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THE DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS USING MODERN CONTROL THEORY (WITH APPLICATIONS TO COUPLED CORE REACTOR CONTROL)

by Charles R. Slivinsky, Donald G. Schultz, and Lynn E. Weaver

Prepared by
UNIVERSITY OF ARIZONA
Tucson, Ariz.
for

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ABSTRACT

Design techniques for linear multivariable systems are considered. Both conventional, frequency-domain techniques and modern, combined frequency-domain, time-domain procedures are considered. Noninteraction is taken as one of the two basic design requirements; the other is that specified subsystem transfer functions be achieved. Conventional methods are quickly shown to have the disadvantage of complexity--both in carrying out the design calculations and in the physical implementation of the compensation.

The bulk of the attention to design is given to the state variable feedback design of multivariable systems. All previous work is summarized, including procedures which make possible the identification of the fixed zeroes of the subsystems of the multivariable system and the number of subsystem poles which are controlled by state variable feedback. By treating each subsystem individually, the designer can apply some of the previously developed knowledge of state variable feedback design of single-input, single-output systems.

A topic which has not been previously studied is the addition of dynamics to the multivariable system

before state variable feedback is applied, for the purpose of improving the system response. Three methods are proposed and analyzed for adding dynamics. The first, Method A, requires that the compensation, or additional dynamics, be placed in the control input channels of the multivariable plant and that all the states of the augmented system be fed back. This method is the preferred one when it works, because of its simplicity. Its most serious drawback is that the plant which results from the addition of compensation by Method A may have lost the ability to be decoupled by state variable feedback, even though it possessed that ability before the compensation was added. Another disadvantage is that there is no sure way of knowing how the structures of the subsystem transfer functions are affected by the added compensation. Thus, the designer has no guide to determining what to put in the compensators.

One important, practical case of Method A is considered in detail; namely, the case where first-order compensators of the same form are added in all the input channels. It is shown that decoupling is never lost by this procedure.

The second method, Method B, is shown to have serious practical problems and is given only a brief treatment.

In Method C the problems of Method A are eliminated by the intermediate step of decoupling the plant before the compensation is added and all the states are fed back. When Method C is used, it is proved that the structure of the final, compensated system is completely determined by the structure of the decoupled plant and the structure of the added compensation. Unlike Method A, the designer now knows what compensation to add in order to meet the design specifications.

Orderly design procedures are presented both for the case where additional compensation is not needed and for the case where it is. For the most part, the design procedures are based on previously known techniques. However, a procedure is presented which allows a savings in computational labor in certain design problems where dynamics are added to the multivariable system.

A practical example of the application of state variable feedback design is given. The specific physical system considered is the coupled-core nuclear reactor, and a three-core linear model is used. Finally, suggestions are given for further research.

CHAPTER 1

INTRODUCTION

In control engineering one studies the problem of forcing some physical system such as a rocket, nuclear reactor, chemical processing plant, or even an economic or social system to behave in a manner which meets prescribed performance specifications. Such diverse physical systems as those mentioned exhibit many similarities, once the mathematical models describing their behavior are found and compared. In fact, much of control engineering involves the study of the behavior of the abstract system models, rather than the specialized study of the physical systems. Presumably, after the control engineer acquires a thorough understanding of the general principles and techniques of control theory, he is ready to make useful contributions to the actual design problems in a variety of fields where physical processes must be controlled.

The specific concern of this study is the design (in the abstract sense discussed above) of systems which have a multiplicity of inputs and outputs, and for which the number of inputs is equal to the number of outputs. Examples of such systems are

1. An aircraft flight control system where typical inputs are the rudder deflection and the aileron deflection, and the outputs are the roll rate and yaw rate of the aircraft.

2. A turboprop jet engine control, where the inputs are the propeller blade angle and the fuel rate, and the outputs are the engine speed and turbine inlet temperature.

3. A set of coupled-core reactors, where the inputs are the control-rod positions and the outputs are the power levels of the individual reactors.

For each of the above examples one input affects more than one output. In the first example, for instance, the rudder setting affects both the yaw of the aircraft and its roll rate. Such multivariable systems are said to be coupled.

Both the terms noninteraction and decoupled are used to describe the situation in which each input of the multivariable system affects one and only one output. Since coupling is usually inherent in the plant, or system before it has been designed, noninteraction is a condition which is part of the design objective. One advantage of choosing noninteraction as a design requirement is that the decoupled system appears to function in the simplest possible manner when seen from the input-output point of view. A further advantage is that once noninteraction is obtained, the multivariable system is reduced to a set of single-input, single-output systems, and the well-established design techniques for such system are applicable. Both these advantages are present when the methods described in this study are used.

The first attempts to formulate design procedures in which noninteraction is required were reported in the 1950's and early 1960's (Boksenbom and Hood, 1949; Povejsil and Fuchs, 1955; Freeman, 1957, 1958; Kavanagh, 1956, 1957, 1958; Horowitz, 1960; Chen, Mathias, and Sautér, 1962). The multivariable system is assumed to be describable by a set of linear differential equations with constant coefficients, and the Laplace transform is used to obtain the corresponding set of linear algebraic equations in the complex frequency variable s. Methods based on this description are known as frequency-domain techniques.

Chapter 2 discusses notation and describes two typical frequency-domain design techniques, designated Configuration I design and Configuration II design. Their basic design objectives are noninteraction and the realization of given transfer function relationships between each input and its corresponding output. The relevant design equations are derived in each case, and the disadvantages of the methods are pointed out; the latter are that it is difficult to carry out the computations which the procedures require and that there is no assurance that the resulting system compensation can be implemented on the physical system. In a sense, the chapter is a "warm-up" because the succeeding chapters describe design techniques that are superior to those of Chapter 2.

Chapter 3 provides a comprehensive treatment of the material on which the main contributions of dissertation are based. The modern, state variable description of the multivariable system in both time- and frequency-domain is used. Again, the design objectives are noninteraction and the realization of transfer function relationships between input-output pairs. The design objectives are achieved by feeding back all the state variables of the system and by coupling in the system inputs; this form of compensation, known as state variable feedback, was studied by Morgan (1963, 1966), Rekasius (1965), Falb and Wolovich (1967 a, b), and Gilbert (1968). The first two of the above authors present results which are superseded by the work of the last three authors.

Falb and Wolovich formulated and proved a necessary and sufficient test for determining whether or not state variable feedback can decouple the multivariable system. They also developed a standard procedure for decoupling and presented a design technique in which certain of the system poles are controlled, but all zeroes are canceled.

Gilbert utilized Falb and Wolovich's standard procedure for decoupling and changed variables in the decoupled system to establish a canonical form for the multivariable system. The use of his canonical form permits the identification of m decoupled subsystems, where m is the number of inputs (and outputs); it also makes it possible to apply the well-developed state variable feedback design technique for single-input, single-output systems discussed by Schultz and Melsa (1967). If state variable feedback is applied to each of the decoupled subsystems, then the subsystem poles can be placed arbitrarily but the zeroes remain fixed. In addition, there are some poles of the multivariable system which cannot be controlled by state variable feedback if decoupling is to be preserved.

The work of Gilbert described in Chapter 3 is notable for its completeness. All that is necessary to design the multivariable system by state variable feedback is given.

The fact that state variable feedback cannot change the order of the system and cannot by itself add new zeroes to the system is a disadvantage because a common design specification is zero velocity-error coefficient, and control over the zeroes is needed to meet this requirement. Other design situations require that poles be added to the system. In single-input, single-output design, additional dynamics are added by inserting compensator networks.

The extension of this technique to multivariable systems is the subject of Chapter 4. This chapter contains the main contributions of the dissertation.

