimal orientation control, do not depend on the coordinate. As a result of the integration, we have

$$P_u = P_u^\circ \cos\nu t - \frac{P_x^\circ}{\nu}\sin\nu t,$$

$$P_v = P_v^\circ \cos\nu t - \frac{P_y^\circ}{\nu}\sin\nu t, \qquad /I.7/$$

$$P_w = P_w^\circ \cos\nu t - \frac{P_z^\circ}{\nu}\sin\nu t.$$

The components of vector $\vec{p}$ change according to the sine rule with one and the same period. Consequently, the hodograph $\vec{p}$ is an ellipse the plane of which passes through the initial coordinate and turns relative to the original coordinate system (Fig. 2). The angles of orientation of the ellipse plane in the $Oxyz$ coordinate system are determined by the initial value of the conjugate variable. In its turn, the latter are found from a solution of the double-pointed boundary value problem for Eqs. (1.1) and (1.6) where P, θ, andϕ are assigned as the conditions of the maximum principle for Eqs. (1.2) and (1.3).

Thus, with optimal motion in a suff-ciently narrow spherical mayer of the central field, the vector , together with thrust vector , do not emerge from some plane translationally shifting in the Oxy inertial system together with the points of variable mass. The mentioned plane is designated the control plane (Fig 3).

PART II. Synthesis of Optimal Flight Control of a Craft Between Close Orbits.

In an analysis of the problem of interplanetary flight in Ref. 2, the so-called translational coordinate system was utilized. Essentially, it consists of the following: the appropriate equations of motion and the bounded conditions are noted in some mobile system which is selected so that it is easy to find a linear approximation to a solution of the problem of optimal flight. With a successful selection of a translational system, a solution of the problem, already in a linear approximation, may yield good precision -- sufficient for all practical purposes.

As a translational system, the coordinate for the problem of optimal flight between orbits of earth and Mars, in Ref. 5, a trihedron was selected, moving in a Keplerian orbit the beginning of which, at the moment of launch, coincided with earth and at the final moment with a projection of Mars in an elliptic plane.

In this work, the plane of motion between the two near-circular orbits is examined in the Oxy translational system the beginning of which moves in a certain average circular orbit during which the Ox axis is directed along the velocity vector, and Oy along the radius vector. Let, at the initial moment of time $t = 0$, the given mass is $m(0) = m_0$, the coordinates and velocity of the craft in the Oxy translational system $u(0) = u_0$, $v(0) = v_0$, $x(0) = x_0$, $y(0) = y_0$ (2.1)
whereupon, this parameter conforms to the motion of the craft along a certain circular orbit. Furthermore, let it be required that such programs be found for a change of magnitude and orientation of the thrust vector in order for time T to convert to a point moving along a near circular orbit and having the coordinates of

$$u(T) = u_1,\ v(T) = v_1,\ x(T) = x_1, \text{and } y(T) = y_1 \qquad (2.2)$$

with minimal consumption of mass. We shall consider that in the process of movement, the deflection of the craft from the initial translational system is low compared to the radius of the translational orbit R_0.

Then, disregarding the members of the series $x^2 + y^2/R_0^2$, one may write an equation for the craft's motion as

$$\dot{u} + 2\omega v = \frac{\tilde{N} u_1 \cos\varphi}{mc},$$

$$\dot{v} - 2\omega u - 3\omega^2 y = \frac{\tilde{N} u_1 \sin\varphi}{mc}, \qquad /2.3/$$

$$\dot{x} = u, \quad \dot{y} = v, \quad \dot{m} = -\frac{\tilde{N} u_1}{c^2},$$

(Refs. 6 and 7), where u and v are projections of relative velocity in a translational system of coordinates, $u_1 = N/N_{max}$ is a dimensionless power, c is the controlled exhaust velocity, ϕ is the dip angle of thrust to the Ox axis, $N = 2N_{max}/M(0)$ and $m = M(t)/M(0)$ is a dimensionless mass of the craft, and in addition

$$m(0) = 1 \qquad (2.4)$$

Considering the optimal equation of the points of variable mass the movement of which is described by the system in Eq. (2.3) and must satisfy the bounded conditions of Eqs. (2.2) and (2.4); then we maximize the functional appearing to be dimensionless final mass $S(T) = m_1$. If, during the solution of this problem, it is considered that the velocity of the exhaust stream may change from 0 to infinity

$$0 < c < \infty$$

then, as was shown in Refs. (7) and (8), the optimal control will be determined by the following correlations:

$$u_1 \equiv 1, \quad \cos\varphi = -\frac{p_u}{\rho}, \quad \sin\varphi = -\frac{p_v}{\rho}, \quad c = -\frac{2m p_m}{\rho},$$

where $\rho = (P_u^2 + P_v^2)^{1/2}$ and $(Pu, Pv, Px, Py,$ and $Pm)$ is the vector of conjugated variables the components of which satisfy the following systems of differential equations:

$$\dot{p}_u = -p_x - 2\omega p_v,$$

$$\dot{p}_v = -p_y + 2\omega p_u, \qquad /2.5/$$

$$\dot{p}_x = 0, \quad \dot{p}_y = -3\omega p_v, \quad \dot{p}_m = \frac{\tilde{N}\rho^2}{2m^3 p_m};$$

$$p_m(T) = -1. \qquad /2.6/$$

In Ref. 7 a formula is introduced which makes it possible for each concrete bounded condition to find programs $c(t)$, $\phi(t)$, solving the original boundary problem. We present it here. First of all, from the system of linear equations:

$$AC = B \tag{2.7}$$

we find a fourdimensional vector C. Elements of matrix A in vector B are computed by the formula presented in Appendix II.1 and II.2.

After finding C_i $, i = 1, \ldots, 4,$ from the system in Eq. (2.7) we find the assumption $t = T$ and the value $J(T)$ by the formula

$$\begin{aligned} J(t) = {} & \frac{3(C_1^2 - C_2^2)}{4\omega}\sin 2\omega t + \frac{3C_1C_2}{2\omega}\cos 2\omega t + \\ & + \frac{4}{\omega}\left(C_1C_3 - \frac{4C_2C_4}{\omega}\right)\sin\omega t + \frac{4}{\omega}\left(C_2C_3 + \frac{4C_1C_4}{\omega}\right)\cos\omega t + \\ & + \left[\frac{12C_4}{\omega}(C_1\sin\omega t + C_2\cos\omega t) + \frac{5}{2}(C_1^2 + C_2^2) + C_3^2 + \frac{4C_4^2}{\omega^2}\right]t + \\ & + 3C_4(C_3 + C_4 t)t^2 - \frac{3C_1C_2}{2\omega} - \frac{4}{\omega}\left(C_2C_3 + \frac{4C_1C_4}{\omega}\right). \end{aligned} \qquad /2.8/$$

Further, we define the constants (Ref. 8)

$$m(T) = \frac{1}{1 + \tilde{N}^{-1}J(T)},$$

$$\gamma = -\frac{\tilde{N}}{2m^2(T)}, \quad \beta = -m^2(T).$$

After that, we compute the four constants

$$C_i^* = \frac{C_i}{\gamma} \qquad (i = 1, \ldots, 4),$$

through which in a clear form we express the solution of the systems of Eqs. (2.3) and (2.5). Corresponding formulas are presented in Appendix II.3 and II.4.

Utilizing the obtained correlation, it is not difficult to find in a clear form programs of control for $\phi(t)$ and $c(t)$ in each concrete problem (Refs. 7 and 8).

The obtained solutions are utilized for computing the flight from orbit of radius I in a near-circular orbit. The time of the flight is considered equal to one-half the period of revolution along the original orbit, and the angle of flight in the inertial system is 180 deg. The translational system of coordinates is assumed to be coincidental with the points of the destination. In

Fig. 4 there is presented a dependence of the functional on the radius of the final orbit. For a comparison, the result of a precise solution of the problem is presented which was obtained by a solution of the boundary problem for an exact system of equations of optimal motion in a central field. The agreement is sufficiently good up to $R_1 = 1.05$.

