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Model Reference Adaptive Design

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Submitted on behalf of

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ABSTRACT

The design of model reference adaptive control systems is investigated in this report. Several reasons for considering the model reference adaptive philosophy when designing control systems and several characteristics of a "good" model reference adaptive algorithm are discussed. Adaptive algorithms are derived for linear systems from two approaches. The first three algorithms are based on the steepest descent or gradient minimization of positive definite integral performance indices. The first algorithm attempts to minimize on-line a weighted integral square plant-model error index while the second algorithm attempts to effect a tradeoff between the system error and the perturbation control effort by minimizing an index that reflects the relative cost of each. An estimate of the optimum step size for gradient adaptation is incorporated into the third algorithm by treating adoptation as a discrete process rather than as a continuous process. The fourth algorithm is derived from a stability argument that follows from Lyapunov's Second Method. These algorithms are applied to two second order examples in order to gain insight into such properties as convergence rate, stability, error-nulling capability, and error-perturbation control tradeoff.

The model reference adaptive control design technique is successfully applied to a large flexible launch vehicle of the Saturn V class. The adaptive controller operates on only the measured outputs of the pitch and pitch-rate gyros and nowhere is it necessary to isolate the elastic bending response from the rigid body response. Simulation studies show that the adaptive controller reduces significantly the sensitivity of the booster to variations in the natural frequency of the first elastic bending mode.

CHAPTER I

INTRODUCTION

1.1 Introduction

As modern control systems become more complex and sophisticated it becomes necessary that they be designed with a built-in flexibility that provides the capability of automatically compensating for parameter and environmental variations that may occur during operation. These variations, which may be deterministic, stochastic, or totally unpredictable, arise from incomplete or inaccurate modeling of the physical process and inadequate knowledge of the hostile environment in which the process operates. The Saturn V booster illustrates both of these cases as the frequencies of the elastic bending modes are not totally predictable and the booster must fly through an unknown wind profile. Accordingly, a great deal of effort has been devoted to the study of self-adaptive self-optimizing, and learning control systems in the past few years.

Aseltine et al[1] and Stromer[2] have compiled extensive bibliographies of early contributions to the field of adaptive control and have attempted to classify these techniques into several categories. However, even to date there does not seem to be a universally accepted definition of an adaptive control system. For the purpose of this report the following definition will be considered applicable:

> An adaptive control system is a system which is capable of monitoring its performance relative to some well-defined criterion and adjusting certain control parameters in a systematic manner such as to approach optimum performance with respect to the chosen criterion.

This definition of an adaptive control system indicates that adaptation is a three-step process: 1) identification, 2) decision, and 3) implementation. These three steps are not always separable but are always present in some form.

The identification process involves obtaining a description of the plant. Several identification schemes have been developed for determining the impulse response, pole-zero pattern and differential equation which characterize a plant[3]. Alternately, the system identification problem can be treated from an index of performance (IP) point of view. An IP has been defined as "a functional relationship involving system characteristics in such a manner that the optimum operating characteristics may be determined from it".[4] The advantage of the IP is that it encompasses into a single number a quality measure for the performance of the system. One well-known IP is integral square error. Definition of a satisfactory IP is an art rather than a science and no adaptive system can be expected to perform better than its IP dictates.

The decision process is closely related to that of identification as the information provided by the latter is used in making any decision regarding system performance with respect to the optimum as defined by the IP. If performance is not adequate, a systematic program of parameter adjustment must be undertaken such as to improve this performance. In most cases this parameter adjustment is not a one-step operation but of an iterative nature such that the optimum is reached gradually.

The final stage, that of implementation, consists of the actual process of modifying the system parameters such as to bring the system "closer" to the optimum conditions. This is most often accomplished by adjusting some type of gain, either in a feedback loop or in a series compensator, or generating an auxiliary control signal.

Adaptive systems can be classified as 1) parameter adaptive or 2) signal-synthesis adaptive. In a parameter adaptive system, a parameter of the controller, such as a feedback gain, is adjusted so as to compensate for unsatisfactory performance. Signal-synthesis adaptation is achieved by generating an auxiliary control signal which when combined with the primary control signal will provide improved performance.

One method of parameter adaptation which has received special attention is the parameter perturbation approach as described by McGrath and Rideout[5] and Eveleigh[6]. If the IP is assumed to be a function of k adaptive parameters, it may be considered as a hypersurface above a k dimension hyperplane. The object is to find values for the k parameters that minimizes the IP. By perturbing the adaptive parameters sinusoidally, the partial derivatives of the IP with respect to the various adaptive parameters can be determined by correlation methods. When each adaptive parameter is adjusted at a rate directly proportional to its corresponding partial derivative, i.e., $\Delta P_i \propto -\frac{\partial IP}{\partial P_i}$, adaptation proceeds in the proper direction towards the minimum. This is essentially a search of the surface along the path of steepest descent. While this method is applicable to a wide class of systems, the choice of the IP is critical as it should have no relative extrema at which the gradient is identically zero but an absolute minimum does not occur. One inherent disadvantage of this method is the degradation in system performance that arises from continually perturbing the system.

A second type of parameter adaptive system that has become quite popular is the model reference adaptive control system. This type of system has been studied from several points of view by Osborn[7] et al, Donalson and Leondes[8], Shackcloth[9], Parks[10], and Dressler[11] among others. The performance criterion for this type of system is chosen as a function of the error between the system and some appropriate model. In references 7 and 8 adaptation again proceeds according to the method of steepest descent. The techniques of Shackcloth, Parks and Dressler, while not requiring the generation of the partial derivatives necessary for the steepest descent methods, do not appear to be applicable to as large a class of systems as is the steepest descent or gradient methods. The merits and pitfalls of several of the most prominent model-reference adaptive techniques are examined in detail in Appendix A. It is to this type of system that the remainder of this report is devoted.

Signal-synthesis adaptation is accomplished by generating an auxiliary control signal which should improve system performance. Systems of this nature incorporate the use of future prediction, based on past operating history, to synthesize a control signal which optimizes system performance one interval at a time. In a signal-synthesis system developed by Groupe and Cassir[12], extrapolation techniques are used for identification and error-predictions at discrete time intervals. The system employs rectangular adaptation pulses of finite duration to minimize a cost-functional of predicted square errors.

1.2 Organization of Report

Chapter II treats model reference adaptive control system design from two distinct viewpoints. First the concepts of the M.I.T. rule of

Osborn[7] et al are considered with some modifications. Secondly, a Lyapunov stability approach is investigated. The design algorithms that are derived are applied to two second order examples in order to obtain a feel for their applicability. Several conclusions regarding the properties of these algorithms are discussed.

In Chapter III the model reference technique is successfully applied to the pitch control of a large flexible launch vehicle. Because of vehicle flexure, the pitch and pitch-rate gyros measure local flexure in addition to rigid body motion. If the elastic bending modes are overly excited, the vehicle will break up. Thus, the control of such a vehicle is of great current interest. The necessity of an adaptive controller arises from the inprecise knowledge of the frequencies of these elastic bending modes. Several schemes have been proposed for attacking this problem. Smyth and Davis[13] have proposed the use of a notch filter with an adjustable center frequency and Lee[14] has suggested the use of redundant gyros to try to cancel the local bending from the measurements. Kezer[15] et al have applied the M.I.T. rule to the flexible booster problem but in so doing have assumed that the normalized bending is measurable. Of these schemes, only the notch filter has had much engineering success and even this method depends on the bending frequencies being higher than the speed of response of the closed loop system. For the present study only first order bending and no slosh modes are included in the booster model. The outputs of the pitch and pitch-rate gyros are assumed to be the only available measurements. The system is subjected to noise in the form of a wind-gust profile.

CHAPTER II

DEVELOPMENT OF ADAPTATION RULES

2.1 Introduction

This chapter begins with a discussion of the characteristics of a "good" model reference adaptive control system and the reasons for considering the model reference technique when designing control systems. The design of model reference adaptive control systems will be treated from two distinct viewpoints. One adaptive algorithm will be derived from a Lyapunov stability argument while several others will be derived from the steepest descent or gradient minimization of positive-definite integral indices. Examples are included to illustrate the application of the various algorithms.

2.2 Description of Model Reference Adaptive Control Systems

This study treats only the class of dynamical systems that can be described by linear ordinary differential equations. The state-space representation of such systems is employed throughout; an excellent reference on this subject is found in DeRusso [21] et al.

The model reference adaptive control system as considered in this study is represented schematically in Figure 2-1. In what follows the characteristics of the adaptive control system can be described by the following linear differential equation:

$$\dot{\underline{x}}_p(t) = A_p(t)\, \underline{x}_p(t) + B_p(t)\, \underline{u}_p(t) \tag{2.2-1}$$

$$\underline{y}_p(t) = C(t)\, \underline{x}_p(t) \tag{2.2-2}$$

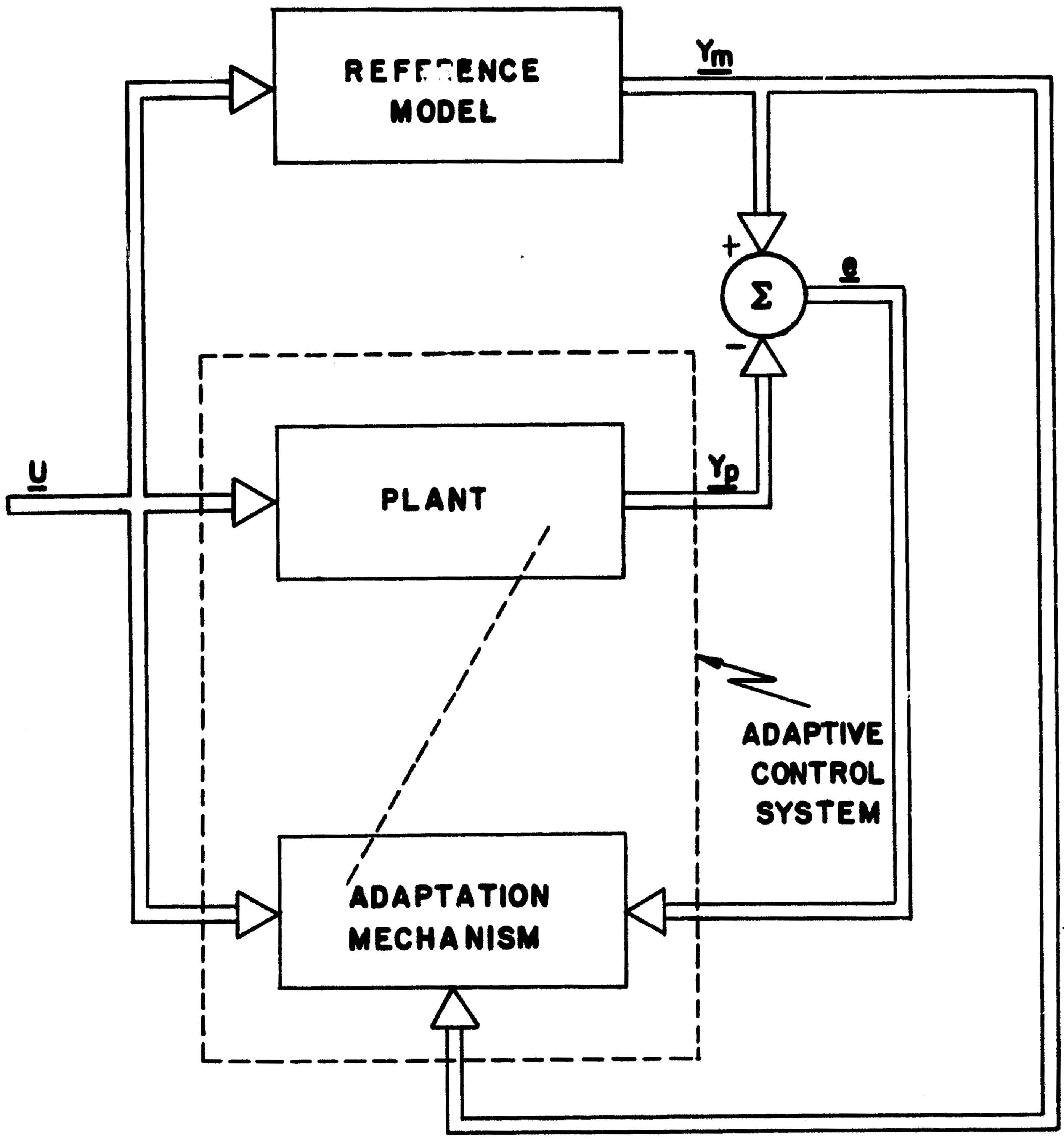

REFERENCE
MODEL
Ym
+
Σ
e
−
U
PLANT
Yp
ADAPTIVE
CONTROL
SYSTEM
ADAPTATION
MECHANISM

FIGURE 2-1

where $\underline{x}_p(t)$ = n - dimensional state vector of the adaptive control system.

$\underline{u}_p(t)$ = m - dimensional input vector to the adaptive control system.

$\underline{y}_p(t)$ = r - dimensional output vector of the adaptive control system.

$A_p(t)$ = n x n state matrix.

$B_p(t)$ = n x m control matrix, and

$C(t)$ = r x n output (measurement) matrix.

It is assumed that an arbitrary number of plant parameters, elements of $A_p(t)$ and $B_p(t)$, vary in an unknown manner but such that the structure of the matrices remains the same.

In classical feedback theory performance criteria are specified in such terms as rise time, overshoot, bandwidth, and stability. For the work that follows it shall be assumed that these criteria can be formulated in terms of a vector linear differential equation that yields the desired input-output relations. This set of differential equations will be referred to as the system reference model and can be considered as an implicit characterization of the performance criterion. This reference model is described by the following:

$$\dot{\underline{x}}_m(t) = A_m(t)\,\underline{x}_m(t) + B_m(t)\,\underline{u}_m(t) \tag{2.2-3}$$

$$\underline{y}_m(t) = C(t)\,\underline{x}_m(t) \tag{2.2-4}$$

where $\underline{x}_m(t)$ = n - dimensional state vector of the model

$\underline{u}_m(t)$ = m - dimensional input vector to the model

$\underline{y}_m(t)$ = r - dimensional output vector of the model

$A_m(t)$ = n x n state matrix

$B_m(t)$ = n x m control matrix

$C(t)$ = r x n output matrix

It is assumed that the order of the adaptive control system and the reference model are equal. If this is not the case, the model can be augmented such that the additional states have little effects on the behavior of the model.

Adaptation can be implemented in either of two ways - the systematic adjustment of the elements of $A_p(t)$ and/or $B_p(t)$ or the systematic synthesis of $\underline{u}_p = \underline{u}_m + \Delta \underline{u}$. The latter approach is used with the gradient minimization concept while the former is more amenable to the Lyapunov stability approach. The actual applicability of these two methods of implementation to realistic systems will be discussed later in this chapter.

2.3 General Design Philosophy

Before proceeding with the development of the adaptive algorithms it is informative to briefly consider two questions relative to model reference adaptive control systems: 1) when and why are such systems necessary and 2) what are the characteristics of a "good" adaptive algorithm? First, as control systems become more advanced and sophisticated it becomes extremely difficult to derive an accurate mathematical model of the plant while at the same time the performance requirements imposed on the plant become more demanding. Tuel [16], Dougherty[17], Rillings[18], and Cassidy[19] have applied the concepts of optimal control to this problem. The basic concept is to define a variable which represents the sensitivity of the plant trajectory to changes in plant parameters. These sensitivity variables are then treated as additional state variables and are included in the cost index that is to be minimized. This technique optimizes, with respect to the chosen performance index, the tradeoff between state response, control effort, and trajectory dispersion. As a result, its best performance may be poorer than true optimal performance but its range of acceptable performance is extended. However, with a precomputed control law it is always possible, even if highly unlikely, for the plant parameters to vary to such an extent as to cause instability. On the other hand, a model reference adaptive control system can always be designed to perform "optimally" at nominal conditions by choosing the nominal plant as the reference model. In addition, adaptation should reduce any trajectory dispersions resulting from both off-nominal parameter values, regardless of the magnitude of these parameter variations, and external disturbances encountered during operation. Figure 2-2 best summarizes the level of performance and range of acceptable performance that can be obtained from 1) optimal control systems designed without sensitivity considerations, 2) optimal control systems designed with sensitivity considerations, and 3) model reference adaptive control systems. In conclusion, there are three principle reasons for considering the model reference technique: 1) no degradation in nominal performance, 2) enhancement of stability, and 3) reduction in effects of external disturbances.

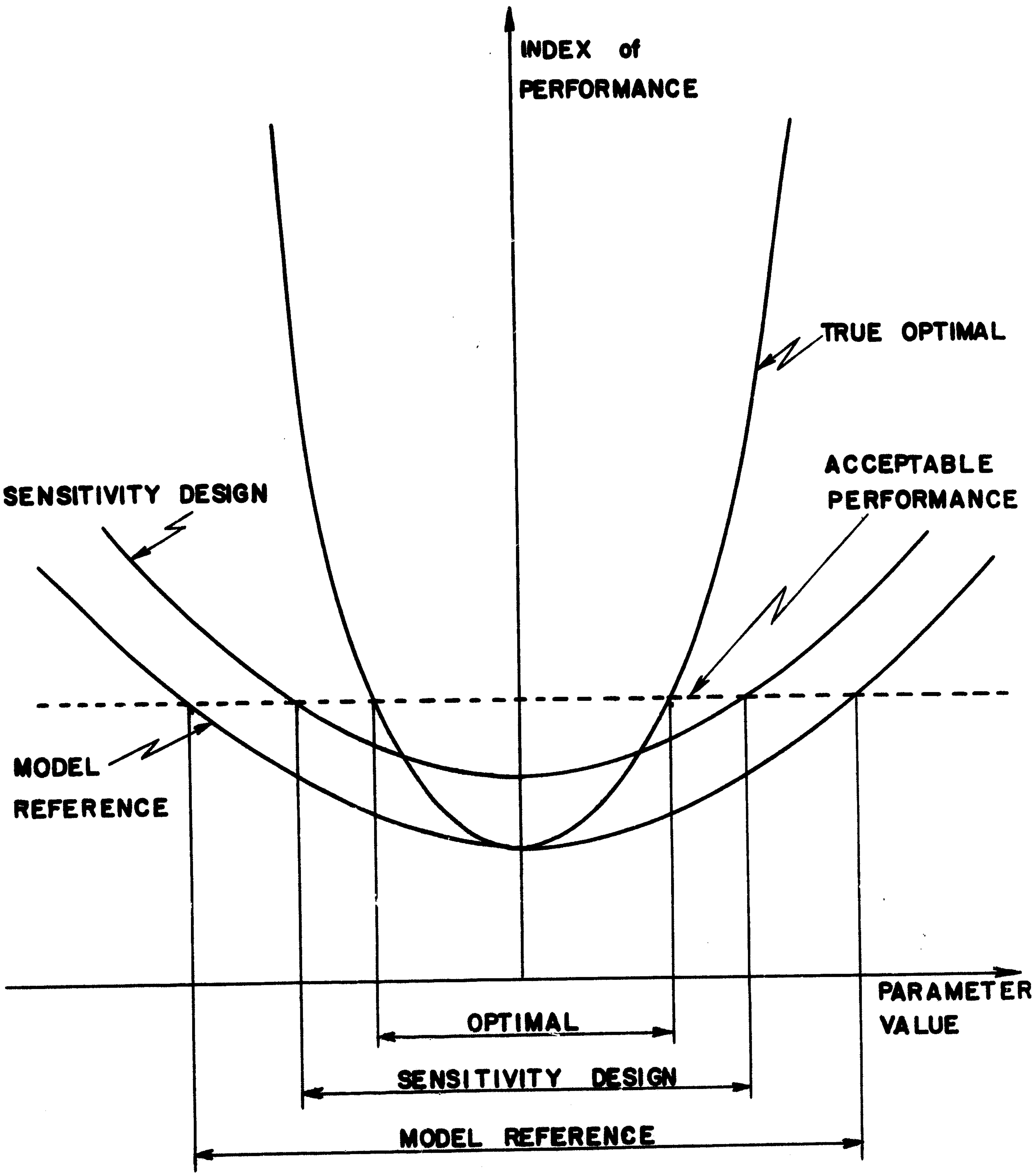

INDEX of PERFORMANCE
TRUE OPTIMAL
SENSITIVITY DESIGN
ACCEPTABLE PERFORMANCE
MODEL REFERENCE
PARAMETER VALUE
OPTIMAL
SENSITIVITY DESIGN
MODEL REFERENCE

FIGURE 2-2

If there is to be no degradation in performance for nominal parameter values, it is necessary that no adaptation occur for zero error. This implies, not too unexpectedly, that any adaptation algorithm must be functionally dependent on the system error such that $f(\underline{e}) = 0$ for $\underline{e} = \underline{0}$. Since one of the reasons for implementing a model reference adaptive controller is to enhance stability, it is important that the plant response converge rapidly to the "optimal" and that the overall system be stable. From a purely practical consideration, any model reference adaptive controller should not be too complex to implement or its value becomes questionable. Thus there are at least four important characteristics of a "good" model reference adaptive algorithm: 1) no adaptation for zero error, 2) rapid convergence to the "optimal", 3) stability of the total system, and 4) simplicity of implementation. It will be seen later that these characteristics are not always totally independent.

2.4 Continuous Gradient Adaptation

One popular criterion for the design of adaptive control systems has been the minimization of the integral-square error of the system - model configuration. This is the criterion that was successfully applied by Osborn[7] et al and led to the well-known M.I.T. rule for model reference adaptive control system design. This section presents some ramifications of the M.I.T. rule as applied to vector linear systems.

2.4.1 $\Delta \underline{u} = -K \underline{y}_p$

Consider the system - model configuration described by Equations 2.2-1 - 2.2-4 and define

$$\underline{\mathcal{E}} = \underline{x}_m - \underline{x}_p \tag{2.4-1}$$

$$\underline{e} = \underline{y}_m - \underline{y}_p = C\,\underline{\mathcal{E}} \tag{2.4-2}$$

In what follows, time dependence of all quantities will not be explicitly stated in order to simplify the notation, but will be assumed unless noted otherwise. Choosing

$$J = \int \underline{e}^T Q\, \underline{e}\, dt \tag{2.4-3}$$

in which Q is a non-negative definite symmetric matrix as an index of performance, a reasonable criterion for successful adaptation is the minimization of J. If the control signal is postulated as $\underline{u}_p = \underline{u}_m - K\,\underline{y}_p$, this minimization reduces to the determination of a value of K such that J is minimized. This minimum occurs when

$$\frac{\partial}{\partial k_{ij}} J = 0 \tag{2.4-4}$$

for all k_{ij}, $k_{ij} \not\equiv 0$. Treating J as a hypersurface in $\underline{\hat{K}}$ space, where $\underline{\hat{K}}$ respresents the non-identically zero element of K in vector form, an on-line search is performed along the surface in a steepest-descent fashion. In other words, $\underline{\hat{K}}$ is adjusted in the direction of the gradient of J with respect to $\underline{\hat{K}}$. Thus

$$\Delta \hat{k}_i \;\alpha\; - \frac{\partial J}{\partial \hat{k}_i} \tag{2.4-5}$$

or $$\dot{\hat{k}}_i \;\alpha\; - \left(\frac{\partial J}{\partial \hat{k}_i}\right)' \tag{2.4-6}$$

From Equation 2.4-3 it is seen that

$$\frac{\partial}{\partial \hat{k}_i} J = 2 \int \underline{e}^T Q \frac{\partial}{\partial \hat{k}_i} \underline{e} \, dt \tag{2.4-7}$$

which results in

$$\dot{\hat{k}}_i = - \beta_i \; \underline{e}^T Q \frac{\partial}{\partial \hat{k}_i} \underline{e} \tag{2.4-8}$$

Since

$$\frac{\partial}{\partial \hat{k}_i} \underline{y}_m = \underline{0} \quad ,$$

$$\frac{\partial}{\partial \hat{k}_i} \underline{e} = - \frac{\partial}{\partial \hat{k}_i} \underline{y}_p = - \frac{\partial}{\partial \hat{k}_i} C \underline{x}_p =$$

$$- C \frac{\partial}{\partial \hat{k}_i} \underline{x}_p = - C \underline{z}_i \tag{2.4-9}$$

Differentiating Equation 2.2-1 with respect to $\hat{k}_i$ results in

$$\frac{\partial}{\partial \hat{k}_i} \dot{\underline{x}}_p = A_p \frac{\partial}{\partial \hat{k}_i} \underline{x}_p + B_p \left(- K C \underline{z}_i - y_{kp} \underline{1}_j\right) \tag{2.4-10}$$

where $\hat{k}_i = K(j,k)$ and $\underline{1}_j$ is a vector with its j^{th} element equal to one and all other elements zero. Appendix B shows that $\frac{\partial}{\partial \hat{k}_i} \dot{\underline{x}}_p = \dot{\underline{z}}_i$

which simplifies Equation 2.4-10 to

$$\dot{\underline{z}}_i = A_p \underline{z}_i - B_p (K \quad C \underline{z}_i + y_{kp} \underline{1}_j) \qquad (2.4\text{-}11)$$

Thus $$\dot{\hat{k}}_i = \beta_i \underline{e}^T Q C \underline{z}_i \qquad (2.4\text{-}12)$$

where $\underline{z}_i$ is the solution of the differential equation given by Equation 2.4-11. A close examination of Equation 2.4-11 indicates that it is a function of A_p and B_p - both unknown. Thus $\hat{k}_i$ is adapted according to an approximation of the true path of steepest descent, namely

$$\dot{\hat{k}}_i = \beta_i \underline{e}^T Q C \hat{\underline{z}}_i \qquad (2.4\text{-}13)$$

where $\hat{\underline{z}}_i$ is the solution of

$$\dot{\hat{\underline{z}}}_i = A_m \hat{\underline{z}}_i - B_m y_{kp} \underline{1}_j \qquad (2.4\text{-}14)$$

and β_i is a convergence factor. The effects of the choice of β_i on system performance will be discussed later in this chapter. It is noted that this adaptation rule satisfies one of the criteria for a "good" adaptive algorithm, namely that no adaptation occurs for zero error. However, the implementation of this algorithm necessitates the generation of the $\hat{\underline{z}}_i$ vectors which might involve some rather involved filtering for high order system. The convergence property will be discussed later in the chapter.

2.4.2 $\Delta \underline{u} = - K \underline{e}$

In the problem formulation of section 2.4.1 the adaptive criterion was selected as the minimization of only a weighted integral square error and no attempt was made to limit the magnitude of the perturbation control term, $\Delta \underline{u}$. In this part, the control will be postulated as $\underline{u}_p = \underline{u}_m - K\,\underline{e}$ and a term reflecting the magnitude of the perturbation control, $\Delta\,\underline{u} = -K\,\underline{e}$, will be included in the performance index. In other words, the criterion for successful adaptation will be the selection of K to minimize

$$J = \int \left[\underline{e}^T Q\,\underline{e} + \Delta\,\underline{u}^T R\,\Delta\,\underline{u} \right] dt \tag{2.4-15}$$

in which Q and R are non-negative definite symmetric matrices. Again letting the vector $\hat{\underline{k}}$ represent the non-zero terms of the feedback matrix K and proceeding in a manner similar to that in section 2.4.1, an adaptation rule of the form

$$\dot{\hat{k}}_i = \beta_i \left[\underline{e}^T (Q + K^T R K)\, C\, \hat{\underline{z}}_i - \underline{e}^T K^T R\, e_k\, \underline{1}_j \right] \tag{2.4-16}$$

can be derived in which $\hat{k}_i = K\,(j,k)$ and $\hat{\underline{z}}_i$ is the vector solution to

$$\dot{\hat{\underline{z}}}_i = A_m\, \hat{\underline{z}}_i + B_m\, (K\, C\, \hat{\underline{z}}_i - e_k\, \underline{1}_j) \tag{2.4-17}$$

This result is derived in Appendix C. It is again seen that no adaptation occurs for zero system error but that this algorithm is somewhat more involved in terms of implementation. The convergence properties of this algorithm will be discussed later in this report.

2.5 Discrete Gradient Adaptation

One of the desired properties of a "good" model reference adaptive algorithm is rapid convergence of the plant trajectory to that of the model. For the continuous gradient adaptation rules of the previous section, the speed of convergence is a function of the β_i's. However, there does not appear to be any reasonable approach for analytically determining the "optimum" values of the β_i's for continuous adaptation. However, if the adaptive parameters are adjusted only at discrete instants of time instead of continuously, an analytical development for the "optimum" choice of the β_i's is possible. Pearson[20] has recently treated the model reference adaption problem in a similar manner but with a somewhat different motivation.

Consider once again the plant-model configuration of Equations 2.2-1 - 2.2-4 with

$$\underline{u}_p = \underline{u}_m - K(i)\, \underline{y}_p \tag{2.5-1}$$

where $K(i)$ is a constant matrix for $iT < t \leq (i+1)\,T$. The basic concept is to monitor the system during the time interval $iT < t \leq (i+1)T$ and determine that value of $K(i)^*$ that would have resulted in the smallest value of

$$J_i = \int_{iT}^{(i+1)T} \underline{e}^T Q\, \underline{e} \;dt \tag{2.5-2}$$

should $K(i)$ have been adjusted in the direction of the gradient of J_i. In other words what value of $\beta(i)$ would have produced the smallest value of J_i had

$$\hat{\underline{k}}(i)^* = \hat{\underline{k}}(i) - \beta(i)\, \underline{G}(i) \tag{2.5-3}$$

been used instead of $\hat{\underline{k}}(i)$ where

$$G_p(i) = \frac{\partial J_i}{\partial \hat{k}_j} \tag{2.5-4}$$

It has been previously shown that

$$\frac{\partial J_i}{\partial \hat{k}_j} = - 2 \int_{iT}^{(i+1)T} \underline{e}^T Q C \underline{z}_j \, dt \tag{2.5-5}$$

where $\underline{z}_j$ is the solution to

$$\dot{\underline{z}}_j = A_p \underline{z}_j - B_p (K C \underline{z}_j + y_{kp} 1_j) \tag{2.5-6}$$

However,

$$\underline{x}_p{}^* = \underline{x}_p + D \Delta \hat{\underline{k}} + \mathcal{O}(\| \Delta \hat{\underline{k}} \|^2) \tag{2.5-7}$$

where $\Delta \hat{\underline{k}} = \hat{\underline{k}}^* - \hat{\underline{k}}$, $\underline{x}_p{}^*$ represents the state trajectory that would have resulted from $\hat{\underline{k}}^*$, $\mathcal{O}(\| \Delta k \|^2)$ represents terms reflecting second and higher order effects, and

$$D = \left[\underline{z}_1 \; \underline{z}_2 \; \cdot \cdot \cdot \right] \tag{2.5-8}$$

From this,

$$\mathcal{E} = \underline{x}_m - \underline{x}_p$$

$$\underline{\mathcal{E}}^* = \underline{x}_m - \underline{x}_p{}^* = \underline{\mathcal{E}} - D \Delta \hat{\underline{k}} \tag{2.5-9}$$

to terms of first order. Choosing

$$\Delta \hat{\underline{k}}(i) = - \beta_i \underline{G}(i) \quad , \tag{2.5-10}$$

$$\underline{\mathcal{E}}^{*T} Q \underline{\mathcal{E}}^* = \underline{\mathcal{E}}^T Q \underline{\mathcal{E}} + 2 \beta(i) \underline{\mathcal{E}}^T Q D \underline{G}(i)$$

$$+ \beta(i)^2 \underline{G}(i)^T D^T Q D \underline{G}(i) \tag{2.5-11}$$

and $$J_i(\beta) = \int_{iJ}^{(i+1)T} \left[\underline{\xi}^T Q \underline{\xi} + 2 \beta \underline{\xi}^T Q D \underline{G} + \beta^2 \underline{G}^T D^T Q D \underline{G} \right] dt \qquad (2.5\text{-}12)$$

The value of β that would result in a minimum of $J_i(\beta)$ can be found by setting $\frac{d}{d\beta} J_i(\beta) = 0$ or

$$\int_{iT}^{(i+1)T} \left[2 \underline{\xi}^T Q D \underline{G} + 2 \beta \underline{G}^T D^T Q D \underline{G} \right] dt = 0 \qquad (2.5\text{-}13)$$

from which

$$\beta = \frac{\frac{1}{2} \underline{G}^T \underline{G}}{\underline{G}^T \left[\int_{iT}^{(i+1)T} D^T Q D \, dt \right] \underline{G}}$$

Thus to first order terms, the value of

$$\underline{\hat{k}}(i)^* = \underline{\hat{k}}(i) - \frac{1}{2} \frac{\underline{G}^T \underline{G}}{\underline{G}^T \left[\int_{iT}^{(i+1)T} D^T G D \, dt \right] \underline{G}} \underline{G} \qquad (2.5\text{-}15)$$

would have resulted in the smallest value of J_i should a gradient type of search be utilized. Unfortunately, the optimum value of $\beta(i)$ is dependent upon A_p and B_p, both of which may be unknown. Thus again an approximation must be made and $\underline{\hat{z}}_i$ is substituted throughout for $\underline{z}_i$, $\underline{\hat{z}}_i$ being derived from

$$\dot{\underline{\hat{z}}}_i = A_m \underline{\hat{z}}_i - B_m \, y_{kp} \, \underline{1}_j \qquad (2.5\text{-}16)$$

While it can no longer be said that the optimum value of $\beta(i)$ is obtained, experience has shown that this approximation is fairly good.

A question naturally arises as to the proper choice of T. From experience it has been found that T should be chosen approximately equal to the settling time of the system. Right away this limits the usefulness of the algorithm as it is difficult to apply it in situations where plant instability may occur. However, for some classes of systems it has been found to reduce significantly the instability problem often associated with gradient forms of adaptation. Again this is at the expense of additional complexity in implementation as the appropriate value of β must be calculated on-line at each adjustment time.

2.6 Lyapunov Adaptation

One of the major difficulties encountered in model reference adaptive control system design has been the determination of the stability properties of the resulting system. Recent work by Shackcloth[9] and Parks[10] has uncovered an interesting new approach to the design of such systems by incorporating Lyapunov's Second Method[21] into the design technique.

Considering once again the plant-model configuration of Equations 2.2-1 - 2.2-4 with A_m and B_m restricted to be time-invariant matrices and maintaining $\underline{u}_p = \underline{u}_m$, the differential equation for $\underline{\varepsilon} = \underline{x}_m - \underline{x}_p$ becomes

$$\dot{\underline{\varepsilon}} = A_m \underline{\varepsilon} + \left[A_M - A_p(t)\right] \underline{x}_p + \left[B_m - B_p(t)\right] \underline{u}_m \qquad (2.6\text{-}1)$$

or

$$\dot{\underline{\varepsilon}} = A_m \underline{\varepsilon} + A \underline{x}_p + B \underline{u}_m \qquad (2.6\text{-}2)$$

with $A = \left[a_{ij}(t) \right]$ and $B = \left[b_{ij}(t) \right]$

Choose as a Lyapunov function the quadradic form

$$V = \underline{\xi}^T Q \ \underline{\xi} + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\alpha_{ij}} \ a_{ij}^2 + \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{1}{\beta_{ij}} \ b_{ij}^2 \qquad (2.6\text{-}3)$$

in which Q is a symmetric positive definite matrix to be determined later, $\alpha_{ij} > 0$, and $\beta_{ij} > 0$. The total time derivative of Equation 2.6-3 is

$$\dot{V} = \underline{\xi}^T \left[Q A_m + A_m^T Q \right] \underline{\xi} + 2 \underline{\xi}^T Q A \underline{x}_p + 2 \underline{\xi}^T Q B \underline{u}_m + 2 \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{1}{\beta_{ij}} \ b_{ij} \ \dot{b}_{ij} + 2 \sum_{i=1} \frac{1}{\alpha_{ij}} \ a_{ij} \ \dot{a}_{ij} \qquad (2.6\text{-}4)$$

But $$\underline{\xi}^T Q A \underline{x}_p = \sum_{i=1}^{n} (\underline{\xi}^T \underline{q}_i) (\underline{a}_i^T \underline{x}_p)$$

$$(2.6\text{-}5)$$

and $$\underline{\xi}^T Q B \underline{u}_m = \sum_{i=1}^{n} (\underline{\xi}^T \underline{q}_i) (\underline{b}_i^T \underline{u}_m)$$

in which $Q = \left[\underline{q}_1 \ \underline{q}_2 \cdots \underline{q}_n \right]$, $A^T = \left[\underline{a}_1, \underline{a}_2 \cdots \underline{a}_n \right]$,

and $B^T = \left[\underline{b}_1, \underline{b}_2 \cdots \underline{b}_n \right]$. If

$$\dot{a}_{ij} = - \alpha_{ij} \ \underline{\xi}^T \ \underline{q}_i \ x_{jp} \qquad (2.6\text{-}6)$$

and $$\dot{b}_{ij} = - \beta_{ij} \underline{\varepsilon}^T \underline{q}_i u_{jm} \tag{2.6-7}$$

$$\dot{V} = \underline{\varepsilon}^T \left[Q A_m + A_m^T Q \right] \underline{\varepsilon} \tag{2.6-8}$$

However, if A_m is the system matrix of a stable model, there exists a unique positive definite symmetric matrix Q which is the solution of

$$A^T Q + Q A^T = - P \tag{2.6-9}$$

in which P is also a symmetric positive definite matrix. With this choice of Q in Equation 2.6-3, V is a positive definite quadradic form while $\dot{V}$ is a negative definite quadradic form. This guarantees that the adaptive system is stable and should operate in the neighborhood of the origin in $\underline{\varepsilon}$-space[21] .