Three methods are discussed in Chapter 4 for adding additional dynamics to the multivariable plant. In Method A compensator networks are added in the input channels of the plant. This method, although it has the advantage of simplicity, does not always give the desired results. Its most serious drawback is that the plant which results from the addition of compensation by Method A may have lost the ability to be decoupled by state variable feedback, even though it possessed that ability before the compensation was added. A further disadvantage is that there is no sure way of knowing how the structures of subsystem transfer functions are affected by the added compensation. Thus, the designer has no guide to determining what to put in the compensators.

One important, practical case of Method A is considered in detail; namely, the case in which first-order compensators of the same form are added in all the input channels; according to Theorem 4.1 decoupling is never lost by this procedure.

The second method, Method B, is shown to have serious practical problems, and is best considered as a step towards Method C.

In Method C the problems of Method A are eliminated by the intermediate step of decoupling the plant before the compensation is added, and all states are fed back. This fact is intuitively plausible but must be proved; a proof is provided in Theorem 4.3. Theorem 4.3 is the central result of the chapter. Although its proof is abstract, its content is easily understood. Basically, the theorem shows that in designing a system by Method C, one knows beforehand that the structure of the final, compensated system is completely determined by the structure of the decoupled plant and the structure of the added compensation. Unlike Method A, the designer now knows what compensation to add in order to meet the design specifications; exactly the same freedom exists in single-input, single-output design problems as that provided by the use of Method C.

Progress through Chapters 3 and 4 reveals that the design procedures for state variable feedback design of multivariable systems--whether or not additional compensation is needed--are quite complicated in comparison with single-input, single-output design. Chapter 5 alleviates the complexity in two ways. First, an orderly design procedure, complete with all relevant formulas, is presented both for the case where additional compensation is not needed and for the case where it is. Second, computationally efficient algorithms are presented for carrying out the design steps by digital computer. Most of Chapter 5 is not new; however, a new idea is presented which allows a savings in computational labor in certain design problems where additional compensation is needed.

In Chapter 6 a practical example of state variable feedback design is given. The specific physical multivariable system considered is the coupled-core nuclear reactor (Weaver, 1968).

Chapter 7 presents the conclusions and suggestions for further research.

CHAPTER 2

CONVENTIONAL, FREQUENCY-DOMAIN TECHNIQUES

In this chapter notation and the means for modeling multivariable systems in both the frequency domain and the time domain are given. Compensation and the design constraint called noninteraction are introduced. Two frequency-domain design techniques are presented as typical of previous efforts to compensate multivariable systems. The basic aim is the demonstration of their inadequacy, as a means for developing perspective and for leading into the state variable design technique of the following chapters.

This chapter provides essential background material but, except for Theorem 2.1, no new results are presented here.

Plant Equations and Notation

Linear, time-invariant multivariable control systems have one or more inputs and one or more outputs; this is the origin of the term "multivariable". The inputs and outputs are related by a set of ordinary, linear differential equations with constant coefficients.

It is assumed in what follows that the number of inputs is the same as the number of outputs. Means for augmenting the multivariable system so that this constraint is satisfied are discussed in Chen et al (1962), for the case where there are fewer outputs than inputs. Very little work has been done on systems which have more outputs than inputs (Leeds and Cox, 1967).

Conventional, frequency-domain design techniques require a mathematical description of the system of the form

$$y(s) = P(s)u(s) \tag{2.1}$$

Here all quantities are Laplace transformed quantities and are functions of the complex frequency variable s.

$y(s)$ is an m-dimensional vector, the output of the system

$u(s)$ is an m-dimensional vector, the control input to the system

$P(s)$ is an m x m matrix, the plant matrix

Lower case letters are used for scalars and vectors. When subscripts or superscripts are used, lower case letters refer to either scalars or elements of vectors which may themselves be either vectors or scalars. Capital letters with subscripts or superscripts denote submatrices of the matrix represented by the same capital letter without the subscript. The superscripts T and -1 are used to denote the transpose and the inverse of a matrix, respectively. The symbol 0 is used for the scalar 0, the null vector, and the null matrix. Vectors and matrices are not underlined because for the most part very few scalars appear in the text and these are always explicitly pointed out. Whenever feasible the notation used is the same as that used in current papers on the subject of the design of multivariable systems using state variable feedback.

As Equation 2.1 indicates, the input and the output are related by a transfer matrix. This is in contrast to single-input, single-output systems where the input and output are related by a transfer function. Each of the elements of the plant matrix P(s) (e.g., $p_{ij}(s)$) is a transfer function.

Modern, time-domain and combined frequency-domain, time-domain design techniques require the mathematical description of the system to have the following form

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad 2.2$$

$$y(t) = Cx(t) \qquad 2.3$$

Here the • indicates differentiation with respect to the time t and

x(t) is an n-dimensional vector, the state of the system

u(t) is an m-dimensional vector, the control input to the system y(t) is an m-dimensional vector, the output of the system

A is an n x n matrix of constants, the system matrix

B is an n x m matrix of constants, the control input matrix

C is an m x n matrix of constants, the output matrix

(Note that y is being used as the symbol for both a function of time, in Equation 2.3, and, in Equation 2.1, for the Laplace transform of itself, now a function of s. Once understood, this usage is not a source of confusion.) The number of states, n, is required to be greater than or equal to the number of control inputs, m. Equation 2.2 is a set of coupled first-order linear differential equations, and Equation 2.3 defines the m outputs of the system as linear combinations of the n states.

Both the frequency-domain and the time-domain representations given above refer to the same physical system; they are merely two different ways of describing it. Equations 2.1, 2.2, and 2.3 are taken as starting points. Methods for modeling physical systems in terms of these types of equations and state variable concepts are discussed in many textbooks, such as Cannon (1967) and Schultz and Melsa (1967).

Forms of Compensation

The basic problem being considered is that of realizing the desired performance in a multivariable system. To accomplish this aim, often the fixed plant must be compensated; i.e., additional physical components such as electronic amplifiers and resistor-capacitor networks must be used to alter the dynamic system performance. Here, the mathematical aspects of the compensation problem rather than the "hardware" aspects are treated.

Three forms of compensation are to be considered and all three are defined in terms of their effect on the control input u. They are

$$u(s) = D(s)[r(s) - y(s)] \qquad 2.4$$

$$u(s) = G(s)r(s) + L(s)y(s) \qquad 2.5$$

$$u(t) = Fx(t) + Gr(t) \qquad 2.6$$

The variable r is an m-dimensional vector, representing the system input, and should not be confused with the control input, u. The matrices D, G, and L are of dimension $m \times m$ and F is $m \times n$. Equations 2.4 and 2.5 apply to the system when it is represented as in Equation 2.1; i.e., in the frequency-domain formulation. Equation 2.6 applies to the state variable formulation of Equation 2.2 and 2.3, and in this case the matrices F and G are assumed to have constant elements.

The first two control inputs given above lead to Configuration I Design and Configuration II Design, respectively. These two frequency-domain design techniques are discussed in this chapter. The third control input applies to the state variable feedback design technique; by far it occupies the bulk of the attention in the chapters which follow.

Once the control input has been chosen, the relationship between the system input r and the output y can be found. This relationship is indicated by the equation

$$y(s) = H(s)r(s) \tag{2.7}$$

where H(s) is an m x m transmission matrix. H(s) is a function of the fixed portion of the system and the control input u. H(s) is to be chosen by the designer to satisfy design specifications such as bandwidths, rise times, and steady-state errors.

Noninteraction

The equations representing a multivariable system are coupled. This means that if one of the system inputs, say r_1, is changed, then not only output y_1 is changed, but in general all the outputs are affected. A great simplification in the apparent operation of the system would be to have noninteraction. With noninteraction each input affects one and only one output. In terms of Equation 2.7 noninteraction can be defined as follows:

> Definition 2.1 A system is said to be noninteracting when the transmission matrix H defined by the equation $y = Hr$ is diagonal and nonsingular.