APPENDIX:

I. The equation for the matrix elements A and vector component B are

$$
\begin{aligned}
a_{11} &= 5T\cos\omega T - \frac{3}{\omega}\sin\omega T,\\
a_{12} &= \frac{6(1-\cos\omega T)}{\omega} - 5T\sin\omega T,\\
a_{13} &= \frac{4\sin\omega T}{\omega} - 3T,\\
a_{14} &= \frac{16(1-\cos\omega T)}{\omega^2} - \frac{9T^2}{2},\\
a_{21} &= \frac{5T}{2}\sin\omega T,\\
a_{22} &= \frac{5T}{2}\cos\omega T - \frac{3\sin\omega T}{2\omega},\\
a_{23} &= \frac{2(1-\cos\omega T)}{\omega},\\
a_{24} &= \frac{2}{\omega}\left(3T - \frac{4}{\omega}\sin\omega T\right),\\
a_{31} &= \frac{1}{\omega}\left[5T\sin\omega T + \frac{8}{\omega}(1-\cos\omega T)\right],\\
a_{32} &= \frac{1}{\omega}\left(5T\cos\omega T - \frac{11}{\omega}\sin\omega T + 6T\right),\\
a_{33} &= \frac{4(1-\cos\omega T)}{\omega^2} - \frac{3T^2}{2},\\
a_{34} &= \frac{16}{\omega^2}\left(T - \frac{1}{\omega}\sin\omega T\right) - \frac{3T^3}{2},\\
a_{41} &= \frac{5}{2\omega}\left(\frac{1}{\omega}\sin\omega T - T\cos\omega T\right),\\
a_{42} &= \frac{1}{\omega}\left[\frac{4}{\omega}(\cos\omega T - 1) + \frac{5T}{2}\sin\omega T\right],\\
a_{43} &= \frac{2}{\omega}\left(T - \frac{1}{\omega}\sin\omega T\right),\\
a_{44} &= \frac{1}{\omega}\left[3T^2 + \frac{8}{\omega^2}(\cos\omega T - 1)\right].
\end{aligned}
\qquad /\text{П.I.}
$$

$$B_1 = u_1 + (3 - 4\cos\omega T)u_0 + 2\sin\omega T \cdot v_0 + 6\omega(1 - \cos\omega T)y_0,$$
$$B_2 = v_1 - 2\sin\omega T \cdot u_0 - \cos\omega T \cdot v_0 - 3\omega\sin\omega T \cdot y_0,$$
$$B_3 = x_1 + \left(3T - \frac{4\sin\omega T}{\omega}\right)u_0 + \frac{2}{\omega}(1 - \cos\omega T)v_0 - x_0 + 6(\omega T - \sin\omega T)y_0, \qquad /II.2/$$
$$B_4 = y_1 + \frac{2}{\omega}(\cos\omega T - 1)u_0 - \frac{\sin\omega T}{\omega}v_0 + (3\cos\omega T - 4)y_0.$$

II. The solution for systems II.6 and II.12 as functions of time is:

$$u(t) = \left(2B_2^* - \frac{3C_2}{\omega} + 5C_1 t\right)\cos\omega t - \left(2B_1^* + \frac{3C_1}{\omega} + 5C_2 t\right)\sin\omega t - $$
$$- \frac{9}{2}C_4 t^2 - 3C_3 t - \frac{3}{2}\omega B_3^* + \frac{4}{\omega^2}C_4,$$
$$v(t) = \left(B_1^* + \frac{5}{2}C_2 t\right)\cos\omega t + \left(B_2^* + \frac{5}{2}C_1 t\right)\sin\omega t + \frac{6}{\omega}C_4 t + \frac{2}{\omega}C_3,$$
$$x(t) = \left(\frac{2B_1^*}{\omega} + \frac{8C_1}{\omega^2} + \frac{5C_2}{\omega}t\right)\cos\omega t + \left(\frac{2B_2^*}{\omega} - \frac{8C_2}{\omega^2} + \frac{5C_1}{\omega}t\right)\sin\omega t -$$
$$- \frac{3}{2}C_4 t^3 - \frac{3}{2}C_3 t^2 + \left(\frac{4C_4}{\omega^2} - \frac{3\omega B_3^*}{2}\right)t + B_4^*,$$
$$y(t) = -\left(\frac{B_2^*}{\omega} - \frac{5C_2}{2\omega^2} + \frac{5C_1}{2\omega}t\right)\cos\omega t + \left(\frac{B_1^*}{\omega} + \frac{5C_1}{2\omega^2} + \frac{5C_2}{2\omega}t\right)\sin\omega t$$
$$+ \frac{3}{\omega}C_4 t^2 + \frac{2}{\omega}C_3 t + B_3^*, \qquad /II.3/$$
$$m(t) = \frac{1}{1 + \tilde{N}^{-1}J(t)},$$
$$P_u(t) = 2C_1^*\cos\omega t - 2C_2^*\sin\omega t + 3C_4^* t + C_3^*,$$
$$P_v(t) = C_1^*\sin\omega t + C_2^*\cos\omega t - \frac{2}{\omega}C_4^*,$$

Whereupon

$$B_1^* = v_0 - \frac{2C_3}{\omega}, \qquad /II.4/$$
$$B_2^* = 2u_0 + 3\omega y_0 - \frac{3C_2}{2\omega} - \frac{8C_4}{\omega^2},$$
$$B_3^* = \frac{2u_0}{\omega} + 4y_0 - \frac{4C_2}{\omega^2} - \frac{8C_4}{\omega^3},$$
$$B_4^* = x_0 - \frac{2v_0}{\omega} - \frac{8C_1}{\omega^2} + \frac{4C_3}{\omega^2}.$$

PART III. An Optimal Landing of a Reactive Craft on the Surface of the Moon and the Optimal Injection of that Craft into the Orbit of an ASM.

In this section we present an investigation of the results of a precise solution of the problem of variation of the landing of a rocket craft on the moon and on its flight from the lunar surface to the orbit of an ASM. The following assumptions were made of the general character:

1. The gravitational field of the moon is central and Newtonian.

2. The orbit of an ASM is circular.

3. The control value P and thrust direction ϕ is noninertial; the magnitude of the thrust is limited to $0 \leq P \leqq P_{max}$.

4. The velocity of the exhaust of the jet stream is constant.

5. The trajectory of the take-off and landing is two-dimensional.

A diagram of the take-off and landing is presented in Fig. 5. Here, Cxy is the Cartesian inertial coordinate system beginning in the center of the moon, axis Oy is presented through the point of landing/take-off. Maximizing the functional serves as the final value of mass m_1. An important parameter is the angular distance ψ_π, the landing part, and ψ_β the take-off.

The equation for the optimal motion consists of Eq. (I.1) for the coordinate u, v, x, y, and m, and the conjugate system (I.4) and the necessary condition for control optimality (1.2) and (1.3) where it follows to assume $\phi = 0$, $Pw = Pz = 0$.

The bounded conditions in the landing problem have the following aspect:

$$\underline{t = 0.}$$
$$H = H_0,\ L = L_0 = R_\Lambda \cdot \psi_n,\ V_\tau = V_{kp}(H_0),\ V_\eta = 0,\ m = 1 \qquad /3.I/$$
$$\underline{t = T_n}$$
$$H = L = 0.\ V_\tau = u_1 = 0,\ V_\eta = v_1 = 0,\ P_{m1} = -1.$$

Analogously for the take-off

$$\underline{t = 0}$$
$$H = L = V_\tau = V_\eta = 0,\quad m = 1, \qquad /3.2/$$
$$\underline{t = T_B}$$
$$H = H_0,\ L = L_1,\ V_\tau = V_{kp}(H_0),\ V_\eta = 0,\ P_m = -1,$$

where $R_0 = R_\Lambda + H_0$. H_0 is the radius and altitude of the orbit, R_Λ is the radius of the moon, L is the selenocentric distance along the surface of the moon, V_η and V_τ are the radial and transversal components of velocity.

Solving the bounded problem in Eq. (1.1)and (1.4) with the boundary conditions in Eq. (3.1) and (3.2), respectively, we find in the initial moment $t = 0$, the values Pu_0, Pv_0, Px_0, Py_0, and Pm. With an allowance for the intergal, $\mathcal{H} = 0$ determines the free time $T_\pi (T_\beta)$, and a number of selected parameters is lowered to four.

The bounded problem was solved by means of a modification of Newton's method (Ref. 9). In addition, the optimal control was automatically determined; i.e., the program for regulating the magnitude and orientation of thrust, and their corresponding optimal trajectory for landing and take-off.

For an account of the results of the numerical calculation, it is convenient to start with a description of the p-trajectory; $\vec{p} = (pu(t)\vec{i} + pv(t)\vec{j})$.

From Fig. 6 it can be seen that for the value ψ_π 100 deg, the p-trajectory is close to the arc of the circle, for ψ_π= 60 deg to the arc of the ellipse, for ψ_π = 10 deg to the linear segment (or arc of the ellipse with great excentricity) which agrees with the theory of a uniform-spherical (Ref. 4) and plane-parallel (Ref. 3) field of gravitation.

In the process of parametric calculations, the altitude of the initial bounded orbit varied from 15 km to 200 km, and the angular distance $\psi_\pi (\psi_\beta)$ varied from 10 to 180 deg. In the indicated range, the optimal trajectories were composed of three sectors: two sectors of maximal thrust (a duration of t_1 and $t_3 - t_2$ each), divided by the passive sector (see Fig. 7 for the landing and Fig. 8 for the take-off). That result also agrees with Ref. 4. In the 15-20 deg interval, the first active sector of the landing trajectory was small in time and characteristic velocity (Fig. 9). The impulse is given for the descent from orbit. Basically, it appears as the second active sector (in which the velocity is dampened to 98 percent). In a range of short distances, the active sector is commensurate to the length and characteristic velocity. The total period of the active sector little depends on the angular distance, and the duration of the maneuvers depend on $\psi_\pi (\psi_\beta)$ almost linearly when $\psi_i \geq 15$ to 20 deg, $i = \pi, \beta$.

For the take-off function, the active sectors change and the second appears to be the correcting.