Equations 2.6-6 and 2.6-7 provide a rule for adapting the individual elements of A_p and B_p. Unless the time-varying nature of A_p and B_p is known, which is not usually the case if adaptation is necessary, the successful implementation of these rules is limited to time-invariant or slowly-time-varying plants. In many linear systems the individual elements of the state and control matrices are not accessible and control must be implemented by a feedback structure. When this is true, the only adjustable parameters are the feedback gains and not the individual state matrix elements. For example, the closed-loop representation of a time-invariant scalar control problem with $u = m - \underline{K}^T \underline{x}_p$ takes the form

$$\dot{\underline{x}}_p = \left[A_p - \underline{b}_p \underline{K}^T \right] \underline{x}_p + \underline{b}_p m \tag{2.6-10}$$

and

$$\dot{\underline{x}}_m = A_m \underline{x}_m + \underline{b}_m r \qquad (.26\text{-}11)$$

For this case it is seen that

$$\dot{a}_{ij} = b_{p_i} \dot{k}_j \qquad (2.6\text{-}12)$$

which will generally result in inconsistent values of $\dot{k}_j$ upon application of Equation 2.6-6. Even for systems in which $\underline{b}_p$ contains only one non-zero element, $b_{pl} \neq 0$, the implementation of the resulting unique k_j ; $j = 1,2,\ldots, n$ may not explicitly guarantee stability just as constraining some of the $\dot{a}_{ij} = 0$, $i \neq \ell$, may not lead to the satisfaction of the conditions for stability. Thus for systems in which the structure allows access only to a set of feedback gains, the adaptation rules of Equations 2.6-6 and 2.6-7 are not directly applicable.

One further limitation of this algorithm is the necessity of measuring all of the states of the system which might be an unrealistic requirement for certain classes of systems. However, for those systems for which this adaptation rule is applicable, it is deserving of prime consideration as little on-line computation is necessary and stability is insured.

2.7 Illustrative Examples

To illustrate the application of the adaptive algorithms derived in sections 2.4, 2.5, and 2.6, two simple second order examples are considered. The results obtained for the various algorithms are compared in terms of time-response and integral-square error.

2.7.1 Example 1

The plant for this example is described by the vector differential equation

$$\dot{\underline{x}}_p = \begin{bmatrix} 0.0 & 1.0 \\ -2.414 & -1.5 \end{bmatrix} \underline{x}_p + \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix} u_p \tag{2.7-1}$$

and the model is described by

$$\dot{\underline{x}}_m = \begin{bmatrix} 0.0 & 1.0 \\ -1.414 & -1.352 \end{bmatrix} \underline{x}_m + \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix} u_m \tag{2.7-2}$$

With no adaptation, this plant is stable but will exhibit a steady-state error for $u_m = 1.0$.

2.7.1-a Lyapunov Adaptation

From Equation 2.6-6 the Lyapunov adaptation rule for this example is

$$\dot{a}_{p21} = \alpha_{21} (q_{12} e_1 + q_{22} e_2) x_{p1} \tag{2.7-3}$$

$$\dot{a}_{p22} = \alpha_{22} (q_{12} e_1 + q_{22} e_2) x_{p2}$$

where Q is the positive definite symmetric matrix solution of

$$A_m^T Q + Q A_m = - I \tag{2.7-4}$$

or

$$Q = \begin{bmatrix} 1.371 & 0.354 \\ 0.354 & 0.631 \end{bmatrix} \tag{2.7-5}$$

Thus $\dot{a}_{p21} = \alpha_{21} (0.5\, e_1 + 0.707\, e_2)\; x_{p1}$

and $\dot{a}_{p22} = \alpha_{22} (0.5\, e_1 + 0.707\, e_2)\; x_{p2}$ (2.7-6)

Simulations of plant responses arising from this adaptation algorithm are shown for various values of $\alpha_{21} = \alpha_{22}$ in Figure 2-3.

2.7.1-b Continuous Gradient Adaptation

For $u_p = u_m - \underline{K}^T \underline{x}_p$, the minimization of

$$J = \int \underline{e}^T \underline{e} \quad dt \tag{2.7-7}$$

by the continuous gradient method yields the adaptation rules, Equation 2.4-12,

$$\dot{K}_1 = \beta\, \underline{e}^T \underline{z}_1 \tag{2.7-8}$$
$$\dot{K}_2 = \beta\, \underline{e}^T \underline{z}_2$$

with $$\dot{\underline{z}}_i = \begin{bmatrix} 0.0 & 1.0 \\ -1.414 & -1.352 \end{bmatrix} \underline{z}_i - \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix} x_{pi} \quad ; \; i = 1,2$$

the forms of the appropriate filters for $\underline{z}_1$ and $\underline{z}_2$. Simulation responses are shown in Figure 2-4 for various values of β.

Choosing the perturbation control $\Delta \underline{u} = - \underline{K}^T \underline{e}$, the steepest descent minimization of

$$J = \int (\underline{e}^T \underline{e} + R\, \Delta\, u^2)\; dt \tag{2.7-9}$$

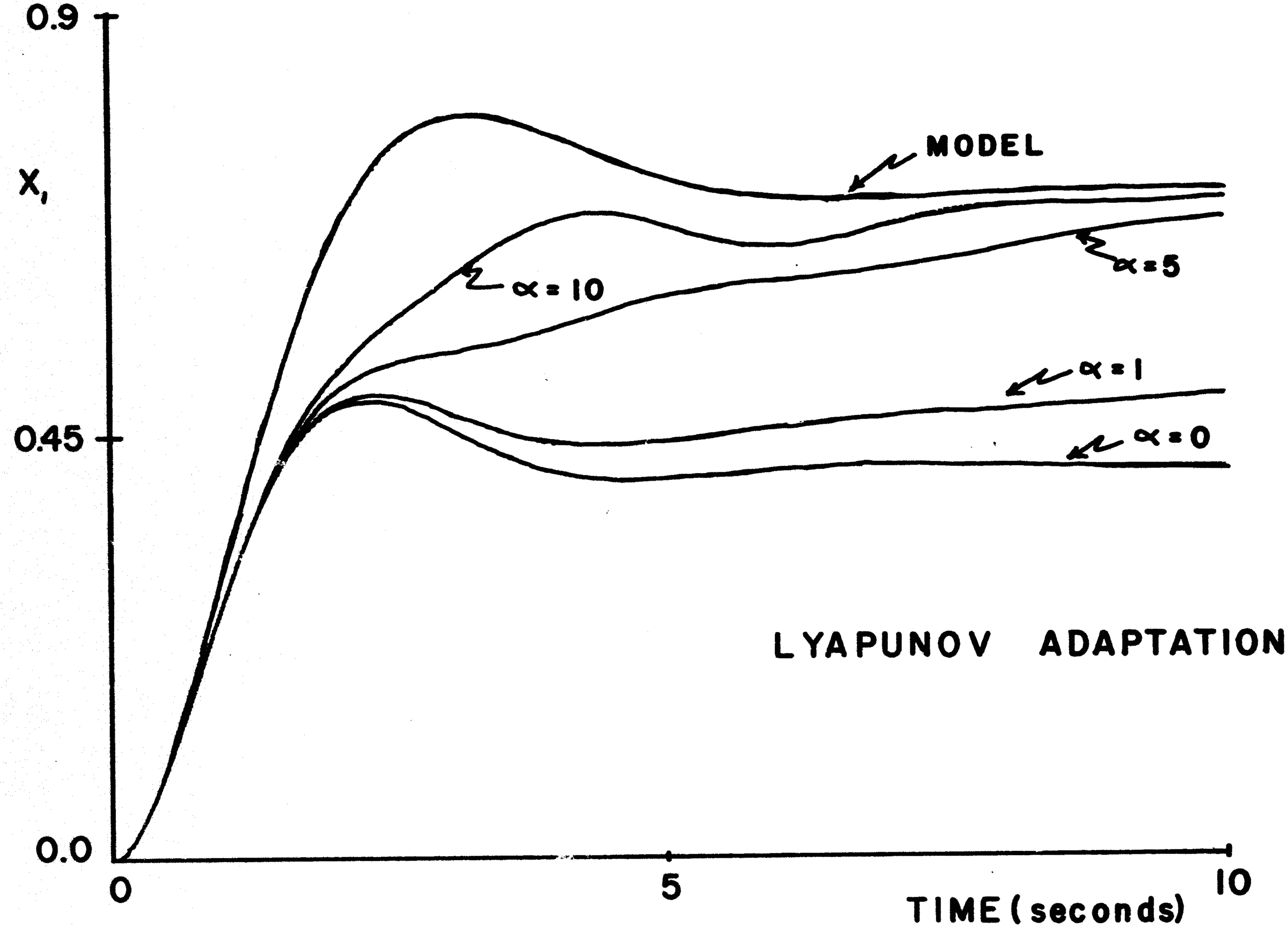

0.9
0.45
0.0
X₁
MODEL
α = 10
α = 5
α = 1
α = 0
LYAPUNOV ADAPTATION
0
5
10
TIME (seconds)

FIGURE 2-3

results in the adaptation rule, Equation 2.4-16,

$$\dot{k}_i = \beta \left\{ \underline{e}^T \left[I + R\,\underline{K}\,\underline{K}^T \right] \underline{z}_i - R\,\underline{e}^T\,\underline{K}\,e_i \right\} \; ; \quad i = 1, 2 \qquad (2.7\text{-}10)$$

where $\underline{z}_1$ and $\underline{z}_2$ are synthesized from filters described by

$$\dot{\underline{z}}_i = \begin{bmatrix} 0.0 & 1.0 \\ -1.414 & -1.352 \end{bmatrix} \underline{z}_i + \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix} \left\{ \underline{K}^T \underline{z}_i - e_i \right\} ; \; i = 1,2 \qquad (2.7\text{-}11)$$

Simulation responses are shown as a function of β in Figure 2-5 for $R = 0$ and in Figure 2-6 for $R = 1$.

2.7.1-c Discrete Gradient

Applying the discrete adaptation rule to the index

$$J_i = \int_{iT}^{(i+1)T} \underline{e}^T \underline{e} \; dt \qquad (2.7\text{-}12)$$

results in the adaptation rule

$$\underline{K}_{i+1} = \underline{K}_i - \beta_i \, \underline{G}_i \qquad (2.7\text{-}13)$$

with

$$\underline{G}_i^T = -2 \int_{iT}^{(i+1)T} \underline{e}^T \left[\underline{z}_1, \underline{z}_2 \right] dt \qquad (2.7\text{-}14)$$

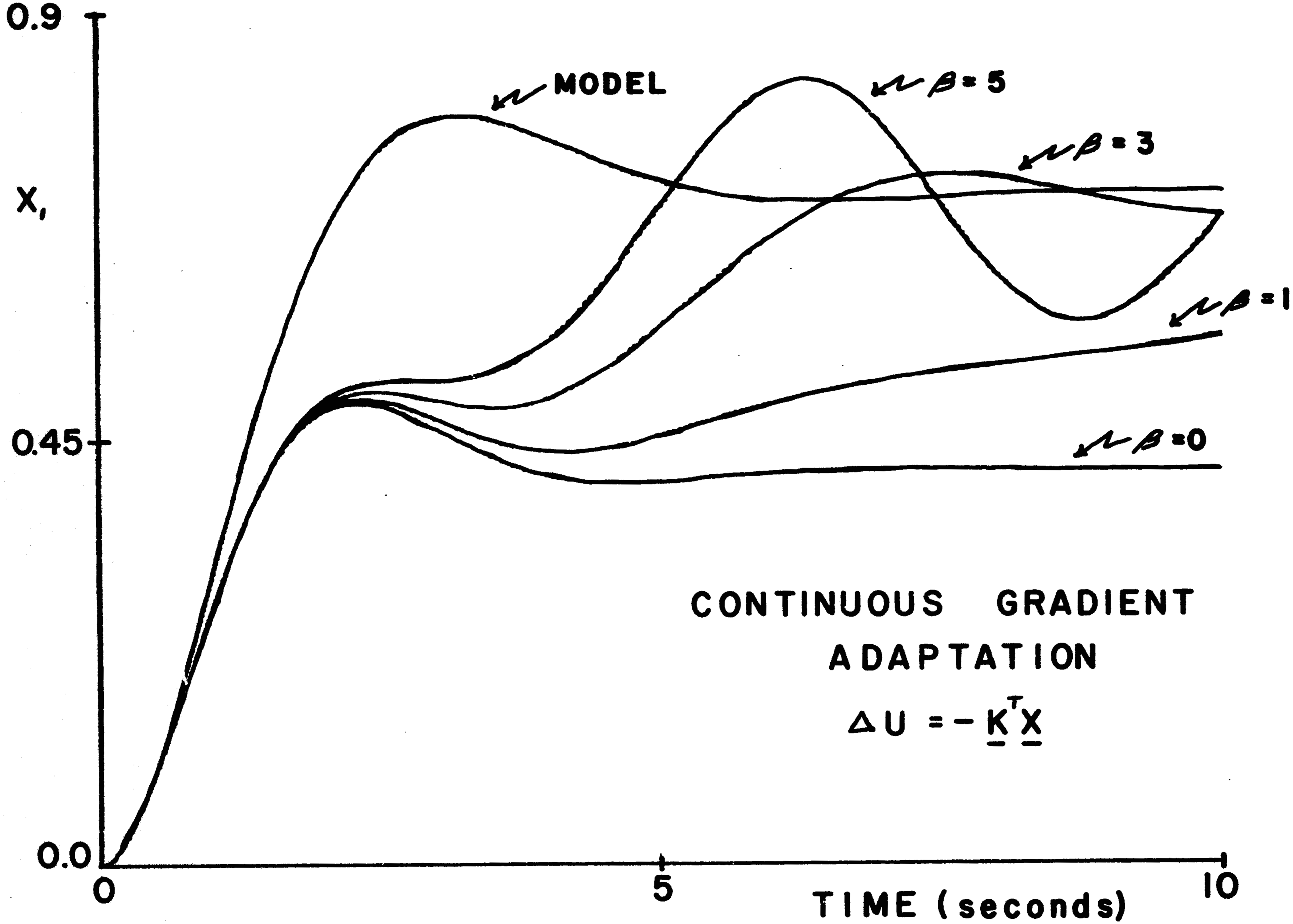

0.9
X₁
0.45
0.0
0
5
10
TIME (seconds)
MODEL
β = 5
β = 3
β = 1
β = 0
CONTINUOUS GRADIENT
ADAPTATION
ΔU = - K^T X

FIGURE 2-4

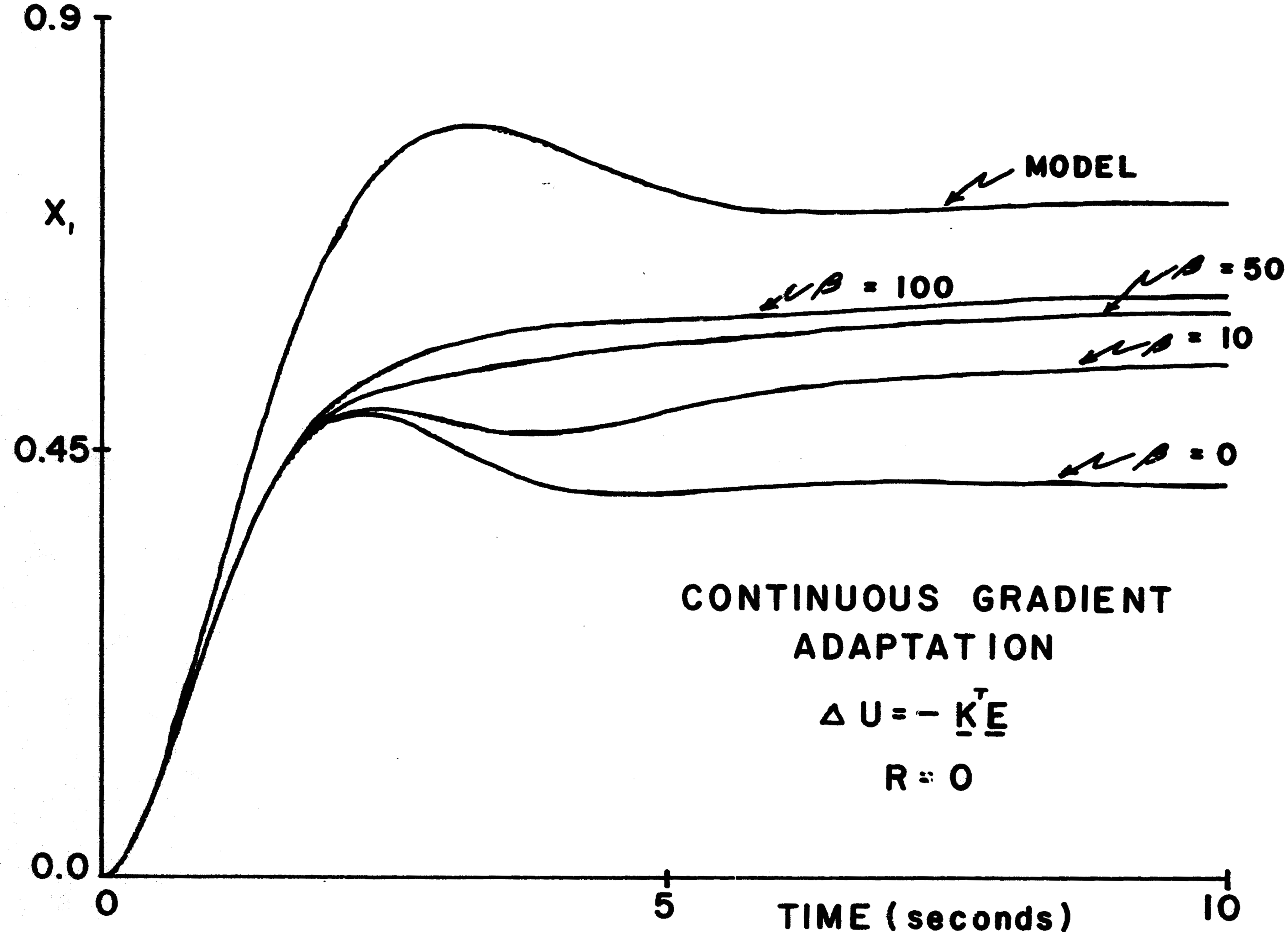

0.9
0.45
0.0
X_1
MODEL
νβ = 100
νβ = 50
νβ = 10
νβ = 0
CONTINUOUS GRADIENT
ADAPTATION
$\Delta U = -\underline{K}^T\underline{E}$
R = 0
0
5
10
TIME (seconds)

FIGURE 2-5

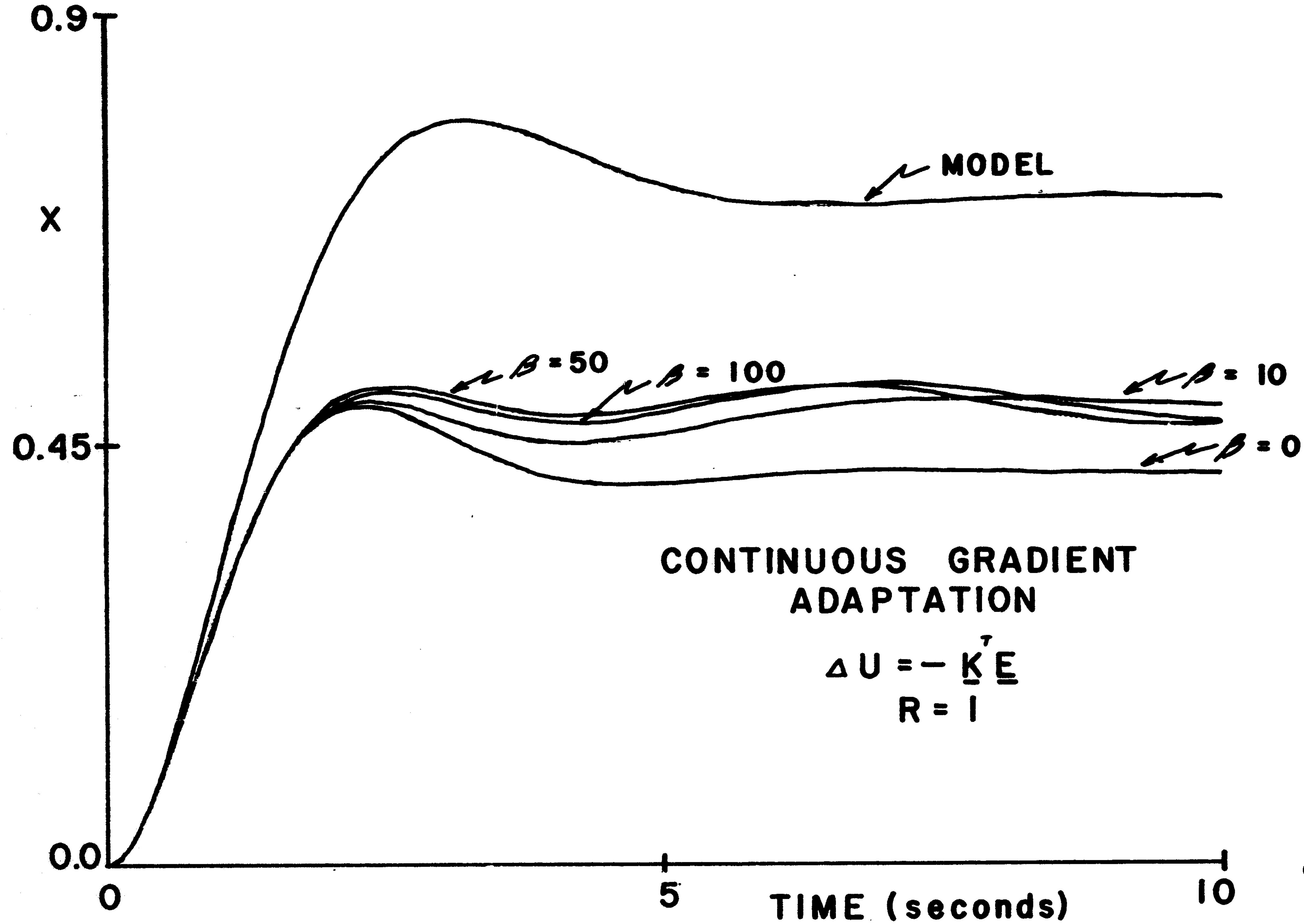

0.9
X
0.45
0.0
MODEL
β = 50
β = 100
β = 10
β = 0
CONTINUOUS GRADIENT ADAPTATION
ΔU = − K^T E
R = 1
0
5
10
TIME (seconds)

FIGURE 2-6

$$\beta_i = \frac{1}{2} \frac{\underline{G}_i^T \underline{G}_i}{\underline{G}_i^T \left[\int_{iT}^{(i+1)T} \left[\underline{z}_1, \underline{z}_2\right]^T \left[\underline{z}_1, \underline{z}_2\right] dt \right] \underline{G}_i} \qquad (2.7\text{-}15)$$

and the $\underline{z}_i$ are the outputs of filters characterized by

$$\dot{\underline{z}}_i = \begin{bmatrix} 0.0 & 1.0 \\ -1.414 & -1.352 \end{bmatrix} \underline{z}_i - \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix} x_{pi}; \; i = 1, 2 \qquad (2.7\text{-}16)$$

Simulation responses are shown for several values of T in Figure 2-7.

The integral-square error for these four cases is tabulated in Tables 2-1 and 2-2.

2.7.2 Example 2

The plant for this example is described by the differential equation

$$\dot{\underline{x}}_p = \begin{bmatrix} 0.4 & 1.6 \\ -2.1 & 4.4 \end{bmatrix} \underline{x}_p + \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix} u_p \qquad (2.7\text{-}17)$$

and its related model by

$$\dot{\underline{x}}_m = \begin{bmatrix} 0.4 & 1.1 \\ -1.6 & -1.9 \end{bmatrix} \underline{x}_m + \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix} u_m \qquad (2.7\text{-}18)$$

Once again the plant is stable but will exhibit a steady-state error for $u_p = u_m = 1.0$. However, unlike in Example 1, it is not possible to totally null this steady-state error with a feedback controller as is discussed in Appendix D. Error nulling can be achieved only if it is possible to

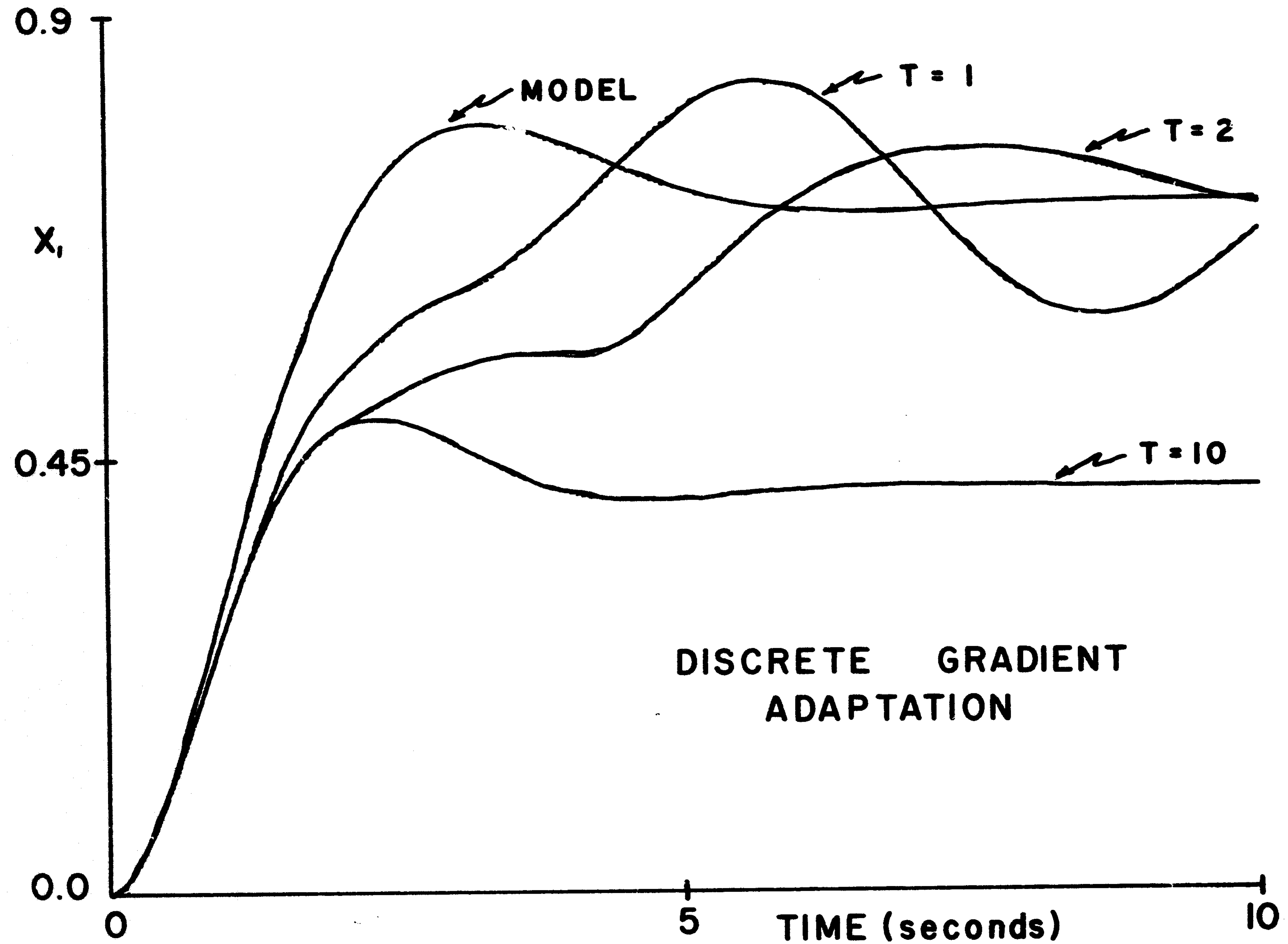

0.9
0.45
0.0
X_1
MODEL
T = 1
T = 2
T = 10
DISCRETE GRADIENT
ADAPTATION
0
5
10
TIME (seconds)

FIGURE 2-7

$$J_s = \int_0^{10} \underline{e}^T \underline{e} \, dt$$

A. Lyapunov

α	J_s
0.0	0.7926
1.0	0.5947
3.0	0.3241
5.0	0.2022
10.0	0.0964
20.0	0.0417

B. Discrete Gradient

T	J_s
10.0	0.7926
2.0	0.2514
1.0	0.1834

TABLE 2-1

$$J_s = \int_0^{10} \underline{e}^T \underline{e} \; dt$$

$$J_{\Delta u} = \int_0^{10} \Delta u^2 \; dt$$

A. Continuous Gradient - $\Delta u = - \underline{K}^T \underline{x}_p$

β	J_s
0.0	0.7926
1.0	0.5186
5.0	0.3423
10.0	0.7606
20.0	1.2050

β	J_s
2.75	0.3009
3.00	0.2932
3.25	0.2888
3.50	0.2878
3.75	0.2899

B. Continuous Gradient - $\Delta u = - \underline{K}^T \underline{e}$

R = 0		
β	J_s	$J_{\Delta u}$
0.0	0.7926	0.0
10.0	0.4571	0.4563
30.0	0.3259	0.9099
50.0	0.2721	1.0910
100.0	0.2090	1.3290

R = 1		
β	J_s	$J_{\Delta u}$
0.0	0.7926	0.0
10.0	0.5593	0.1794
50.0	0.5060	0.2340
100.0	0.5057	0.2282
200.0	0.5032	0.2274

TABLE 2-2

independently adjust the state matrix elements a_{p12}, a_{p21}, and a_{p22}.

2.7.2-a Lyapunov Adaptation

If the individual elements of the plant state matrix are independently accessible, the Lyapunov adaptation algorithm of Eq. 2.6-6 can be applied and gives

$$\begin{aligned} \dot{a}_{p12} &= \alpha_{12} (e_1 q_{11} + e_2 q_{12}) x_{p2} \\ \dot{a}_{p21} &= \alpha_{21} (e_1 q_{12} + e_2 q_{22}) x_{p1} \\ \dot{a}_{p22} &= \alpha_{22} (e_1 q_{12} + e_2 q_{22}) x_{p2} \end{aligned} \qquad (2.7\text{-}19)$$

where $Q = \begin{bmatrix} 2.39 & 0.91 \\ 0.91 & 0.79 \end{bmatrix}$

is the solution of

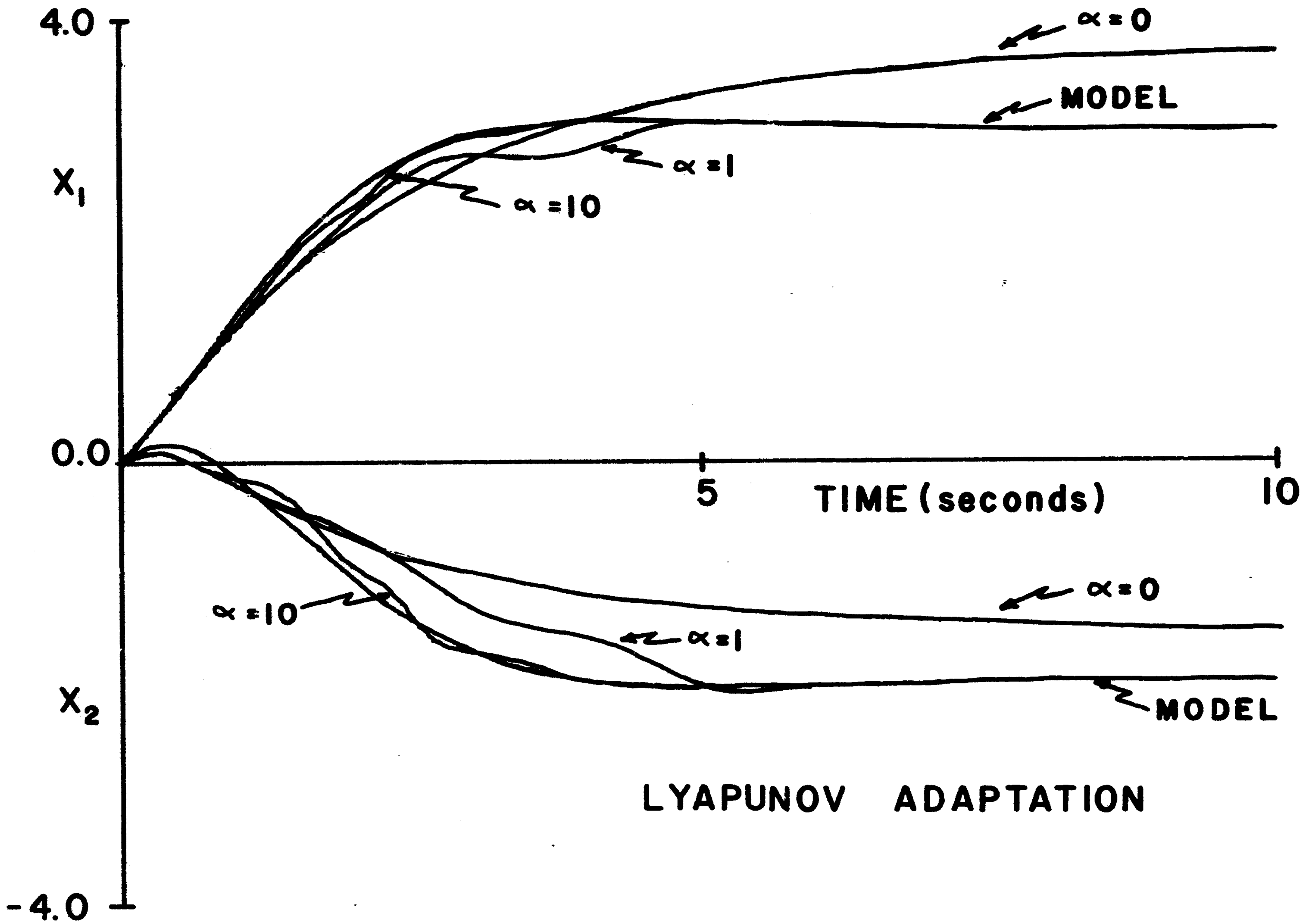
$$A_m^T Q + Q A_m = - I \qquad (2.7\text{-}20)$$

Simulation of plant responses arising from this adaptation rule are shown in Figure 2-8 for various values of $\alpha_{12} = \alpha_{21} = \alpha_{22} = \alpha$.

2.7.2-b Continuous Gradient Adaptation

In those situations in which the state matrix elements are not independently accessible it may be convenient to postulate a feedback structure for the perturbation control signal. For $\Delta u = - \underline{K}^T \underline{x}_p$, the on-line minimization of

$$J = \int \underline{e}^T \underline{e} \, dt \qquad (2.7\text{-}21)$$



4.0
α=0
MODEL
α=1
α=10
X_1
0.0
5
TIME (seconds)
10
α=0
α=10
α=1
X_2
MODEL
LYAPUNOV ADAPTATION
-4.0

FIGURE 2-8

gives the adaptation rule, Eq. 2.4-12,

$$\dot{K}_i = \beta_i \, \underline{e}^T \underline{z}_i \; ; \qquad i = 1, 2 \tag{2.7-22}$$

with $\underline{z}_1$ and $\underline{z}_2$ synthesized from filters characterized by

$$\dot{\underline{z}}_i = \begin{bmatrix} 0.4 & 1.1 \\ -1.6 & -1.9 \end{bmatrix} \underline{z}_i - \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix} x_{pi} \tag{2.7-23}$$

Simulation responses were obtained for two cases: 1) adaptation of K_1 only and 2) adaptation of both K_1 and K_2. Since the responses of these two cases were similar in nature, only those for case 1 are shown in Figure 2-9 for various values of β_1.

Postulating the perturbation control signal as $\Delta u = - \underline{K}^T \underline{e}$ and the performance index as

$$J = \int (\underline{e}^T \underline{e} + R \, \Delta u^2) dt \tag{2.7-24}$$

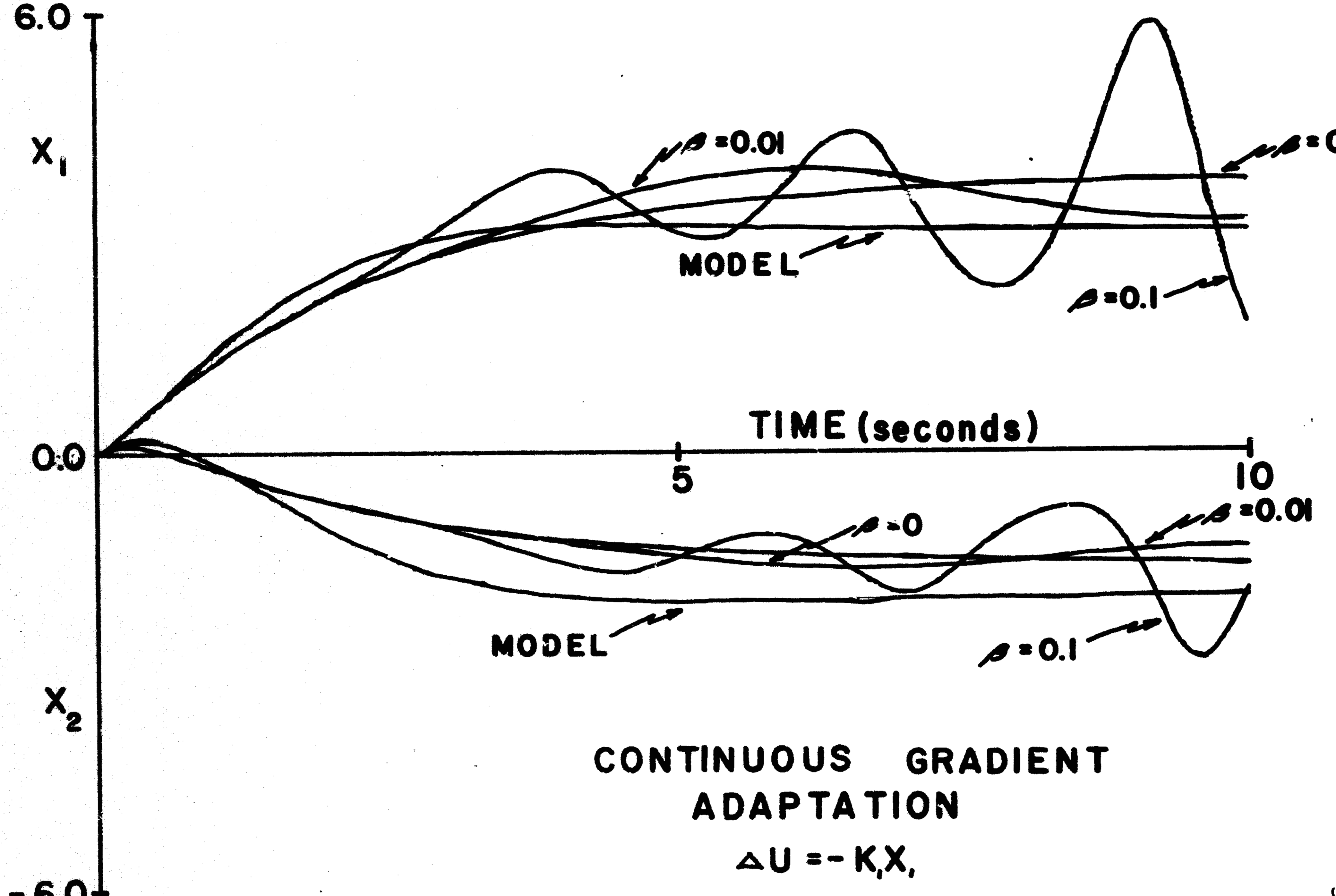
gives the adaptation rule, Eq. 2.4-16,

$$\dot{K}_i = \beta_i \left\{ \underline{e}^T (I + R \, \underline{K} \, \underline{K}^T) \, \underline{z}_i - R \, \underline{e}^T \underline{K} \, e_i \right\} \; ; \quad i = 1, 2 \tag{2.7-25}$$

with $\underline{z}_1$ and $\underline{z}_2$ derived from the appropriate filters. Simulation responses were obtained for $R = 0$ and $R = 1$ for several values of β. Since these responses were relatively insensitive to the value of R, only those for $R = 0$ are shown in Figure 2-10.