Nonsingularity is necessary to insure that none of the diagonal elements of H is zero.

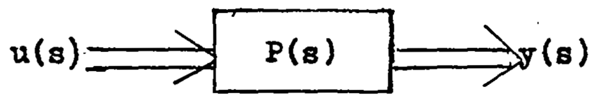
This definition of noninteraction coincides with that of Gilbert (1968) and is equivalent to the one given by Falb and Wolovich (1967a); it follows the intent of one of the earliest papers on multivariable control systems: Boksenbom and Hood (1949).

Multivariable systems are inherently interacting. In a jet engine, for example, when a control input such as the fuel flow rate is changed, both the engine speed and the engine temperature change. Now when the system has been compensated for noninteraction, a change in the input corresponding to the control input fuel flow rate would cause all the control inputs to change in such a manner that only the single output corresponding to flow rate will change. Thus from an input-output point of view the system possesses noninteraction, while from a control input-output point of view it is still interacting.

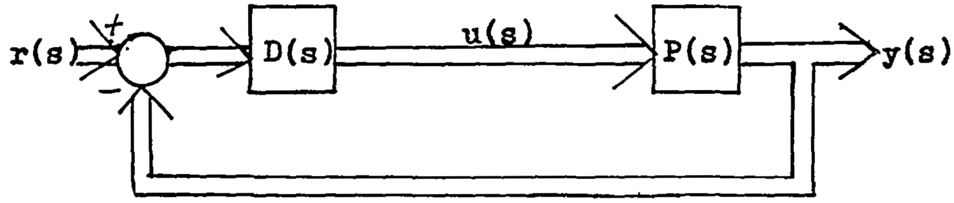
Noninteracting multivariable systems can be conceived as consisting of a collection of individual subsystems, each of which has a single input and a single output. Dealing with single-input, single-output subsystems has two advantages: First, the problem of specifying performance requirements is simplified, Second, each subsystem can be treated separately. The problem of designing to meet the specifications is then a more tractable one because there are many design techniques for single-input, single-output systems. In the design procedures discussed in the following chapters the criterion of noninteraction is taken as the fundamental design requirement to be met.

Configuration I Design

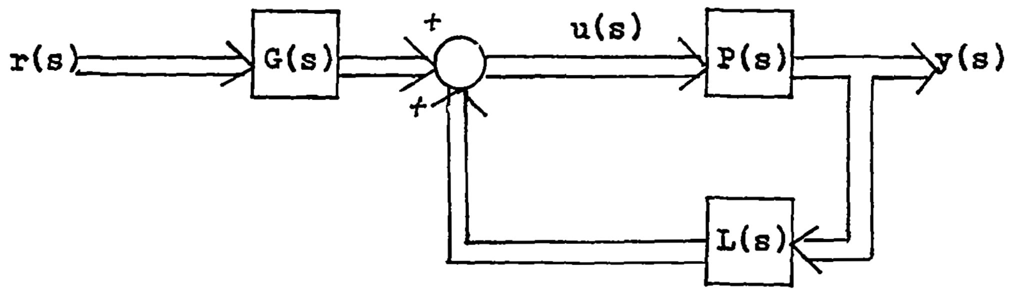
For Configuration I the form of compensation to be used is that given in Equation 2.4 and shown in Figure 2.1(b) (where the double lines are used to indicate vector quantities). The matrix D(s) is the unknown compensation transfer function matrix. Configuration I has been discussed by many authors, but perhaps the bulk of the theory is presented in the paper and the attendant discussions given in Chen et al (1962), and the papers by Povejsil and Fuchs (1955), Mathias (1963), Gilbert (1963), and Chen (1968 a, b).



(a) The Fixed Plant



(b) Compensated System for Configuration I Design



(c) Compensated System for Configuration II Design

Figure 2.1 Conventional, Frequency-Domain Design Techniques

Substituting Equation 2.4 into the plant equation, 2.1, gives

$$y(s) = P(s)D(s)[r(s) - y(s)] \qquad 2.8$$

and solving Equation 2.8 for y(s) yields

$$y(s) = [I + P(s)D(s)]^{-1}P(s)D(s)r(s) \qquad 2.9$$

where I is the m x m unit matrix. The transmission matrix relating the input r(s) and the output y(s) is thus

$$H(s) = [I + P(s)D(s)]^{-1}P(s)D(s) \qquad 2.10$$

The particular case of most interest is the one where noninteraction is given as one of the design criteria. In this case it is possible to find an expression for H(s) which shows clearly that the multivariable system can be regarded as a set of single-input, single-output subsystems.

Let the loop gain matrix N(s) be defined by the equation

$$N(s) = P(s)D(s) \qquad 2.11$$

Substituting N(s) into Equation 2.10 gives

$$\begin{aligned} H(s) &= [I + N(s)]^{-1}N(s) \\ &= (N^{-1}(s)[I + N(s)])^{-1} \\ &= [N^{-1}(s) + I]^{-1} \qquad 2.12 \end{aligned}$$

For noninteraction H(s) must be a diagonal matrix. From the above equation H(s) will be diagonal if N(s) is a diagonal matrix. In fact, if N(s) is given by

$$N(s) = \begin{bmatrix} n_{11}(s) & 0 & \cdots & 0 \\ 0 & n_{22}(s) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & n_{mm}(s) \end{bmatrix} \qquad 2.13$$

then Equation 2.12 becomes

$$H(s) = \begin{bmatrix} \frac{n_{11}(s)}{1+n_{11}(s)} & 0 & \cdots & 0 \\ 0 & \frac{n_{22}(s)}{1+n_{22}(s)} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{n_{mm}(s)}{1+n_{mm}(s)} \end{bmatrix} \qquad 2.14$$

Equation 2.14 shows that the multivariable system consists of m subsystems, each of which has one input, one output, a loop gain transfer function $n_{ii}(s)$, and unity feedback. The $n_{ii}(s)$ depend on both P(s) and D(s) and are to be selected to give a satisfactory response from input r_i to output y_i.

Once the matrix N(s) has been selected, the elements of D(s) can be found. Let $P^{-1}(s)$ be expressed as

$$P^{-1}(s) = \frac{\tilde{\mathscr{P}}(s)}{\det P(s)} \qquad 2.15$$

where det P(s) is the determinant of P(s) and $\tilde{\mathscr{P}}(s)$ is the adjoint matrix of P(s), (Nering, 1963), having elements $\tilde{\mathscr{P}}_{ij}(s)$. Then from the equation which defines N(s), Equation 2.11,

$$D(s) = P^{-1}(s)N(s)$$

$$= \frac{\tilde{\mathscr{P}}(s)}{\det P(s)} N(s) \qquad 2.16$$

In terms of the elements of D(s), Equation 2.16 becomes

$$d_{ij}(s) = \frac{1}{\det P(s)} \sum_{k=1}^{m} \tilde{\mathscr{P}}_{ik} n_{kj}(s) \qquad 2.17$$

or since N(s) is a diagonal matrix,

$$d_{ij}(s) = \frac{\tilde{\mathscr{P}}_{ij}(s)}{\det P(s)} n_{jj}(s) \qquad 2.18$$

In summary, when noninteracting systems are to be designed using Configuration I, the procedure is the following. First, a set of loop gain transfer functions is chosen to give the desired input-output relationships for each of the m input-output pairs, thereby determining N(s). This step can be accomplished by using the standard design techniques such as Bode plots, root locus diagrams, and Nyquist diagrams. Second, the elements of D(s) are found by using Equation 2.18.