In the development ψ_π, ψ_β, $\psi_i \geq 20$ deg, the optimal program for pitch tends to the linear function t. In addition, the average value $d\theta/dt$ with increase ψ_i approaches the angular velocity of the revolution of the satellite of the moon at zero altitude. In Fig. 10 there is shown an average angular velocity of pitch (where such an average has meaning). In the trifle force t_1, the angle of orientation in the sector of descent from orbit changes insignificantly and a corresponding dependence is presented in Fig. 11 for the coordinate system associated with the horizon in a point of orbital descent. With an increase of ψ_π the impulse converges to a transversal, at least the increase of the altitude of the orbit is a deflection from the transversal calculation.

With a small value for distance, the optimal program $\theta(t)$ is essentially nonlinear. Such an evolution of the orientation is explained by the plot above in a variable type p-trajectory (see also Refs. 3 and 4).

In Fig. 12, the trajectory of descent is presented (the second active sector).

The sum of the calculations is the magnitude of the final mass after landing on the moon (and in the orbit of an ASM) is presented in Fig. 13. The functional m_1 quickly increases reaching ψ_i = 30 to 40 deg of asymptotic value. The latter little depends on the altitude of the orbit and thrust units of the craft (Fig. 14) beginning with a value of the initial (ground) reactive acceleration of 0.5.

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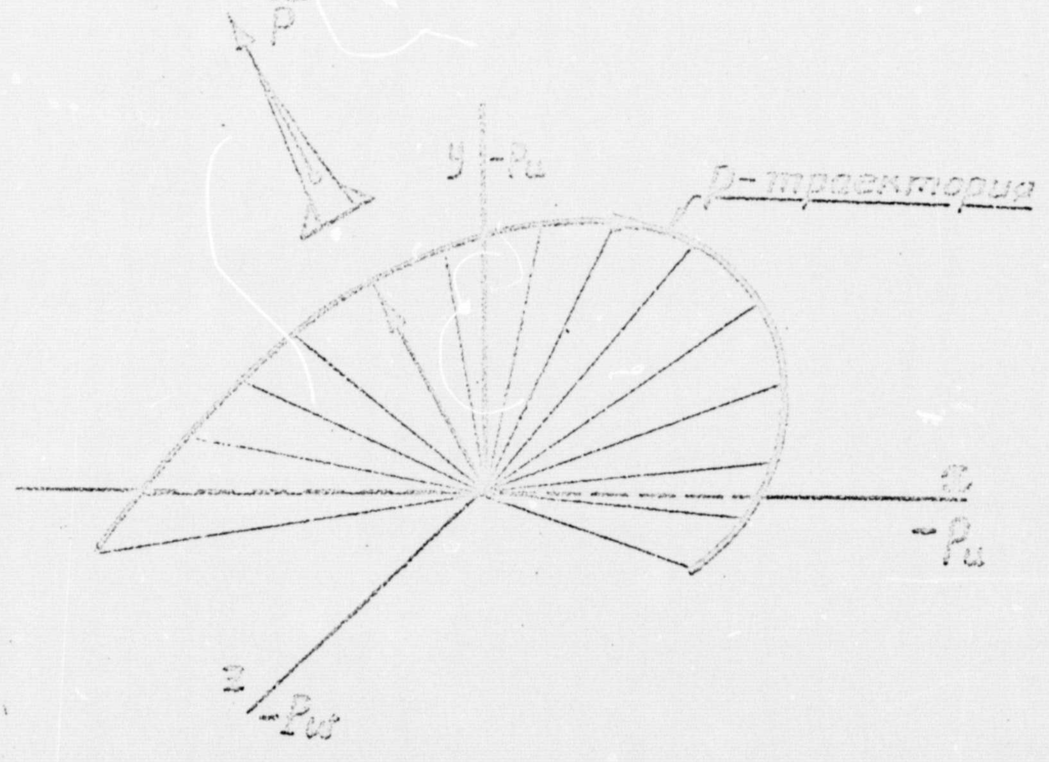
* Translator's Note: The translator does not know the meaning of these acronyms; MAF, however, probably means *International Federation for Aeronautics (Astronautics?)*.

**The year of publication was not included in this reference. The year of publication was probably 1962 (assumed from the data in Ref. 3 of this paper).



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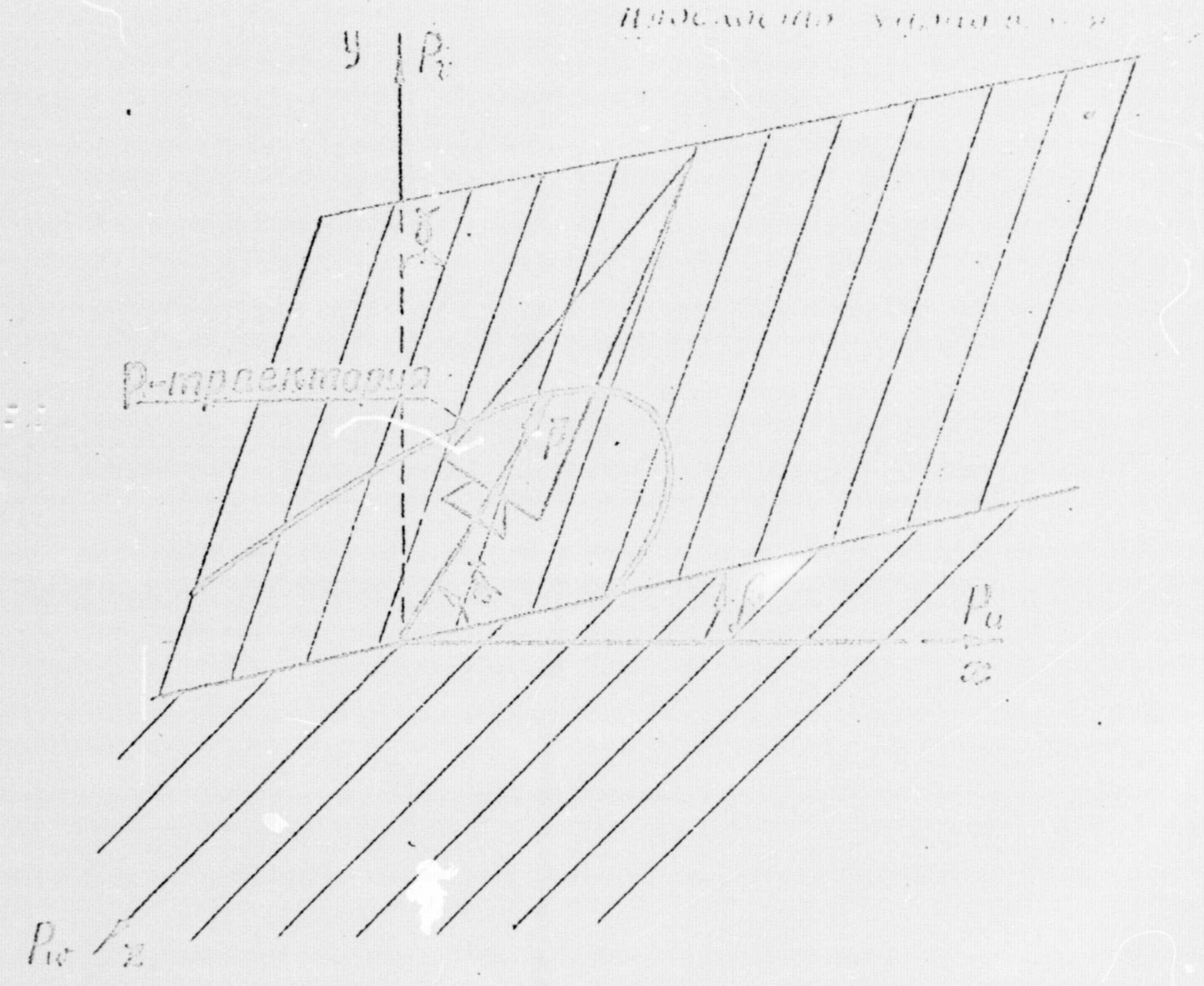