2.7.2-c Discrete Adaptation

The discrete adaptation rule of Eqs. 2.5-15 and 2.5-16 was applied to the performance index



CONTINUOUS GRADIENT ADAPTATION

$\Delta U = -K_1 X_1$

FIGURE 2-9

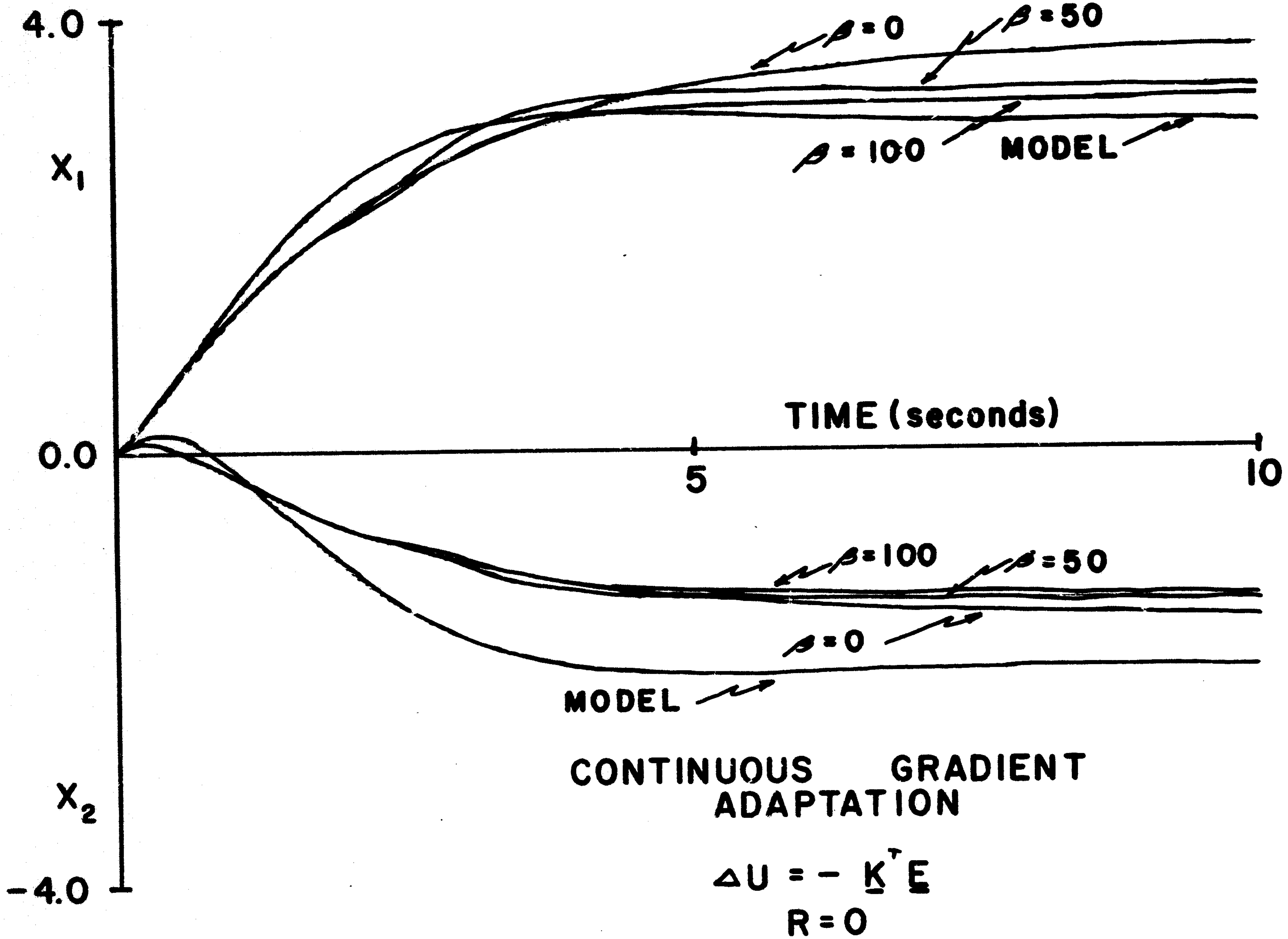

4.0
0.0
-4.0
X_1
X_2
TIME (seconds)
5
10
β=0
β=50
β=100
MODEL
β=100
β=50
β=0
MODEL
CONTINUOUS GRADIENT ADAPTATION
$\Delta U = -\underline{K}^T \underline{E}$
R=0

FIGURE 2-10

$$J_i = \int_{iT}^{(i+1)T} \underline{e}^T \underline{e} \, dt \qquad (2.7\text{-}26)$$

for the cases in which 1) only K_1 was adjusted and 2) both K_1 and K_2 were adjusted. Since the simulation responses for the two cases were once again similar in nature, only those for the second case are shown in Fig. 2-11 for various values of T.

2.8 Convergence Rate, Stability, and Error Nulling

The interaction of the rate of convergence and stability plays an important role in the design of model reference adaptive control systems. Although only linear plants have been considered in this study, the addition of an adaptive control loop results in a non-linear system. In what follows, a model reference control system will be considered stable if the plant output converges to that value which satisfies the design criterion.

The convergence rate of the Lyapunov adaptation rule is seen from the two examples of the previous section to be dependent on the value of the α_{ij} terms while similar dependence has been found on the β_{ij} terms. Since this adaptation rule is derived from a stability consideration, system stability is guaranteed as long as the necessary assumptions remain valid. When this is not the case, further investigation, in all probability of a simulation nature, may be necessary to determine the range of convergence factors for which stability can be expected. Figures 2-3 and 2-8 illustrate the degree of convergence that can be achieved by this adaptation rule.

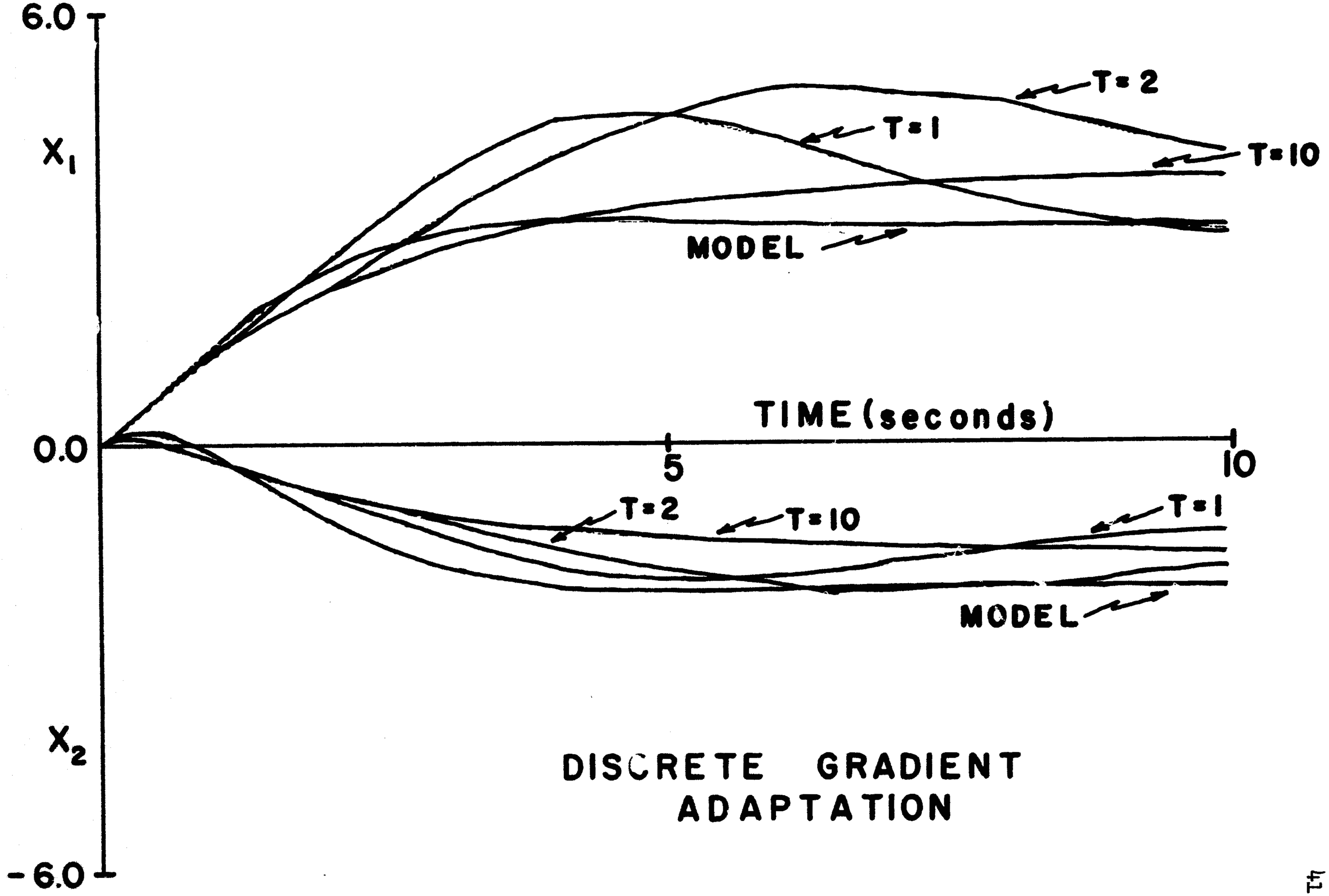

6.0
X_1
T = 2
T = 1
T = 10
MODEL
TIME (seconds)
0.0
5
10
T = 2
T = 10
T = 1
MODEL
X_2
DISCRETE GRADIENT
ADAPTATION
-6.0

FIGURE 2-11

The effect of the convergence factor or step size on the gradient adaptation rules can best be illustrated by referring to Figure 2-12. Figure 2-12a shows a typical response pattern for a conservatively low-gain system in which many steps are required but the optimum is finally achieved. Attempting to increase the rate of convergence by increasing the gain can produce the response pattern of Figure 2-12b in which instability is a definite possibility. Figure 2-12c shows a compromise between low-gain and high-gain operation and illustrates the trade-off between the rate of convergence and stability. While Figure 2-12 is based on discrete adaptation, a similar effect can be expected for continuous adaptation. In Figure 2-4 and Table 2-2 it is seen that for $\beta = 1$ the plant trajectory of Example 1 is slowly converging to the model trajectory, for $\beta = 3$ fair convergence has been achieved, and for $\beta > 5$ the response is diverging from the optimum. Thus it is seen that the value of β can be a critical factor in the design of model reference adaptive control systems by the continuous gradient rule of Eq. 2.4-12.

One characteristic of the adaptation rule of Eq. 2.4-16 is that plant-model error nullity is never possible as the perturbation control signal is a function of this error. However, this should not be too alarming since it is not, in general, possible to null this error for forced linear systems as is shown in Appendix D. One class of systems for which this error can be nulled is that for which the plant and model represent the scalar n^{th} order linear differential equation in vector notation. Example 1 is a member of this class and it is easily seen that the plant model error can be nulled for $\Delta u = x_{p1} + 0.148\, x_{p2}$.

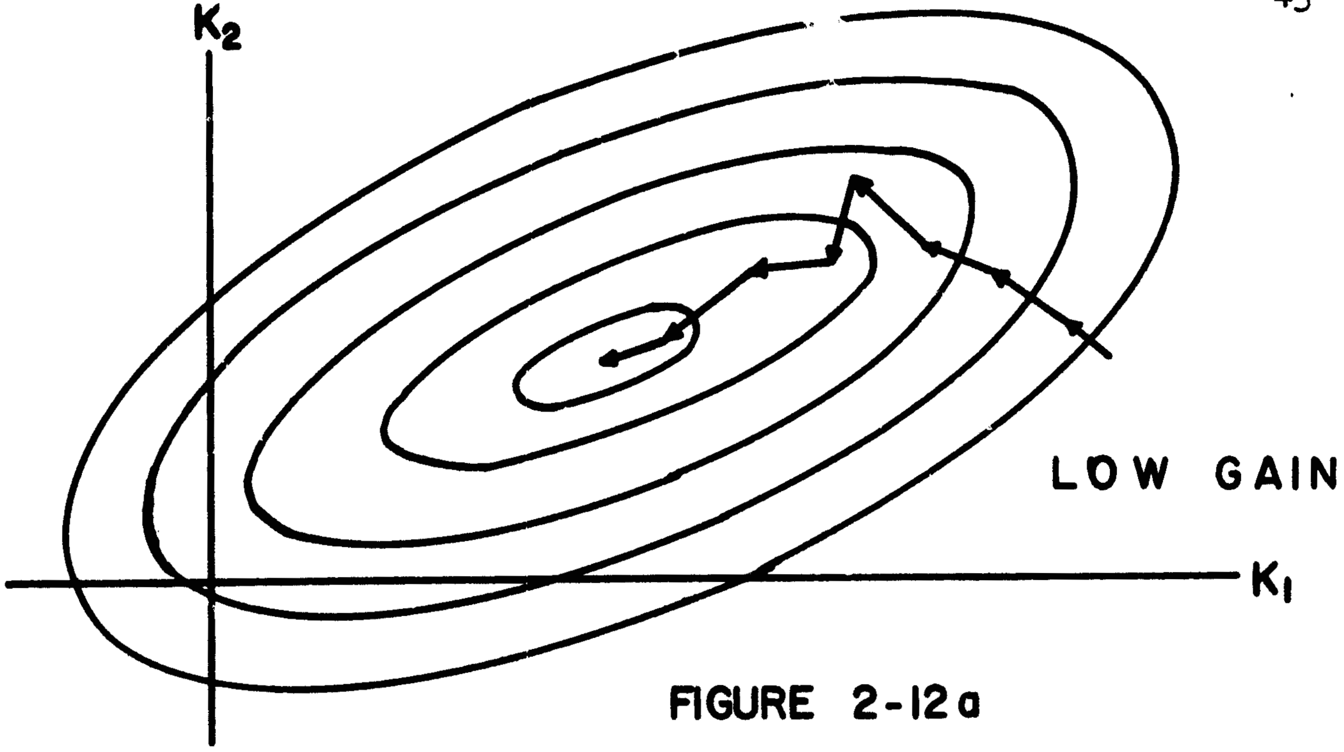


FIGURE 2-12a

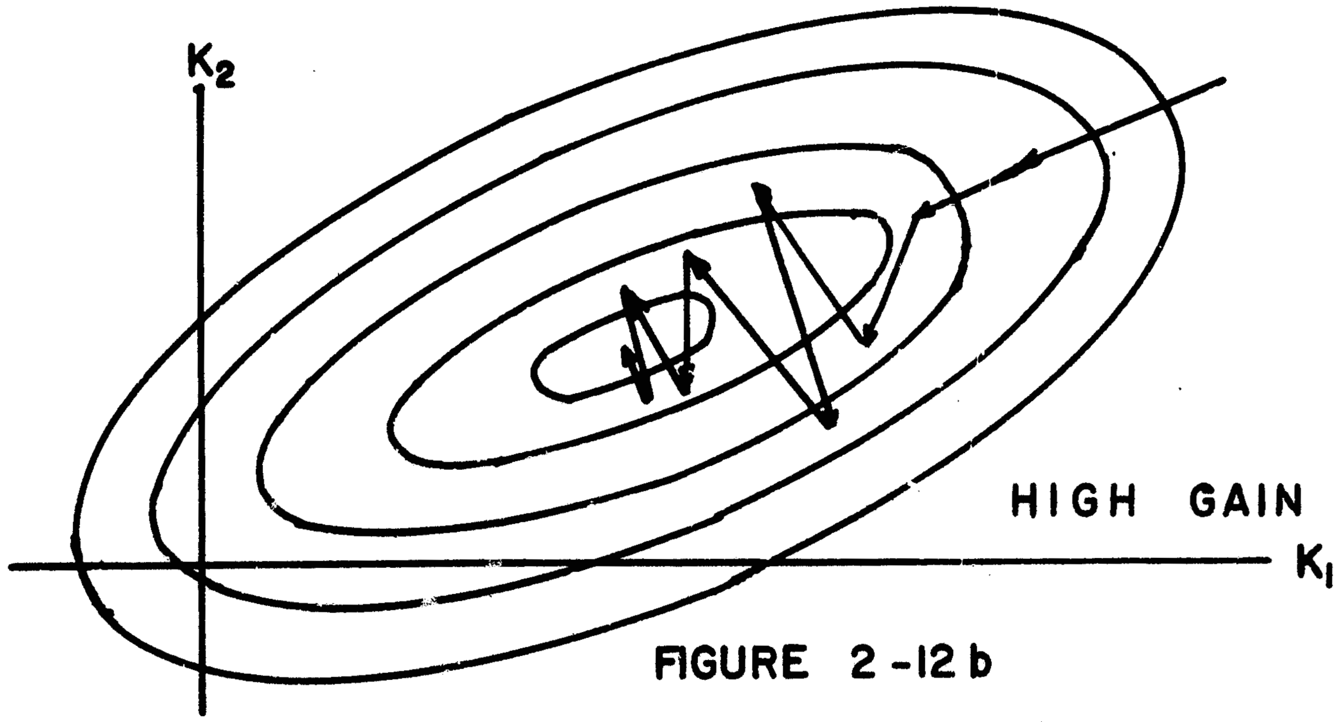


FIGURE 2-12b

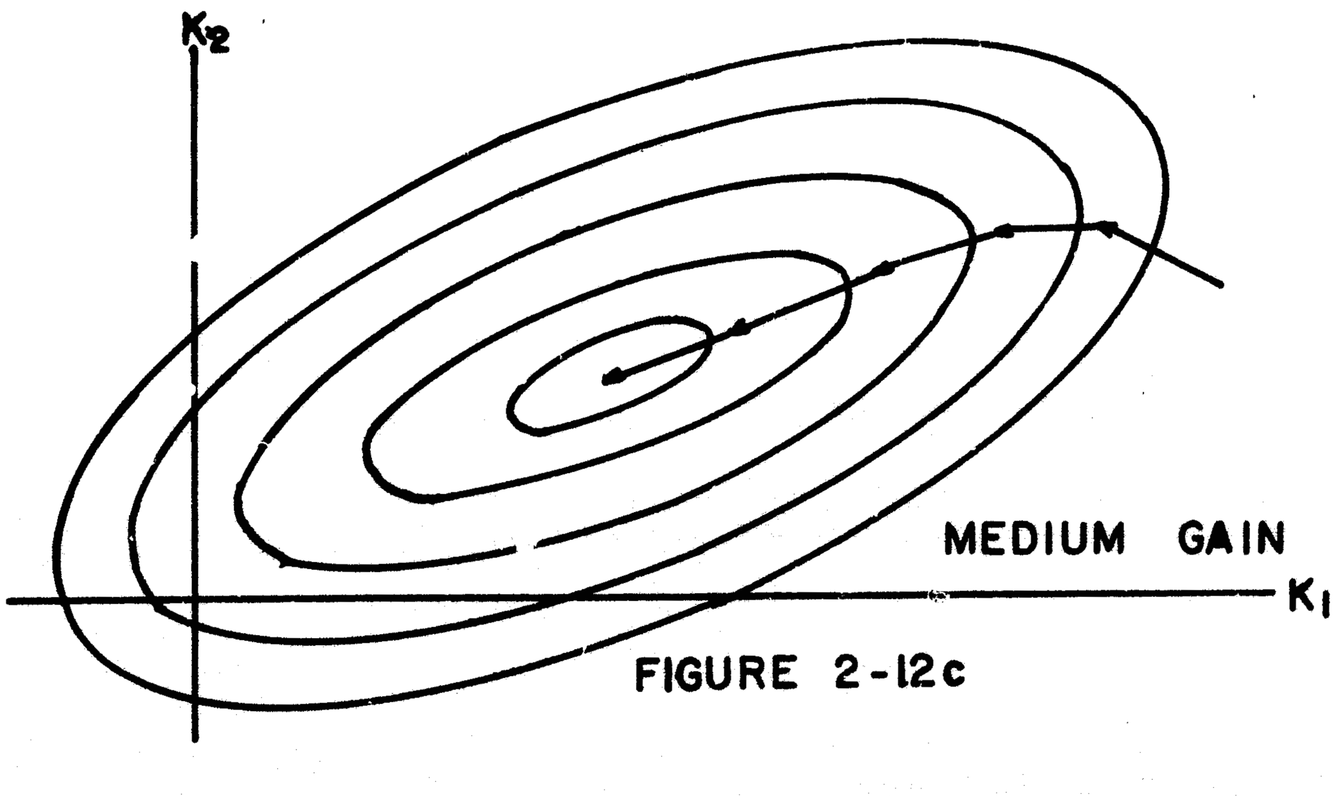


FIGURE 2-12c

Figures 2-4 and 2-7 indicate that nullity is approached for appropriate values of β and T while Figures 2-5 and 2-6 illustrate the residual error for $\Delta u = -\underline{K}^T\underline{e}$. The trade-off between state error and perturbation control effort for the latter case is also illustrated in Figures 2-5 and 2-6 and in Table 2-2b. It is seen that increasing R from zero to one in the performance index of Equation 2.7-9 reduces the amount of perturbation control effort but at the same time the state error increases. Thus it appears that the inclusion of the perturbation control weighting term in the performance index serves the purpose which was intended.

It is not possible to null the plant-model error for Example 2 if a feedback controller is postulated. In fact, the steady-state ratio $\frac{x_{p1}}{x_{p2}} = -2.4$ regardless of the perturbation control Δu . From this relation it is determined that the values of $x_{p1} = 3.26$ and $x_{p2} = -1.36$ minimize the index $J = \int \underline{e}^T\underline{e}\ dt$ in the steady-state. The values of x_{p1} and x_{p2} achieved by gradient adaptation with $\Delta u = -\underline{K}^T\underline{e}$ are very close to these optimum values. Unlike the foregoing, the continuous gradient rule for $\Delta u = -\underline{K}^T\underline{x}_p$ and the discrete gradient rule give oscillatory results. One possible reason for this was originally thought to be the decision to adjust two gains and the resulting non-uniqueness of a value of $\underline{k}$ to minimize the criterion. However, later simulations for the adjustment of a single gain, i.e., $\Delta u = -k_1 x_{p1}$, showed little improvement. Hence, it is felt that the adaptive rules for $\Delta u = -\underline{K}^T\underline{x}_p$ are more sensitive to the convergence factor β than are similar rules for $\Delta u = -\underline{K}^T\underline{e}$

2.9 Conclusions

Several adaptation algorithms have been developed and applied to simple examples in this chapter. From these and other examples the following conclusions have been drawn:

1) The Lyapunov adaptation rule should receive prime consideration for use with those systems in which the necessary state and control matrix elements are independently accessible and in which all the states are available since it is the simplest to implement and stability is guaranteed.

2) For those systems in which adaptation is necessary but for which nulling the plant-model error either is not of prime importance or is not possible, the continuous gradient adaptation rule for $\Delta \underline{u} = - K \underline{e}$ is recommended, in spite of the additional implementation complexity, as it has been found to be less sensitive to the value of the convergence factor β .

3) In those situations in which the plant is known to be stable and in which continuous monitoring of the process is possible but continuous adaptation is not necessary, the discrete adaptation rule, despite its complexity, merits consideration.

4) Regardless of which adaptation rule is finally chosen for a particular situation, the importance of a detailed simulation study in the design procedure cannot be overly stressed.

CHAPTER III

CASE STUDY

3.1 Introduction

The pitch control of a large flexible launch vehicle of the Saturn V class has been chosen to demonstrate the application of the model reference design philosophy to a system of current engineering significance. A linear perturbation model of the Saturn V is developed and a nominal control law is specified. An adaptive control loop based on the continuous gradient method is designed to accommodate for any degradation in performance arising from variations in the system parameters. The overall system is tested by a digital computer simulation of the time-varying model. This model is excited by a worst case design wind which is so constructed as to excite any instabilities that are inherent in the system.

3.2 Overview of the Problem[22]

As launch vehicles become progressively larger and more complex it likewise becomes progressively more difficult to develop precise mathematical models of these vehicles. With the current length to diameter ratio of better than ten to one, a launch vehicle of the Saturn V class cannot be considered rigid but must be treated as a free-free beam with a controllable torque applied at one end. This control torque is exerted by gimballing the four outer engines of the booster vehicle. As a consequence of this engine gimballing, the elastic bending modes of the flexible vehicle are excited. If these bending modes are not controlled the structural integrity of the vehicle may be exceeded and the vehicle destroyed.

Until now large "shake tables" have been constructed to dynamically test the vehicles. Such testing has produced bending profiles from which such characteristics as mode shapes and mode natural frequencies may be determined. The tremendous size of the Saturn V-Apollo configuration shown in Figure 3.1 makes this procedure just marginally possible and the next generation of launch vehicles will probably render it useless. Also the current trend is to employ the same basic launch vehicle for the boost phase of several different missions and it is not feasible to shake test every configuration. Thus the bending characteristics, most notably the natural frequency of each mode, may not be known accurately enough for successful control of the vehicle. This is one reason for considering a model reference adaptive control loop.

The control of a launch vehicle is further complicated by the inherent aerodynamic instability of the rigid body mode. This arises from the center of pressure being forward of the center of gravity, a condition that is encountered for all but a few seconds of the flight as is shown by Figure 3.2. The aerodynamic forces tend to rotate the vehicle and thus continuous gimballing of the engines is necessary to keep the vehicle in nominal orientation.

One further effect that is not considered in this development is fuel sloshing which occurs as fuel is expended from the tanks. For completeness, Figure 3.3 shows the frequency spectrum of the Saturn V-Apollo configuration during the boost phase. The spread in the frequencies of the various modes results from the time-varying nature of the problem.

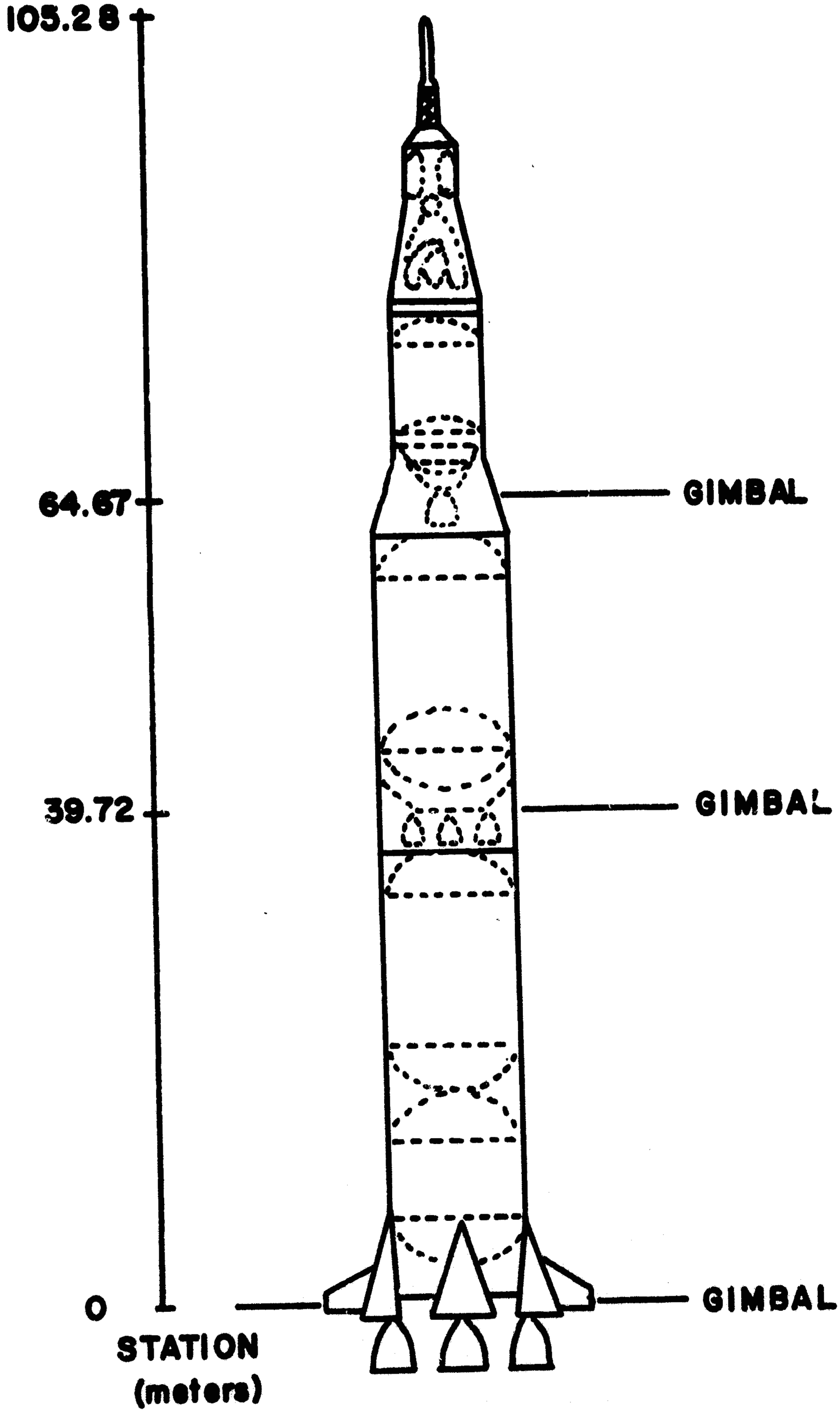


FIG. 3-1 VEHICAL CONFIGURATION

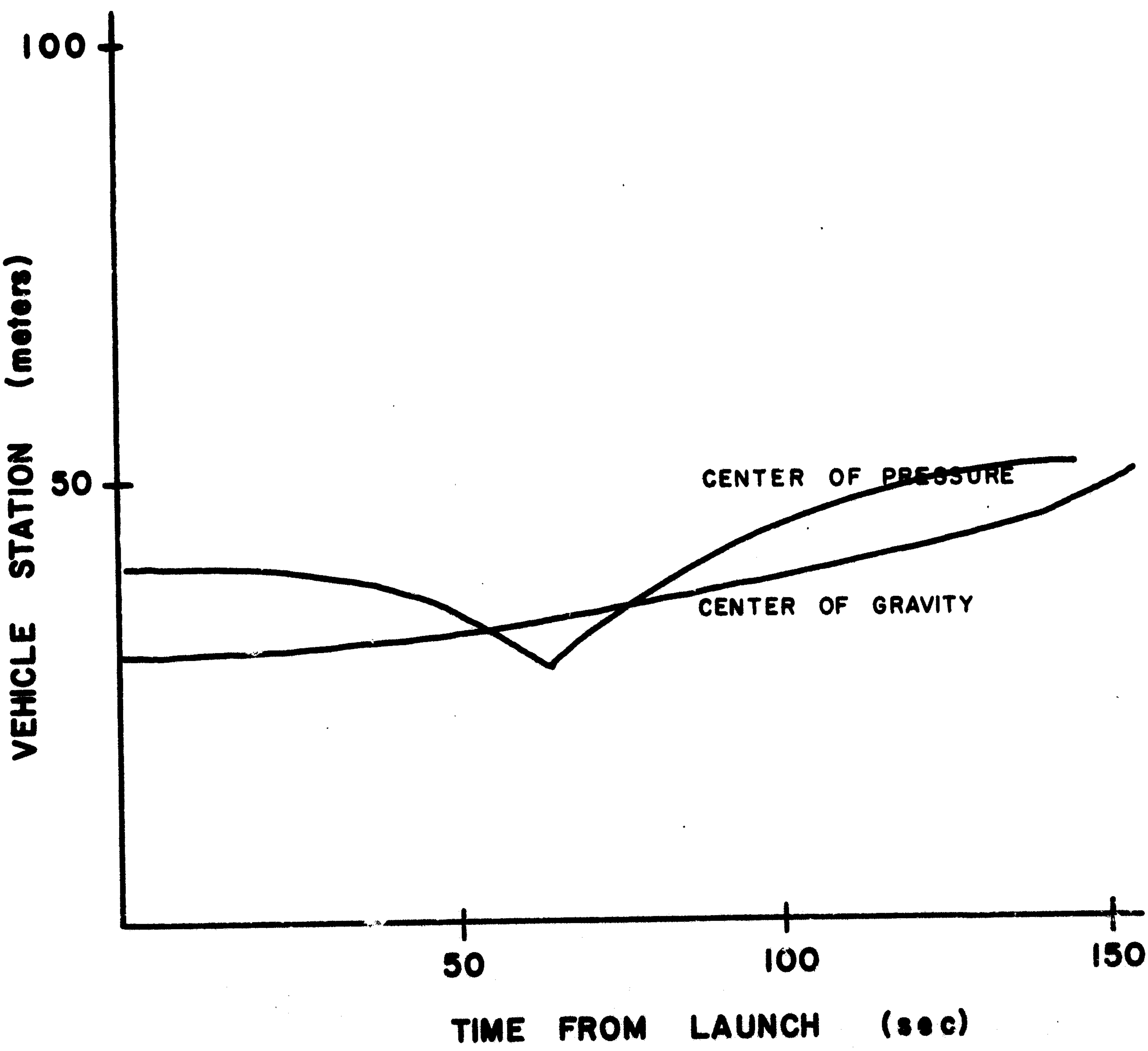


FIG. 3-2 CENTERS OF PRESSURE AND GRAVITY

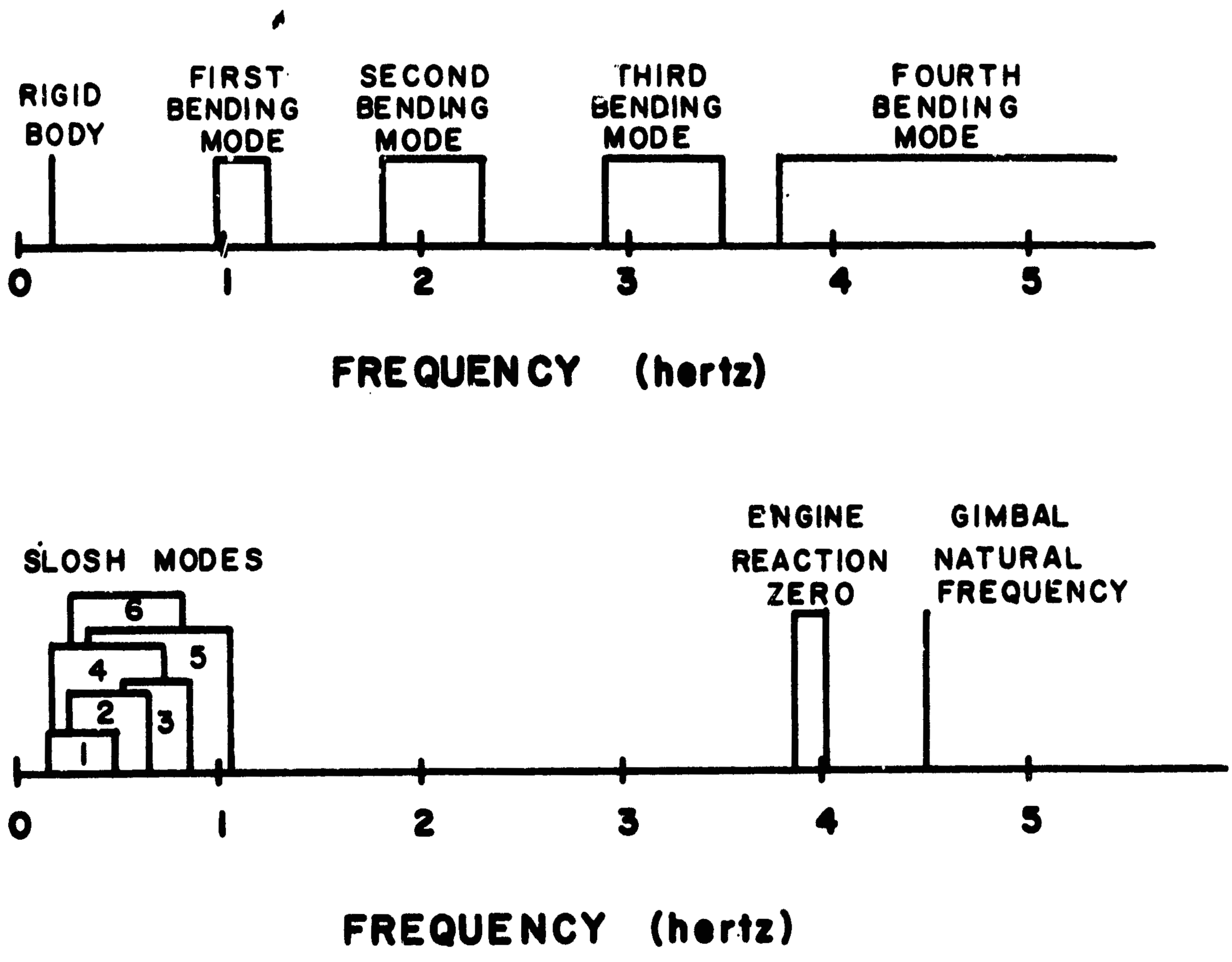


FIG. 3-3 BOOSTER FREQUENCY SPECTRUM

In summary, the control problem under consideration consists of the control of a time-varying aerodynamically unstable vehicle in which the measured pitch and pitch rates are the superposition of rigid body motion and elastic bending motion, the latter often characterized by inaccurate parameter values.