A disadvantage of this approach can be seen by examining Equation 2.18, which shows that the scheme is basically one of cancellation. The term $d_{ij}(s)$ is found by multiplying the desired $n_{jj}(s)$ by an element of $P^{-1}(s)$; in effect the plant is being "canceled out" and new dynamics are being inserted in its place. Cancellation is never exact because the plant is never known exactly. The scheme also suffers from a computational standpoint, as finding $P^{-1}(s)$ requires taking the inverse of a matrix having elements which are functions of the literal variable s.

In the paper by Chen et al (1962) cancellation and right-half plane poles in P(s) are discussed, and constraints on D(s) are given. Unfortunately, when these constraints are incorporated the resulting D(s) may have a very complicated structure. The example in the last reference cited required the capability of synthesizing a D(s) having both poles and zeroes in the right-half plane. Such compensators have no practical value.

More recently, Gilbert (1963) showed that the general problem of cancellation and the effects of unstable transfer functions in P(s) are best clarified by using concepts which are defined in terms of the state variable representation. This discussion of Configuration I in terms of state variable concepts is continued in the papers by Chen (1968 a, b), where a means for determining stability is given.

Configuration II Design

For Configuration II the form of compensation to be used is indicated in Figure 2.1(c) and the equation

$$u(s) = L(s)y(s) + G(s)r(s) \qquad 2.5$$

Here $L(s)$ and $G(s)$ are the unknown compensation matrices. Configuration II has been discussed by Kavanagh (1956, 1957, 1958), Gibson (1963) and Rekasius (1965).

Substituting Equation 2.5 into Equation 2.1 and solving for $y(s)$ yields the expression for the transmission matrix, as

$$H(s) = [I - P(s)L(s)]^{-1}P(s)G(s) \qquad 2.19$$

Multiplying both sides of Equation 2.19 by $I - P(s)L(s)$ and transposing terms gives

$$H(s) = P(s)L(s)H(s) + P(s)G(s) \qquad 2.20$$

Given the desired $H(s)$, one must solve the m^2 scalar equations in Equation 2.20 (one for each element of $H(s)$) for the $2m^2$ unknown elements of $L(s)$ and $G(s)$.

The overabundance of unknowns can be considered both a boon and a burden. It is a boon, for example, when some of the inputs are noise or disturbance inputs and cannot be manipulated; then some of the elements of $G(s)$ would be constrained to be 0, reducing the number of unknowns. Other situations in which the number of unknowns is reduced are discussed in Kavanagh (1956). The overabundance of unknowns is burdensome because it hampers the formulation of exact procedures for solving for $L(s)$ and $G(s)$; the computational problem is further compounded by the fact that each of the so-called unknowns is itself a transfer function which may have several unknown paramaters.

Configuration II design can be recast in the modern, state variable framework. This is instructive because it shows how Configuration II design is related to the state variable feedback technique. Taking the Laplace transform of Equations 2.2 and 2.3, while assuming zero initial conditions, gives

$$sx(s) = Ax(s) + Bu(s) \qquad 2.21$$

$$y(s) = Cx(s) \qquad 2.22$$

Now the control input u(s) is given by

$$\begin{aligned} u(s) &= L(s)y(s) + G(s)r(s) \\ &= L(s)Cx(s) + G(s)r(s) \end{aligned} \qquad 2.23$$

If noninteraction is required, necessary and sufficient conditions can be given for the existence of a noninteracting control. The requirement is that P(s) be nonsingular, as Theorem 2.1 indicates. (The proof is adapted from the proof of a theorem given in Rekasius (1965).)

Theorem 2.1 A necessary and sufficient condition for the existence of a noninteracting control for Configuration II is that P(s) be nonsingular.

Proof. Rearranging terms in Equation 2.23 yields

$$L(s)Cx(s) = u(s) - G(s)r(s) \qquad 2.24$$

Solving Equation 2.21 for x(s) and multiplying the result by L(s)C gives

$$L(s)Cx(s) = L(s)C(sI - A)^{-1}Bu(s) \qquad 2.25$$

A comparison of Equations 2.24 and 2.25 reveals that

$$u(s) - G(s)r(s) = L(s)C(sI - A)^{-1}Bu(s) \qquad 2.26$$

or

$$u(s) = [I - L(s)C(sI - A)^{-1}B]^{-1}G(s)r(s) \qquad 2.27$$

Now from the equation

$$y(s) = P(s)u(s) \qquad 2.1$$

and Equation 2.27 there results

$$y(s) = P(s)[I - L(s)C(sI - A)^{-1}B]^{-1}G(s)r(s) \qquad 2.28$$

so that

$$H(s) = P(s)[I - L(s)C(sI - A)^{-1}B]^{-1}G(s) \qquad 2.29$$

To prove necessity it must be shown that $P(s)$ is nonsingular if a noninteracting control can be found, Under the assumption of noninteraction, $H(s)$ in Equation 2.29 is a nonsingular diagonal matrix. Taking its inverse leads to the desired result; namely, that $P(s)$ is nonsingular. Sufficiency is proved by letting $L(s) = 0$ in Equation 2.29 and solving for the compensation matrix $G(s)$ in terms of $H(s)$ and $P^{-1}(s)$.

Configuration II suffers from the same problems of Configuration I; namely, it is again necessary to invert matrices which are functions of s, and there is no guarantee that the compensation can be implemented with a reasonable amount of equipment, if at all. In fact, if the scheme in the proof of Theorem 2.1 is utilized, then $L(s) = 0$ and $G(s) = P^{-1}(s)H(s)$. A glance at Figure 2.1(c) shows that this scheme has no feedback around the compensation; it is basically an open-loop cancellation scheme.

Instead of feeding back the system outputs and coupling in the inputs, as indicated by Equation 2.23, a better scheme uses the control law

$$u(s) = F(s)x(s) + G(s)r(s) \qquad 2.30$$

Here all the states, rather than just certain linear combinations of states, are being utilized. Theorem 2.1 still holds, and the same practical problems are present in the design procedure.

Morgan (1963) proposed the use of a control input of the type given in Equation 2.30--with the restriction that F and G be constant matrices. The multivariable system is now said to be compensated by state variable feedback, the subject of the remainder of this work.

Conclusions

Two representative conventional techniques for designing multivariable systems have been presented. For the case of noninteraction, formal procedures are given for carrying out the design process. Basically, the disadvantages of these techniques are that it is difficult to perform the required computations and that the resulting compensation matrices may be difficult or impossible to implement in a physical system.

Both the computational problem and the problem of implementing the design arise because of the great generality of the problem formulation. No restrictions are placed on the compensation matrices, so that the number of possible parameters which could be present is unlimited. Under these circumstances it is to be expected that the formulation of feasible systematic computational procedures would be difficult and that desirable compensation matrices could not be expected with any degree of regularity.

In the following chapters the state variable feedback design technique is treated. The form of compensation is limited right at the start to a structure that permits the application of linear algebra and matrix theory to a far greater extent than is possible with conventional design techniques. As a result the problems of computation and ease of physical implementation are greatly relieved, in exchange for a loss of generality in the form of the closed loop system which can be achieved.

CHAPTER 3

STATE VARIABLE FEEDBACK DESIGN

This chapter is intended to provide an up-to-date account of the status of state variable feedback design of multivariable systems. Except for computational aspects, which are discussed in Chapter 5, the presentation is sufficiently complete to enable one to design physical systems by this technique. The primary concern of this dissertation is the formulation of design techniques and computational procedures for the case where state variable feedback alone is not sufficient for meeting the design requirements. The chapter must be understood if the main contributions of the dissertation are to be understood and assessed, but no new results appear here.