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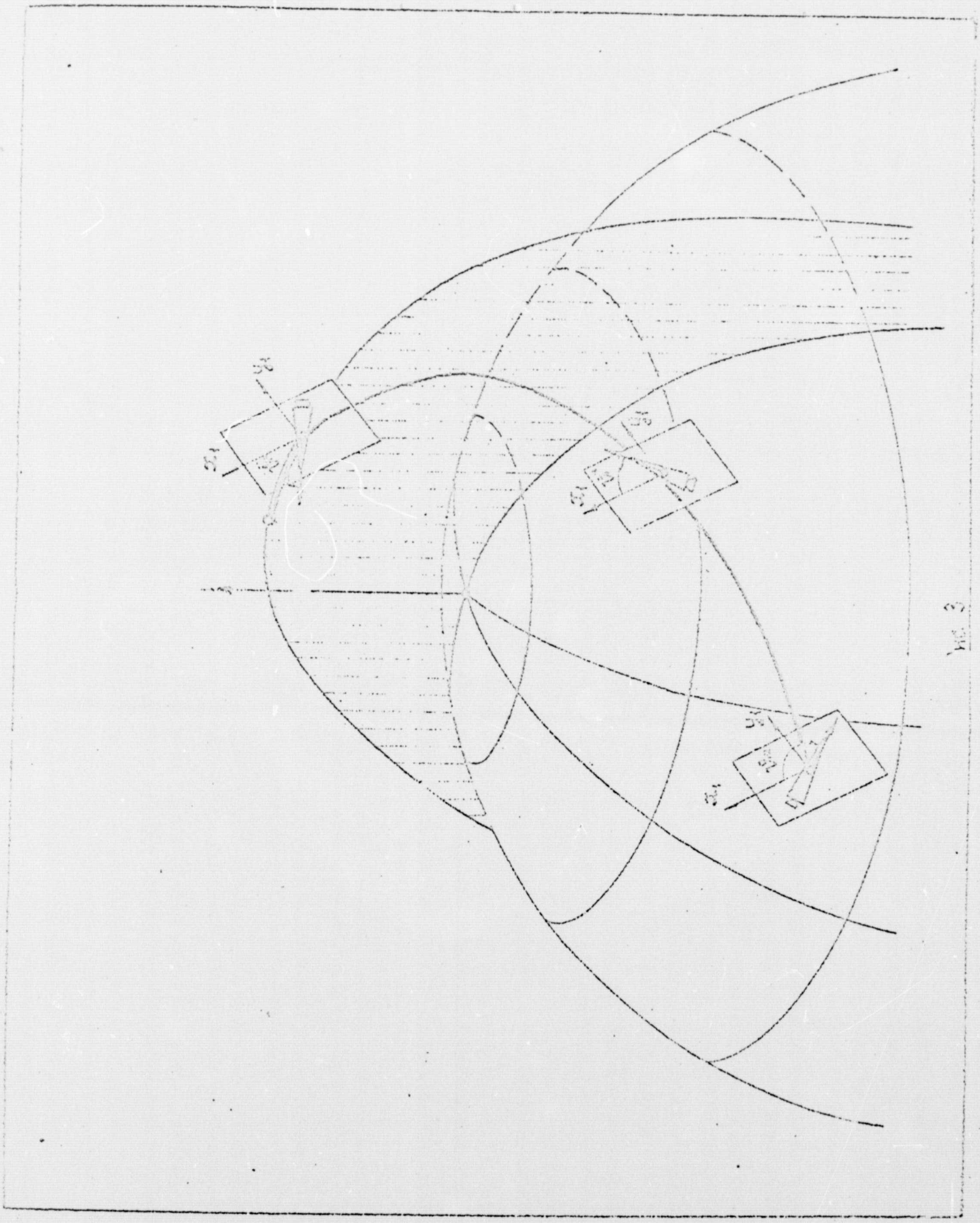

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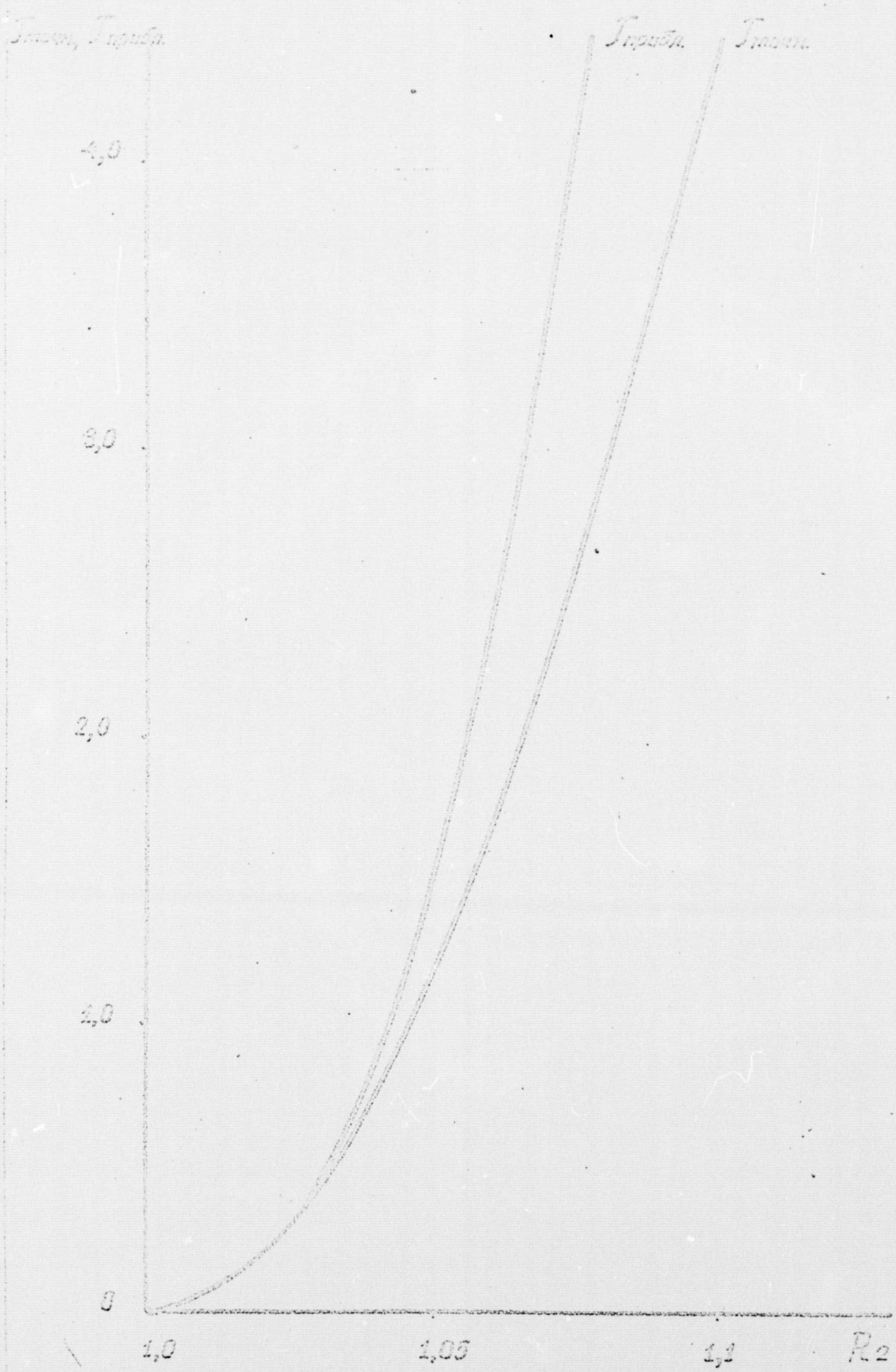

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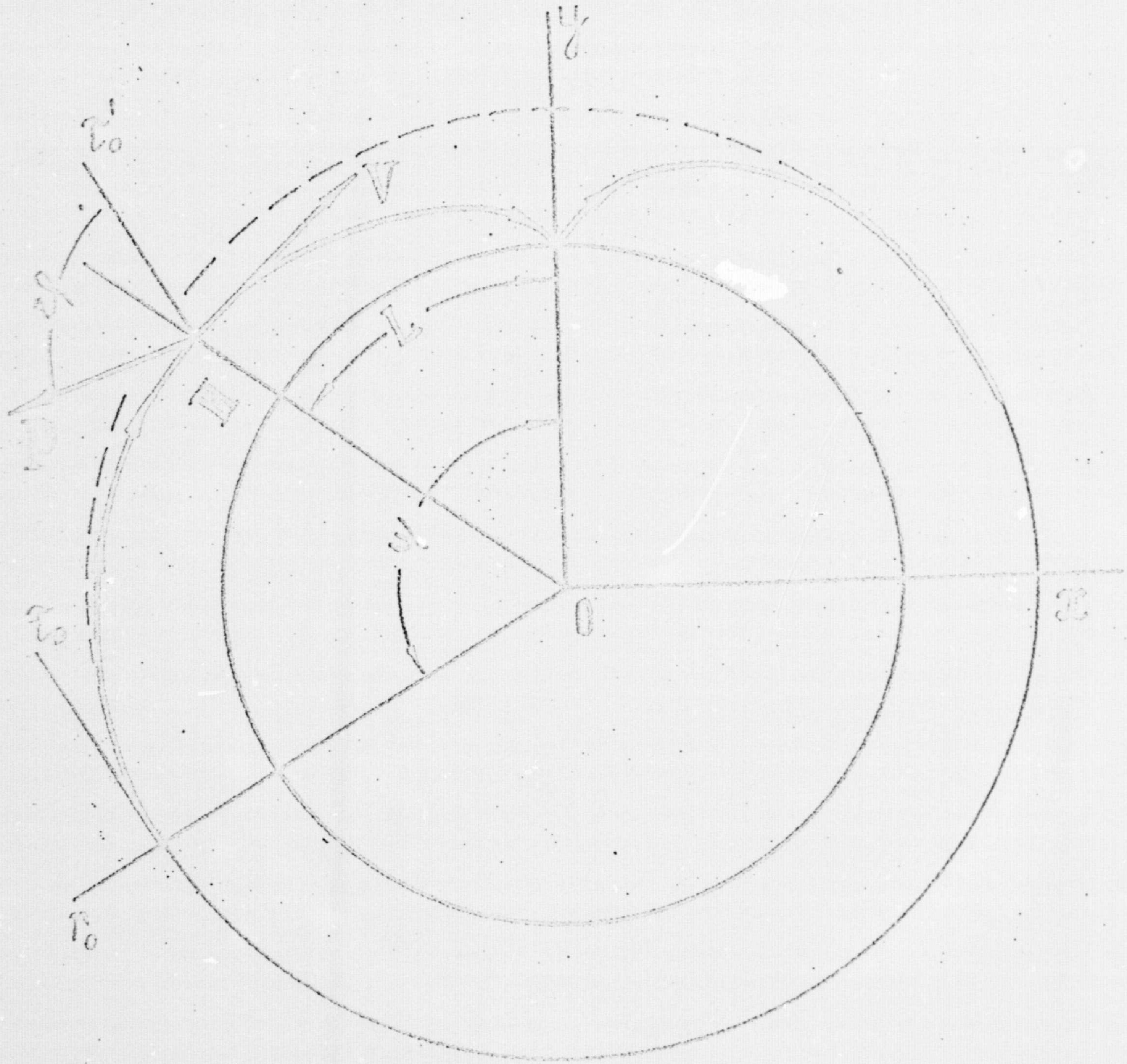