3.3 Equations of Motion [22, 23]

The first step in the development of a model reference adaptive control system for the Saturn V is the derivation of the linearized perturbation equations for the vehicle. First the rigid body equations are derived for the pitch plane under the assumption of a flat earth.

The orientation of the missile in the pitch plane is shown in Figure 3-4. Three sets of axes are necessary to describe the motion of the vehicle in this plane. The first coordinate system has its origin at the launch point with its X and Y axes aligned with the local horizontal and local vertical respectively. This is the inertial coordinate system. The X_n - Y_n coordinate system is defined relative to the reference trajectory as follows: the X_n axis is directed tangential to the nominal trajectory and the Y_n axis is perpendicular to it in the pitch plane. The degree of freedom along the X_n axis is eliminated by allowing the coordinate system to accelerate with the vehicle center of gravity in the X_n direction. The third set of axes moves with the origin at the vehicle center of gravity. In this body-fixed coordinate system the x axis lies along the center line of the vehicle with the y axis perpendicular to it.

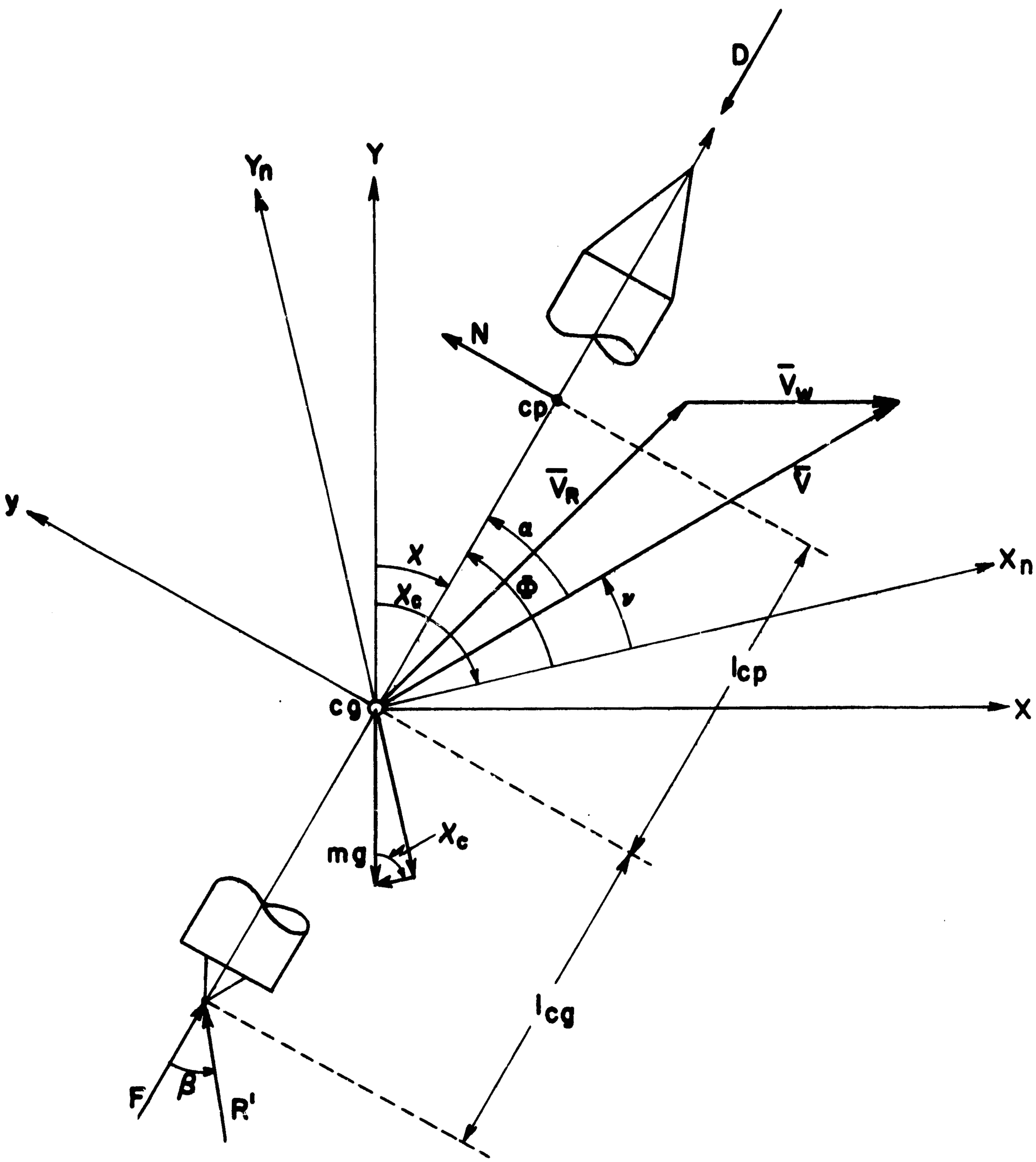


FIG. 3-4 FREE BODY DIAGRAM

The forces acting on the vehicle can be decomposed in the X_n and Y_n directions as follows:

$$F_{X_n} = (F + R' \cos\beta - D)\cos\varphi - N\sin\varphi$$
$$- R'\sin\beta\sin\varphi - m\,g\cos\chi_c \qquad (3.3\text{-}1)$$

$$F_{Y_n} = (F + R'\cos\beta - D)\sin\varphi + N\cos\varphi$$
$$+ R'\sin\beta\cos\varphi - m\,g\sin\chi_c \qquad (3.3\text{-}2)$$

Similarly the torques can be summed about the center of gravity to give

$$I\ddot{\varphi} = -N\,\ell_{cp} - R'\,\ell_{cg}\sin\beta \qquad (3.3\text{-}3)$$

with $\ell_{cp} = x_{cg} - x_{cp}$ and $\ell_{cg} = x_{cg} - x_\beta$. The angle χ_c is the pitch command angle and is determined by the mission profile.

The velocity of the vehicle with respect to the $X_n - Y_n$ coordinate system is

$$\bar{V} = V\cos\gamma\,\bar{i} + V\sin\gamma\,\bar{j} \qquad (3.3\text{-}4)$$

from which the acceleration of the vehicle is

$$\bar{a} = \dot{V}\cos\gamma\,\bar{i} + V\cos\gamma\,\frac{d\bar{i}}{dt} - V\sin\gamma\,\dot{\gamma}\,\bar{i}$$
$$+ \dot{V}\sin\gamma\,\bar{j} + V\sin\gamma\,\frac{d\bar{j}}{dt} + V\cos\gamma\,\dot{\gamma}\,\bar{j} \qquad (3.3\text{-}5)$$

However, $\frac{d\bar{i}}{dt} = \bar{\omega}\times\bar{i}$, $\frac{d\bar{j}}{dt} = \bar{\omega}\times\bar{j}$, and $\bar{\omega} = -\dot{\chi}_c\,\bar{k}$. With these expressions, Equation 3.3-5 reduces to

$$\bar{a} = \left[\dot{V}\cos\gamma - V\sin\gamma\,\dot{\gamma} + V\sin\theta\,\dot{\chi}_c\right]\bar{i}$$
$$+ \left[\dot{V}\sin\gamma + V\cos\gamma\,\dot{\gamma} - V\cos\theta\,\dot{\chi}_c\right]\bar{j} \qquad (3.3\text{-}6)$$

Noting that

$$\ddot{X}_n = \frac{d}{dt}(\dot{X}_n) = \frac{d}{dt}(V\cos\gamma) = \dot{V}\cos\gamma - V\sin\gamma\,\dot{\gamma} \qquad (3.3\text{-}7)$$

and

$$\ddot{Y}_n = \frac{d}{dt}(\dot{Y}_n) = \frac{d}{dt}(V \sin \gamma) = \dot{V} \sin \gamma + V \cos \gamma \dot{\gamma} \quad ,$$

Equation 3.3-6 further reduces to

$$\bar{a} = \left[\ddot{X}_n + V \sin \gamma \dot{\psi}_c \right] \bar{i} + \left[\ddot{Y}_n - V \cos \gamma \dot{\psi}_c \right] \bar{j} \qquad (3.3\text{-}8)$$

From Newton's Law, $\bar{F} = m \bar{a}$, the final equations of motion of the vehicle in terms of the X - Y coordinate system reduces to

$$m \left[\ddot{X}_n + V \sin \gamma \dot{\psi}_c \right] = (F + R' \cos \beta - D) \cos \phi$$
$$- N \sin \phi - R' \sin \beta \sin \phi - m g \cos \psi_c \qquad (3.3\text{-}9)$$

and

$$m \left[\ddot{Y}_n - V \cos \gamma \dot{\psi}_c \right] = (F + R' \cos \beta - D) \sin \phi$$
$$+ N \cos \phi + R' \sin \beta \cos \phi - m g \sin \psi_c \qquad (3.3\text{-}10)$$

These equations can be linearized by making the usual small angle approximations that $\sin \psi \approx x$ and $\cos \psi \approx 1$. Hence

$$\ddot{X}_n = \frac{F + R' - D}{m} - \frac{N}{m} \phi - g \cos \psi_c - V \theta \dot{\psi}_c \qquad (3.3\text{-}11)$$

$$\ddot{Y}_n = \left(\frac{F + R' - D}{m}\right) \phi + \frac{N}{m} + \frac{R'}{m} \beta + V \dot{\psi}_c - g \sin \psi_c \qquad (3.3\text{-}12)$$

Since the degree of freedom along the X_n axis has been eliminated, Equation 3.3-11 need not be considered further.

Launch vehicles are usually programmed to fly a "gravity turn" trajectory which is characterized by

$$\dot{\psi}_c = \frac{g \sin \psi_c}{V} \qquad (3.3\text{-}13)$$

in which case the last two terms of Equation 3.3-12 cancel.

The aerodynamic force, N, of Equation 3.3-12 is proportional to the angle of attack and is given by

$$N = N' \alpha \qquad (3.3\text{-}14)$$

Substituting Equation 3.3-14 into Equation 3.3-12, allowing for a "gravity turn" trajectory and letting $T = F + R'$ gives

$$\ddot{Y}_n = \left(\frac{T-D}{m}\right) \varphi + \frac{N'}{m} \alpha + \frac{R'}{m} \beta \qquad (3.3\text{-}15)$$

Making similar small angle approximations on Equation 3.3-3 gives the pitch angle equation

$$\ddot{\Phi} = -\left(\frac{N' \ell_{cp}}{I}\right) \alpha - \left(\frac{R' \ell_{cg}}{I}\right) \beta \qquad (3.3\text{-}16)$$

One final equation relating pitch angle and angle of attack may be obtained from Figure 3.3-4 by again making small angle approximations. This relation is

$$\alpha - \alpha_w = \varphi - \frac{\dot{Y}}{V} \qquad (3.3\text{-}17)$$

Equations 3.3-15, 3.3-16 and 3.3-17 completely describe the linearized rigid body motion of the Saturn V about its nominal trajectory.

The form of the equations describing the elastic bending effects is that of a linear oscillator driven by a forcing function proportional to the gimbal angle β . These equations are written in terms of normalized coordinates such that the deformation at any station along the vehicle is given by the value of the normal coordinate multiplied by the mode shape coefficient for that station. This equation is

$$\ddot{\eta}_i + 2 \zeta_i \omega_i \dot{\eta}_i + \omega_i^2 \eta_i = \frac{R' Y_i(X_\beta)}{m_i} \beta \qquad (3.3\text{-}18)$$

3.4 Wind Disturbance

The only external disturbance acting on the above model of the booster in flight is wind. The wind alters the apparent angle of attack by an amount α_w. This can be related to the vehicle velocity and the wind velocity by examining Figure 3-5 which is a detailed version of Figure 3-4 for $\alpha = \phi = 0$. Considering only horizontal winds, it is seen that

$$\alpha_w = \frac{V_w \cos \chi_c}{V - V_w \sin \chi_c} \qquad (3.4\text{-}1)$$

where V_w is the wind velocity. Using nominal values of V and χ_c a wind angle of attack profile can be constructed from the synthetic design wind speed profile shown in Figure 3-6. This design wind has wind magnitudes that exceed those of 95% of the measured winds in the May-November reporting period at Cape Kennedy, Florida.[24] In addition a gust was added in the region of expected maximum dynamic pressure. This gust will tend to excite any unstable mode of the vehicle. The resulting wind induced angle of attack is shown in Figure 3-7 and is the external disturbance that is used on all time-varying simulations of the booster.

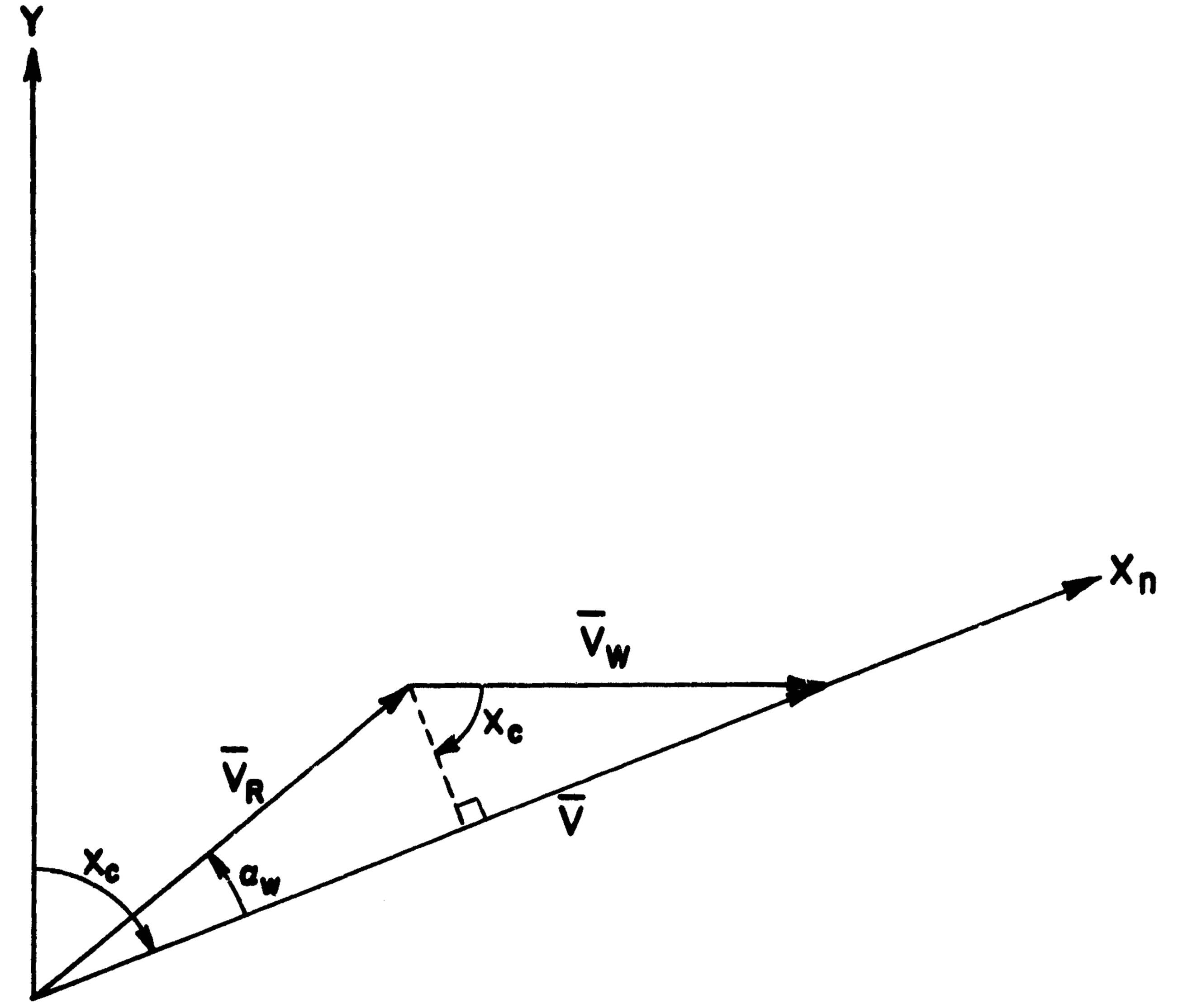


FIG. 3-5 WIND ANGLE OF ATTACK

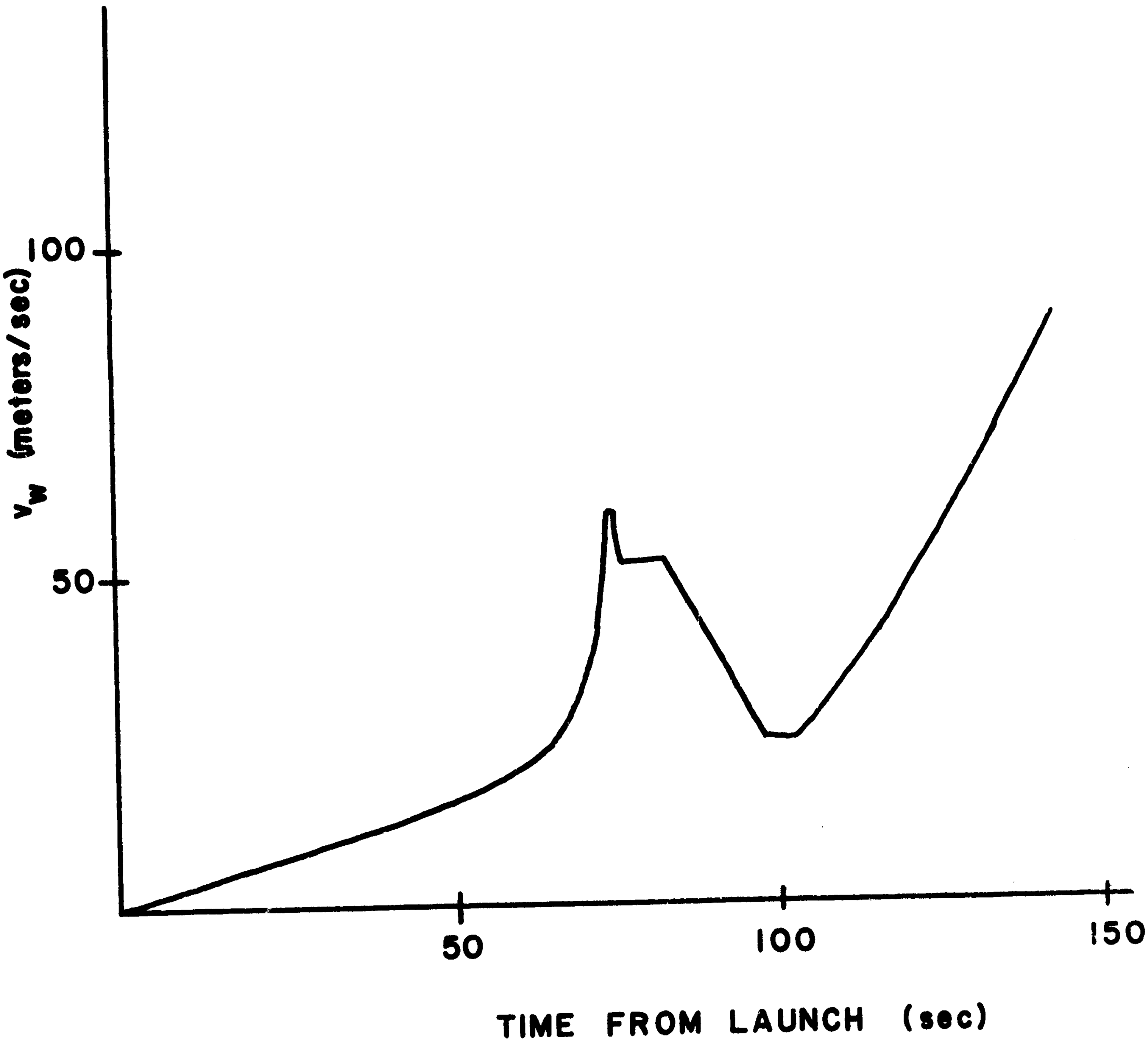


FIG. 3-6 SYNTHETIC WIND

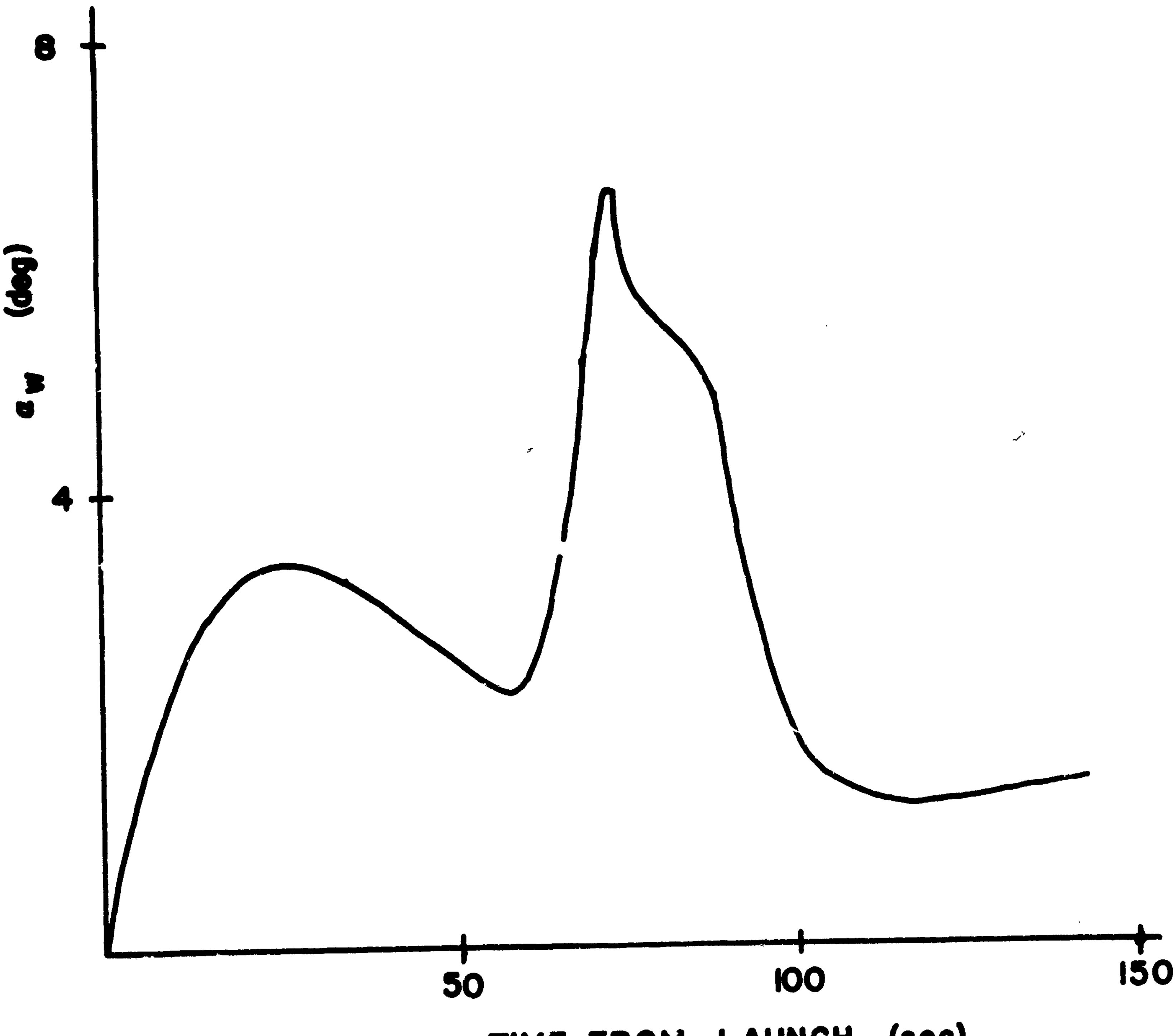


FIG. 3—7 WIND ANGLE OF ATTACK

3.5 The Control Law

The design of a linear control law for a flexible launch vehicle is complicated by the fact that position and rate gyros measure local pitch and pitch rate which are a superposition of rigid body motion and elastic bending motion. The outputs of these gyros can be represented by

$$\varphi_D = \varphi + \sum_i Y_i'(x_D)\,\eta_i \qquad (3.5\text{-}1)$$

$$\dot{\varphi}_{RG} = \dot{\varphi} + \sum_i Y_i'(X_{RG})\,\dot{\eta}_i \qquad (3.5\text{-}2)$$

in which $Y_i'(X_D)$ and $Y_i'(X_{RG})$ represent the mode shapes of the respective stations.

While about a decade of frequency separate the rigid body mode and the first bending mode, several of the slosh modes are centered around the rigid body frequency. Although consideration of the slosh modes is beyond the scope of the present work, Rillings[18] has suggested that these modes be accounted for by restricting the cutoff frequency of any series compensating filter to be above one hertz. With this restriction it is felt that any additional phase shift would not affect the stability of the slosh modes.

The control law for the work that follows will be analogous to that determined by Rillings in his analog sensitivity design treatment of the booster problem. This control law consists of a constant gain feedback controller and a series compensating filter. The filter is described by

$$\frac{\beta(s)}{\beta_c(s)} = \frac{50}{s^2 + 10\, s + 50} \qquad (3.5\text{-}3)$$

in which β is the gimbal angle and β_c the control input which is synthesized from the measured signals or

$$\beta_c = 0.8\, \varphi_D + 0.8\, \dot{\varphi}_{RG} \qquad (3.5\text{-}4)$$

Rillings found that this control law resulted in "optimal" performance of the booster with one bending mode for nominal parameter values. However, when the natural frequency of the bending mode was decreased to 80% of nominal, the booster became unstable. The model reference technique will be employed in an attempt to alleviate this condition of instability that arises with variation in the natural frequency of the bending mode.

3.6 State Equations of the Booster

Equations 3.3-15 - 3.3-23 completely describe the linearized perturbation model of the Saturn booster. Equations 3.3-15 and 3.3-17 can be combined to give

$$\dot{\alpha} = -\left(\frac{T-D}{m} - \frac{\dot{V}}{V}\right)\varphi + \dot{\varphi} - \left(\frac{N'}{m} + \frac{\dot{V}}{V}\right)\alpha$$

$$- \frac{R'}{m}\,\beta + \left(\frac{\dot{V}}{V}\,\alpha_\omega + \dot{\alpha}_\omega\right) \qquad (3.6\text{-}1)$$

These equations can be represented in state variable form by defining the state equation

$$\underline{\dot{x}} = A\,\underline{x} + \underline{b}\,\beta_c + \underline{u} \qquad (3.6\text{-}2)$$

and the output equation

$$\underline{y} = C\,\underline{x} \qquad (3.6\text{-}3)$$

where $\underline{x}$ is a state vector, β_c the scalar control signal, A the vehicle state matrix, $\underline{b}$ the controller vector, $\underline{u}$ the disturbance vector, $\underline{y}$ the output vector, and C the output measurement matrix. These are given by

$$\underline{x} = \left[\varphi, \dot{\varphi}, \alpha, \eta_1, \dot{\eta}_1, \beta, \dot{\beta}\right]^T \tag{3.6-4}$$

$$\underline{b} = \left[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -50\right]^T \tag{3.6-5}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{N'\ell_{CG}}{I} & 0 & 0 & -\frac{R'\ell_{CG}}{I} & 0 \\ -\left(\frac{T-D}{mV} - \frac{\dot{V}}{V}\right) & 1 & -\left(\frac{N'}{mV}+\frac{\dot{V}}{V}\right) & 0 & 0 & -\frac{R'}{mV} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\omega_1^2 & -2\zeta_1\omega_1 & \frac{R'Y_1(X_\beta)}{m_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -50 & -10 \end{bmatrix}$$

(3.6-6)

$$\underline{u} = \left[0 \quad 0 \quad \frac{\dot{V}}{V}\alpha_\omega + \dot{\alpha}_\omega \quad 0 \quad 0 \quad 0 \quad 0\right]^T \tag{3.6-7}$$

$$\underline{y} = \left[\varphi_D \quad \dot{\varphi}_{RG}\right]^T \tag{3.6-8}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & Y'_1(X_D) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & Y'_1(X_{RG}) & 0 & 0 \end{bmatrix} \tag{3.6-9}$$

$$\beta_c = \begin{bmatrix} 0.8 & 0.8 \end{bmatrix} \underline{y} = - \underline{K}^T \underline{y} \qquad (3.6\text{-}10)$$

In order to restrict the problem to manageable size while retaining a meaningful plant description only the first elastic bending mode has been included. A table of the time-varying values of the above matrix elements is included in Appendix E.

3.7 Model Reference Design

Rillings[18] found that his optimal gains were so sensitive that instability occurred for variations of less than 20% from the nominal value of the natural frequency of the first elastic bending mode for the Saturn V. The necessity of accommodating for such parameter sensitivity is the motivation for considering a model reference adaptive control loop, hereinafter referred to as the "outer-loop", in addition to the "inner-loop" which is based on Rillings' optimal gains.

The first question that must be considered is which of the design algorithms of Chapter II is the most appropriate for application to the booster. The Lyapunov design algorithm can be eliminated from consideration on two accounts: first, the seven states of the assumed booster model are not all measurable and secondly, the elements of the plant state matrix are not independently accessible as a feedback control law has been specified. One characteristic of the discrete gradient adaptation rule is that no adaptation occurs during the monitoring interval. Thus if a condition causing instability arises during this time interval, the booster might destroy itself before a gain adjustment could be made. Since stability of the booster control system is a major consideration, the discrete gradient algorithm was also eliminated leaving the choice to one of the two continuous gradient rules. The simulations of Chapter II indicate that the continuous gradient rule for the case in which the perturbation control signal is a linear function of the plant model error is the less sensitive to the value of the convergence factor β; hence, this approach was selected. However, as will be seen later, when some engineering restrictions are considered, the two continuous gradient rules become almost identical.

In what follows, the reference model of the booster, not to be confused with the linearized perturbation model of the booster upon which this entire analysis is based, will be described by Equations 3.6-2 through 3.6-10 for nominal values of the matrix parameter or

$$\dot{\underline{x}}_m = \left[A^* - \underline{b}\,\underline{K}^T C\right] \underline{x}_m + \underline{u}_m \qquad (3.7\text{-}1)$$

$$\underline{y}_m = C\,\underline{x}_m \qquad (3.7\text{-}2)$$

in which the asterisk denotes the nominal value. The actual plant can be described in a similar manner by

$$\dot{\underline{x}}_p = \left[A - \underline{b}\,\underline{K}^T C\right] \underline{x}_p + \underline{b}\,\hat{\beta}_c + \underline{u}_p \qquad (3.7\text{-}3)$$

$$\underline{y}_p = C\,\underline{x}_p \qquad (3.7\text{-}4)$$

in which some of the elements of A, namely the natural frequency of the first bending mode, are not precisely known and $\hat{\beta}_c$ represents the perturbation control signal. Defining the error vector as

$$\underline{e} = \underline{y}_m - \underline{y}_p \tag{3.7-5}$$

the perturbation control signal is postulated as

$$\hat{\beta}_c = - k_{a1}\, e_1 - k_{a2}\, e_2 \tag{3.7-6}$$

Selecting as a performance index

$$J = \int (\underline{e}^T \underline{e} + R\, \hat{\beta}_c^{\,2})\, dt \tag{3.7-7}$$

the adaptation rule, Equation 2.4-16, is

$$\dot{K}_{ai} = \beta_i \left[\underline{e}^T (I + \underline{K}_a\, R\, \underline{K}_a^T)\, C\, \underline{z}_i - \underline{e}^T \underline{K}_a\, R\, e_i \right] \; ; \quad i = 1, 2 \tag{3.7-8}$$

in which $\underline{z}_1$ and $\underline{z}_2$ are synthesized from filters described by

$$\dot{\underline{z}}_i = \left[A^* - \underline{b}\, \underline{K}^T \right] \underline{z}_i - \underline{b}\, (\underline{K}_a^{\,T}\, C\, \underline{z}_i - e_i) \quad ; \quad i = 1, 2 \tag{3.7-9}$$

A block diagram of this adaptive control system is shown in Figure 3-8. To evaluate the performance of this system, the seven-state time varying model of the Saturn V was simulated on an I.B.M. 360/50 digital computer. This simulation consists of integrating a system of 32 differential equations - seven for the plant, seven for the model, seven

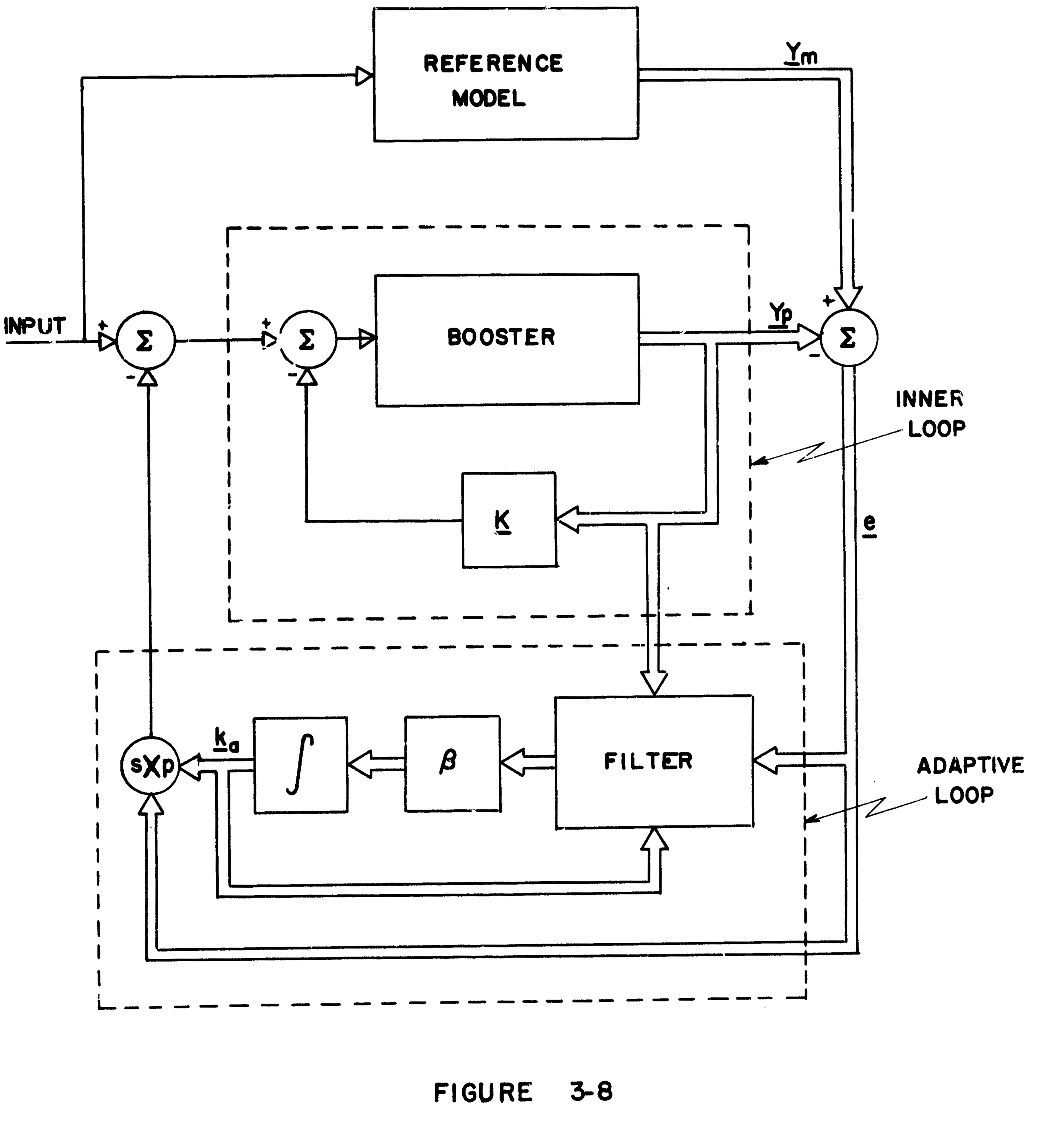

REFERENCE MODEL
Ym
INPUT
+
Σ
-
+
Σ
-
BOOSTER
Yp
+
Σ
-
INNER LOOP
K
e
ka
sXp
∫
β
FILTER
ADAPTIVE LOOP

FIGURE 3-8

for each of the two filters, two for the adaptive gains K_{a1} and K_{a2} and two for the value of the criterion. Three different cases were considered.