The state variable feedback form of compensation was first proposed by Morgan (1963, 1964); his work is discussed in the first section of this chapter. Also discussed in the same section are the contributions of Rekasius (1965). Only a brief account of their work is given because later developments have more general application.

The second section is a presentation of the work of Falb and Wolovich (1967 a, b). These authors formulated a simple test for determining whether or not the multivariable system can be decoupled by using state variable feedback. They also provided the formulas for decoupling the system into a form in which some of the system poles can be arbitrarily placed, but all of the zeroes are canceled. The results here are not completely satisfactory because the formulas do not provide for the greatest possible design freedom. However, the work of Falb and Wolovich serves as the basis for understanding and using the most recent contribution to the development of the design technique; namely, the work of Gilbert (1968).

The third section of this chapter is devoted to Gilbert's results. He provides a thorough and complete treatment which relates the multivariable problem to the design of single-input, single-output systems by state variable feedback. This section is the culmination of all previous work on state variable feedback design; system behavior which has previously been unaccounted for is explained, the question of system stability is made clear, and the limitations of design by state variable feedback alone are given. The chapter is concluded with a summary.

Early Design Efforts

In the state variable formulation the plant equations are assumed to be in the form

$$\dot{x} = Ax + Bu \qquad 3.1$$

$$y = Cx \qquad 3.2$$

and the control input is taken to be

$$u = Fx + Gr \qquad 3.3$$

the vectors x, y, and u are functions of time, but this dependence is no longer being shown explicitly. Recall that there are n states, m inputs, and m outputs.

Morgan (1963) is responsible for the introduction of the form of the above control law u given in Equation 3.3. His approach to the problem requires making a linear change of variables that puts the system into the simpler form

$$\dot{x} = Ax + Bu \qquad 3.4$$

$$y = [I \quad 0]x \qquad 3.5$$

where I is the m x m identity matrix and 0 is the m x (n - m) null matrix. In order to avoid cumbersome notation, the same symbols are being used for the new representation; the linear transformation relating the original variables to the variables in Equations 3.4 and 3.5 is discussed later.

The simplification is that the m system outputs are now equal to the first m state variables. This fact is best utilized if the A, B, and F matrices are partitioned into submatrices as follows:

$$A = \begin{bmatrix} A_{11}(m x m) & A_{12}(m x(n-m)) \\ A_{21}((n-m) x m) & A_{22}((n-m) x (n-m)) \end{bmatrix} \quad B = \begin{bmatrix} B_1(m x m) \\ B_2((n-m) x m) \end{bmatrix}$$

$$F = [F_1(m x m) \quad F_2(m x (n-m))]$$

If the above partitioning scheme is used and the control law of Equation 3.3 is substituted into Equation 3.4, the state equations become

$$\dot{x} = \begin{bmatrix} A_{11}+B_1F_1 & A_{12}+B_1F_2 \\ A_{21}+B_2F_1 & A_{22}+B_2F_2 \end{bmatrix} x + \begin{bmatrix} B_1G \\ B_2G \end{bmatrix} r \qquad 3.6$$

Noninteraction is achieved by isolating the first m states. Let

$$A_{12} + B_1F_2 = 0 \qquad 3.7$$

$$A_{11} + B_1F_1 = A_{11}^c \qquad 3.8$$

$$B_1G = B_1^c \qquad 3.9$$

where A_{11}^c and B_1^c are nonsingular diagonal matrices. The F and G which satisfy the above equations are

$$F_1 = B_1^{-1}(A_{11}^c - A_{11}) \qquad 3.10$$

$$F_2 = -B_1^{-1}A_{12} \qquad 3.11$$

$$G = B_1^{-1}B_1^c \qquad 3.12$$

provided that B_1 is a nonsingular matrix. For the F and G of Equations 3.10 - 3.12, Equation 3.6 is

$$\dot{x} = \begin{bmatrix} A_{11}^c & 0 \\ A_{21}^c+B_2F_1 & A_{22}+B_2F_2 \end{bmatrix} x + \begin{bmatrix} B_1^c \\ B_2G \end{bmatrix} r \qquad 3.13$$

Clearly the first m state equations (and thus the m output equations) are uncoupled first-order differential equations, and each subsystem has the transfer function

$$\frac{y_i(s)}{r_i(s)} = \frac{b^c_{ii}}{s - a^c_{ii}} \qquad i = 1, 2, \ldots, m \qquad 3.14$$

where b^c_{ii} and a^c_{ii} are the ith diagonal elements of the matrices B^c_1 and A^c_{11}, respectively.

There are two drawbacks to Morgan's method. First, only the sufficient condition that B_1 be nonsingular is given, and this allows only first-order subsystems to be obtained. In most multivariable systems it is to be expected that such simple subsystem responses will not be typical, so that B_1 is singular for a typical system, and the method is inapplicable. Second, even if the method applies, the state equations involving the derivatives $\dot{x}_i$, $i = m+1, \ldots, n$, are not under the designer's control, as Equation 3.13 indicates. The compensation matrices F and G are completely determined by the specification of the first m state equations; therefore, there is no way of insuring in advance that the last n-m state equations have satisfactory responses or are even stable.

An improvement over Morgan's technique was developed by Rekasius (1965). Here, his treatment of first-order subsystems is given, and its extensions are discussed.

Suppose the response of the multivariable system is specified by the designer to be

$$V_0\overset{\circ}{y} + V_1 y = W_1 r \qquad 3.15$$

where V_0, V_1, and W_1 are m x m diagonal matrices which must be specified by the designer. Once again first-order uncoupled subsystems with no zeroes are being sought.

The expression for y is given by Equation 3.2; differentiating this equation with respect to time gives

$$\overset{\circ}{y} = C\overset{\circ}{x} \qquad 3.16$$

The expression for $\overset{\circ}{x}$ is given in Equation 3.1; after the control input of Equation 3.3 is substituted in Equation 3.1, there results

$$\overset{\circ}{x} = (A + BF)x + BGr \qquad 3.17$$

so that Equation 3.16 becomes

$$\overset{\circ}{y} = C(A + BF)x + CBGr \qquad 3.18$$

Now if Equations 3.2 and 3.18 are substituted into Equation 3.15 and two terms in x are combined, the result is

$$[V_0C(A + BF) + V_1C]x + V_0CBGr = W_1 r \qquad 3.19$$

The above equation is satisfied if the coefficient of x is equal to 0,

$$V_0C(A + BF) + V_1C = 0 \qquad 3.20$$

and if

$$V_0CBG = W_1 \qquad 3.21$$

These last two equations can be solved for F and G, as

$$F = -(V_0CB)^{-1}(V_0CA + V_1C) \qquad 3.22$$

$$G = (V_0CB)^{-1}W_1 \qquad 3.23$$

If V_0 is nonsingular, then Rekasius' sufficient condition for first-order noninteracting subsystems is that CB be a nonsingular matrix.

In order to show the equivalency of the results of Morgan and Rekasius, it is necessary to recall that Morgan required the first m state variables to be equal to the m outputs. Since most systems do not satisfy this condition in their original variables, a linear change of variables must be made. Let the new variables be $\bar{x}$ and let the required transformation be defined by

$$\bar{x} = Tx \qquad 3.24$$

Morgan (1963) showed that T is given by

$$T = \begin{bmatrix} C \\ C^* \end{bmatrix} \qquad 3.25$$

Here C is the system output matrix, which is assumed to be of rank m; C* is any (n - m) x n matrix which results in a nonsingular T.