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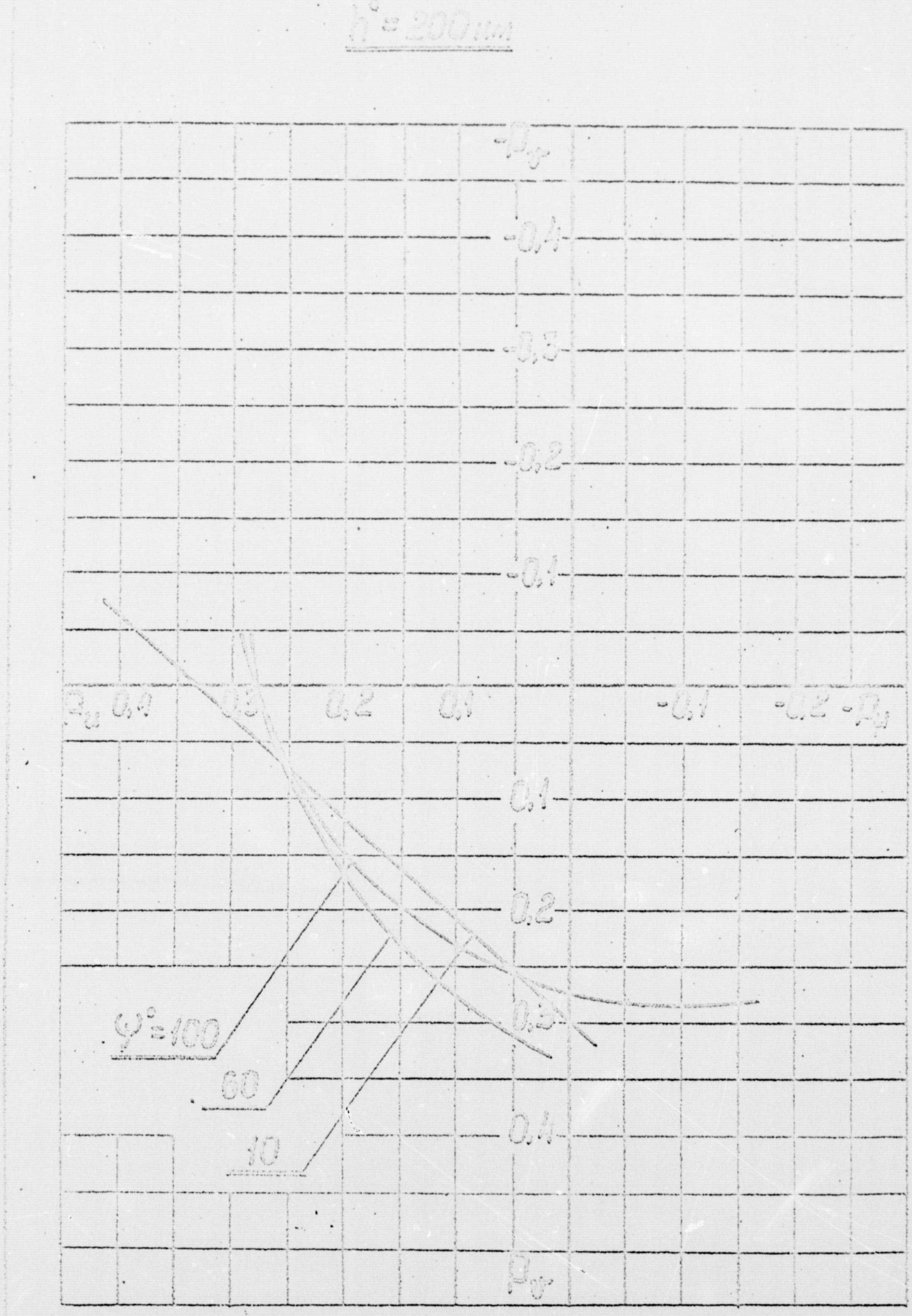
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$-0{,}3$
$-0{,}2$
$-0{,}1$
p_u 0,4
0,3
0,2
0,1
$-0{,}1$
$-0{,}2$ $-p_u$
0,1
0,2
0,3
0,4
$\varphi^\circ = 100$
60
10
p_v