Everything, including the filters, was considered to be time-varying in the first case and it was further assumed that the same wind excited the plant and the reference model. Simulations were made for $R = 1$ and for values of $\beta_1 = \beta_2 = \beta$ in the range 100 to 5000. It is known that the unadapted booster is unstable for $\omega_1 = 0.8\ \omega_1{}^*$ and it is seen that the response of the booster with adaptation becomes more acceptable as the value of β is increased. For $\beta = 100$, the value of $J_s = \int_0^{140} y_p{}^T y_p \, dt$ is 25.963 and the maximum value of $\eta_1 = 0.597$ meters while for $\beta = 5000$, $J_s = 11.398$ and $\eta_{1\,max} = 0.168$ meters. For β in the range 1000 to 5000, the adaptive gains K_{a1} and K_{a2} converge respectively to values in the neighborhood of -0.125 and -0.385; the major difference being that convergence is achieved at about 120 seconds for $\beta = 1000$ while convergence is achieved about 10 seconds earlier for $\beta = 5000$. This does not mean that 100 seconds is needed for convergence since very little adaptation occurs before the elastic bending response becomes prominent at about 100 seconds. A simulation of the model is shown in Figures 3-9 and 3-10 and a simulation of the adaptive control system for $\beta = 5000$ in Figures 3-11 and 3-12. While a few cycles of high frequency bending occur in the neighborhood of 100 seconds into the flight, this oscillation quickly damps out once the adaptive loop has sensed this unstable condition. This is a major improvement over the instability that occurs when no adaptation is considered.

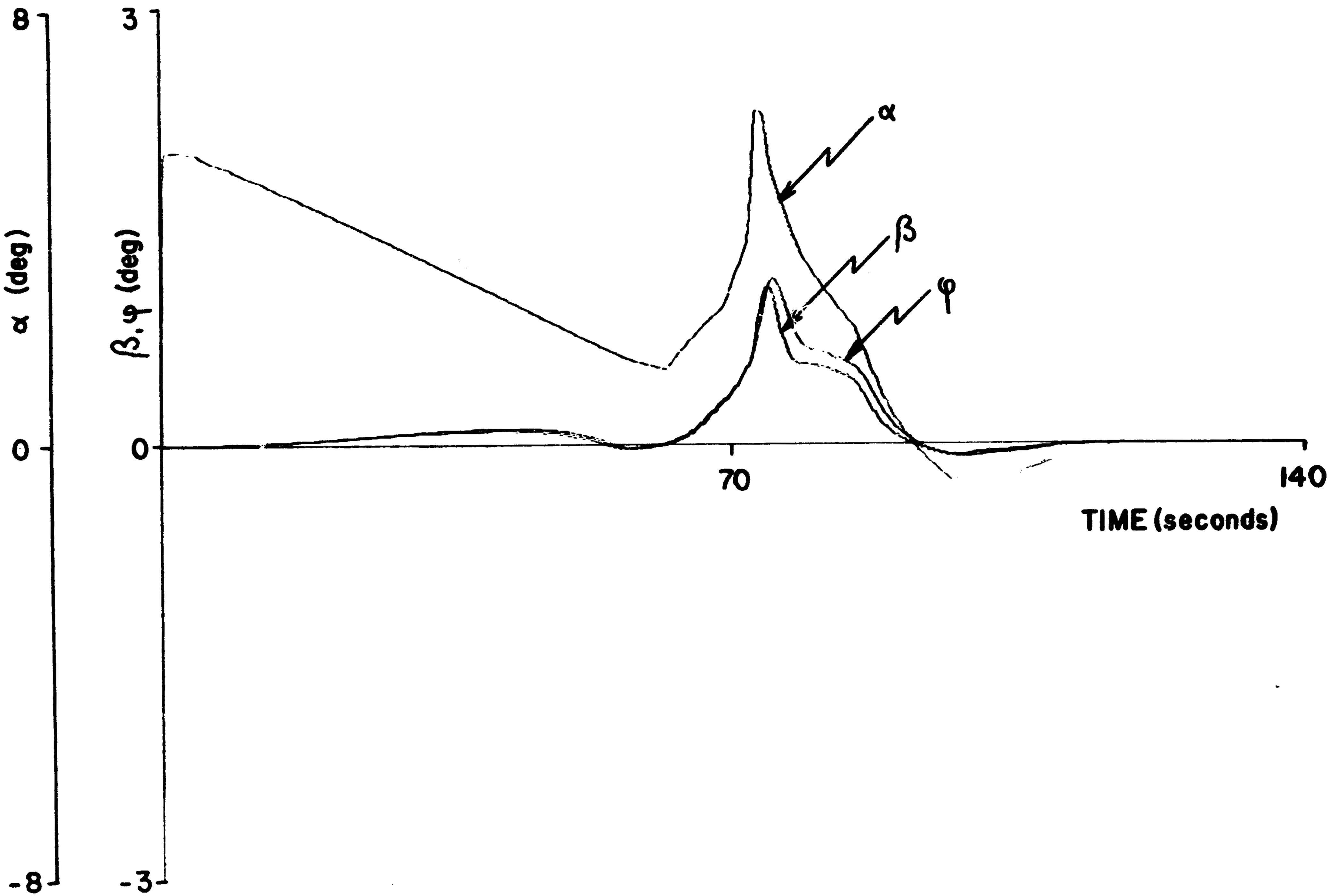


FIGURE 3-9 MODEL NOMINAL RESPONSE

RIGID BODY

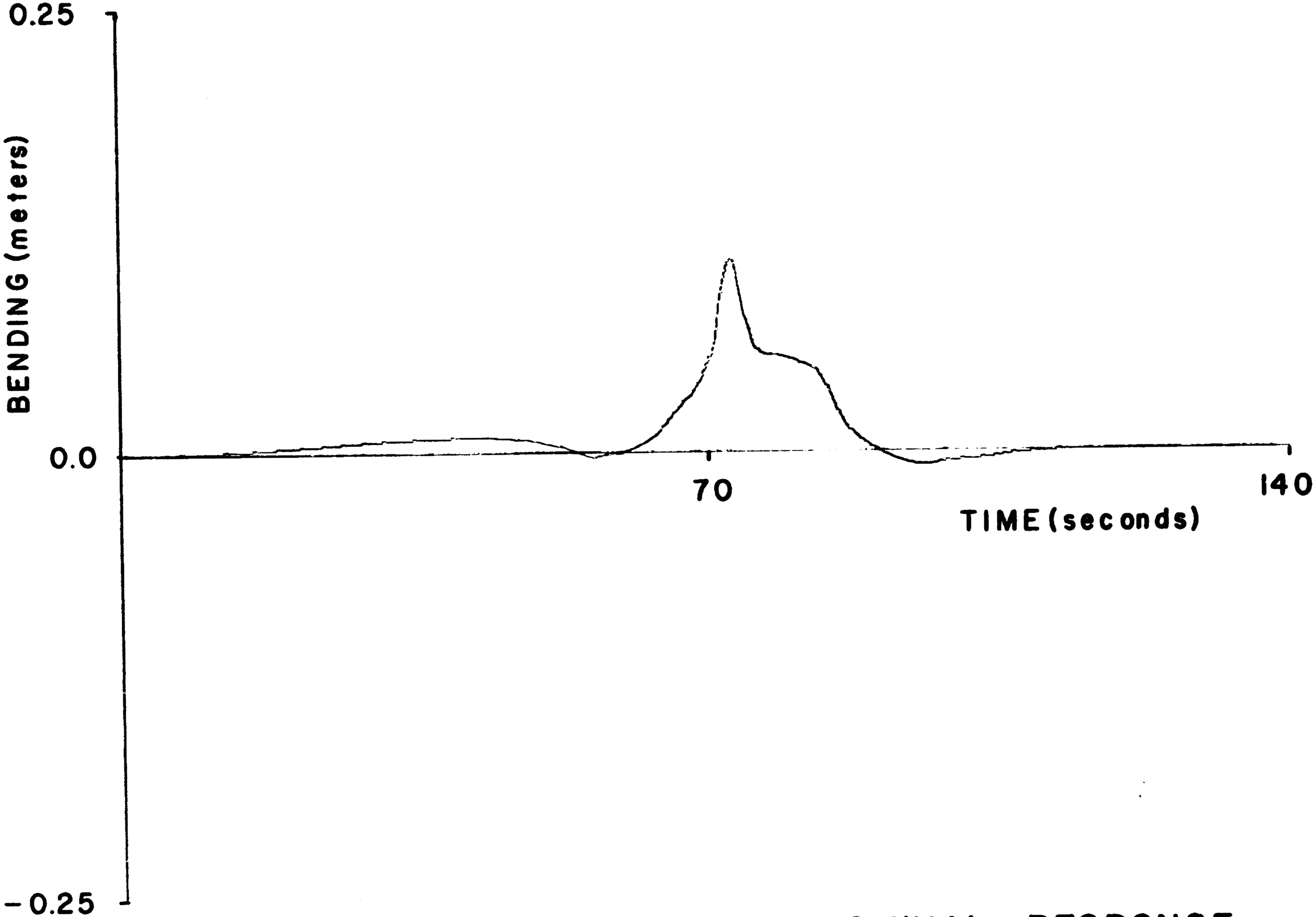


FIGURE 3-10 MODEL NOMINAL RESPONSE
BENDING

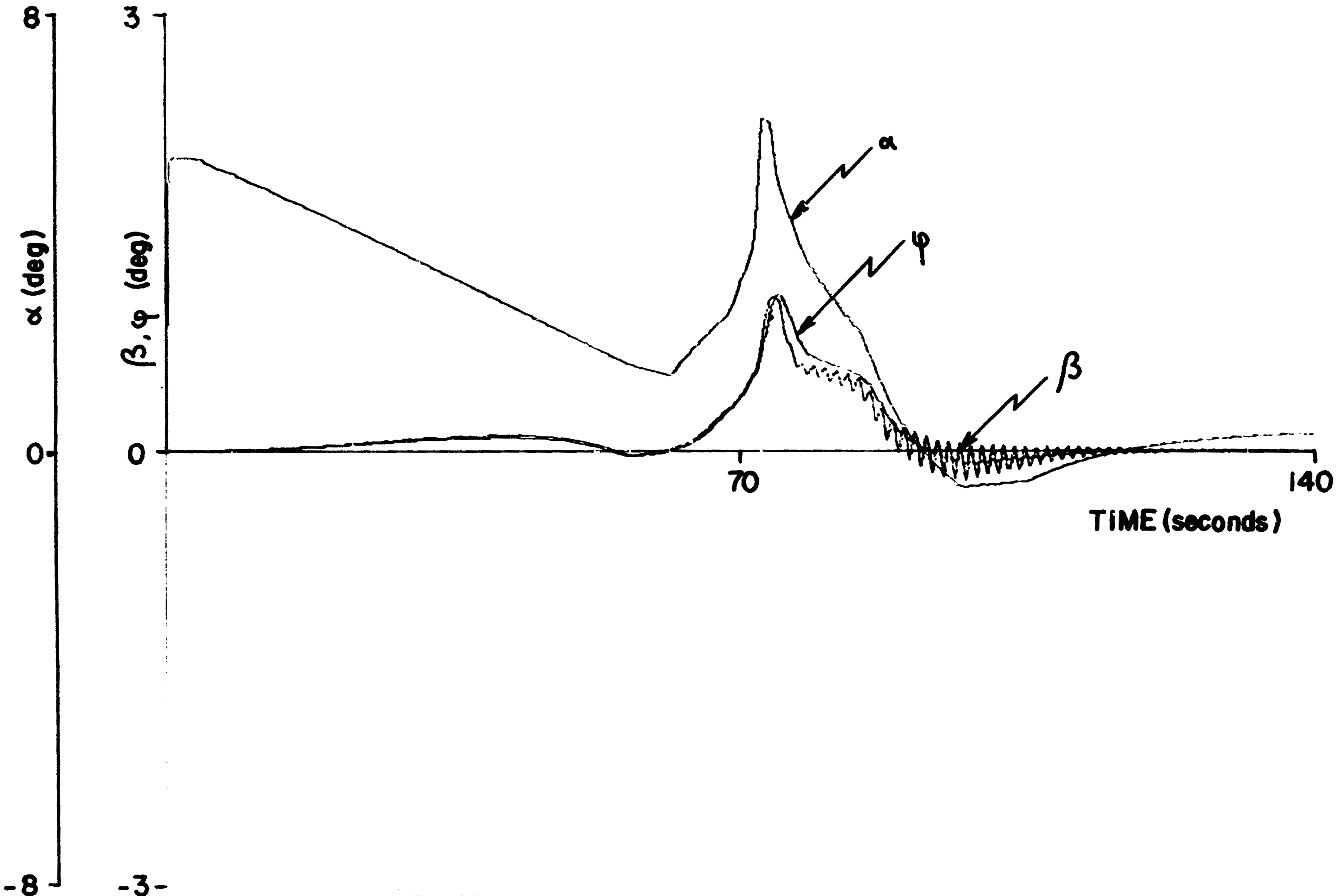


FIGURE 3-11 MODEL REFERENCE $E = Y_m - Y_p$ $0.8\,\omega_1^*$

RIGID BODY

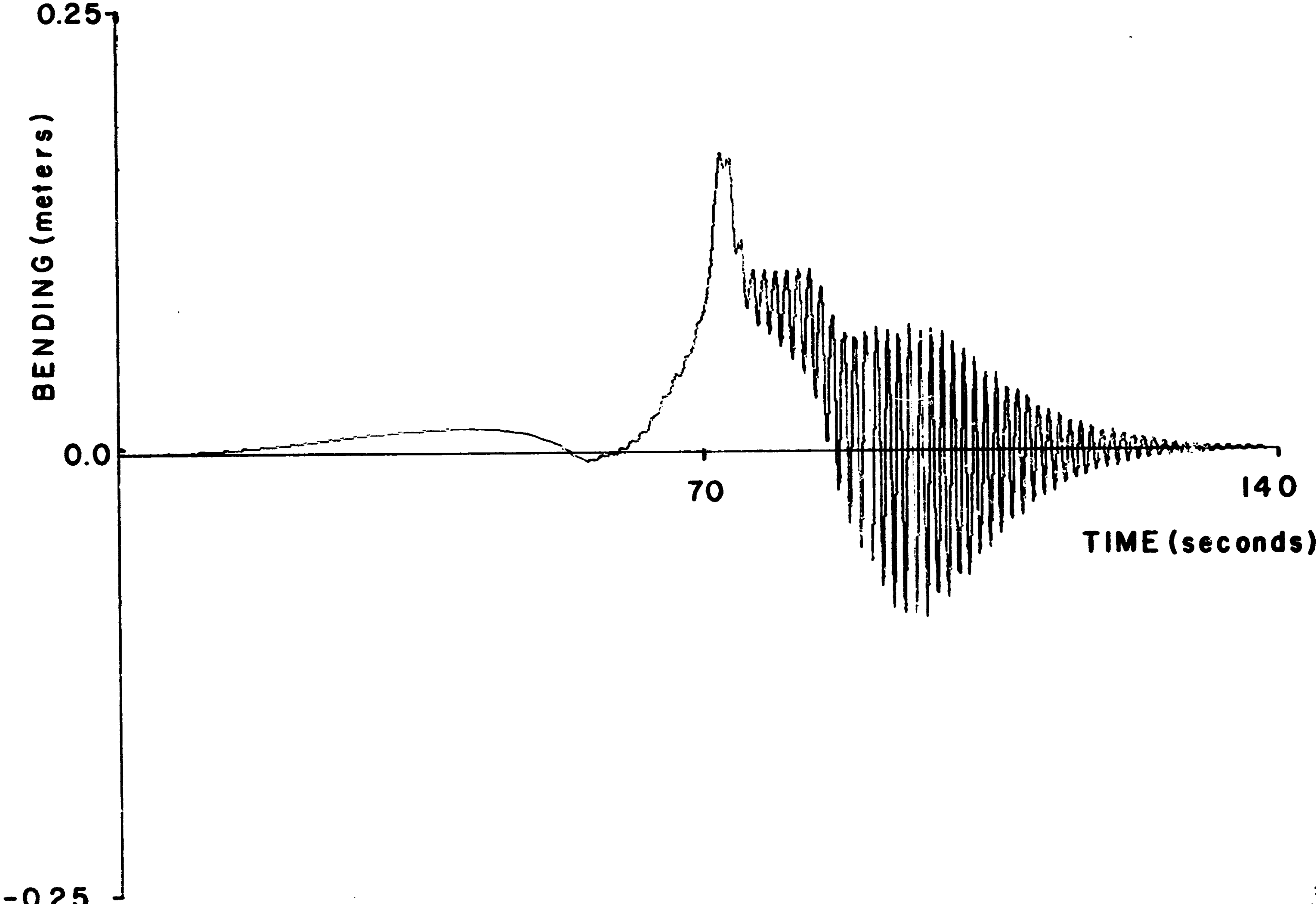


FIGURE 3-12 MODEL REFERENCE $E = Y_m - Y_p$ $0.8\,\omega_1^*$

BENDING

Since an important consideration in the design of model reference controllers is the complexity of the final system, it would be desirable to eliminate the necessity of time-varying filters. With this in mind a second set of simulations was made with the filters of Equation 3.7-9 designed at $t = 80$ seconds, a time found to be representative of the booster during the critical period of maximum dynamic pressure. The simulations with these time-invariant filters were found to differ very little from those for which the filters were time-varying. For example, the simulation for $\beta = 5000$ yielded a value of $J_s = 11.422$ versus $J_s = 11.398$ for the corresponding fully time-varying simulation. As a result of this set of simulations it is felt that acceptable performance can be obtained with the use of time-invariant filters.

It was assumed in the first two cases that the same wind excited both the plant and the model. Since it is very difficult to measure the actual wind encountered in flight, the foregoing may not be a valid assumption. With this in mind a third set of simulations was studied in which it was assumed that the reference model perfectly followed the reference trajectory. In other words, it was assumed that the reference model encountered no external disturbances in which case the output $\underline{y}_m$ is identically zero and the error signal becomes the negative of the plant output. Simulations based on this error definition and time-invariant gradient filters indicate that acceptable performance is achieved. The simulations for $\omega_1 = \omega_1^*$, $\omega_1 = 0.9\ \omega_1^*$, and $\omega_1 = 0.8\ \omega_1^*$ are shown in Figures 3-13 through 3-18 for $\beta = 5000$. These simulations are compared with those for 1) the "optimal" inner-loop alone, and 2) the desensitized inner-loop designed by Rillings[18] in Table 3-1 and Figure 3-19.