For the new variables $\bar{x}$ the control input matrix $\bar{B}$ can be written in terms of T and B;

$$\bar{B} = TB$$

$$= \begin{bmatrix} C \\ C^* \end{bmatrix} B = \begin{bmatrix} CB \\ C^*B \end{bmatrix} \qquad 3.26$$

Now recall that Morgan's partitioning of B was the following

$$\bar{\bar{B}} = \begin{bmatrix} \bar{\bar{B}}_1 \\ \bar{\bar{B}}_2 \end{bmatrix} \qquad 3.27$$

Thus the requirement that $\bar{\bar{B}}_1$ be nonsingular is the same as Rekasius' requirement that CB be nonsingular. The equivalence of Morgan's and Rekasius' work has been noted previously (Falb and Wolovich, 1967 a) but not shown explicitly. For future reference, the symbols $\bar{\bar{x}}$ and T are used in the succeeding chapters in a different context.

The above procedure of Rekasius has been extended by him in considering an uncoupled output equation of the form

$$V_0y^{(k)} + V_1y^{(k-1)} + \ldots + V_ky = W_1r^{(k-1)} + \ldots + W_kr \qquad 3.28$$

where the superscripts indicate time derivatives, and all the V_i and W_j are diagonal matrices. Equation 3.28 provides for the realization of higher-order subsystem responses. The same procedure that was employed in the first-order case can be used to derive formulas similar to those of Equations 3.20 and 3.21. Such formulas again provide sufficient conditions for the realization of the chosen decoupled response. The drawback is that there is no assurance that a realizable response has been chosen or even that the system can be decoupled at all.

Rekasius' procedure is one of trial-and-error, with no guarantee of success.

Both Morgan's and Rekasius' work contribute a certain amount of understanding to the multivariable design problem. However, the developments to be described in the next two sections relegate the earlier work to a position of historical value only.

The Work of Falb and Wolovich

In this section the work of Falb and Wolovich (1967 a, b) is discussed. They are responsible for finding a necessary and sufficient condition for decoupling and for formulating a procedure for obtaining a less restricted class of compensated systems than those of Morgan and Rekasius.

Recognizing that the transfer functions of the different subsystems comprising the multivariable system are generally different from one another in structure, Falb and Wolovich treat each subsystem separately. Let C_i denote the ith row of the output matrix C, and let the scalars $d_1, d_2, \ldots, d_m$ be given by

$$d_i = \min[j \,;\, C_iA^jB \neq 0,\ j = 0, 1, \ldots, n-1] \qquad 3.29$$

or $$d_i = n-1 \text{ if } C_iA^jB = 0 \text{ for all } j \qquad 3.30$$

To find d_i the successive row matrices C_iB, C_iAB, ..., $C_iA^{d_i}B$ must be formed. The scalar d_i is the smallest integer such that $C_i A^{d_i} B \neq 0$; it is shown later that d_i+1 represents the pole-zero excess for the ith subsystem. A simple calculation and the use of the definition of d_i yield the results

$$C_i(A + BF) = C_iA$$
$$C_i(A + BF)^2 = C_iA^2$$
$$\vdots$$
$$C_i(A + BF)^{d_i} = C_iA^{d_i}$$
$$C_i(A + BF)^{d_i+1} = C_iA^{d_i+1} + C_iA^{d_i}BF \qquad 3.31$$

Consider the individual output equations and the effects of state variable feedback. From Equation 3.2,

$$y_i = C_ix \qquad 3.32$$

where y_i is a scalar and is the ith component of the output vector y. Equation 3.32 is now successively differentiated until the (d_i+1)th derivative is reached, and at each step Equation 3.31 is used to simplify the resulting expressions.

Taking the time derivative of Equation 3.32 yields

$$\dot{y}_i = C_i\dot{x} \qquad 3.33$$

The expression for $\dot{x}$ is found in Equation 3.1, but after state variable feedback is applied, $\dot{x}$ becomes

$$\dot{x} = (A + BF)x + BGr \qquad 3.34$$

If Equation 3.34 is substituted into 3.33 and Equations 3.29 and 3.31 are used for simplification, the result is that

$$\dot{y}_i = C_iAx \qquad 3.35$$

Continuing the above procedure for higher derivatives of y_i gives

$$y_i^{(2)} = C_iA^2x$$
$$\vdots$$
$$y_i^{(d_i)} = C_iA^{d_i}x$$
$$y_i^{(d_i+1)} = [C_iA^{d_i+1} + C_iA^{d_i}BF]x + C_iA^{d_i}BGr \qquad 3.36$$

Since there are m outputs, m scalar equations of the above form can be written.

Define the m-dimensional vector y^*, the m x m matrix B^*, and the m x n matrix A^* as follows:

$$y^* = \begin{bmatrix} y_1^{(d_1+1)} \\ y_2^{(d_2+1)} \\ \vdots \\ y_m^{(d_m+1)} \end{bmatrix} \qquad B^* = \begin{bmatrix} C_1A^{d_1}B \\ C_2A^{d_2}B \\ \vdots \\ C_mA^{d_m}B \end{bmatrix} \qquad A^* = \begin{bmatrix} C_1A^{d_1+1} \\ C_2A^{d_2+1} \\ \vdots \\ C_mA^{d_m+1} \end{bmatrix} \qquad 3.37$$

Using these definitions all m scalar equations of the form of Equation 3.36 can be combined into the vector equation

$$y^* = (A^* + B^*F)x + B^*Gr \qquad 3.38$$

It is desired to decouple the multivariable system by using the as yet unknown compensation matrices F and G. Choose

$$F = F^* = -B^{*-1}A^* \qquad 3.39$$
$$G = G^* = B^{*-1} \qquad 3.40$$

Then Equation 3.38 becomes

$$y^* = r \qquad 3.41$$

so that F^* and G^* decouple the system. Although these decoupling matrices are not the only F and G which decouple the system, they play a prominent role in design.

The above development served as the intuitive basis for the definition of the matrix B^* and for the formulation of the decoupling theorem, designated Theorem 3.1 below. Theorem 3.1 has been proved by Falb and Wolovich (1967 a), and in a different manner by Gilbert (1968).

> Theorem 3.1 Let B^* be the m x m matrix defined in Equation 3.37. Then there is a pair of matrices F and G which decouple the multivariable system described by Equations 3.1 and 3.2 if and only if B^* is nonsingular. Furthermore, if the pair F,G are a decoupling pair, then $G = B^{*-1}\Omega$ where the m x m matrix Ω is diagonal and nonsingular.

Theorem 3.1 provides a simple test for determining whether or not a system can be decoupled by state variable feedback. The condition is more restrictive than the condition for decoupling with Configuration II design (where F and G are not constant matrices but could be frequency dependent), discussed in the previous chapter. There the requirement was that the plant matrix P(s) be nonsingular.

Gilbert (1968) finds a particular system in which P(s) is nonsingular but B* is singular. For that example, Configuration II design permits the decoupling of the system while state variable feedback design does not. In fact, Gilbert indicates that if B* is singular but P(s) is nonsingular, then states can be always added to the system in such a manner that the new system can be decoupled by state variable feedback.

Several disadvantages are present when F* and G* are used for decoupling the multivariable system. As Equations 3.37 and 3.41 indicate, the transfer function for the ith subsystem is

$$\frac{y_i(s)}{r_i(s)} = \frac{1}{s^{d_i+1}} \qquad 3.42$$

This equation indicates that the subsystem gain has been made unity and that d_i+1 of the subsystem poles are at the origin. These are highly impractical features. Not shown explicitly are the cancellations of the subsystem zeroes by subsystem poles; such cancellation is an inherent characteristic of the use of F* and G*.