Рис. 6

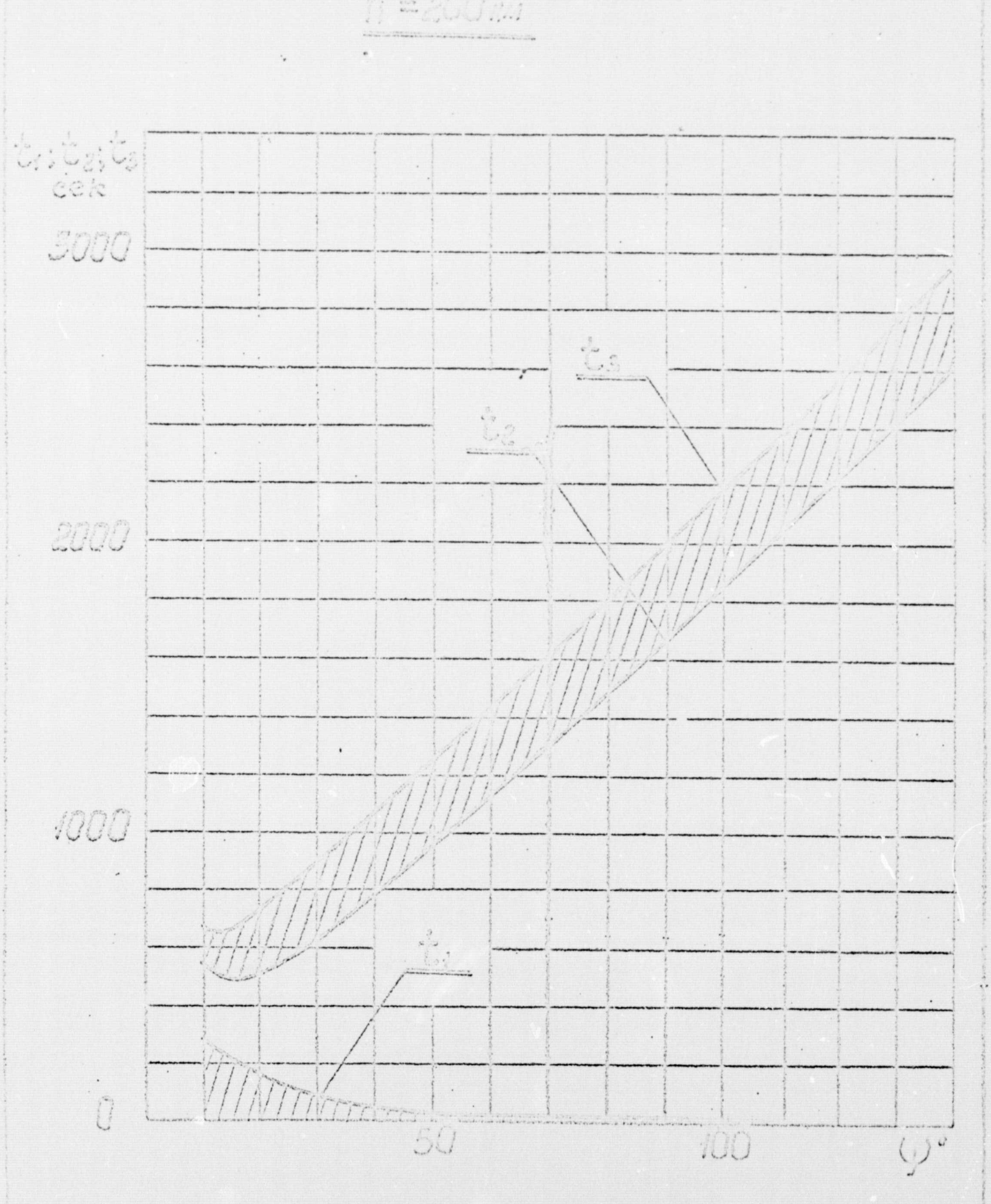
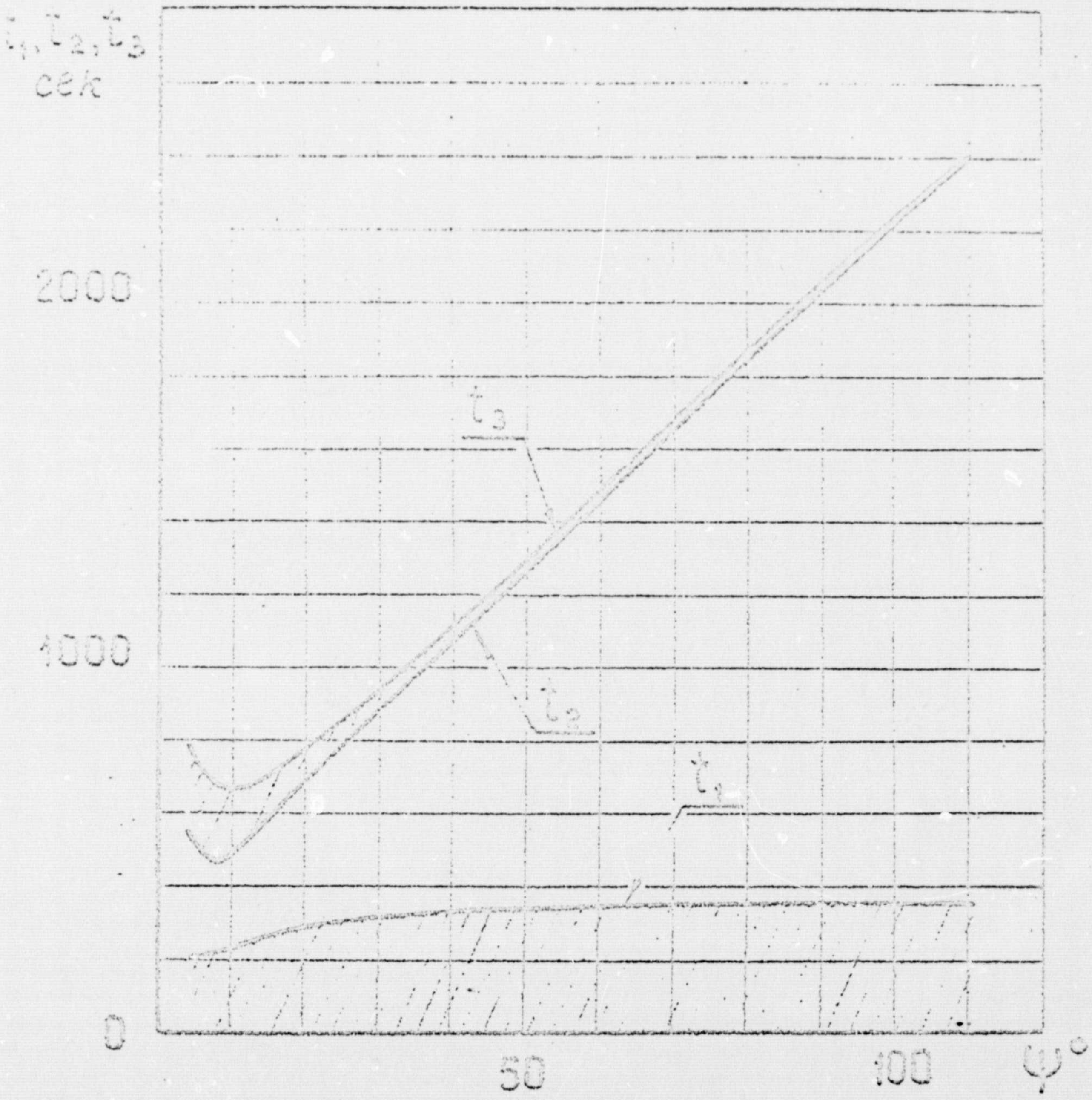


Рис. 7

$\overset{\circ}{H} = 200$ км



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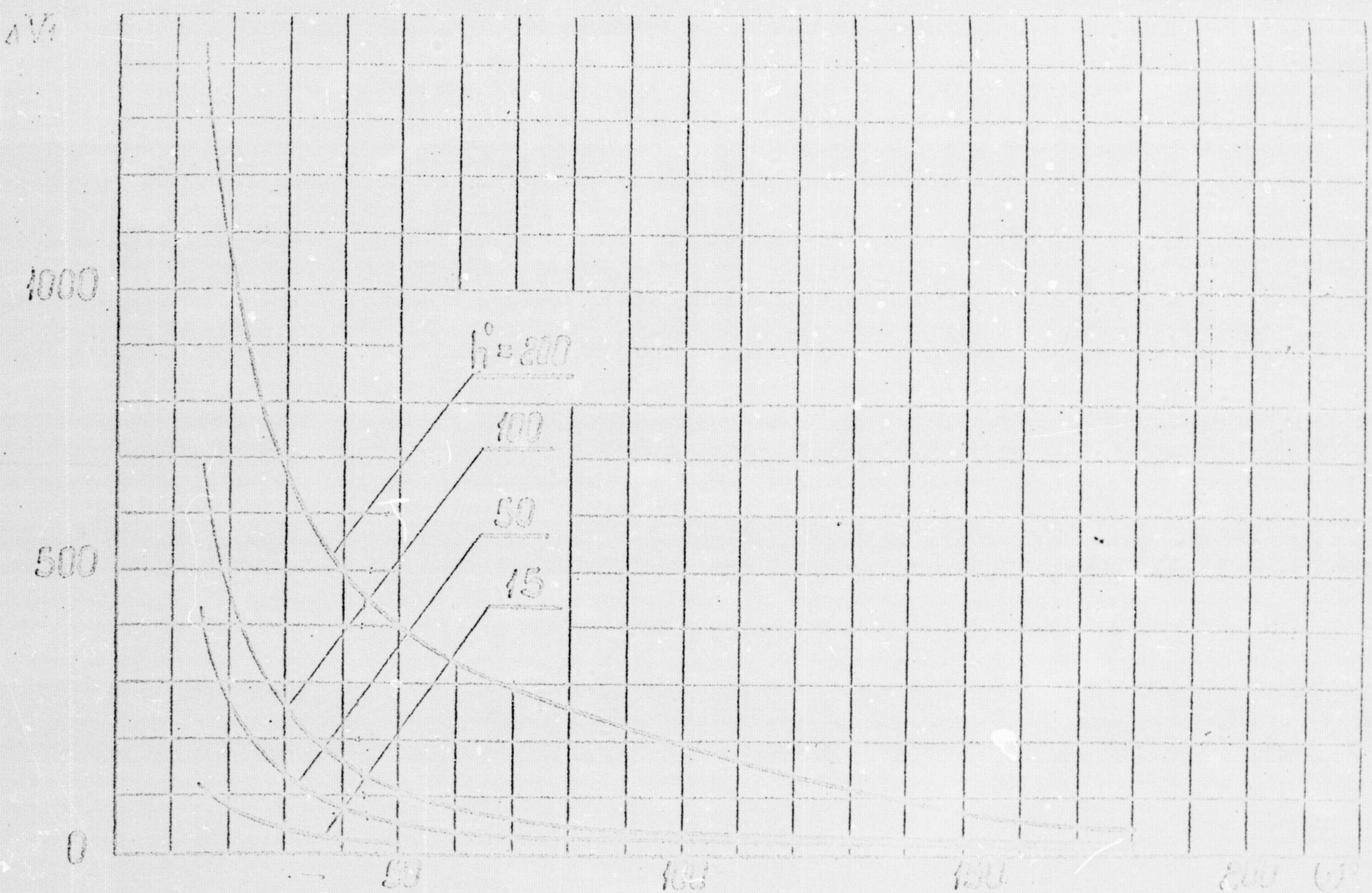