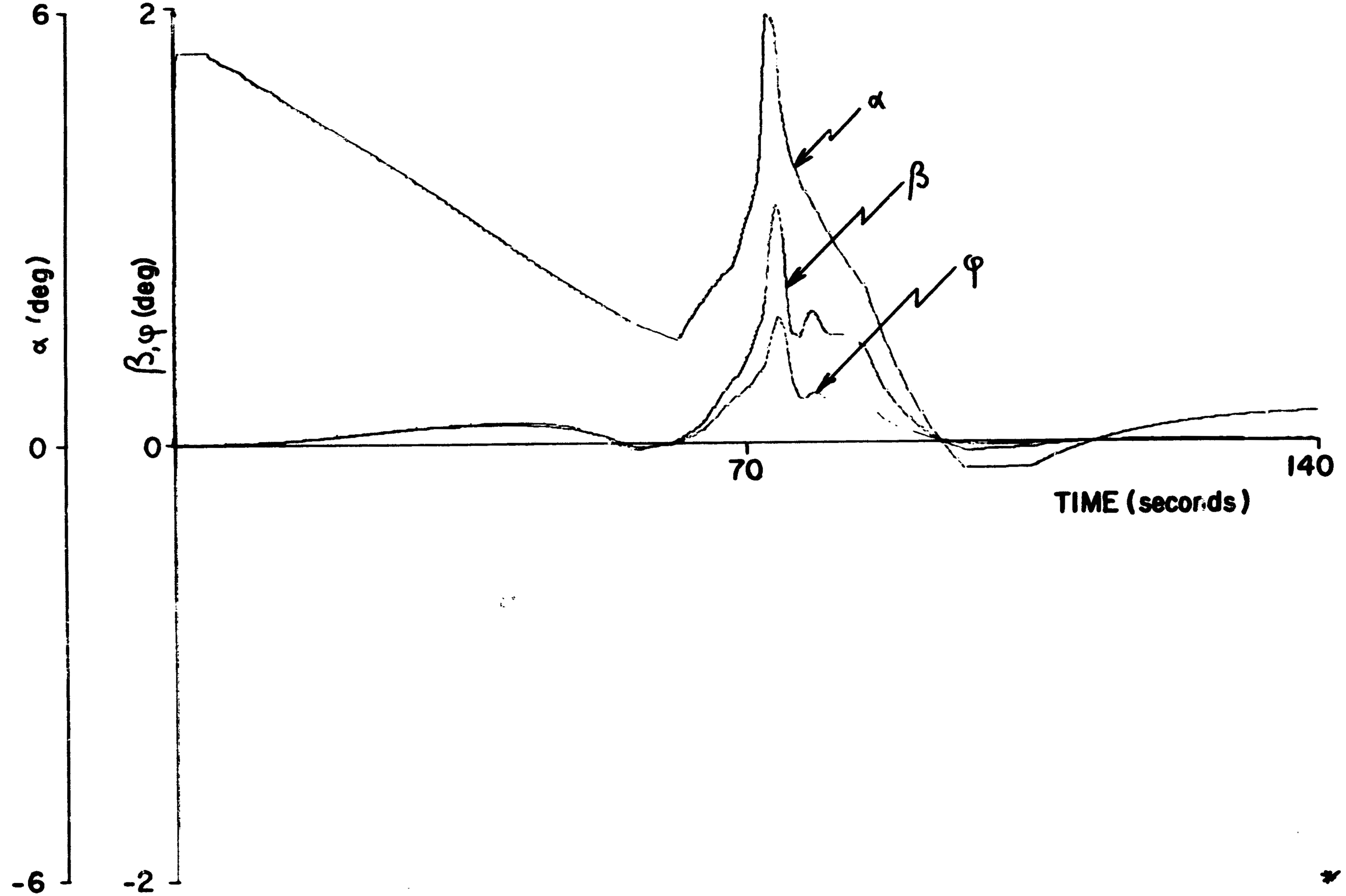


FIGURE 3-13 MODEL REFERENCE $E = -Y_p$ W_1^*
RIGID BODY

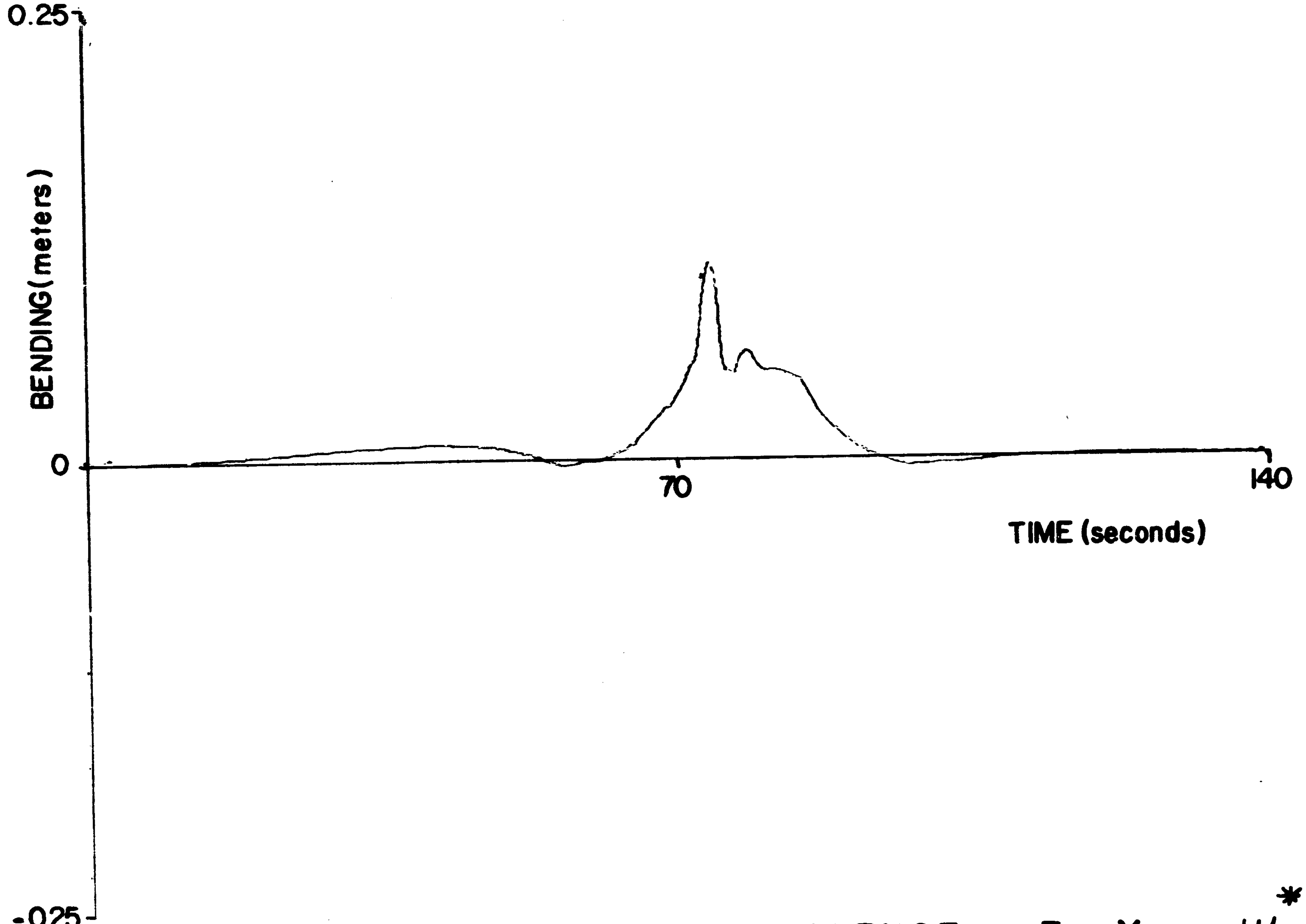


FIGURE 3-14 MODEL REFERENCE $E = -Y_p$ ω_1^*
BENDING

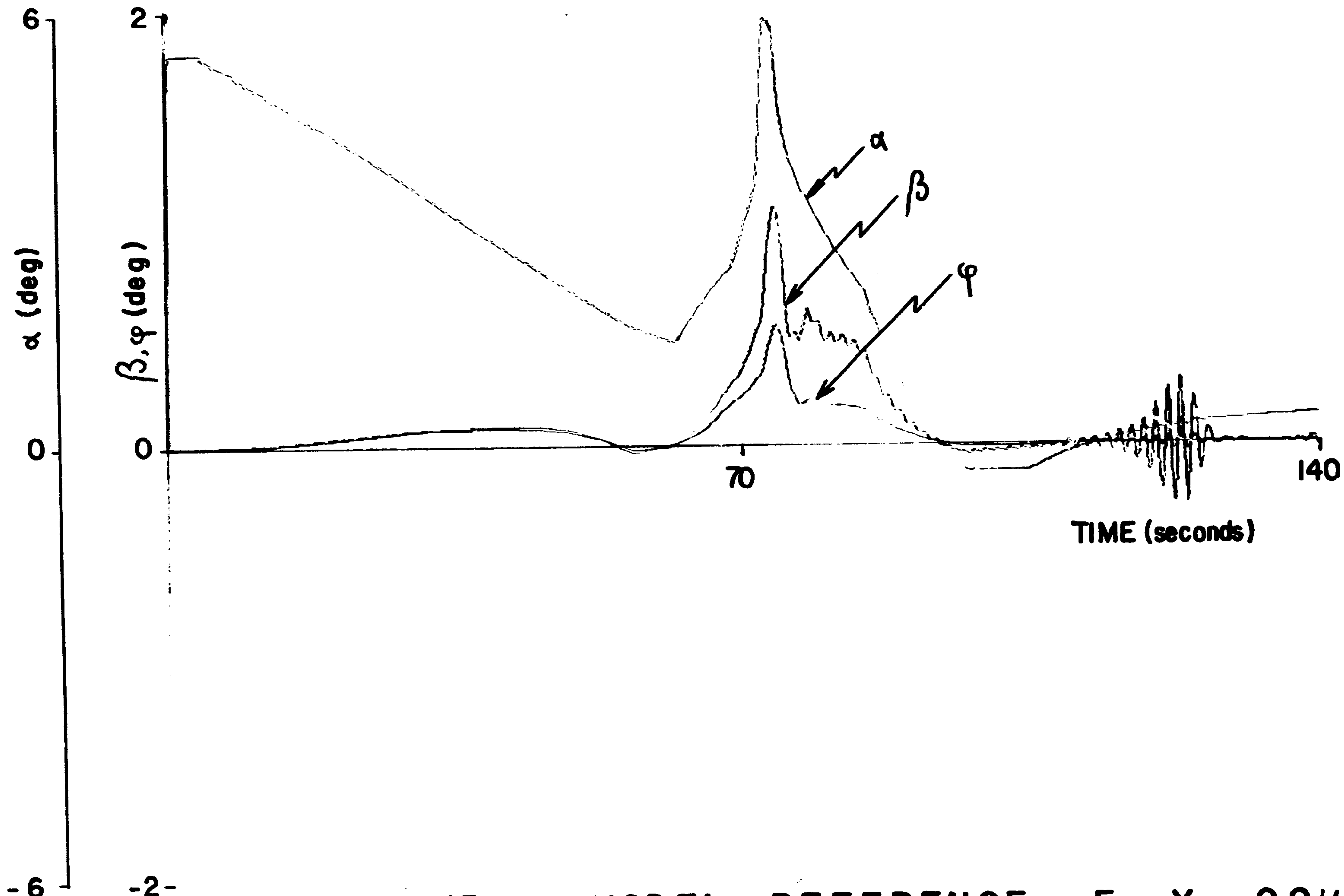


FIGURE 3-15 MODEL REFERENCE $E = -Y_p$ $0.9\,\omega_1^*$

RIGID BODY

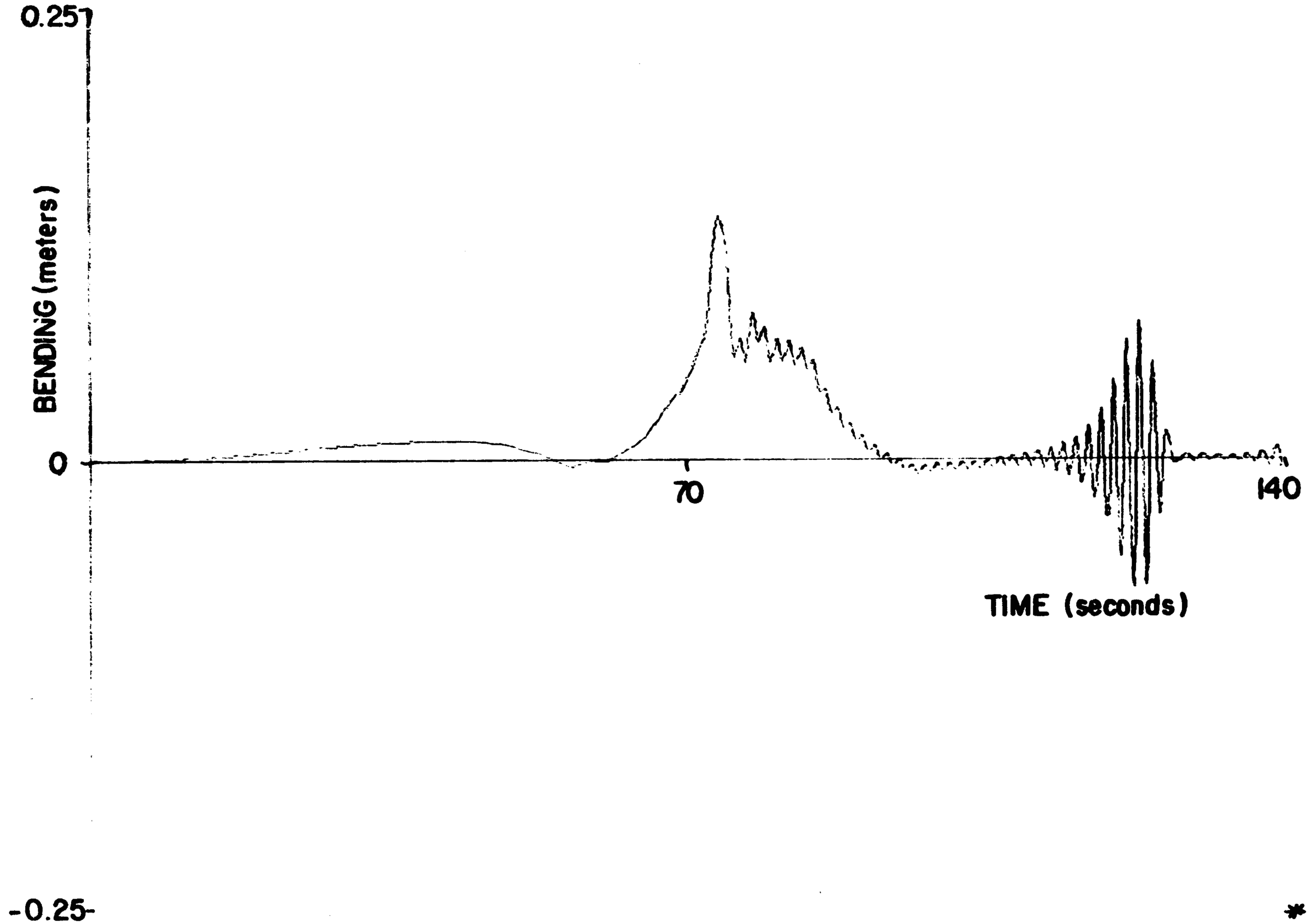


FIGURE 3-16 MODEL REFERENCE $E=-Y_p$ $0.9\omega_1^*$
BENDING

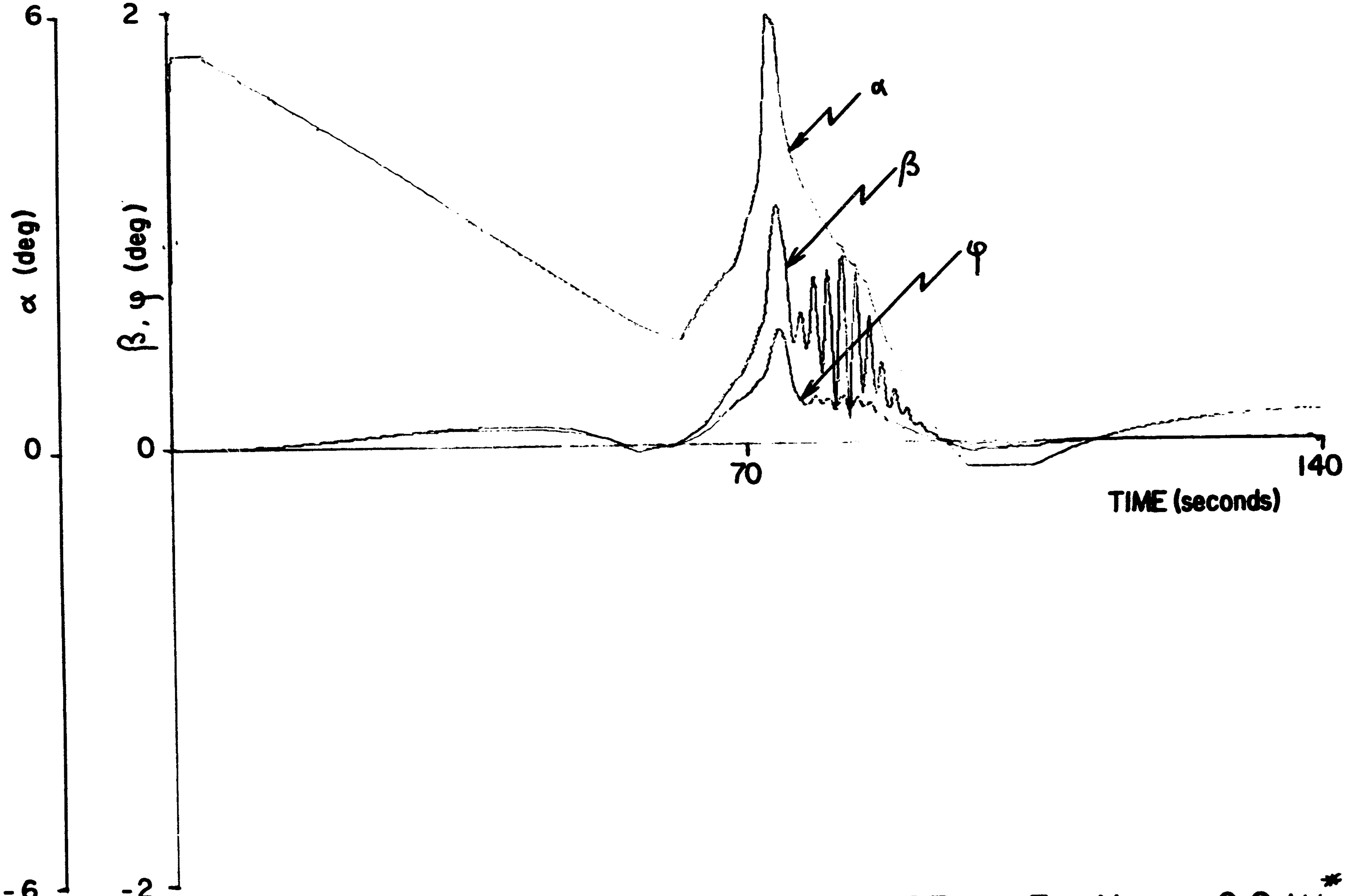


FIGURE 3-17 MODEL REFERENCE $E = -Y_p$ $0.8\,\omega_1^*$
RIGID BODY

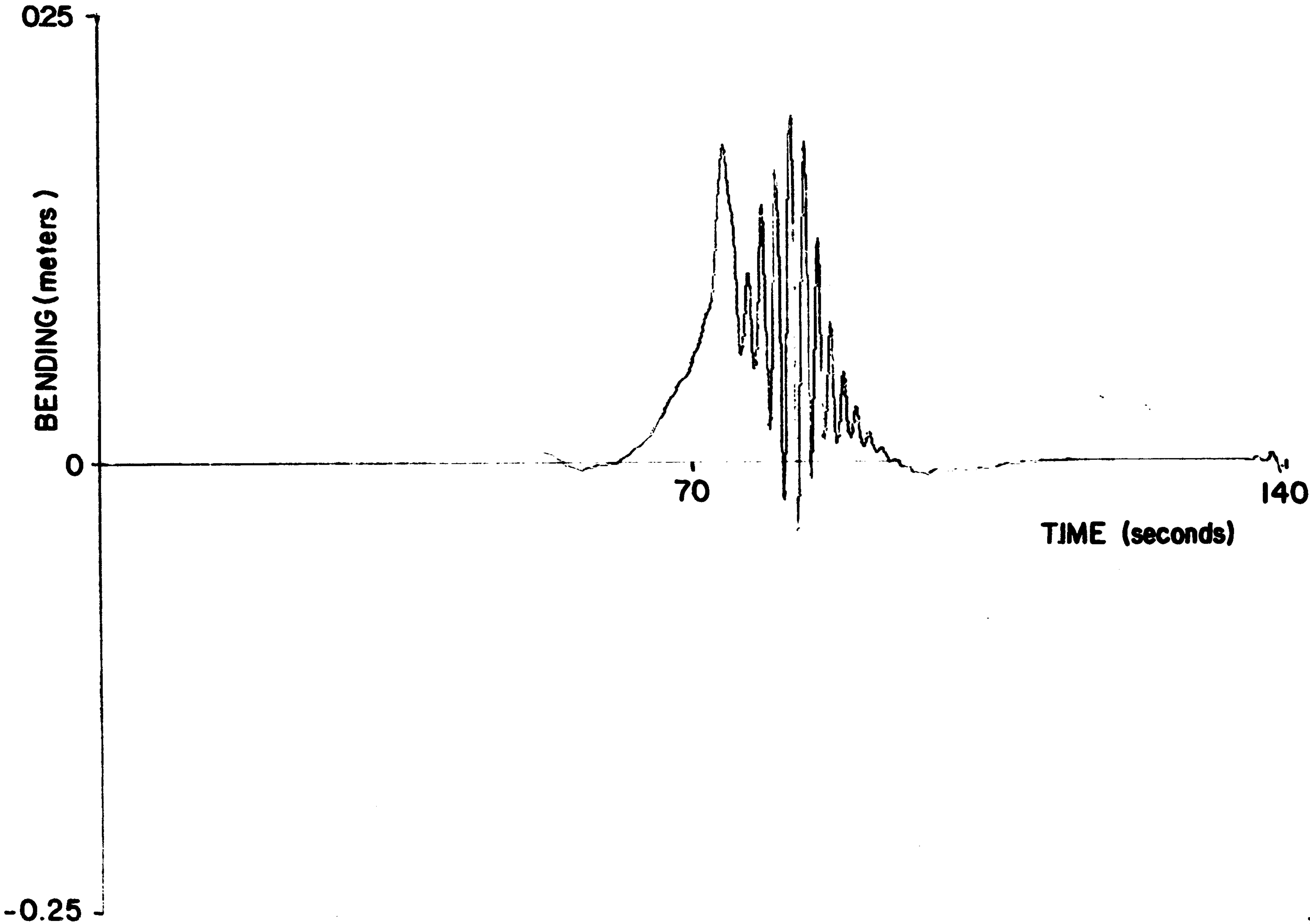
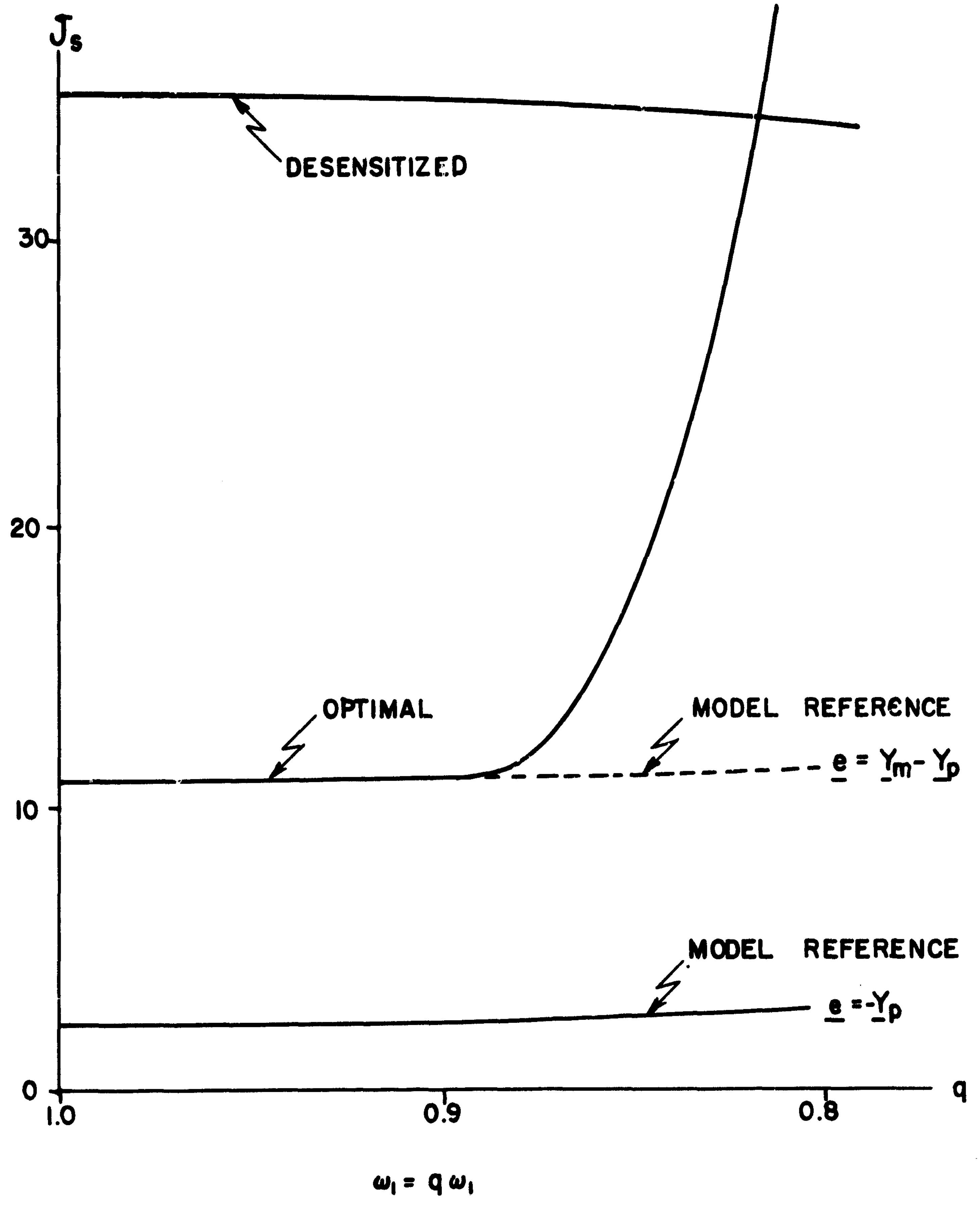


FIGURE 3-18 MODEL REFERENCE $E=-Y_p$ $0.8\omega_1^*$
BENDING



J_s
DESENSITIZED
30
20
OPTIMAL
MODEL REFERENCE
$\underline{e} = \underline{Y}_m - \underline{Y}_p$
10
MODEL REFERENCE
$\underline{e} = -\underline{Y}_p$
0
1.0
0.9
0.8
q
$\omega_1 = q\,\omega_1$

FIGURE 3-19

OPTIMAL INNER LOOP				
	ϕ_{max}	α_{max}	l_{max}	J_s
$\omega_1 = \omega_1^*$	1.14°	6.11°	0.109m	11.22
$\omega_1 = 0.9\ \omega_1^*$	1.11°	6.10°	0.133m	11.16
$\omega_1 = 0.8\ \omega_1^*$	UNSTABLE	UNSTABLE	UNSTABLE	68613

DESENSITIZED INNER LOOP				
	ϕ_{max}	α_{max}	l_{max}	J_s
$\omega_1 = \omega_1^*$	2.21°	6.52°	0.120m	35.40
$\omega_1 = 0.9\ \omega_1^*$	2.18°	6.51°	0.148m	35.18
$\omega_1 = 0.8\ \omega_1^*$	2.13°	6.49°	0.189m	34.88

MODEL REFERENCE				
	ϕ_{max}	α_{max}	l_{max}	J_s
$\omega_1 = \omega_1^*$	0.574°	5.85°	0.108m	2.45
$\omega_1 = 0.9\ \omega_1^*$	0.551°	5.84°	0.134m	2.59
$\omega_1 = 0.8\ \omega_1^*$	0.520°	5.83°	0.191m	2.72

TABLE 3-1

3.8 Discussion of Results

The difference in performance of the system with and without adaptation demonstrates the effectiveness of the model reference adaptive design philosophy. With the experience gained from many computer simulations it is possible to make some remarks about the results obtained.

Probably the most noticeable characteristic of the model reference simulations is the presence of several cycles of high frequency elastic bending oscillation for off-nominal values of the natural frequency. The peak magnitude of this bending at the gimbal plane is 0.2 meters which corresponds to 1.6 meters at the foremost station of the vehicle. It is felt that such oscillation will be inherent in any model reference design which operates on only the outputs of the pitch and pitch rate gyros as the adaptive controller must sense the instability of the bending mode from these signals which contain both bending and rigid body information before adaptation can proceed. It is seen that the oscillations damp out quickly once adaptation begins.

It is felt that the significant reduction in the value of the criterion J_s for the case in which the reference model was assumed to follow the nominal trajectory is due to the reduction of the maximum pitch angle from about 1.5 degrees for the desensitized gains of Rillings[18] to about 0.5 degrees for the model reference design. This reduction in the maximum pitch angle is not unexpected as the major information content of the gyro outputs is related to pitch and pitch rate.

From an engineering viewpoint, the complexity of the adaptive controller is greatly reduced by the finding that time-invariant gradient filters are adequate for successful performance. However, there still remains some question as to whether or not the adaptive system "buys" enough improvement in performance from an engineering viewpoint as to offset the additional complexity.

In the first series of simulations in which the same wind was assumed to excite both the reference model and the plant, it was observed that the adaptive gains, K_{a1} and K_{a2} , converged to values in the neighborhood of -0.125 and -0.385 respectively. Taking into consideration the "inner-loop" gains of $K_1 = -0.8$ and $K_2 = -0.8$, the overall control law becomes $\beta_c = 0.125\, y_{m1} + 0.385\, y_{m2} + 0.675\, y_{p1} + 0.515\, y_{p2}$. It is interesting to note that the resulting plant gains of -0.675 and -0.515 occupy the same region in gain space as those found by both Rillings[18] and Cassidy[19] in their optimal sensitivity analyses. Thus it would appear that the model reference algorithm under investigation converges to a single set of gains independent of the value of the convergence factor β and that these gains are in agreement with those found by other design techniques.

CHAPTER IV

SUMMARY AND RECOMMENDATIONS

4.1 Summary

The design of model reference adaptive control systems has been investigated in this report. Several reasons for considering the model reference adaptive philosophy when designing control systems and several characteristics of a "good" model reference adaptive algorithm are discussed. Adaptive algorithms are derived for linear systems from two approaches. The first three algorithms are based on the steepest-descent or gradient minimization of positive definite integral performance indices. The first algorithm attempts to minimize on-line a weighted integral square plant-model error index while the second algorithm attempts to effect a trade-off between the system error and the perturbation control effort by minimizing an index that reflects the relative cost of each. An estimate of the optimum step size for gradient adaptation is incorporated into the third algorithm by treating adaptation as a discrete process rather than as a continuous process. The fourth algorithm is derived from a stability argument that follows from Lyapunov's Second Method. These algorithms are applied to two second order examples in order to gain insight into such properties as convergence rate, stability, error-nulling capability, and error-perturbation control effort tradeoff.

The model reference adaptive control design technique was successfully applied to a large flexible launch vehicle of the Saturn V class.

The continuous gradient adaptation algorithm in which the perturbation control signal is postulated as a linear function of the plant-model error was chosen for application. This adaptive system operates on only the measured outputs of the pitch and pitch rate gyros and nowhere is it necessary to isolate the elastic bending response from the rigid body response. Simulation studies show that this system reduces significantly the sensitivity of the booster to variations in the natural frequency of the first elastic bending mode. Subsequent simulations indicate that acceptable performance can be achieved with time-invariant gradient filters, designed for an appropriate flight time, thus removing the necessity of implementing time-varying filters in the controller. The encouraging results obtained in this study suggest that the philosophy of model reference adaptive control system design merits further investigation with reference to applicability to large flexible launch vehicles.

4.2 Recommendations for Future Work

There are several possibilities for further investigation into the theory of model reference adaptive control system design.

It was assumed in this study that the plant and reference model were the same order. It would be of practical interest to investigate the conditions under which a plant can track a lower order model. For example, a nineteenth order of the Saturn V is obtained upon considering four elastic bending modes, three slosh modes, and a second order filter in addition to the rigid body mode. Hence, it might be expedient to consider a reference model with fewer states in order to reduce the complexity of the gradient filters.

In many launch missions it is desirable to limit the lateral drift of the vehicle. Unfortunately, under certain conditions a drift minimum control system can cause excessive structural loading of the vehicle. Thus it is often necessary to switch from a drift minimum control system to a load relief control system during the period of high dynamic pressure. It is felt that the discrete gradient adaptation rule would afford sufficient time to obtain a good indication of vehicle performance and that gain adjustments could be made at the proper time to provide load relief.

It would be interesting to attempt to determine whether or not there is any correlation between the values to which the adaptive gains converge for a given performance criterion and those that are obtained from optimal control theory for a similar index.

This study has been limited to linear plants. While the concept of minimizing a positive definite integral performance index by the gradient method can be directly extended to non-linear systems, the design of the necessary gradient filters becomes less well defined. The design of such filters merits further study.

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Appendix A

Survey of Model Reference Adaptive Control

The philosophy and merits of model reference adaptive control was discussed in Chapter 2 of this report. Design algorithms based on Lyapunov's Second Method and the minimization of an integral performance index by the gradient method were developed and applied to representative problems. For the purpose of completeness, several additional, existing adaptation rules are discussed in this Appendix.

1. Dynamics of a First-Order Model Reference Adaptive System[1,2,3]

In order to gain some insight into the performance of higher-order model reference adaptive systems, consider the first order plant

$$T \dot{x} + (b + K_{FB}) x = g(t) \qquad \text{A-1.1}$$

which is to follow the model

$$x_m = g(t) \qquad \text{A-1.2}$$

The structural diagram of this plant-model combination and four possible variations of an adaptive system are shown in Figure A-1.

From Figure A-1 it is seen that three adjustable parameters K_1, K_g, and K_f have been incorporated into the system. The general law of variation of these parameters is assumed to be of the form

$$K_i = K_{oi} \pm K_{ui} \int e_i \, dt + K_{ei} \, e_i \qquad \text{A-1.3}$$

where K_{oi}, K_{ui}, and K_{ei} are constant coefficients, e_i is an error computed from some well-defined rule, and the plus sign is associated with K_g and K_f while the minus sign is associated with K_1.

Consider first the case in which $K_{ui} = 0$. If the e_i are defined as

$$
\begin{aligned}
e_1 &= (g - x) \operatorname{sgn} x \\
e_g &= (g - x) \operatorname{sgn} g \qquad \text{A-1.4} \\
e_f &= (g - x) \operatorname{sgn} \dot{x}
\end{aligned}
$$

the dynamical equations for the various configurations of Figure A-1 can be written. For example, upon setting K_{FB} and K_{oi} equal to one, the equations for the system of Figure A-1c are

$$
\begin{aligned}
T \dot{x} + (b + K_1) x &= K_g \, g(t) \\
K_1 &= 1 - K_{e1} (g - x) \operatorname{sgn} x \qquad \text{A-1.5} \\
K_g &= 1 + K_{eg} (g - x) \operatorname{sgn} g
\end{aligned}
$$

Two modes of operation are of interest; namely $g(t) = 0$ and $g(t) = g_o > 0$. Upon defining $y = T\dot{x}$, the x-y phase-plane trajectory equation for Equation A-1.5 becomes

$$
\begin{aligned}
y = g_o - (b + 1) x + K_{eg} \, g_o (g_o - x) \\
+ K_{e1} \, x (g_o - x) \operatorname{sgn} x \qquad \text{A-1.6}
\end{aligned}
$$

For $g = 0$, equation A-1.6 reduces to

$$y = - (b + 1) x - K_{e1} x^2 \operatorname{sgn} x \qquad \text{A-1.7}$$

which is plotted in Figure A-2 for the four cases 1) $b = 0$, 2) $b > 0$, 3) $-1 < b < 0$, and 4) $b < -1$. The points of equilibrium of this system are those for which $y = 0$. For trajectories 1, 2, and 3, the origin of phase space is of stable equilibrium. For $b < -1$, the condition $y = 0$ can be satisfied by three values of x. However, only for points O_1 and O_2 is y of proper sign to return the system to equilibrium. Since at these points of stable equilibrium $x \neq 0$ for $y = 0$, the introduction of adaptation has made an unstable system $(b < -1)$ stable at the expense of a steady-state error. For $g = g_o > 0$, defining the variables $\mathcal{E} = g_o - x$ and $y_1 = T\dot{\mathcal{E}}$ yields $\mathcal{E} - y_1$ phase trajectory equations for equation A-1.5 of the form

$$y_1 = -\mathcal{E} + b\,(g_o - \mathcal{E}) - K_{eg}\, g_o\, \mathcal{E} - K_{el}\,(g_o - \mathcal{E})\,\mathcal{E}\ ; \quad \mathcal{E} < g_o$$

A-1.8

$$y_1 = -\mathcal{E} + b\,(g_o - \mathcal{E}) - K_{eg}\, g_o\, \mathcal{E} + K_{el}\,(g_o - \mathcal{E})\,\mathcal{E}\ ; \quad \mathcal{E} > g_o$$

The phase-space plot of these equations is shown in Figure A-3 for the various ranges of b. The stable equilibrium points are O_1, O_2, O_3 and O_4 respectively which again implies stability for any value of b. However, for $b \neq 0$, there is always a steady-state error, the magnitude of which is dependent on b.

Similiar analyses can be carried out for the remaining configurations of Figure A-1 and for error definitions differing from those of Equation A-1.4. Since it is indicated in Reference 1 that the best results can be expected from the configuration of Figure A-1c and the error definitions of Equation A-1.4, attention has been and will continue to be focused only on this system.

The effect of integral adaptation can be examined by adjusting only K_1 and setting $K_{e1} = 0$. The dynamical equation A-1.5 for $g = 0$ reduces to

$$T \dot{x} + (b + 1)\, x + K_{u1}\, x\, z = 0 \qquad \text{A-1.9}$$

where

$$z = \int x \operatorname{sgn} x \, dt \qquad \text{A-1.10}$$

and results in a x-z phase-plane representation of Equation A-1.9 in which

$$\frac{dx}{dz} = - \frac{(b+1)\, x + K_{u1}\, xz}{T\, x \operatorname{sgn} x} \qquad \text{A-1.11}$$

In the phase plot of Equation A-1.11, Figure A-4, it is seen that the value of z will tend to increase for any initial value of x ; consequently for free motion, K_1 will increase continuously. For $g = g_o > 0$, Equation A-1.5 becomes

$$T \dot{x} + bx = \mathcal{E} + K_{u1}\, xz \qquad \text{A-1.12}$$

upon defining $\mathcal{E} = g_o - x$ and $z = \int \mathcal{E} \operatorname{sgn} x \, dt$. The $\mathcal{E}$-z phase trajectories of Equation A-1.12 have

$$\frac{d\mathcal{E}}{dz} = - \frac{\mathcal{E} - b(g_o - \mathcal{E}) + K_{u1}\, (g_o - \mathcal{E})\, z}{T\, \mathcal{E} \operatorname{sgn}(g_o - \mathcal{E})} \qquad \text{A-1.13}$$

The isoclines (curves of equal $\frac{d\mathcal{E}}{dz}$) are hyperbolas with a singular common asymptote $\mathcal{E} = g_o$ and two asymptotes parallel to the $\mathcal{E}$ - axis as is seen in Figure A-5. Since for $\mathcal{E} > g_o$ all motion is toward the boundary $\mathcal{E} = g_o$, system stability may be determined by examining only the $\mathcal{E} < g_o$ region. This is best accomplished by considering the $\mathcal{E}$-z* plane in which $z^* = z - b/K_{u1}$ and

$$\frac{d\mathcal{E}}{dz^*} = \frac{d\dot{\mathcal{E}}}{dz} = - \frac{\mathcal{E} + K_{u1}\,(g_o - \mathcal{E})\,z^*}{T\,\dot{\mathcal{E}}\,\text{sgn}\,(g_o - \mathcal{E})} \qquad \text{A-1.14}$$

It is now seen that the zero isocline, $z^* = - \frac{\mathcal{E}}{K_{u1}(g_o - \mathcal{E})}$, passes through the second and fourth quadrants for any b and that the trajectory will always twist toward the equilibrium state $\dot{\mathcal{E}} = 0$ or $x = g$. Thus when K_1 is adjusted solely on the basis of an integral law, a stable system with zero steady-state error is obtained for any value of b. However, in free motion, the integral accumulates causing K_1 to be set incorrectly. This condition can be remedied by using an error algorithm that takes into consideration any dead-band of the system.

A similiar analysis shows that including the term $K_{e1} \neq 0$ in the adaptation of K_1 results in improved stability and an improved transient response. When the adaptation of K_g is considered, it is found that the inclusion of the integral term in the adaptation law reduces system stability, impairs the transient response, but does result in a zero steady-state error.

While the results of this section are only valid for first-order linear systems, the analysis provided by the phase-plane technique should offer valuable insight into the stability and steady-state error difficulties that might be expected in higher-order model reference systems and also into the reasoning behind the choice of adaptive algorithms.

2. M.I.T. Rule

The original model reference adaptive control design algorithm was developed by Osburn and Whitaker[4] for single input-single output, linear, time-invariant systems. The algorithm is based on the on-line minimization of an integral-square error performance index. If the response error is defined as the difference between the system output and the output of an appropriate model of the system, $E = x_p - x_m$, the performance index is given by

$$PI = \int E^2 dt = \int (x_p - x_m)^2 \, dt \qquad \text{A-2.1}$$

The variation of PI with the change in a system control parameter K has the general characteristic shown in Figure A-6. The desired parameter value corresponds to the minimum of this PI vs K curve or the point where the slope of the curve is zero,

$$\frac{\partial}{\partial K} PI = \frac{\partial}{\partial K} \int E^2 \, dt = 0 \qquad \text{A-2.2}$$

When the operating value of K differs from that for which the optimality condition of Equation A-2.2 is satisfied, a well-defined technique for adjusting the value of K is required. Defining $(EQ)_K = \frac{\partial}{\partial K}$ P.I. , the design objective is to drive $(EQ)_K$ towards zero. Choosing $\Delta K \;\alpha\; (EQ)_K$, the adaptation rule becomes

$$\dot{K} = -\beta \frac{d}{dt} (EQ)_K \qquad \text{A-2.3}$$

which is readily observed to be nothing other than adaptation based on the gradient of the performance index. Interchanging the order of differentiation and integration reduces Equation A-2.3 to

$$\dot{K} = -2\beta E \frac{\partial E}{\partial K} = -\beta' E \frac{\partial E}{\partial K} \qquad \text{A-2.4}$$

Noting that x_m is independent of K, $\frac{\partial E}{\partial K} = \frac{\partial x_p}{\partial K}$. The implementation of Equation A-2.4 requires the synthesis of $z = \frac{\partial x_p}{\partial K}$. This can be accomplished by either of two methods: straight-forward partial differentiation of the differential equation for x_p or by block diagram manipulation. For example, consider the system of Figure A-7 where

$$e_s = K e_2$$

$$\delta e_s = e_2 \delta K$$

$$\delta x_p = \frac{G}{G_1} e_2 \delta K$$

with $G = \frac{x_p(s)}{q(s)}$.

Since G(s) is unknown, $G_m(s) = \frac{x_m(s)}{q(s)}$ is substituted for G(s) based on the assumption that $G_m(s) \approx G(s)$ for an appropriate model. Thus

$$\frac{z(s)}{e_2(s)} = \frac{G_m}{G_1} \qquad \text{A-2.5}$$

represents the transfer function from which $z = \frac{\partial x_p}{\partial K}$ is synthesized.

One disadvantage of the M.I.T. Rule is that it can lead to an overall system that is unstable. As an example of this consider the system shown in Figure A-9 in which the adaptive parameter is K_c. The differential equation for the error E is

$$b_2 \ddot{E} + b_1 \dot{E} + E = (K - K_v K_c) r(t) \qquad \text{A-2.6}$$

and the adaptive equation based on the M.I.T. Rule is

$$\dot{K}_c = -\beta E \frac{\partial E}{\partial K_c} \qquad \text{A-2.7}$$

If $r(t) = R\, U_{-1}(t)$ and the adaptive loop is closed with the system in steady-state, $E = (K - K_v K_c)\, R$ and

$$\frac{\partial E}{\partial K_c} = -K_v R = -K_v x_m$$

or

$$\dot{K}_c = \beta E K_v x_m = \beta' E x_m \qquad \text{A-2.8}$$

Differentiating Equation A-2.6 and substituting Equation A-2.8 for $\dot{K}_c$ results in the third-order differential equation

$$b_2 \dddot{E} + b_1 \ddot{E} + \dot{E} + \beta' R K_v x_m E = 0 \qquad \text{A-2.9}$$

From the Routh Hurwitz criterion, Equation A-2.9 has a pole in the R.H.P. whenever

$$K_v \beta' R x_m > \frac{b_1}{b_2} \qquad \text{A-2.10}$$

which can result in the instability of Equation A-2.9.

In conclusion, the M.I.T. Rule can be easily implemented for linear time-invariant systems to yield good adaptation provided care is taken to determine the regions of stability.

3. Donalson's Algorithm

The minimization of a quadradic function of the system error and its derivatives by a steepest-descent method is the basis of an adaptation algorithm derived by Donalson[5]. Although the algorithm can be applied to all single input-single output, time-invariant, linear systems, an appreciation for the development can best be obtained by considering the system of Figure A-8. The dynamic equation for the plant is

$$\ddot{x}_p + a_1 \dot{x}_p + g_2 x_p = r \qquad \text{A-3.1}$$

with $a_1 = g_1 + k_1$ while for the model

$$\ddot{x}_m + \bar{g} \dot{x}_m + g_2 x_m = r \qquad \text{A-3.2}$$

Three assumptions are basic to the derivation:

1) g_1 varies slowly compared to the basic time-constants of the system,

2) g_1 varies slowly compared to the rate at which k_1 is adjusted, and

3) k_1 is adjusted at a rate that is rapid when compared to the rate at which any function of E and its derivatives changes due to changes in r.

With $\delta = a_1 - \bar{g}$, it is readily apparent that any function $f(E, \dot{E}, \ddot{E})$ is implicitly a function of δ. Thus $f(E, \dot{E}, \ddot{E})$ can be thought of as a surface in the Euclidian space of f and δ; because of assumption 3, $f(E, \dot{E}, \ddot{E})$ can be treated solely as a function of δ. The adjustment of δ is made so as to describe an instantaneous steepest-descent trajectory along the surface of $f(E, \dot{E}, \ddot{E})$ in the $f - \delta$ space; the path of steepest-descent being the one for which the maximum decrease in $f(\delta)$ results at every step. This is accomplished by choosing $\Delta \delta$ proportional to the negative of the gradient of $f(\delta)$ or

$$\Delta \delta \;\alpha\; - \frac{\partial f}{\partial \delta} \qquad \text{A-3.3}$$

Since $\Delta \delta = \Delta a_1$, A3-3 becomes

$$\Delta a_1 \ \alpha \ - \frac{\partial f}{\partial a_1} \qquad \text{A-3.4}$$

At this point it becomes evident that the implementation of Equation A-3.4 requires an explicit knowledge of a_1 and thus g_1 while the objective is to develop an algorithm which does not require the knowledge of g_1. Consequently, an alternative approach is necessary.

Now treat a_1 as fixed, $\bar{g}$ as variable, and adjust $\bar{g}$ such that δ approaches zero. This requires that

$$\Delta \bar{g} \ \alpha \ - \frac{\partial f}{\partial \bar{g}} \qquad \text{A-3.5}$$

If δ is assumed to be small as compared to $\bar{g}$, δ is changed by adding $\Delta \bar{g}$ to $\bar{g}$. Since the objective is not to change $\bar{g}$ but a_1, the same change can be obtained by subtracting $\Delta \bar{g}$ from a_1. This line of reasoning results in an adaptive algorithm of the form

$$\Delta a_1 \ \alpha \ \frac{\partial f}{\partial \bar{g}} \qquad \text{A-3.6}$$

or

$$\dot{a}_1 = \frac{\partial f}{\partial \bar{g}} \qquad \text{A-3.7}$$

which is a good approximation to Equation A-3.4 as long as δ is small.

As an example, consider

$$f(E, \dot{E}, \ddot{E}) = \frac{1}{2} (\bar{g}_o E + \bar{g}_1 \dot{E} + \bar{g}_2 \ddot{E})^2 \qquad \text{A-3.8}$$

The adaptation rule, with $z = \frac{\partial x_m}{\partial \bar{g}}$ and with assumption 2, becomes

$$\dot{k}_1 \approx \dot{a}_1 = - (\bar{g}_o E + \bar{g}_1 \dot{E} + \bar{g}_2 \ddot{E}) (\bar{g}_o z + \bar{g}_1 \dot{z} + \bar{g}_2 \ddot{z})$$

Upon differentiating Equation A-3.2 with respect to $\bar{g}$ and then interchanging the order of differentiation, it is found that z satisfies the linear, nonhomogeneous differential equation

$$\ddot{z} + \bar{g} \dot{z} + g_2 z = - \dot{x}_m \qquad \text{A-3-10}$$

which is of the same form as that describing the model. The adaptation rule Equation A-3.9 is easily implemented once Equation A-3.10 is solved and E, $\dot{E}$, and $\ddot{E}$ are measured.

As long as the three assumptions remain valid and k_1 is close to its optimal value, this technique should provide correct adaptation. When k_1 is not close to its optimal value, no such statement can be made without an extensive stability analysis. The basic idea described here can be readily extended to general linear physical processes with a single input and output. To be noted, however, is the necessity of measuring E and all its derivatives unless the function to be minimized is independent of these derivatives. Also it appears that the model must be of order at least as great as the highest derivative found in $f(E, \dot{E}, \ldots)$.

4. Dressler's Algorithm

The adaptive design techniques described thus far require a certain amount of on-line computation in the synthesis of $\frac{\partial E}{\partial K}$ for the M.I.T. rule and $z = \frac{\partial \underline{x}_m}{\partial \bar{g}}$ for the Donalson algorithm. R.M. Dressler[6] has developed a technique that reduces significantly the amount of on-line computation necessary. This technique is applicable to systems described by linear differential equations of the form

$$\dot{\underline{x}}_p = A_p(t)\, \underline{x}_p + B_p(t)\, \underline{u} \qquad \text{A-4.1}$$

$$y_p = \underline{C}^T \underline{x}_p$$

which are subjected to a performance criterion that can be formulated in terms of the response of the time-invariant linear differential equation

$$\dot{\underline{x}}_m = A_m \underline{x}_m + B_m \underline{u} \qquad \text{A-4.2}$$

$$y_m = \underline{C}^T \underline{x}_m$$

The basic philosophy of Dressler's development is to first obtain an explicit functional dependence of the performance error, $e(t) = y_p(t) - y_m(t)$, on the adaptive parameters and then to determine conditions relating the incremental error, $\Delta e(t) \triangleq e(t + \Delta t) - e(t)$, and successful adaptation. It is assumed that Δt is positive and sufficiently small that any change in $\Delta e(t)$ is due only to the adjustment of the adaptive parameters and not to variations in the plant parameters, input or model response. It is further assumed that $A_p(t)$ and A_m differ by only a "small" amount and similarly for $B_p(t)$ and B_m. The significance of these assumptions is apparent upon considering the development of the adaptation rule which is briefly presented as follows.

The solution to Equation A-4.2 is

$$\underline{x}_m(t) = \Phi_m(t-t_o)\ \underline{x}_m(t_o) + \int_{t_o}^{t} \Phi_m\ (t-\tau)\ B_m\ \underline{u}(\tau)\ d\ \tau \qquad \text{A-4.3}$$

with $\Phi_m(t) = \exp\left[A_m\ t\right]$. Assuming that

$$A_p(t) = A_m + \delta A_\delta(t) \qquad \text{A-4.4}$$

$$B_p(t) = B_m + \delta B_\delta(t)$$

it can be shown[6] that

$$\underline{x}_p(t) = \Phi_m\ (t-t_o)\underline{x}_p\ (t_o) + \int_{t_o}^{t} \Phi_m(t-\tau)\ B_m\ \underline{u}(\tau)\ d\ \tau$$

$$+ \delta \int_{t_o}^{t} \Phi_m\ (t-\tau) \left[B_\delta(\tau)\ \underline{u}(\tau) + A_\delta(\tau) \left\{ \Phi_m(\tau - t_o)\ \underline{x}_p(t) \right.\right.$$

$$\left.\left. + \int_{t_o}^{t} \Phi_m\ (\tau - \xi)\ B_m\ \underline{u}\ (\xi) d\xi \right\} \right]\ d\tau + 0\ (\delta^2) \qquad \text{A-4.5}$$

in which $0\ (\delta^2)$ represents those terms containing second and higher orders of δ. From the definition of $e(t)$, Equations A-4.3 - A-4.5, and neglecting $0(\delta^2)$ based on the assumptions,

$$e(t) = \underline{c}^T \Phi_m\ (t-t_o)\ \underline{\xi}\ (t_o) + \underline{c}^T \int_{t_o} \Phi_m\ (t-\tau)$$

$$\left[\delta B_\delta\ (\tau)\ \underline{u}\ (\tau) + \delta\ A_\delta\ (\tau)\ \underline{x}_m\ (\tau) \right] \underline{d}\ \tau \qquad \text{A-4.6}$$

The design objective is taken as

$$\Delta\ e(t)\ e(t) \leq 0 \qquad \text{A-4.7}$$

Substituting Equation A-4.6 into Equation A-4.7 and rearranging results in

$$\Delta\ e(t) = h(t) + \Delta_1\ e(t) \qquad \text{A-4-8}$$

where $h(t)$ contains only terms that are not affected by adaptation for $t' > t$ and

$$\Delta_1\ e(t) = \frac{(\Delta t)^2}{2} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{C}^T\ \underline{\Phi}_{mi}\ (\Delta t)\ \dot{a}_{ij}(t) \right.$$

$$\left. x_{mj}(t) + \sum_{i=1}^{n} \sum_{j=1}^{r} \underline{C}^T\ \underline{\Phi}_{mi}\ (\Delta t)\ \dot{b}_{ij}(t)\ u_j(t) \right\} \qquad \text{A-4.9}$$

with $a_{ij}(t)$ and $b_{ij}(t)$ representing respectively the elements of $\delta A_\delta(t)$ and $\delta B_\delta(t)$. For Equation A-4. to hold, it is necessary that $h(t)\ e(t) + \Delta_1 e(t)\ e(t) \leq 0$. It is clear that $\Delta\ e_1(t)e(t) \leq 0$ if

$$\dot{a}_{ij} = -\ u'_{ij}\ \underline{C}^T\ \underline{\Phi}_{mi}\ (\Delta t)\ \ x_{mj}(t)\ e(t) \qquad \text{(A-4.10}$$

$$\dot{b}_{ij} = -\ \gamma'_{ij}\ \underline{C}^T\ \underline{\Phi}_{mi}\ (\Delta t)\ \ u_j(t)\ \ e(t)$$

where u'_{ij} and γ'_{ij} are positive constants. By choosing the adaptive gains $u_{ij} = u'_{ij}\ \underline{C}^T\ \underline{\Phi}_{mi}\ (\Delta t)$ and $\gamma_{ij} = \gamma'_{ij}\ \underline{C}^T\ \underline{\Phi}_{mi}\ (\Delta t)$ large enough, the term $\Delta_1 e(t)\ e(t)$ can be made to dominate $h(t)$. Hence, the adaptation equations are taken to be

$$\dot{a}_{ij} = - u_{ij} \; x_{mj} (t) \; e(t) \qquad \text{A-4.11}$$

$$\dot{b}_{ij} = - \gamma_{ij} \; u_j \; (t) \; e(t)$$

It can be seen from Equation A-4.11 that the only on-line computation that must be performed in the implementation of this algorithm is the calculation of the model state - $\underline{x}_m (t)$. Because of the nature of model reference adaptive control systems, this is the minimum amount of computation that can be expected. This is probably the main advantage of the algorithm.

As is true with all the design techniques discussed thus far, the adaptation rules of Equation A-4.11 provide effective adaptation as long as the basic assumptions remain valid. However, the effect of the adaptive gains on system stability and rate of adaptation must be examined in detail for the particular system under consideration. For example, in the system of Figure A-9 Dressler's adaptation rule for K_c is

$$\dot{K}_c = u \, R \, E \qquad \text{A-4.13}$$

for which a Routh-Hurwitz analysis indicates instability for

$$u > \frac{b_1}{b_2 \, K_v \, R} \qquad \text{A-4.14}$$

This is an excellent example of the trade-off between stability and rate of adaptation that is of critical importance in the overall design procedure. One possible disadvantage of Dressler's rule is the structure that requires the adaptation of the individual elements of $\delta A_\delta (t)$ and $\delta B_\delta (t)$, some of which might not be accessible in a physical multi-variable system in which control is implemented by means of a feedback structure.