Falb and Wolovich show that subsystem gain can be added and the d_i+1 poles not used for cancellation of the subsystem zeroes can be shifted from the origin by choosing

$$F = F^* + B^{*-1}\left[\sum_{k=0}^{\delta} M_k C A^k\right] \qquad 3.43$$

$$G = B^{*-1}\Omega \qquad 3.44$$

where $\delta = \max[d_1, d_2, \ldots, d_m]$ and the M_k are diagonal matrices; i.e.,

$$M_k = \mathrm{diag}[m_{11}^k, m_{22}^k, \ldots, m_{mm}^k] \tag{3.45}$$

If this F and G are chosen, the equation for y^* becomes

$$y^* = \sum_{k=0}^{\delta} M_k C A^k x + \omega_i r \tag{3.46}$$

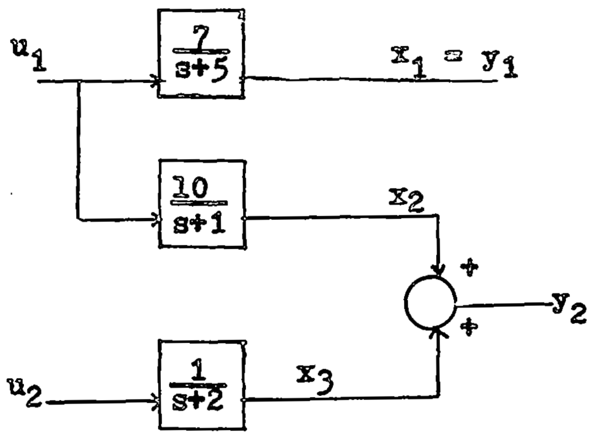
so that the subsystem transfer functions, after cancellation of the subsystem zeroes, are of the form

$$\frac{y_i(s)}{r_i(s)} = \frac{\omega_i}{s^{d_i+1} - m_{d_i d_i}^i s^{d_i} - \ldots - m_{00}^i} \tag{3.47}$$

The transfer function of Equation 3.47 is the most general subsystem transfer function which can be achieved by using the results of Falb and Wolovich; it is not the most general response which can be achieved by state variable feedback, as the following example, Example 3.1, illustrates. Next, Example 3.2 cites another aspect of system behavior which cannot be predicted or fully explained on the basis of the theory presented so far.

Example 3.1

Consider the multivariable system whose block diagram and state equations are shown in Figure 3.1(a) and (b), respectively. The matrix B^* must be found in order to determine whether state variable feedback can decouple the system. In this particular example C_1B



(a) Block Diagram

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 10 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(b) State Variable Representation

Figure 3.1 Example 3.1

$$C_1B = [1 \quad 0 \quad 0]\begin{bmatrix} 7 & 0 \\ 10 & 0 \\ 0 & 1 \end{bmatrix} \qquad C_2B = [0 \quad 1 \quad 1]\begin{bmatrix} 7 & 0 \\ 10 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= [7 \quad 0] \qquad\qquad = [10 \quad 1]$$

$$B^* = \begin{bmatrix} 7 & 0 \\ 10 & 1 \end{bmatrix} \qquad B^{*-1} = \begin{bmatrix} \frac{1}{7} & 0 \\ \frac{-10}{7} & 1 \end{bmatrix}$$

(c) Test for Decoupling

$$C_1A = [1 \quad 0 \quad 0]\begin{bmatrix} -5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad C_2A = [0 \quad 1 \quad 1]\begin{bmatrix} -5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= [-5 \quad 0 \quad 0] \qquad\qquad = [0 \quad -1 \quad -2]$$

$$A^* = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -1 & -2 \end{bmatrix}$$

$$F^* = -B^{*-1}A^* = -\begin{bmatrix} \frac{1}{7} & 0 \\ \frac{110}{7} & 1 \end{bmatrix}\begin{bmatrix} -5 & 0 & 0 \\ 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} \frac{5}{7} & 0 & 0 \\ -\frac{50}{7} & 1 & 2 \end{bmatrix}$$

$$G^* = B^{*-1}$$

(d) Calculation of F^* and G^*

Figure 3.1 Example 3.1 (Continued)

$$H(s,F^*,G^*) = \begin{bmatrix} \frac{s(s+1)}{s^2(s+1)} & 0 \\ 0 & \frac{s(s+1)}{s^2(s+1)} \end{bmatrix}$$

(e) H(s,F*,G*)

$$H(s,F,G) = \frac{\begin{bmatrix} 7g_{11}(s+1)(s+2-f_{23}) & 0 \\ 0 & g_{22}[(s+5)(s+1-10f_{12})-7f_{11}(s+1)] \end{bmatrix}}{[(s+5)(s+1-10f_{12})-7f_{11}(s+1)](s+2-f_{23})}$$

(f) H(s,F,G)

Figure 3.1 Example 3.1 (Continued)

and C_2B are both nonzero (see Figure 3.1(c)) so that, in conformity with Equation 3.29, d_1 and d_2 are 0. The matrix B* is, according to Equation 3.31,

$$B^* = \begin{bmatrix} C_1B \\ C_2B \end{bmatrix} \qquad 3.48$$

As Figure 3.1(c) shows, B* is a nonsingular matrix and so the system can be decoupled.

Figure 3.1(d) illustrates how the matrices F* and G* are formed. As an intermediate step A* is found by using the equation

$$A^* = \begin{bmatrix} C_1A \\ C_2A \end{bmatrix} \qquad 3.49$$

where, according to Equation 3.29, the power of A in the expression for each row of A* is 1 because d_1 and d_2 are both 0. The decoupling matrices F* and G* are calculated according to Equations 3.39 and 3.40; the calculations are shown in Figure 3.1(d). Figure 3.1(e) gives the transmission matrix H(s,F*,G*), relating the input r to the output y. Here, the cancellation of subsystem zeroes discussed previously is shown. After cancellation, each subsystem has the transfer function $\frac{1}{s}$, in agreement with Equation 3.42.

At this stage there is no way of telling how the factor $\frac{s+1}{s+1}$ in each of the diagonal elements of H(s,F*,G*) came about. It appears that in either subsystem 1 or subsystem 2 or in both, a subsystem zero is being canceled by a subsystem pole at the same location, but the use of F* and G* does not allow the identification of the specific subsystem or subsystems containing the zero. This feature is a drawback of the approach of Falb and Wolovich because zeroes influence the system response and one usually wants the option of retaining them or canceling them, whichever results in the better response.

The most general transmission matrix for this problem was worked out by the brute force method.[1] That is, H(s,F,G) was calculated for a completely arbitrary F and G and then the criterion of noninteraction was imposed. The result is shown in Figure 3.1(f). A careful look at the diagonal elements of H(s,F,G) reveals that subsystem 1 has arbitrary gain, a zero at $s = -1$, and two arbitrary poles; thus there is a zero in subsystem 1 which need not be canceled. Subsystem 2 has arbitrary gain and one arbitrary pole. The use of F* and G* has caused the zero in subsystem 1 to be canceled. Subsystem 2 has no zeroes.