Фиг. 9

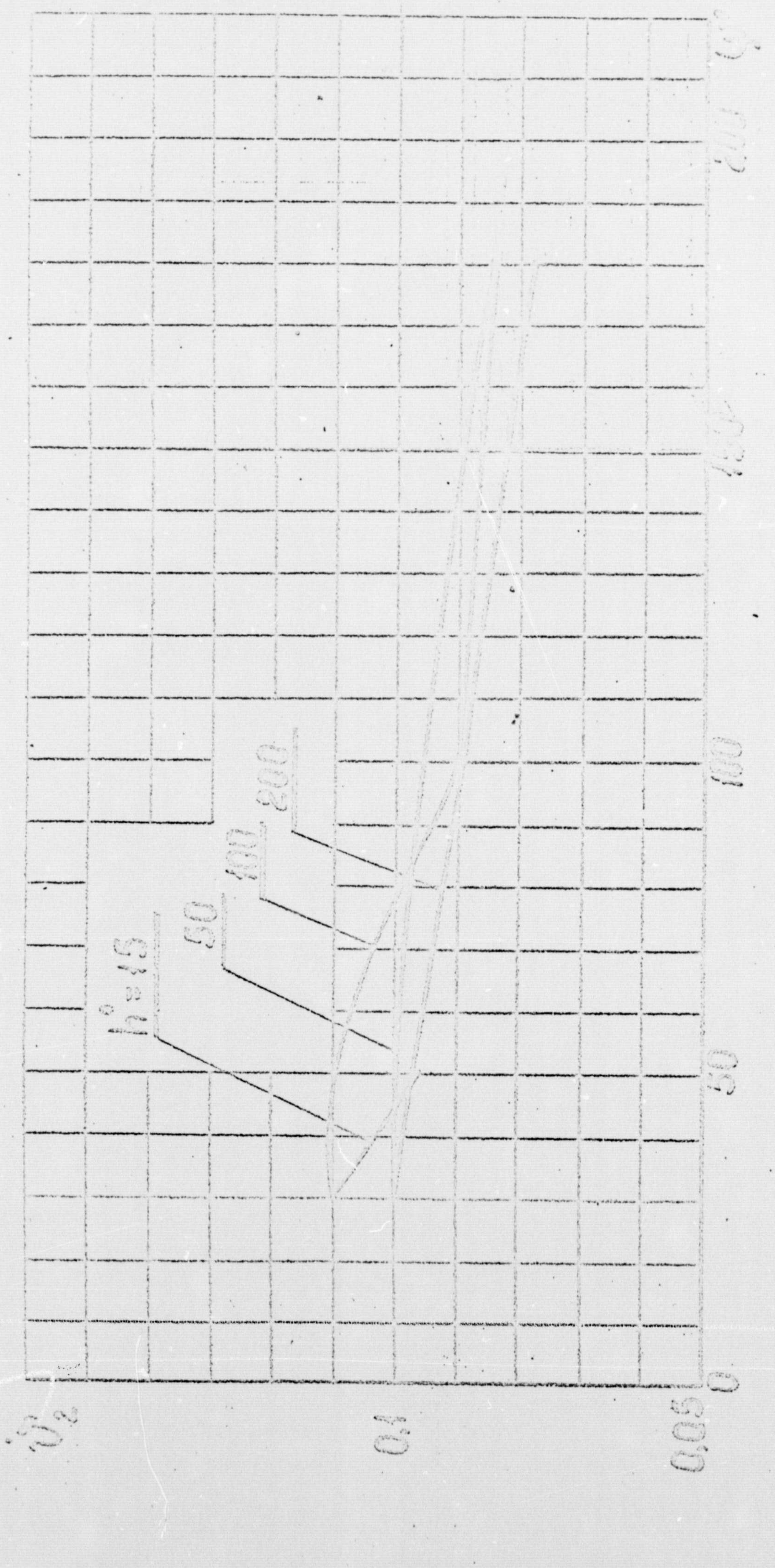
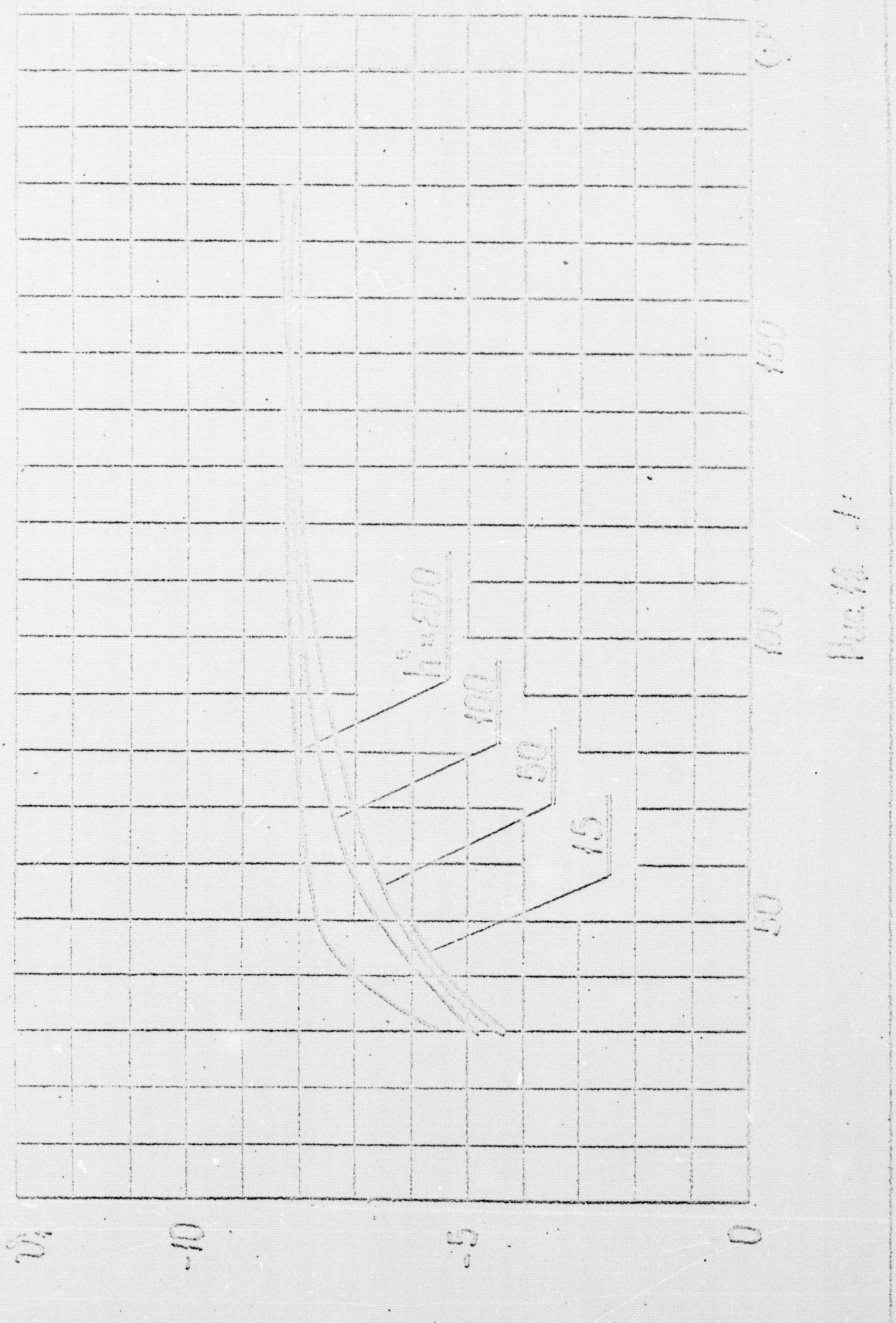