5. Lyapunov Design

The necessity of an extensive stability analysis in conjunction with the implementation of any of the three design techniques examined thus far has been repeatedly emphasized. It has been postulated[7,8] that this stability analysis can be circumvented by designing the adaptive system by a Lyapunov approach. The general philosophy is to determine a positive-definite quadradic function of the system error, its derivatives and any adaptive elements which has a total time derivative which can be made negative-definite by properly choosing the adaptation rule. This guarantees system stability.

For example,[7] consider the system of Figure A-9 and the positive-definite quadradic function

$$V = \frac{b_1}{b_2^2} e^2 + \frac{b_1}{b_2} \dot{e}^2 + \lambda x^2 \qquad \text{A-5.1}$$

with $x = K - K_v K_c$. The total time derivative of Eq. A-5.1 is

$$\dot{V} = -2 \left(\frac{b_1}{b_2} \right)^2 \dot{e}^2 + 2 \left(\frac{b_1}{b_2^2} \right) \dot{e} \, x \, R + 2\lambda x \dot{x} \qquad \text{A-5.2}$$

which reduces to

$$\dot{V} = -2 \left(\frac{b_1}{b_2} \right)^2 \dot{e}^2 \qquad \text{A-5.3}$$

upon choosing $\dot{x} = -K_v \dot{K}_c = -\dfrac{b_1 \dot{e} R}{\lambda b_2^2}$.

The negative-semi-definiteness of Equation A-5.3 insures the stability of the system[9]; however, there may exist a steady-state error as $\dot{K}_c = B' \dot{e} R$ and is ndent of the system error. Lyapunov functions slightly different of Equation A-5.1 can be found that yield an adaptation rule antees asymptotic stability of the system.

A more general model reference system[8] is shown in Fig. A-10. The differential equation for the plant is

$$\left\{ D^n + (a_{11} + K_v h_1) D^{n-1} + \dots + (a_{n1} + K_v h_n) \right\} x_p = K_v K_c R \qquad \text{A-5.4}$$

and that of the model is

$$\left\{ D^n + a_1 D^{n-1} + \dots + a_n \right\} x_m = K R \qquad \text{A-5.5}$$

Defining $E = x_m - x_p$, $y_o = K - K_v K_c$, and $y_i = a_i - (a_{i1} + K_v h_i)$; $i = 1, 2, \dots n$, the differential equation for E becomes

$$\left\{ D^n + a_1 D^{n-1} + \dots + a_n \right\} E = y_o R - y_1 D^{n-1} x_p - \dots - y_n x_p \qquad \text{A-5.6}$$

Choosing

$$V = \underline{e}^T H \underline{e} + \frac{y_o}{B_o}^2 + \dots \frac{y_n}{B_n}^2 \qquad \text{A-5.7}$$

as a Lyapunov function where H is the Hermite matrix of the homogeneous part of Eq. A-5.5 and $\underline{e} = \left[E, \dot{E}, \ldots \right]^T$, the total time derivative of Eq. A-5.7 is

$$\dot{V} = -2 \; Z_n^2 + 2Z_n \left(R \; y_o - y_1 \; D^{n-1} \; x_p - \ldots - y_n \; x_p\right)$$

$$+ 2 \left(\frac{y_o \dot{y}_o}{B_o} + \ldots + \frac{y_n \dot{y}_n}{B_n} \right) \qquad \text{A-5.8}$$

Z_n being defined as $a_1 \; D^{n-1} \; E + a_3 \; D^{n-3} \; E + \ldots$. If

$$\dot{y}_o = - B_o \; Z_n \; R$$

$$\dot{y}_1 = B_1 \; Z_n \; D^{n-1} \; x_p \qquad \text{A-5.9}$$

$$\dot{y}_n = B_n Z_n \; x_p \quad ,$$

$$\dot{V} = -2 \; Z_n^{\;2} \qquad \text{A-5.10}$$

If all of the B_i are positive and H is positive definite, i.e., the model is stable, V is positive definite and $\dot{V}$ is negative semi-definite resulting in a stable system. Furthermore, if K_v is positive and varies slowly (if at all) and the a_{i1} vary slowly, Equation A-5.9 reduces to

$$\dot{K}_c = B_o' \; Z_n \; R$$

$$\dot{h}_i = - B_i' \; Z_n \; D^{n-i} \; x_p \qquad \text{A-5.11}$$

This Lyapunov design technique can be extended to systems in which the plant is of higher order than the model and to plants containing numerator zeroes. The basic shortcoming of this technique is the necessity of measuring not only the system output but all of its derivatives, often not available in a physical system. It is important to note the derivation of A-5.11 is based on the slow variation of the plant parameters K_v, a_{11}, ... and that stability is only guaranteed when these assumptions are valid. However, it is reasonable to assume that this Lyapunov approach will be dependable even if these assumptions are not strictly satisfied.

6. Stability of Model Reference Adaptive Control Systems

Probably the single most important aspect of model reference adaptive control is whether or not the physical plant output converges to that of the model and the rate at which this convergence takes place. This is identical to determining the conditions for the stability of the differential equation for the system error. This differential equation is generally non-linear and time-dependent and any stability analysis presents a rather formidable problem. Previously in this appendix, the Routh-Hurwitz criterion has been applied for determining conditions for stability. However, this method is applicable to only the simplest of single input-single output adaptive systems and will not be pursued further.

Donalson[5] and Dressler[6] have applied the Second Method of Lyapunov[9] to the determination of stability conditions for model reference adaptive systems. To illustrate the application of this method, consider the first order process[6] described by the differential equation.

$$\dot{x}_p(t) = \left[- \alpha (t) - \tilde{f} (t) \right] x_p(t) + u(t) \qquad \text{A-6.1}$$

and its associated model described by

$$\dot{x}_m(t) = - \hat{f} \; x_m(t) + u(t) \; ; \quad \hat{f} > 0 \qquad \text{A-6.2}$$

Equation A-6.1 can be rewritten as

$$\dot{x}_p(t) = \left[- \hat{f} + f(t) \right] \; x_p(t) + u(t) \qquad \text{A-6.3}$$

by defining $f(t) = \hat{f} - \alpha(t) - \tilde{f}(t)$ and the adaptation rule based on Dressler's algorithm is

$$\dot{\alpha}(t) = u' \; x_m(t) \; e(t); \quad u' > 0 \qquad \text{A-6.4}$$

The coupling between the control system, Equation A-6.3, and the adaptation mechanism, Equation A-6.4, can be described by the two-dimensional state vector

$$\underline{\beta} = \begin{bmatrix} f(t) \\ e(t) \end{bmatrix} \qquad \text{A-6.5}$$

Recalling the assumption that $\dot{\tilde{f}}(x) \approx 0$, the state differential equation becomes

$$\dot{\underline{\beta}} = \begin{bmatrix} 0 & -u \; x_m(t) \\ x \; (t) & - \hat{f} \end{bmatrix} \underline{\beta} + \begin{bmatrix} 0 \\ f \; e \end{bmatrix} \qquad \text{A-6.6}$$

The equilibrium state $\underline{\beta} = \underline{0}$ requires $e(t) = 0$ and $f(t) = 0$ or $\alpha(t) + \tilde{f}(t) = \hat{f}$. The stability of this equilibrium point can be investigated by considering as a Lyapunov function

$$V = u' \; e^2 + f^2 \qquad \text{A-6.7}$$

The total time derivative of Equation A-6.7

$$\dot{V} = -2\, u'\, e^2\, (\hat{f} - f) \qquad \text{A-6.8}$$

is negative-definite only for $\hat{f} > f$. Thus the equilibrium point $\underline{\beta} = \underline{0}$ is stable for $\hat{f} > f$ and as long as this condition is satisfied, the adaptive mechanism of Equation A-6.4 will tend to null any error between the plant and the model. However, as Lyapunov's Second Method yields only a sufficient condition for stability, nothing definite can be said about stability for $\hat{f} < f$.

The major problem in applying Lyapunov's Second Method to high order systems is the lack of any well-defined methods for constructing suitable Lyapunov functions. This problem is compounded in the study of model reference systems as the state vector that must be considered is of order equal to the sum of the plant states and the adaptive elements. However, this method seems to be the only presently available technique, in theory at least, to determine the regions of stability for such systems.

7. Summary

Several methods for the design of model reference adaptive control systems have been reviewed with the aim of providing insight into the philosophy of each. The advantages and disadvantages of each method have been discussed briefly as has the type of system to which each is applicable. It is important to reiterate once again the importance of a thorough stability analysis or simulation study in the [illegible] design procedure.

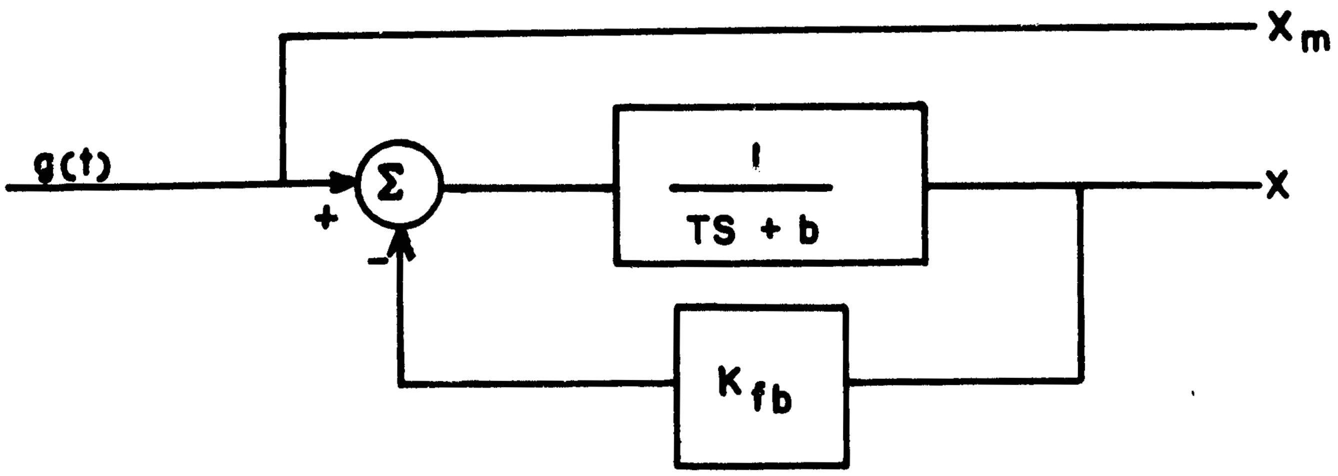

FIG. A- 1a

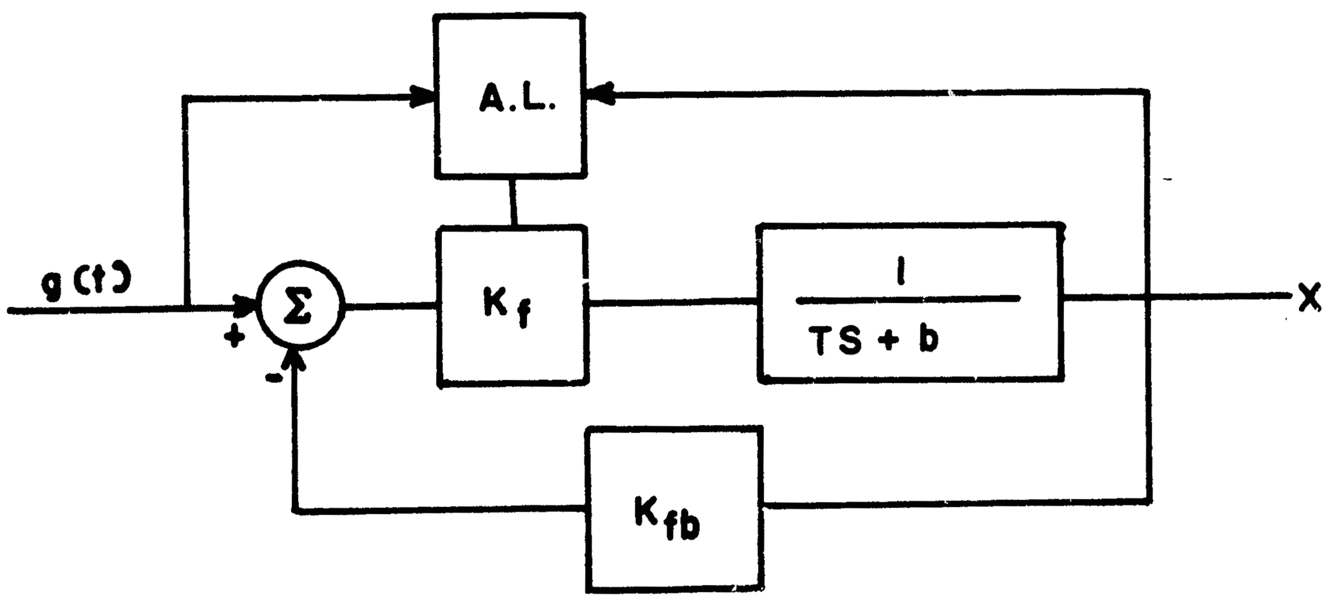

FIG. A- 1b

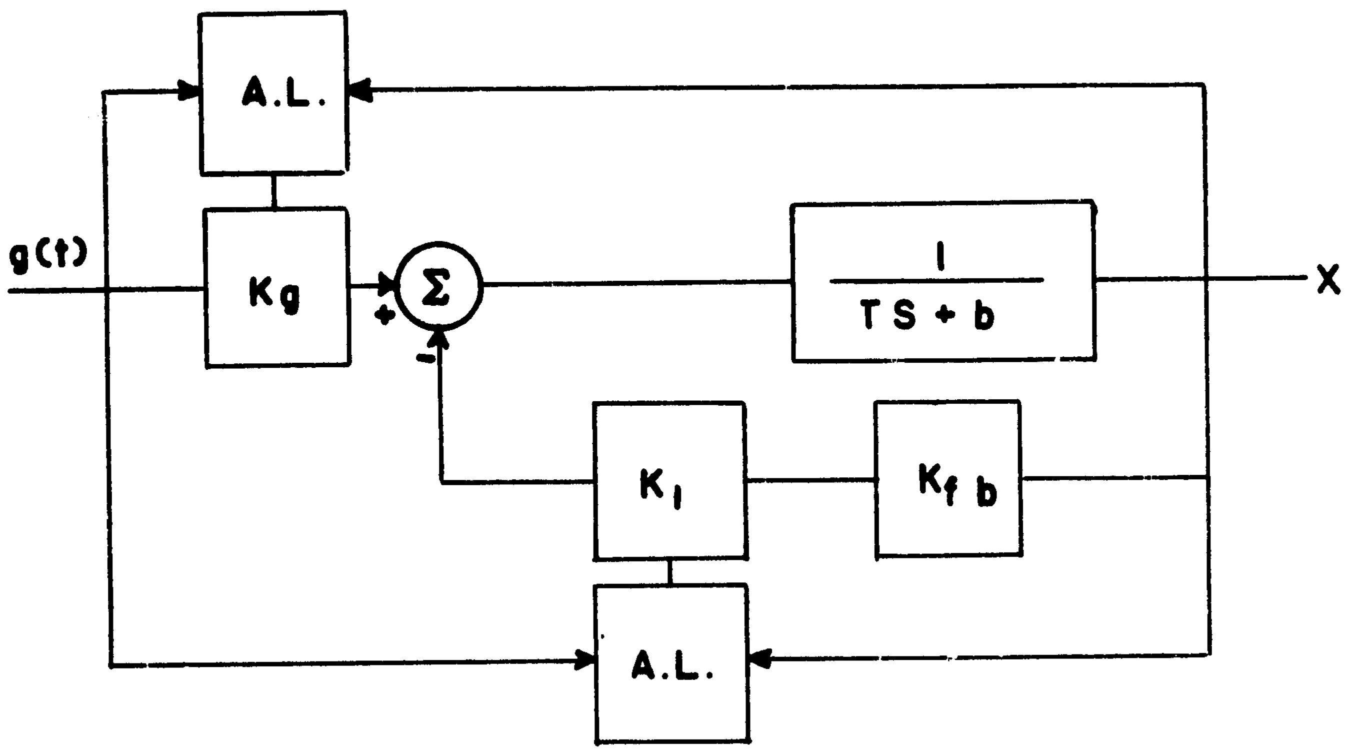

FIG. A-1c

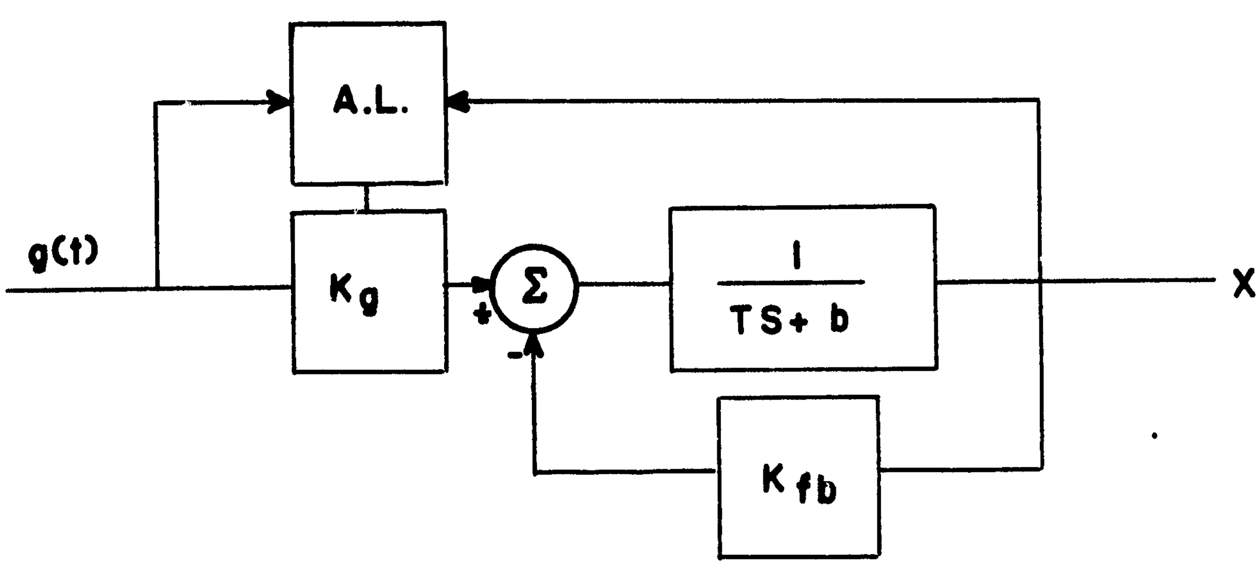

FIG. A-1d

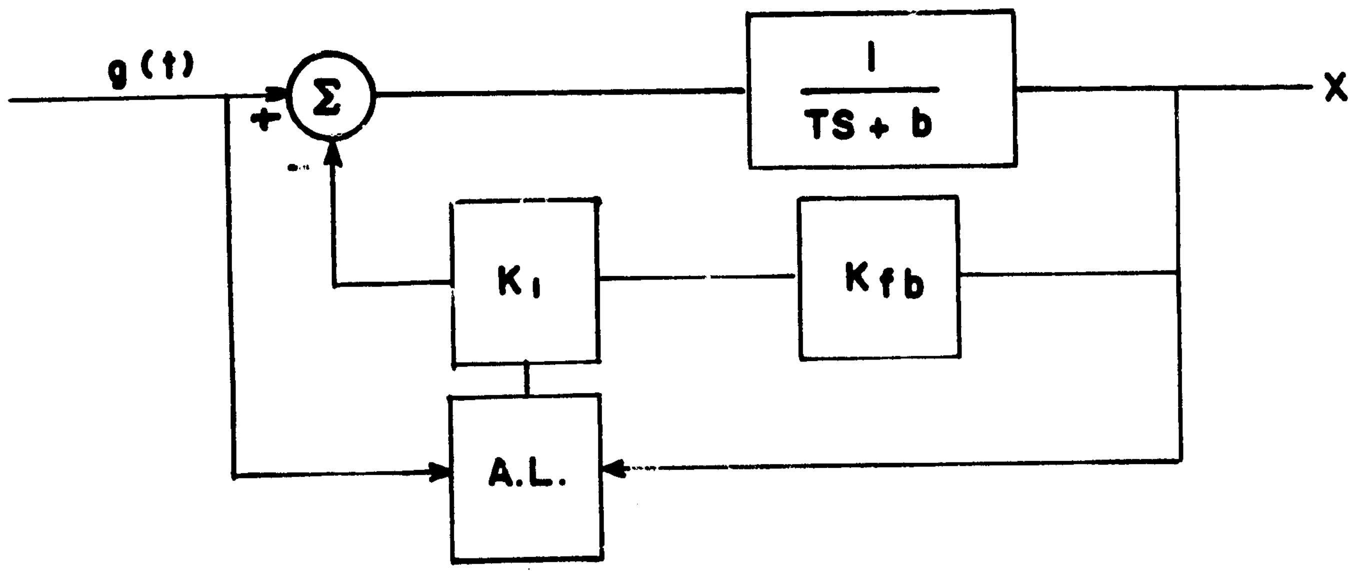

FIG. A-1e

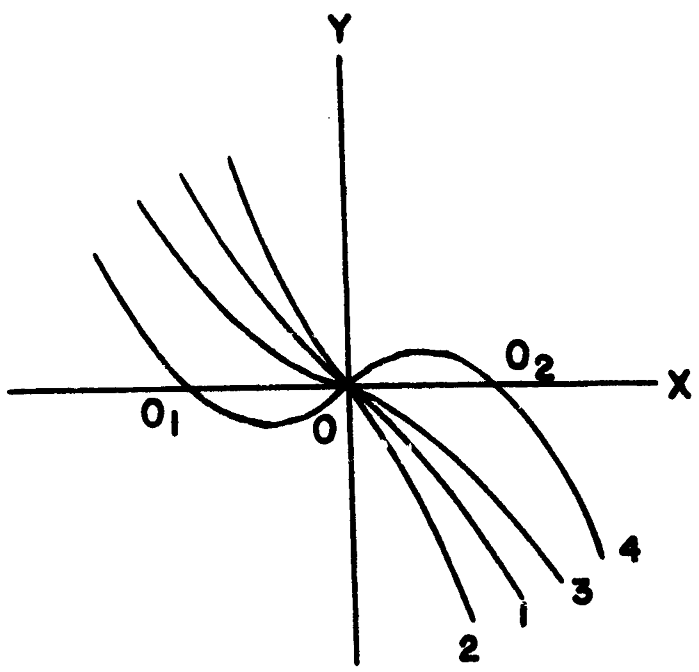

FIG. A-2

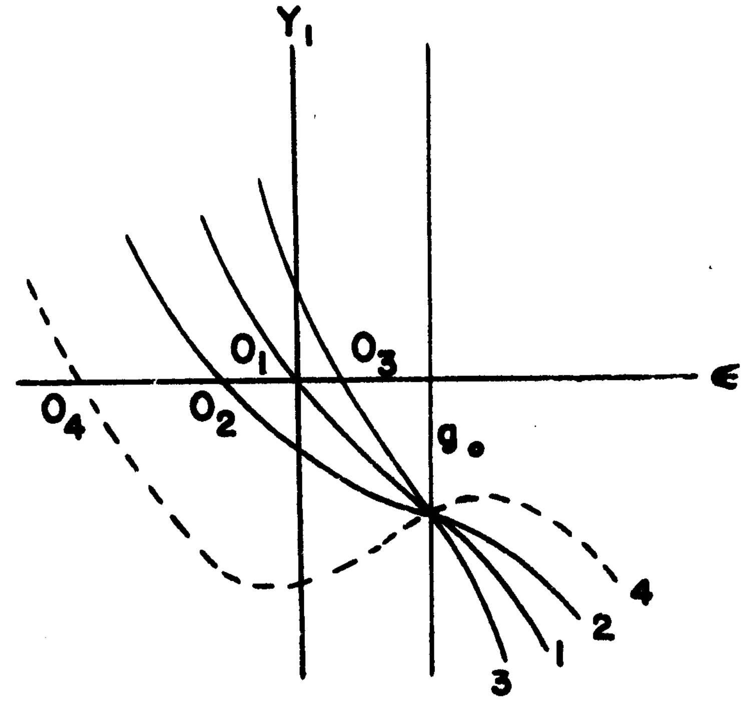

FIG. A-3

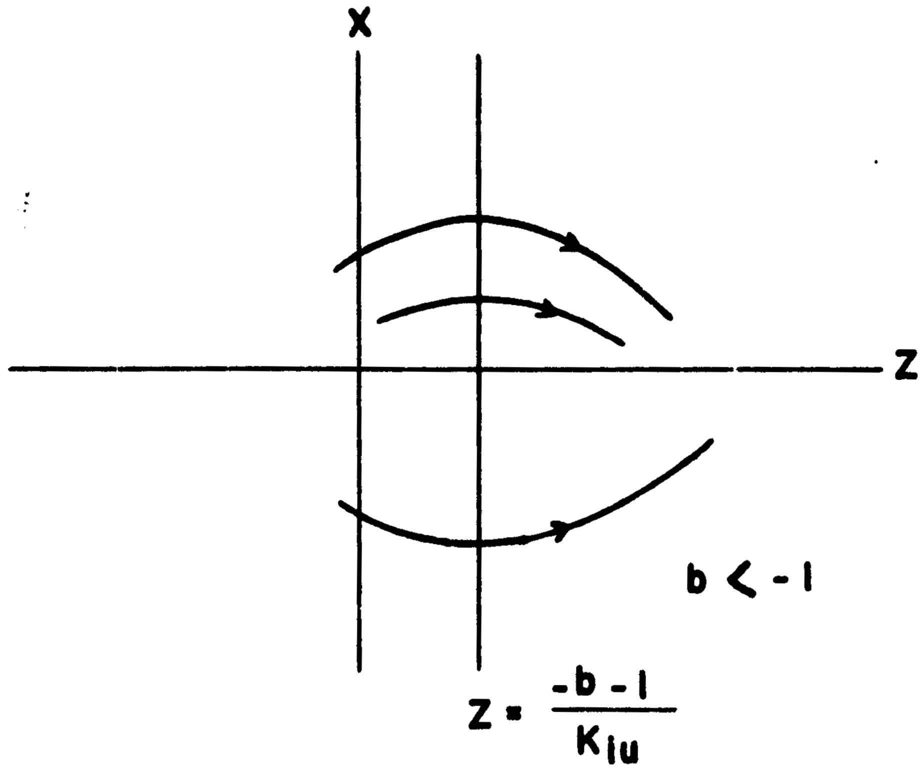

FIG. A-4

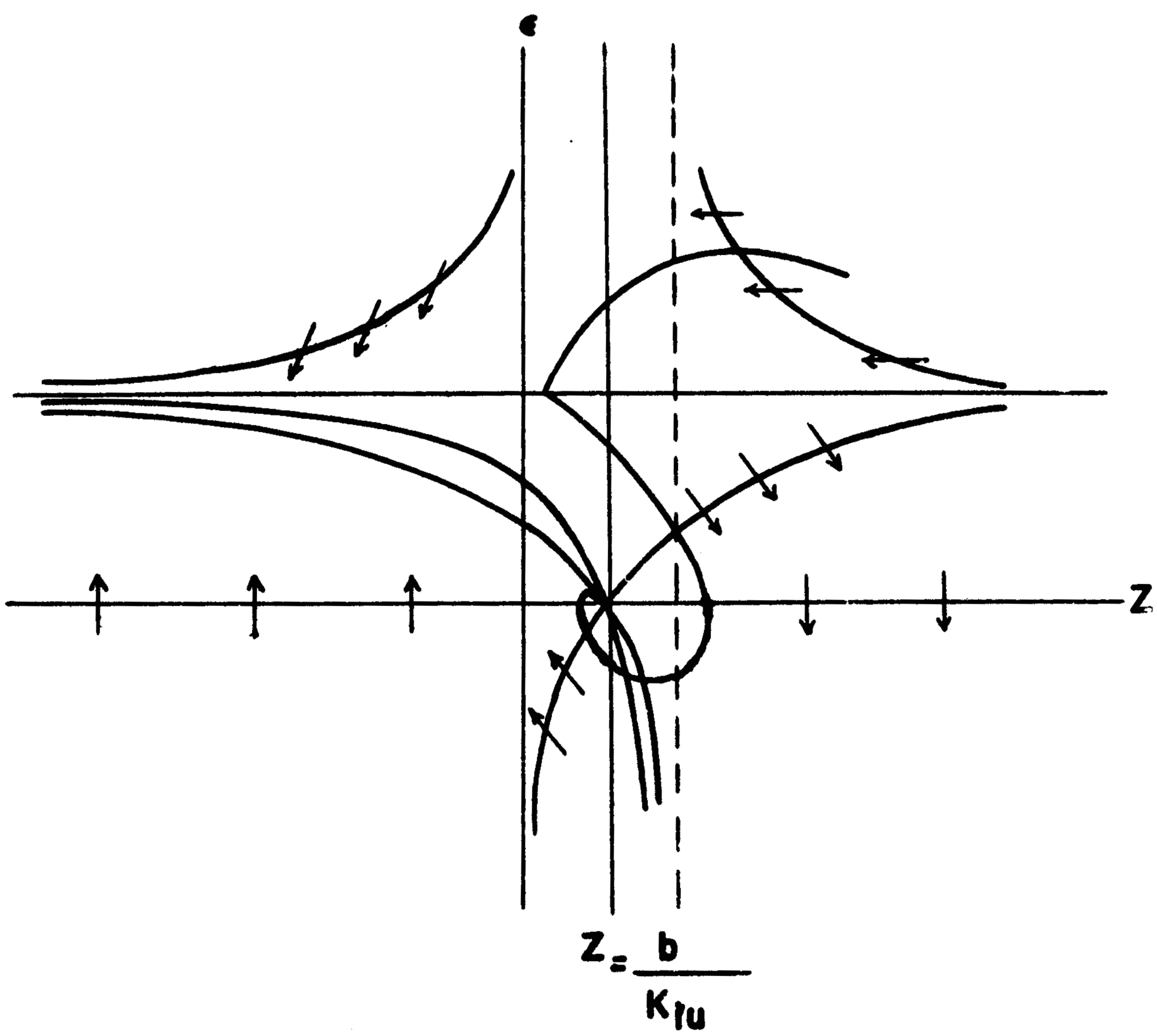

FIG. A - 5

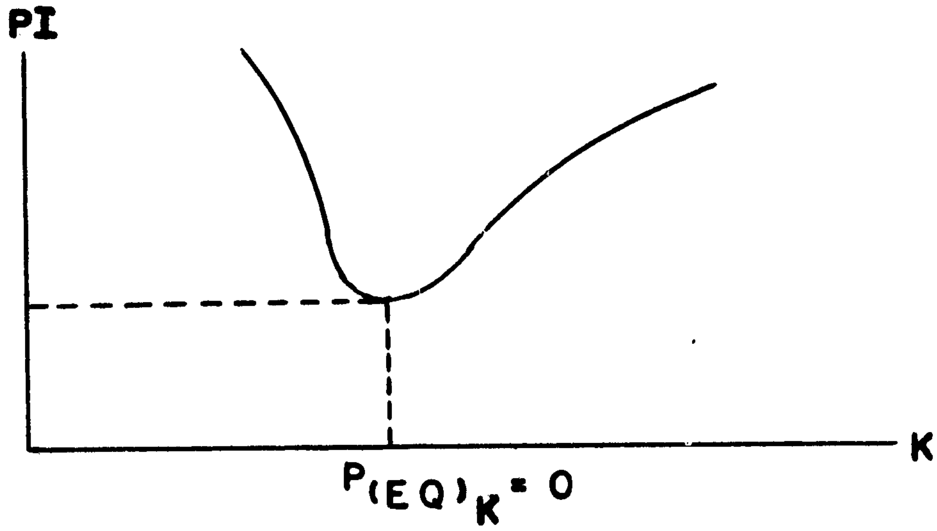

FIG. A-6

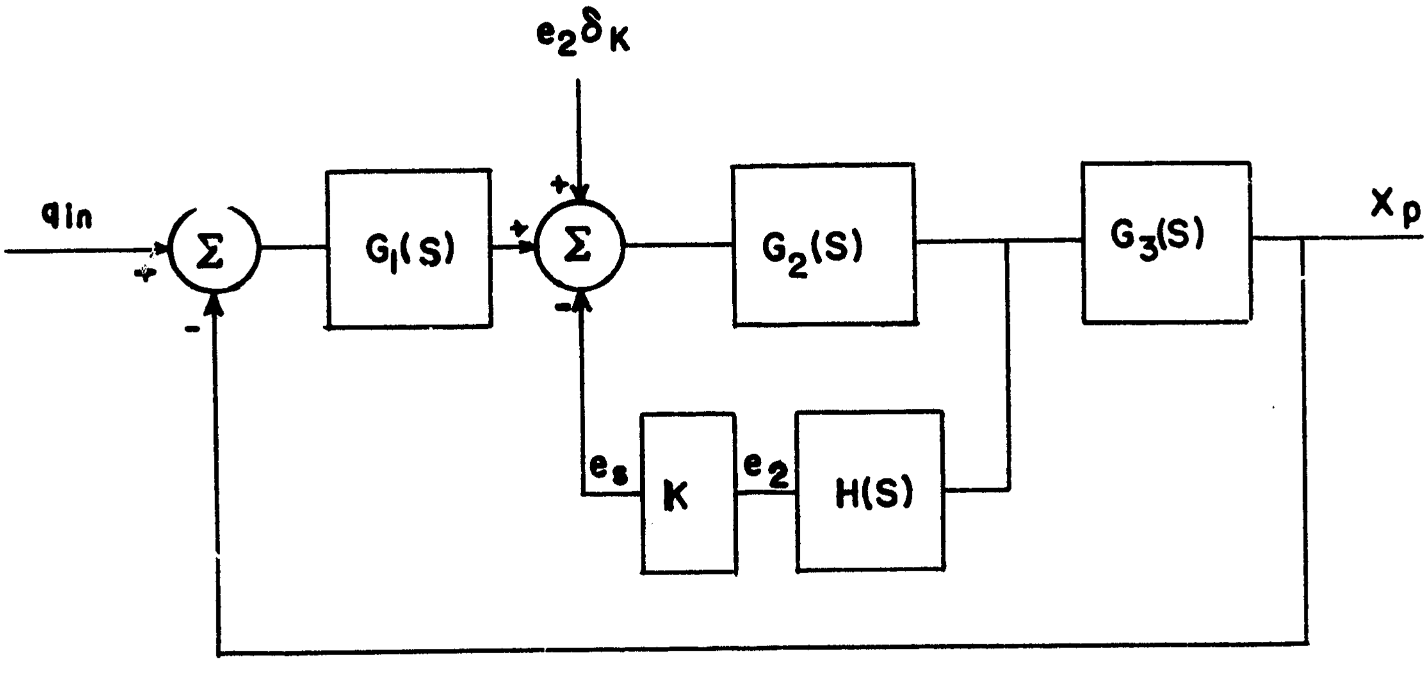

FIG. A-7

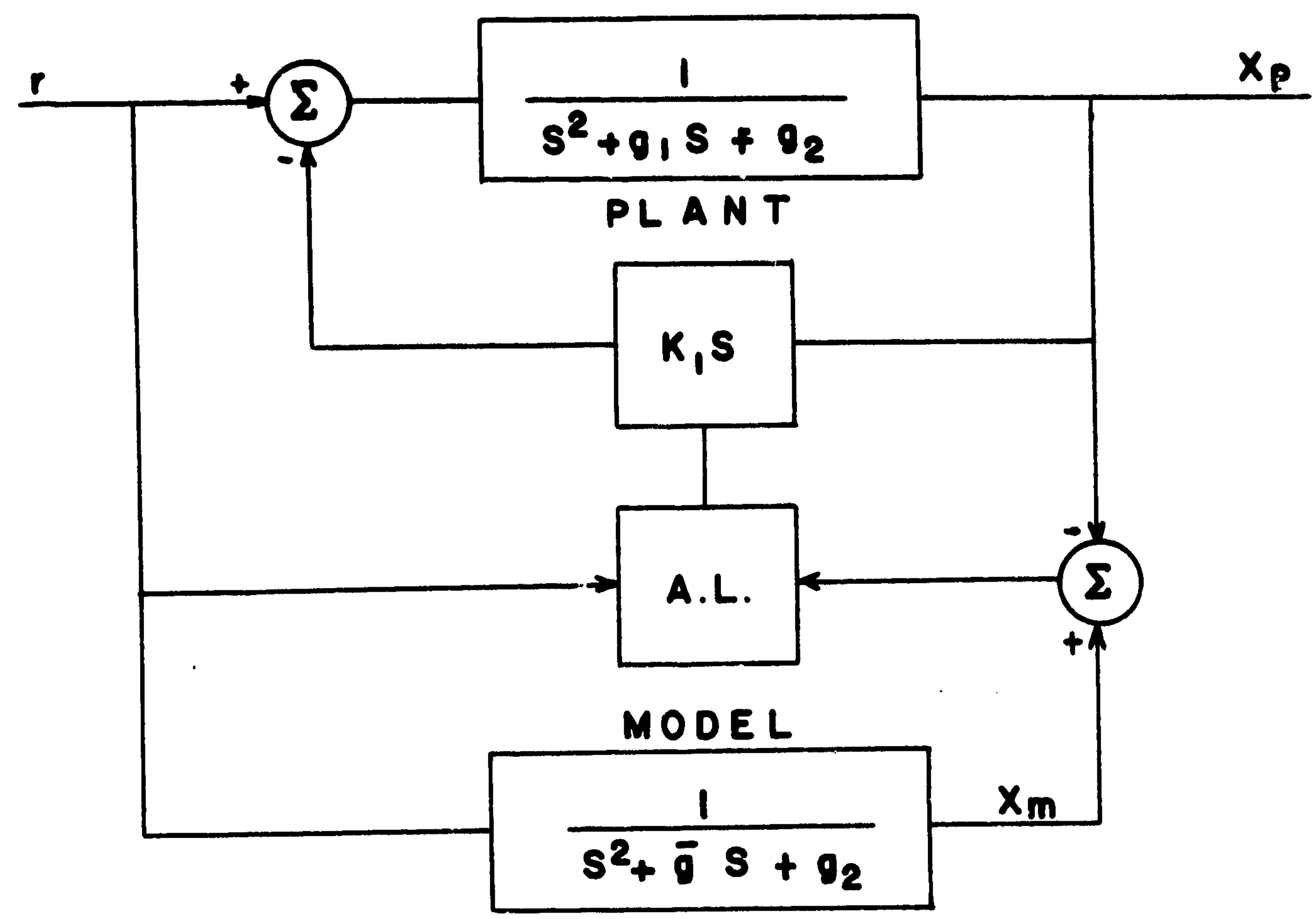
r
+
-
1
$S^2 + g_1 S + g_2$
PLANT
Xp
$K_1 S$
A.L.
-
+
MODEL
1
$S^2 + \bar{g} S + g_2$
Xm