Fig. 10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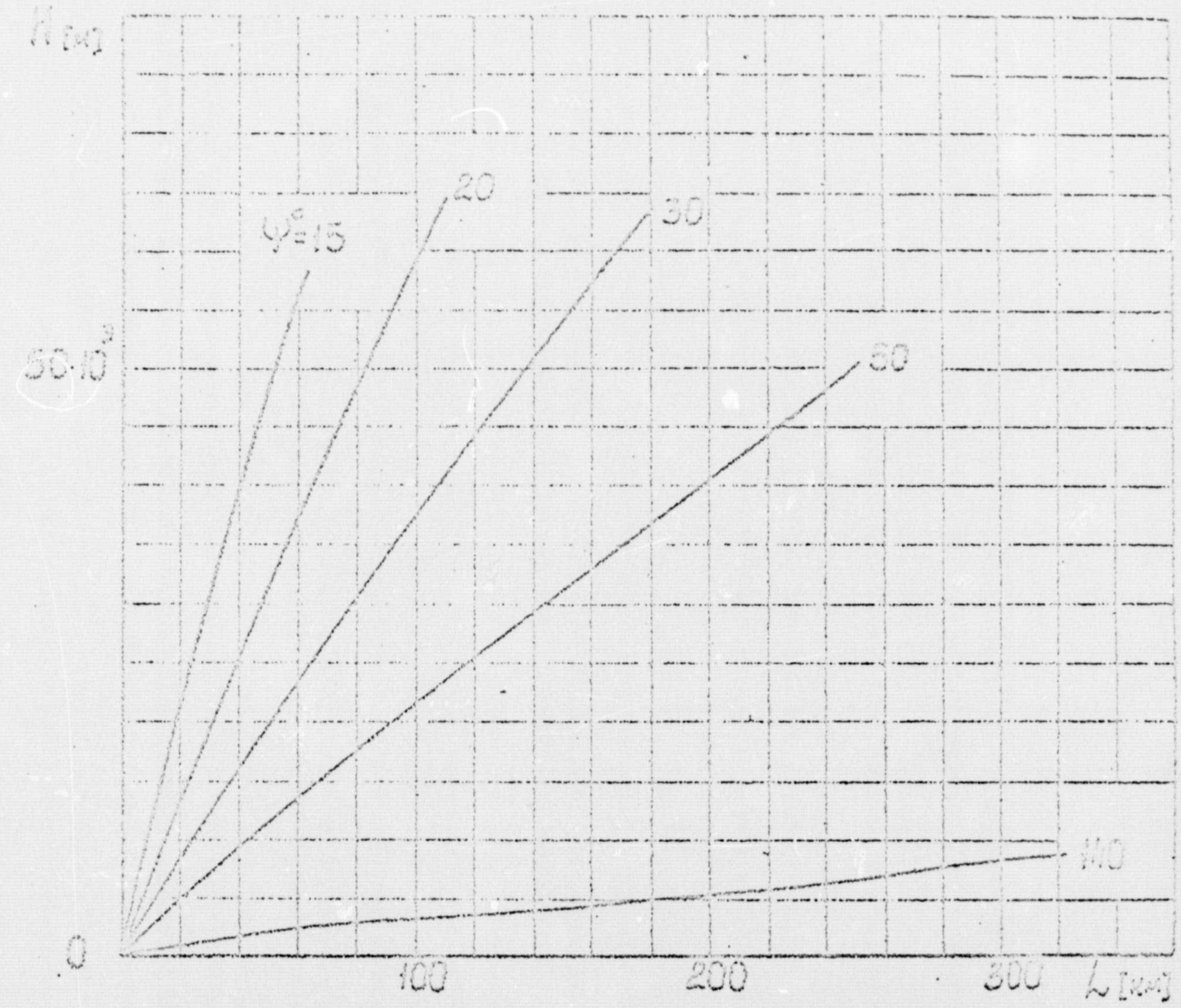
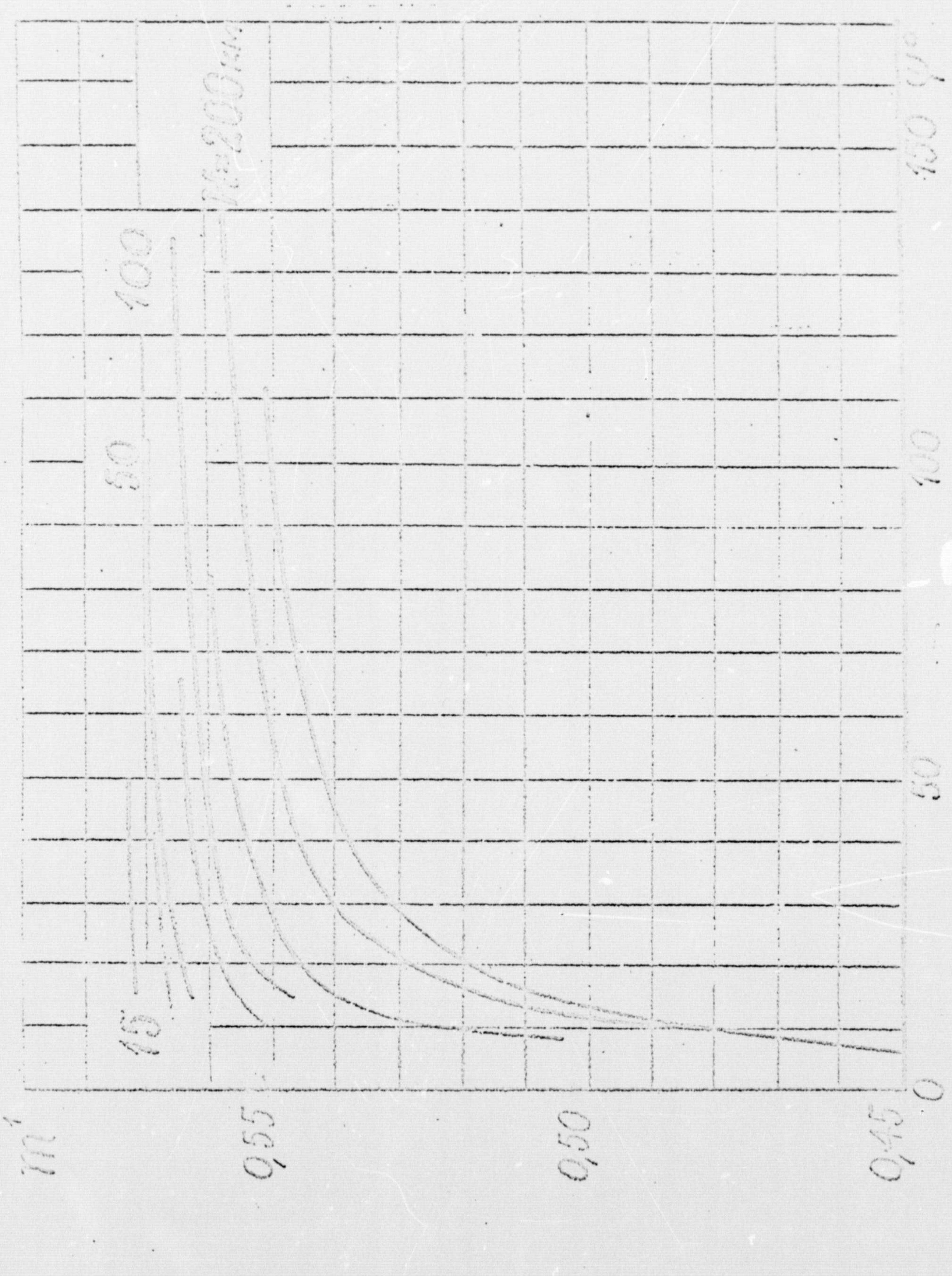


Рис. 12



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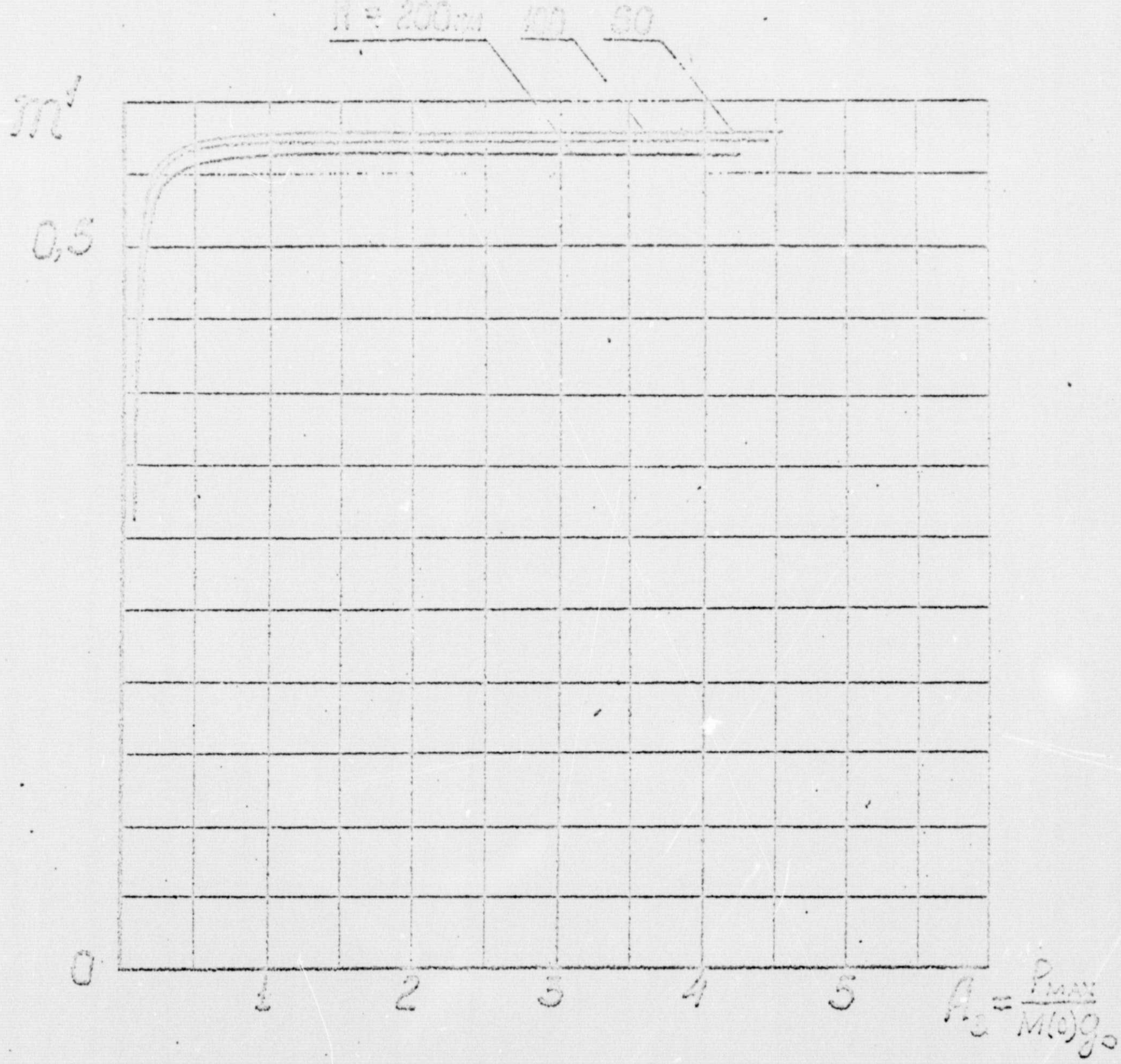

Рис. 14