FIG. A-8

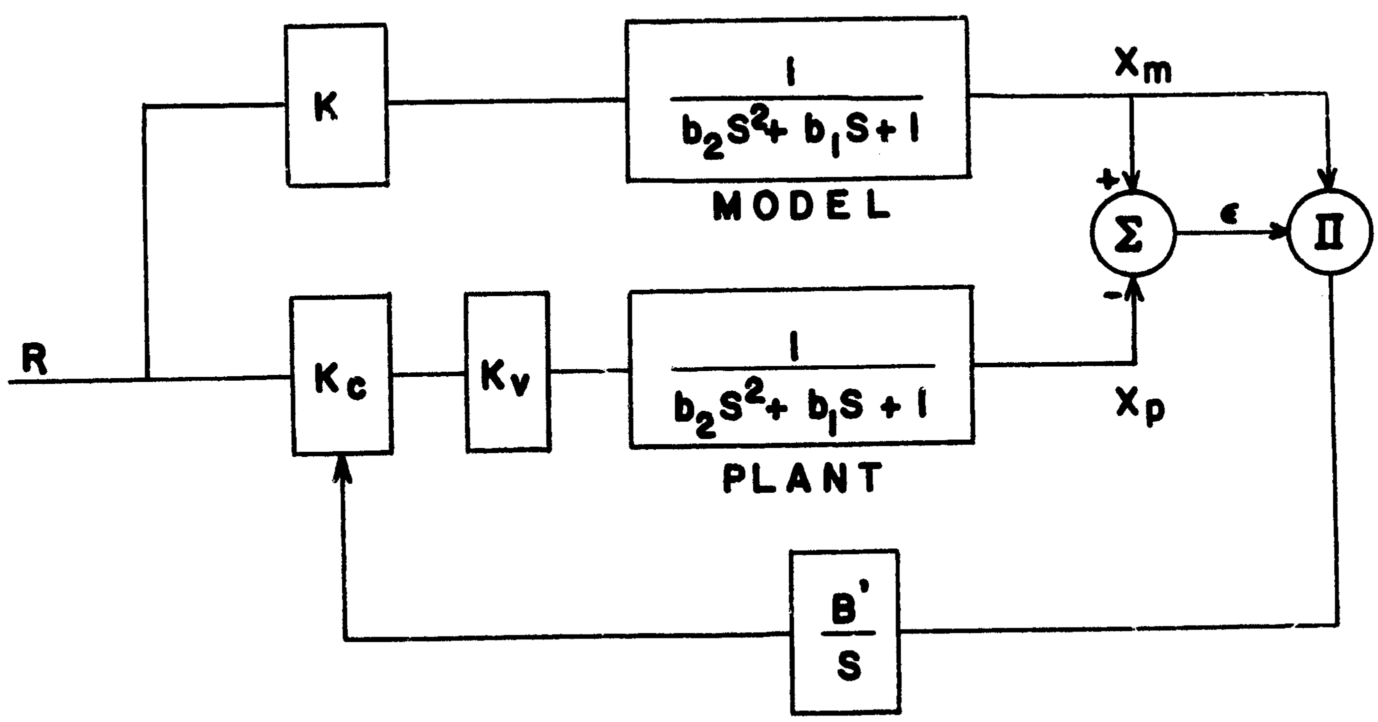
K
1
$b_2 S^2 + b_1 S + 1$
MODEL
Xm
+
-
ε
R
Kc
Kv
1
$b_2 S^2 + b_1 S + 1$
PLANT
Xp
B'
S

FIG. A-9

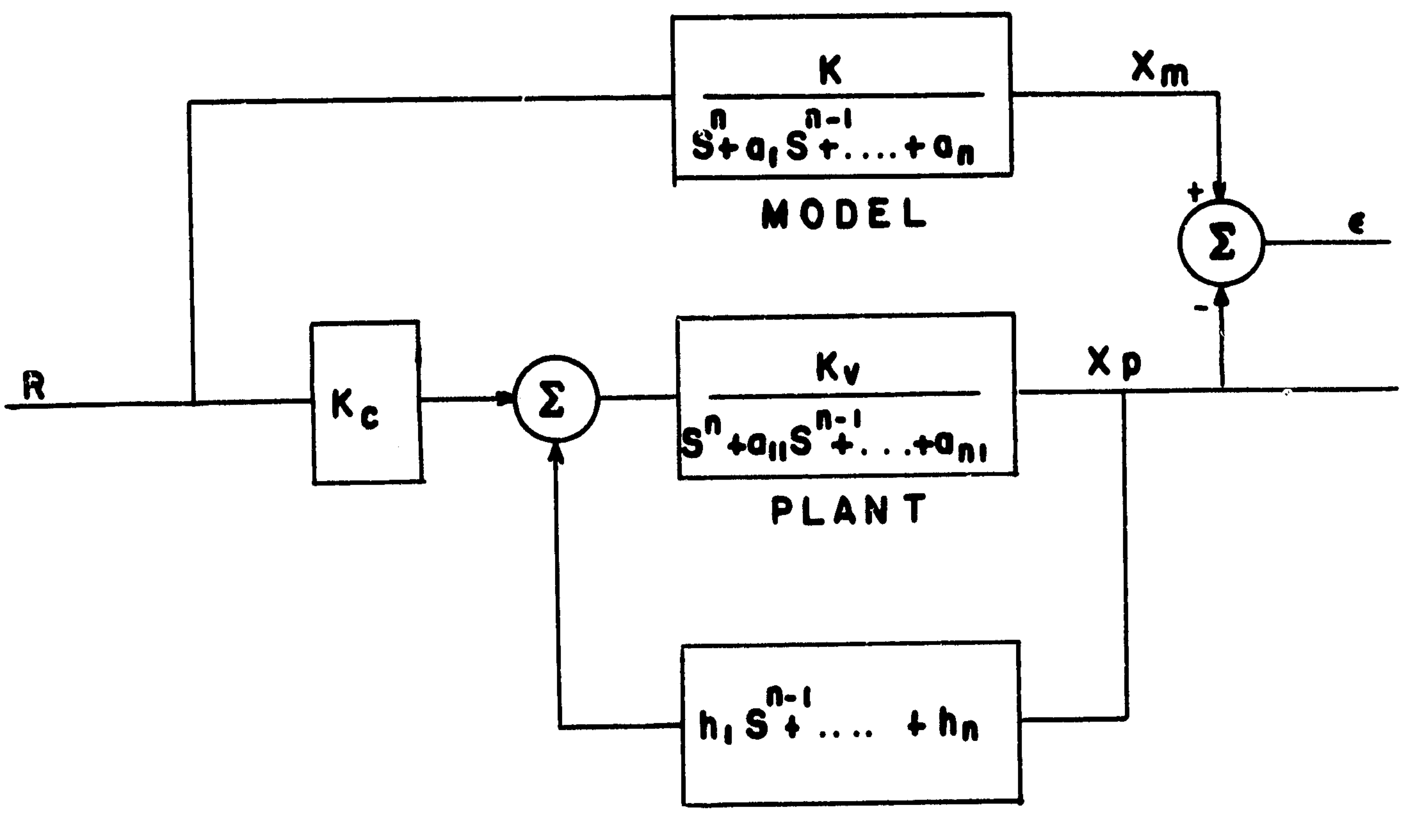
K
$S^n + a_1 S^{n-1} + \ldots + a_n$
MODEL
Xm
+
Σ
-
ε
R
Kc
Σ
Kv
$S^n + a_{11} S^{n-1} + \ldots + a_{n1}$
PLANT
Xp
$h_1 S^{n-1} + \ldots + h_n$

FIG.A-10

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Appendix B

Proof that $\dot{z}_i = \frac{\partial}{\partial \hat{k}_i} = \dot{\underline{x}}$

$$\underline{z}_i = \frac{\partial}{\partial \hat{k}_i} \underline{x} \qquad \text{B-1}$$

$$\dot{\underline{z}}_i = \frac{\partial}{\partial t} \frac{\partial}{\partial \hat{k}_i} \underline{x} \qquad \text{B-2}$$

$$\dot{\underline{x}} = \frac{\partial}{\partial t} \underline{x} \qquad \text{B-3}$$

$$\frac{\partial}{\partial \hat{k}_i} \dot{\underline{x}} = \frac{\partial}{\partial \hat{k}_i} \frac{\partial}{\partial t} \underline{x} \qquad \text{B-4}$$

$$\therefore \quad \dot{\underline{z}}_i - \frac{\partial}{\partial \hat{k}_i} \dot{\underline{x}} = \frac{\partial}{\partial t} \frac{\partial}{\partial \hat{k}_i} \underline{x} - \frac{\partial}{\partial \hat{k}_i} \frac{\partial}{\partial t} \underline{x} \qquad \text{B-5}$$

But for linear systems of the form

$$\dot{\underline{x}}(t) = A(t)\, \underline{x}(t) + B(t) \left\{ \underline{u}(t) - K(t)\, \underline{x}(t) \right\} \qquad \text{B-6}$$

in which $A(t)$, $B(t)$, $K(t)$ and $\underline{u}(t)$ are continuous functions,

$$\underline{x}(t) = \Phi(t,t_o)\, \underline{x}(t_o) + \int_{t_o}^{t} \Phi(t,\tau)\, B(\tau) \left\{ \underline{u}(\tau) - K(\tau)\underline{x}(\tau) \right\} d\tau \qquad \text{B-7}$$

From this,

$$\frac{\partial}{\partial \hat{k}_i} \underline{x}(t) = - \int_{t_o}^{t} \Phi(t,\tau)\, B(\tau)\, x_j(\tau)\, \underline{1}_k \, d\tau \qquad \text{B-8}$$

for $\hat{k}_i$ - K (k,j) and

$$\frac{\partial}{\partial t} \frac{\partial}{\partial \hat{k}_i} \underline{x}(t) = - A(t) \int_{t_o}^{t} \Phi(t,\tau)\, B(\tau)\, x_j(\tau)\, \underline{1}_k \, d\tau$$

$$- B(\tau)\, x_j(\tau)\, \underline{1}_k \qquad \text{B-9}$$

Now since $\underline{x}(t)$, $\frac{\partial}{\partial t} \underline{x}(t)$, $\frac{\partial}{\partial \hat{k}_i} \underline{x}(t)$, and $\frac{\partial}{\partial t} \frac{\partial}{\partial \hat{k}_i} \underline{x}(t)$ are continuous,

$$\frac{\partial}{\partial \hat{k}_i} \frac{\partial}{\partial t} \underline{x}(t) = \frac{\partial}{\partial t} \frac{\partial}{\partial \hat{k}_i} \underline{x}(t) \qquad \text{B-10}$$

or

$$\dot{\underline{z}}_i = \frac{\partial}{\partial \hat{k}_i} \dot{\underline{x}} \qquad \text{B-11}$$

Appendix C

Derivation of Equation 2.4-16

Plant: $$\dot{\underline{x}}_p = A_p \underline{x}_p + B_p \underline{u}_p \qquad \text{C-1}$$

$$\underline{y}_p = C \underline{x}_p$$

Model: $$\dot{\underline{x}}_m = A_m \underline{x}_m + B_m \underline{u}_m \qquad \text{C-2}$$

$$\underline{y}_m = C \underline{x}_m$$

Error: $$\underline{\mathcal{E}} = \underline{x}_m - \underline{x}_p \qquad \text{C-3}$$

$$\underline{e} = y_m - y_p$$

$$e = C \underline{\mathcal{E}}$$

Control: $$\underline{u}_p = \underline{u}_m + \Delta \underline{u} \qquad \text{C-4}$$

$$\underline{u}_p = \underline{u}_m - K \underline{e}$$

Index: $$J = \frac{1}{2} \int \left[\underline{e}^T Q \underline{e} + \Delta \underline{u}^T R \Delta \underline{u} \right] dt \qquad \text{C-5}$$

The minimization of J with respect to $\Delta \underline{u}$ by the method of steepest descent requires the determination of the gradient of J with respect to $\hat{k}$, the vector representation of the elements of K that can be adjusted. With this in mind, it is seen that

$$\frac{\partial J}{\partial \hat{k}_i} = \int \left[\underline{e}^T Q \frac{\partial \underline{e}}{\partial \hat{k}_i} + \Delta \underline{u}^T R \frac{\partial \Delta \underline{u}}{\partial \hat{k}_i} \right] dt \qquad \text{C-6}$$

This expression can be simplified by observing that

$$\frac{\partial \underline{e}}{\partial \hat{k}_i} = - \frac{\partial \underline{y}_p}{\partial \hat{k}_i} = - C \frac{\partial \underline{x}_p}{\partial \hat{k}_i} = - C \underline{z}_i \qquad \text{C-7}$$

and that

$$\frac{\partial \Delta \underline{u}}{\partial \hat{k}_i} = K C \underline{z}_i - e_k \underline{1}_j \qquad \text{C-8}$$

for $\hat{k}_i = K(j, k)$. Thus

$$\frac{\partial J}{\partial \hat{k}_i} = - \int \left\{ \underline{e}^T \left[Q + K^T R K \right] C \underline{z}_i - \underline{e}^T K^T R \underline{1}_j e_k \right\} dt \qquad \text{C-9}$$

Differentiating the differential equation describing the plant partially with respect to $\hat{k}_i$ results in

$$\frac{\partial}{\partial \hat{k}_i} \dot{\underline{x}}_p = A_p \frac{\partial}{\partial \hat{k}_i} \underline{x}_p + B_p \left[K C \underline{z}_i - e_k \underline{1}_j \right]$$

or

$$\underline{z}_i = \left[A_p + B_p K C \right] \underline{z}_i - B_p \underline{1}_j e_k \qquad \text{C-10}$$

Thus the ideal adaptation rule for minimizing this cost index by the path of steepest descent is

$$\dot{\hat{k}}_i = \beta_i \left\{ \underline{e}^T \left[Q + K^T R K \right] C \underline{z}_i - \underline{e}^T K^T R \underline{1}_j e_k \right\} \qquad \text{C-11}$$

for $\hat{k}_i = K\ (j, k)$. However, $\underline{z}_i$ is a function of A_p and B_p, both of which may be unknown. Thus, an approximation is made and the adaptation rule is assumed to be

$$\dot{\hat{k}}_i = \beta_i \left\{ \underline{e}^T \left[Q + K^T R K \right] C \hat{\underline{z}}_i - \underline{e}^T K^T R \underline{1}_j e_k \right\} \qquad \text{C-12}$$

with $\hat{\underline{z}}_i$ the solution of

$$\dot{\hat{\underline{z}}}_i = \left[A_m + B_m K C \right] \hat{\underline{z}}_i - B_m \underline{1}_j e_k \qquad \text{C-13}$$

Appendix D

Two Observations on the Convergence of Linear Model Reference Trajectories

The intent of this appendix is to illustrate two interesting and important observations concerning the convergence of plant trajectories to those of an associated model for general linear, time-invarient systems.

Consider first the plant described by the differential equation

$$\dot{\underline{x}}_p = A_p \underline{x}_p + \underline{b}\, u_p \qquad \text{D-1}$$

which is to be designed to track the model described by

$$\dot{\underline{x}}_m = A_m \underline{x}_m + \underline{b}\, u_m \qquad \text{D-2}$$

Assuming that $u_p = u_m + \Delta u$, it is seen that the differential equation for the error, $\underline{e} = \underline{x}_m - \underline{x}_p$, is

$$\dot{\underline{e}} = A_m \underline{e} + \left[A_m - A_p\right] \underline{x}_p - \underline{b}\, \Delta u \qquad \text{D-3}$$

If the error is nulled and is to remain nulled, $\dot{\underline{e}} = \underline{e} = \underline{0}$ or

$$\left[A_m - A_p\right] \underline{x}_p - \underline{b}\, \Delta u = \underline{0} \qquad \text{D-4}$$

Equation D-4 yields n equations for Δu which are not generally consistent. Thus it is not, in general, possible to totally null the error between plant and model for a linear system. Examination of Equation D-4 indicates two possible conditions for which the error can be nulled: 1) when the plant and model state matrices are in phase-variable form and 2) when a stable regulator problem is considered.

Consider now the case for which $\Delta u = - \underline{K}^T \underline{x}_p$. Under steady-state conditions,

$$\dot{\underline{x}}_p = \left[A_p - \underline{b}\, \underline{K}^T \right] \underline{x}_p + \underline{b}\ u_m = 0 \qquad \text{D-5}$$

From this it is seen that for $u_m = 1.0$

$$\frac{x_i}{x_j} = \frac{\left| A_p - \underline{b}\, \underline{K}^T \right|_i}{\left| A_p - \underline{b}\, \underline{K}^T \right|_j} = \frac{\left| A_p \right|_i}{\left| A_p \right|_j} \qquad \text{D-6}$$

where $\left| A_p \right|_i = \left| \underline{a}_1\ \underline{a}_2\ \cdots\ \underline{a}_{i-1}\ \underline{b}\ \underline{a}_{i+1}\ \cdots\ \underline{a}_n \right|$. Thus the ratio $x_i : x_j$ is independent of the feedback gain matrix and dependent only on the plant parameters A_p and $\underline{b}$. A similar result is obtained for $\Delta u = - \underline{K}^T \underline{e}$.

For the plant considered in Example 2,

$$A_p = \begin{bmatrix} 0.4 & 1.6 \\ -2.1 & -4.4 \end{bmatrix} \quad \text{and} \quad \underline{b} = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix}$$

which results in $\frac{x_1}{x_2} = -2.4$. Since $x_{1mss} = 3.0$ and $x_{2mss} = -2.0$,

$$E^2 = e_1^2 + e_2^2 = (3.0 - x_{1pss})^2 + (-2.0 - x_{2pss})^2$$

$$E^2 = (3.0 + 2.4\, x_{2pss})^2 + (-2.0 - x_{2pss})^2$$

$$E^2 = 6.76\, x_{2pss}^2 + 18.4\, x_{2pss} + 13.0 \qquad \text{D-7}$$

This expression is minimized for

$$\frac{dE^2}{dx_{2pss}} = 13.52\, x_{2pss} + 18.4 = 0 \qquad \text{D-8}$$

or $x_{2pss} = -1.36$ and $x_{1pss} = 3.26$. Hence, these values of x_{1p} and x_{2p} would result in minimum integral square if the observation time is relatively long. It is interesting to note that the continuous gradient adaptation rule with $\Delta u = -\underline{K}^T\underline{e}$ achieves values of x_{1p} and x_{2p} that are very close to these optimum values.

Appendix E

Mathematical Description of the Saturn V Booster

E.1 Definition of Symbols

A	Cross-sectional reference area
$C_{N\alpha}$	Drag coefficient
D	Drag force
I	Pitch plane moment of inertia about the vehicle center of gravity
ℓ_{CG}	Distance from vehicle center of gravity to gimbal point, i.e., $X_{CG} - X_{\beta}$
ℓ_{CP}	Distance from vehicle center of gravity to center of pressure, i.e., $X_{CG} - X_{CP}$
m	Total mass of vehicle
m_i	Generalized mass of i^{th} bending mode
N	Aerodynamic force
N'	Aerodynamic force coefficient, i.e., $N' = C_{N\alpha} Aq$
q	Dynamic pressure
R'	Thrust of control engines
T	Total thrust of engines
V	Velocity of vehicle
V_R	Velocity relative to wind
V_W	Velocity of wind
X_{CG}	Station of center of gravity
X_{CP}	Station of center of pressure
X_{β}	Station of gimbal

X_D	Station of position gyro
X_{RG}	Station of rate gyro
$Y_i(X)$	Normalized displacement of the i^{th} bending mode at station X
$Y_i'(X)$	Normalized slope at station X due to the i^{th} bending mode, i.e., $\frac{d}{dx} Y_i(X)$
Y	Direction normal to reference
α	Angle of attack
α_ω	Angle of attack due to wind
β	Total engine deflection
ζ_i	Damping ratio of i^{th} bending mode
η_i	Generalized displacement of i^{th} bending mode
ψ	Attitude angle
ω_i	Natural frequency of i^{th} bending mode

E.2 Mathematical Model

The time-varying model of the Saturn V used in this study can be represented by Eqs. 3.6-2 through 3.6-10. The values of the time-varying elements of the A and C matrices were calculated at intervals of four seconds and are tabulated on the following pages. Linear interpolation was used to determine values of the coefficients for times other than those listed.

TIME (SEC)	THRUST (N)	DRAG (N)	VELOCITY (M/SEC)	Q (N/M)	CNALPHA
0.00	3.38509E 07	3.53040E 04	0.00000E-01	0.00000E-01	4.60000E 00
4.00	3.38626E 07	4.09890E 04	1.02900E 01	6.25000E 01	4.60000E 00
8.00	3.38989E 07	5.97750E 04	2.15800E 01	2.73500E 02	4.60000E 00
12.00	3.39620E 07	9.35160E 04	3.39400E 01	6.69800E 02	4.60000E 00
16.00	3.40534E 07	1.33627E 05	4.74200E 01	1.28890E 03	4.61000E 00
20.00	3.41742E 07	1.78357E 05	6.21300E 01	2.16840E 03	4.62000E 00
24.00	3.43254E 07	2.24660E 05	7.81700E 01	3.34370E 03	4.63000E 00
28.00	3.45071E 07	2.59467E 05	9.56900E 01	4.84720E 03	4.68000E 00
32.00	3.47186E 07	2.82588E 05	1.14890E 02	6.70820E 03	4.69000E 00
36.00	3.49584E 07	2.84786E 05	1.35970E 02	8.94460E 03	4.70000E 00
40.00	3.52241E 07	3.16057E 05	1.59150E 02	1.15600E 04	4.76000E 00
44.00	3.55119E 07	3.83726E 05	1.84610E 02	1.45280E 04	4.80000E 00
48.00	3.58171E 07	5.02170E 05	2.12520E 02	1.77990E 04	4.90000E 00
52.00	3.61339E 07	7.73218E 05	2.42930E 02	2.12750E 04	5.08000E 00
56.00	3.64560E 07	1.14050E 06	2.75840E 02	2.48160E 04	5.38000E 00
60.00	3.67763E 07	1.71583E 06	3.11270E 02	2.82650E 04	5.70000E 00
64.00	3.70882E 07	2.02296E 06	3.49260E 02	3.14300E 04	5.62000E 00
68.00	3.73853E 07	2.06992E 06	3.90520E 02	3.41980E 04	5.38000E 00
72.00	3.76622E 07	1.91976E 06	4.35880E 02	3.64170E 04	5.08000E 00
76.00	3.79139E 07	1.76683E 06	4.85710E 02	3.76990E 04	4.50000E 00
80.00	3.81348E 07	1.61020E 06	5.40220E 02	3.76130E 04	4.40000E 00
84.00	3.83214E 07	1.44991E 06	5.99450E 02	3.58840E 04	4.82000E 00
88.00	3.84723E 07	1.24739E 06	6.63670E 02	3.26150E 04	4.30000E 00
92.00	3.85877E 07	1.02566E 06	7.33130E 02	2.83000E 04	4.29000E 00
96.00	3.86746E 07	8.28414E 05	8.07950E 02	2.42770E 04	4.28000E 00
100.00	3.87389E 07	6.70182E 05	8.88270E 02	2.05580E 04	4.26000E 00
104.00	3.87795E 07	5.43185E 05	9.77135E 02	1.74090E 04	4.19000E 00
108.00	3.88201E 07	4.16187E 05	1.06600E 03	1.42600E 04	4.12000E 00
112.00	3.88411E 07	3.26879E 05	1.16715E 03	1.17617E 04	4.04000E 00
116.00	3.88621E 07	2.37570E 05	1.26830E 03	9.26339E 03	3.96000E 00
120.00	3.88724E 07	1.83636E 05	1.38300E 03	7.50129E 03	3.93000E 00
124.00	3.88828E 07	1.29702E 05	1.49770E 03	5.73920E 03	3.90000E 00
128.00	3.88877E 07	9.87330E 04	1.62800E 03	4.59475E 03	3.93000E 00
132.00	3.88927E 07	6.77640E 04	1.75830E 03	3.45030E 03	3.96000E 00
136.00	3.88950E 07	5.08205E 04	1.90690E 03	2.76475E 03	4.13000E 00
140.00	3.88973E 07	3.38770E 04	2.05550E 03	2.07920E 03	4.30000E 00

TIME (SEC)	MASS (KG)	XCG (M)	XCP (M)	LCP (M)	IXX (KG-M)
0.00	2.76205E 06	2.73900E 01	3.62103E 01	-8.82031E 00	8.50078E 08
4.00	2.70854E 06	2.73900E 01	3.61097E 01	-8.71973E 00	8.48227E 08
8.00	2.65511E 06	2.74000E 01	3.60091E 01	-8.60915E 00	8.46271E 08
12.00	2.60150E 06	2.74200E 01	3.58080E 01	-8.38797E 00	8.44119E 08
16.00	2.54799E 06	2.74400E 01	3.56068E 01	-8.16679E 00	8.41937E 08
20.00	2.49447E 06	2.74800E 01	3.55062E 01	-8.02621E 00	8.39623E 08
24.00	2.44095E 06	2.75200E 01	3.53051E 01	-7.78505E 00	8.37146E 08
28.00	2.38743E 06	2.75700E 01	3.51039E 01	-7.53387E 00	8.34599E 08
32.00	2.[illegible]3392E 06	2.76400E 01	3.48021E 01	-7.16212E 00	8.31848E 08
36.00	[illegible]040E 06	2.77200E 01	3.42992E 01	-6.57921E 00	8.28952E 08
40.00	2.22688E 06	2.78100E 01	3.38969E 01	-6.08687E 00	8.25766E 08
44.00	2.17337E 06	2.79100E 01	3.32934E 01	-5.38336E 00	8.22359E 08
48.00	2.11985E 06	2.80300E 01	3.21869E 01	-4.15694E 00	8.18694E 08
52.00	2.06633E 06	2.81700E 01	3.01753E 01	-2.00526E 00	8.14624E 08
56.00	2.01282E 06	2.83200E 01	2.72583E 01	1.06169E 00	8.10440E 08
60.00	1.95930E 06	2.84800E 01	2.79624E 01	5.17593E-01	8.05866E 08
64.00	1.90578E 06	2.86700E 01	3.07788E 01	-2.10876E 00	8.00737E 08
68.00	1.85227E 06	2.88800E 01	3.50033E 01	-6.12329E 00	7.95254E 08
72.00	1.79875E 06	2.91100E 01	3.78197E 01	-8.70966E 00	7.89244E 08
76.00	1.74523E 06	2.93800E 01	4.02337E 01	-1.08537E 01	7.82626E 08
80.00	1.69172E 06	2.96600E 01	4.16419E 01	-1.19819E 01	7.75421E 08
84.00	1.63820E 06	2.99700E 01	4.33518E 01	-1.33818E 01	7.67645E 08
88.00	1.58468E 06	3.03200E 01	4.45588E 01	-1.42388E 01	7.58893E 08
92.00	1.53117E 06	3.07100E 01	4.51623E 01	-1.44523E 01	7.49251E 08
96.00	1.47765E 06	3.11500E 01	4.52629E 01	-1.41129E 01	7.38585E 08
100.00	1.42413E 06	3.16300E 01	4.53635E 01	-1.37335E 01	7.26793E 08
104.00	1.37061E 06	3.21900E 01	4.55143E 01	-1.33243E 01	7.13040E 08
108.00	1.31710E 06	3.27500E 01	4.56652E 01	-1.29152E 01	6.99287E 08
112.00	1.26358E 06	3.34400E 01	4.56149E 01	-1.21749E 01	6.82292E 08
116.00	1.21006E 06	3.41300E 01	4.55646E 01	-1.14346E 01	6.65296E 08
120.00	1.15655E 06	3.49950E 01	4.57155E 01	-1.07205E 01	6.44063E 08
124.00	1.10303E 06	3.58600E 01	4.58664E 01	-1.00064E 01	6.22830E 08
128.00	1.04951E 06	3.69500E 01	4.66208E 01	-9.67078E 00	5.95878E 08
132.00	9.95996E 05	3.80400E 01	4.73752E 01	-9.33516E 00	5.68926E 08
136.00	9.42480E 05	3.94400E 01	4.91354E 01	-9.69537E 00	5.33912E 08
140.00	8.88963E 05	4.08400E 01	5.08956E 01	-1.00556E 01	4.98897E 08

TIME	A(2,3)	A(3,1)	A(3,3)	A(2,6)	A(3,6)
0.00	-0.00000E-01	-3.89717E 01	-1.00000E 01	-8.72556E-01	-3.92183E 01
4.00	2.34665E-04	-9.51120E-01	-2.63210E-01	-8.74762E-01	-9.71986E-01
8.00	1.01622E-03	-4.53424E-01	-1.38907E-01	-8.78044E-01	-4.73305E-01
12.00	2.43095E-03	-2.88416E-01	-9.79386E-02	-8.82565E-01	-3.07714E-01
16.00	4.57627E-03	-2.06503E-01	-7.81349E-02	-8.87882E-01	-2.25471E-01
20.00	7.60375E-03	-1.57548E-01	-6.69383E-02	-8.94789E-01	-1.76404E-01
24.00	1.14311E-02	-1.24987E-01	-6.01712E-02	-9.02719E-01	-1.43915E-01
28.00	1.62591E-02	-1.01944E-01	-5.58516E-02	-9.11922E-01	-1.20837E-01
32.00	2.15078E-02	-8.46425E-02	-5.30970E-02	-9.22882E-01	-1.03582E-01
36.00	2.64925E-02	-7.11557E-02	-5.14360E-02	-9.35202E-01	-9.01960E-02
40.00	3.22049E-02	-6.02938E-02	-5.05306E-02	-9.49016E-01	-7.95108E-02
44.00	3.62459E-02	-5.14219E-02	-4.99302E-02	-9.64189E-01	-7.08068E-02
48.00	3.51612E-02	-4.40860E-02	-4.96738E-02	-9.81029E-01	-6.36027E-02
52.00	2.11236E-02	-3.78825E-02	-4.96559E-02	-9.99619E-01	-5.75869E-02
56.00	-1.38871E-02	-3.26468E-02	-5.00528E-02	-1.01913E 00	-5.25287E-02
60.00	-8.21618E-03	-2.79962E-02	-5.04673E-02	-1.03976E 00	-4.82414E-02
64.00	3.69351E-02	-2.43069E-02	-4.94450E-02	-1.06234E 00	-4.45763E-02
68.00	1.12482E-01	-2.10898E-02	-4.79278E-02	-1.08613E 00	-4.13469E-02
72.00	1.62098E-01	-1.82865E-02	-4.60360E-02	-1.11129E 00	-3.84289E-02
76.00	1.86803E-01	-1.57953E-02	-4.27376E-02	-1.13864E 00	-3.57815E-02
80.00	2.03047E-01	-1.36615E-02	-4.06825E-02	-1.16693E 00	-3.33820E-02
84.00	2.39399E-01	-1.18564E-02	-3.96747E-02	-1.19690E 00	-3.12185E-02
88.00	2.08929E-01	-1.02619E-02	-3.57209E-02	-1.22966E 00	-2.92647E-02
92.00	1.85940E-01	-8.86826E-03	-3.31805E-02	-1.26529E 00	-2.75001E-02
96.00	1.57643E-01	-7.70149E-03	-3.09094E-02	-1.30489E 00	-2.59155E-02
100.00	1.31395E-01	-6.22695E-03	-2.93635E-02	-1.34873E 00	-2.44987E-02
104.00	1.08228E-01	-5.52359E-03	-2.73510E-02	-1.40055E 00	-2.31645E-02
108.00	8.61555E-02	-5.01682E-03	-2.56583E-02	-1.45446E 00	-2.21193E-02
112.00	6.73236E-02	-4.78110E-03	-2.38922E-02	-1.52292E 00	-2.10694E-02
116.00	5.00602E-02	-4.68305E-03	-2.23819E-02	-1.59491E 00	-2.02576E-02
120.00	3.89616E-02	-4.04329E-03	-2.16080E-02	-1.68970E 00	-1.94422E-02
124.00	2.85525E-02	-3.60106E-03	-2.09329E-02	-1.79097E 00	-1.88293E-02
128.00	2.32690E-02	-2.85572E-03	-2.06856E-02	-1.92912E 00	-1.82079E-02
132.00	1.78008E-02	-2.33237E-03	-2.04568E-02	-2.08038E 00	-1.77667E-02
136.00	1.64635E-02	-2.27325E-03	-1.98447E-02	-2.29853E 00	-1.73135E-02
140.00	1.43081E-02	-2.35353E-03	-1.93036E-02	-2.54732E 00	-1.70297E-02

TIME	A(5,4)	A(5,5)	A(5,6)	C(1,4)	C(2,5)
0.00	-3.89433E 01	-1.24809E-01	1.99244E 02	1.50000E-02	7.00000E-03
4.00	-3.92419E 01	-1.25287E-01	2.00990E 02	1.50000E-02	7.00000E-03
8.00	-3.95337E 01	-1.25752E-01	2.02669E 02	1.50000E-02	7.00000E-03
12.00	-3.98190E 01	-1.26205E-01	2.04929E 02	1.50000E-02	7.00000E-03
16.00	-4.01126E 01	-1.26669E-01	2.07314E 02	1.50000E-02	7.00000E-03
20.00	-4.04555E 01	-1.27209E-01	2.08774E 02	1.50000E-02	7.00000E-03
24.00	-4.07920E 01	-1.27737E-01	2.11242E 02	1.50000E-02	7.00000E-03
28.00	-4.10170E 01	-1.28089E-01	2.13627E 02	1.50000E-02	7.00000E-03
32.00	-4.13153E 01	-1.28554E-01	2.16306E 02	1.50000E-02	7.00000E-03
36.00	-4.16228E 01	-1.29031E-01	2.18875E 02	1.50000E-02	7.00000E-03
40.00	-4.19477E 01	-1.29534E-01	2.21923E 02	1.50000E-02	7.00000E-03
44.00	-4.22249E 01	-1.29961E-01	2.24808E 02	1.50000E-02	7.00000E-03
48.00	-4.25358E 01	-1.30439E-01	2.27871E 02	1.50000E-02	7.00000E-03
52.00	-4.27984E 01	-1.30841E-01	2.30702E 02	1.50000E-02	7.00000E-03
56.00	-4.30866E 01	-1.31281E-01	2.34065E 02	1.50000E-02	7.00000E-03
60.00	-4.33510E 01	-1.31683E-01	2.37979E 02	1.50000E-02	7.00000E-03
64.00	-4.36078E 01	-1.32073E-01	2.41442E 02	1.50000E-02	7.00000E-03
68.00	-4.38572E 01	-1.32450E-01	2.44845E 02	1.50000E-02	7.00000E-03
72.00	-4.41072E 01	-1.32827E-01	2.47996E 02	1.50000E-02	7.00000E-03
76.00	-4.43580E 01	-1.33204E-01	2.51663E 02	1.50000E-02	7.00000E-03
80.00	-4.46681E 01	-1.33668E-01	2.54610E 02	1.50000E-02	7.00000E-03
84.00	-4.49289E 01	-1.34058E-01	2.58204E 02	1.50000E-02	7.00000E-03
88.00	-4.51988E 01	-1.34460E-01	2.61315E 02	1.50000E-02	7.00000E-03
92.00	-4.54526E 01	-1.34837E-01	2.65393E 02	1.50000E-02	7.00000E-03
96.00	-4.56902E 01	-1.35189E-01	2.69548E 02	1.50000E-02	7.00000E-03
100.00	-4.59368E 01	-1.35553E-01	2.73145E 02	1.50000E-02	7.00000E-03
104.00	-4.62183E 01	-1.35968E-01	2.78995E 02	1.50000E-02	7.00000E-03
108.00	-4.64921E 01	-1.36370E-01	2.84057E 02	1.50000E-02	7.00000E-03
112.00	-4.68182E 01	-1.36848E-01	2.91524E 02	1.50000E-02	7.00000E-03
116.00	-4.71542E 01	-1.37338E-01	2.99606E 02	1.50000E-02	7.00000E-03
120.00	-4.74827E 01	-1.37815E-01	3.10272E 02	1.50000E-02	7.00000E-03
124.00	-4.79427E 01	-1.38481E-01	3.19662E 02	1.50000E-02	7.00000E-03
128.00	-4.83701E 01	-1.39097E-01	3.42572E 02	1.50000E-02	7.00000E-03
132.00	-4.89926E 01	-1.39989E-01	3.74430E 02	1.50000E-02	7.00000E-03
136.00	-4.97874E 01	-1.41120E-01	4.02145E 02	1.50000E-02	7.00000E-03
140.00	-5.07675E 01	-1.42503E-01	4.34076E 02	1.50000E-02	7.00000E-03