The most general form of response which can be realized by the methods of Falb and Wolovich is found by using the F and G of Equations 3.43 and 3.44. In this

1. Falb and Wolovich (1967 b) give another method for carrying out the procedure. However, it is difficult to apply. Still another method is discussed in the following section.

case the subsystem transfer functions, after cancellation, are

$$\frac{y_1(s)}{r_1(s)} = \frac{\omega_1}{s - m_{11}^0} \qquad 3.50$$

$$\frac{y_2(s)}{r_2(s)} = \frac{\omega_2}{s - m_{22}^0} \qquad 3.51$$

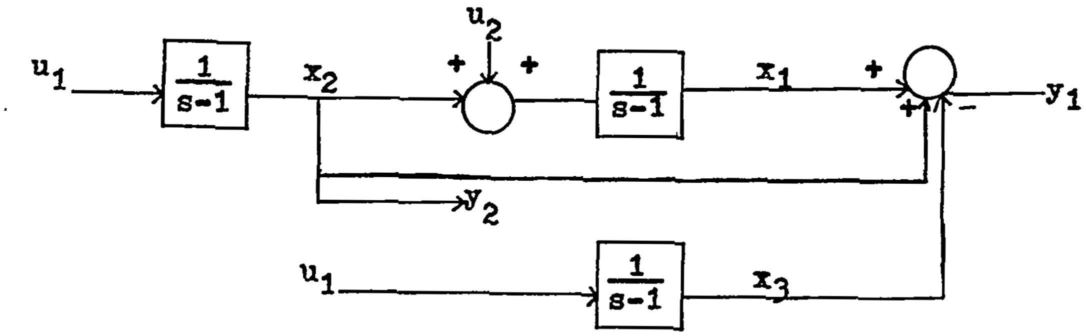
where m_{11}^0 and m_{22}^0 are the arbitrary diagonal elements of M_0. Again, the zero in subsystem 1 has been canceled so that only a first-order response can be achieved.

A type of behavior which has not yet been encountered is illustrated by the following example, taken from Falb and Wolovich (1967 b).

Example 3.2

Consider the system whose block diagram and state equations are shown in Figure 3.2(a) and (b). The most general F which decouples the system and the corresponding transmission matrix are shown in Figure 3.2(c). Here each subsystem transfer function has the factor $\frac{s-1}{s-1}$ even for the most general F. In this instance state variable feedback cannot stabilize the system.

This particular example is uncontrollable (Kalman, 1960), but even for controllable systems there are examples in which some of the roots of the characteristic polynomial are not effected by state variable feedback. If any such roots are in the right half plane, then the



(a) Block Diagram

$$\dot{x} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} x$$

(b) State Equations

$$F = \begin{bmatrix} 0 & f_{12} & f_{13} \\ f_{21} & f_{22} & -1-f_{22} \end{bmatrix}$$

$$H(s,F,G^*) = \frac{\begin{bmatrix} (s-1)(s-f_{12}-f_{13}-1) & 0 \\ 0 & (s-1(s-f_{12}-1) \end{bmatrix}}{(s-1)(s-f_{12}-1)(s-f_{12}-f_{13}-1)}$$

(c) Transmission Matrix

Figure 3.2 Example 3.2

multivariable system is unstable and cannot be stabilized by state variable feedback alone.

The features of Falb and Wolovich's work can now be summarized. First, the authors give a test that determines whether or not state variable feedback can decouple the multivariable system; this test is highly useful and practical because it can be programmed readily on a digital computer. Second, they give easily programmed formulas for particular F and G matrices that decouple the system. The use of these matrices precludes ever having uncanceled zeroes in any of the subsystems; their use does allow the specification of d_i+1 poles of each subsystem, but the remaining subsystem poles are used for canceling the subsystem zeroes. Third, Falb and Wolovich characterize the class of all G matrices in Theorem 3.1 and describe in their papers a cumbersome method for characterizing the class of all F matrices which decouple; with this information they show the class of compensated systems for several simple examples. Finally, as Example 3.2 indicates, state variable feedback does not in every case permit the stabilization of the system.

The Work of Gilbert

Consider the possibility of decomposing the uncompensated multivariable system into m subsystems which have static coupling between each possible pair. Such a canonical decomposition would be useful if it were possible to achieve because then one could conceive of a design procedure in which part of the compensation was used to compensate the subsystems to give the desired response, and the remaining part of the compensation was used to destroy the coupling between subsystems. The logical way of achieving this structure is through a change of variables. However, even though many canonical forms are available for multivariable systems (e.g., Luenberger (1966) and Asseo (1968)), none has been found which accomplishes the desired results.

Conceived in the light of the above discussion, the approach of Gilbert (1968) involves decoupling the system first, and then a canonical form is sought. This two-step procedure is not equivalent to a change of variables (because the decoupling process changes the system dynamics) although a change of variables is involved in the procedure.

Gilbert's concept of an Integrator Decoupled (ID) system is needed.

Definition 3.1 A multivariable system is Integrator Decoupled if $B^* = \Lambda$, where Λ is diagonal and nonsingular, and $C_i A^{d_i+1} = 0$, $i = 1, 2, \ldots, m$.

A specific example of an ID system is the system which results when Falb and Wolovich's F* and G* are used for decoupling. In fact, in this text the term ID system always means the particular ID system resulting from the use of F* and G*. The proof that F* and G* lead to an ID system is given in Gilbert (1968), where it is also shown that the subsystem transfer functions for any Integrator Decoupled system are of the form

$$\frac{y_i(s)}{r_i(s)} = \frac{\lambda_i}{s^{d_i+1}} \qquad i = 1, \ldots, m \qquad 3.52$$

Suppose F* and G* are used for decoupling the multivariable system; since the system is now decoupled, one might conjecture that it is possible to find a set of state variables in which the fact that the system is decoupled is clearly evident. This transformation of variables was found by Gilbert. Of course, the system of interest is not the ID system but rather the original, coupled plant, and furthermore the response of the ID system, as given by Equation 3.52, is not the one that is desired. Gilbert shows that the ID system can be recompensated by state variable feedback to achieve the desired response, and then the two sets of F and G matrices can be used in finding the compensation matrices for the original system which give the same transfer matrix from r to y. The formal development of these ideas follows.

Let S[A,B,C] represent the multivariable system defined by Equations 3.1 and 3.2, and let [F,G] be the control law or set of state variable feedback compensation matrices for S[A,B,C].

Definition 3.2 The multivariable systems S[A,B,C] and $S_1[A_1,B_1,C_1]$ are control law equivalent (CLE) if a one-to-one correspondence between [F,G] and $[F_1,G_1]$ can be found such that for this correspondence $H(s,F,G) = H_1(s,F_1,G_1)$, where $H(s,F,G)$ and $H_1(s,F_1,G_1)$ are the transmission matrices relating the output y to the input r for systems S[A,B,C] and $S_1[A_1,B_1,C_1]$, respectively.

In the present case S[A,B,C] represents the original system and $S_1[A_1,B_1,C]$ represents the ID system, found by compensating S[A,B,C] with [F*,G*]. The state equations for S[A,B,C] are the familiar ones of Equations 3.1 and 3.2. The equations for $S_1[A_1,B_1,C]$ are

$$\dot{x} = (A + BF^*)x + BG^*u \qquad 3.53$$

$$y = Cx \qquad 3.54$$

Expressions for F* and G* are given by Equations 3.39 and 3.40. When these expressions are used in Equation 3.53, the equation becomes

$$\dot{x} = (A - BB^{*-1}A^*)x + BB^{*-1}u \qquad 3.55$$

Suppose F_1 and G_1 are the matrices used for compensating the ID system so that it has a transmission matrix which meets the design specifications. The control input for $S_1[A_1, B_1, C]$ is thus

$$u = F_1 x + G_1 r \qquad 3.56$$

and Equation 3.55 becomes

$$\dot{x} = (A - BB^{*-1}A^{*} + BB^{*-1}F_1)x + BB^{*-1}G_1 r \qquad 3.57$$

The corresponding expression for $S[A, B, C]$ is

$$\dot{x} = (A + BF)x + BGr \qquad 3.58$$

The systems $S[A, B, C]$ and $S_1[A_1, B_1, C]$ will have identical responses if Equations 3.57 and 3.58 are identical. This requires that

$$F = B^{*-1}(F_1 - A^{*})$$

$$= B^{*-1}F_1 + F^{*} \qquad 3.59$$

$$G = B^{*-1}G_1 \qquad 3.60$$

By this development it is established that one can work with either the original system or the ID system and still retain the same design freedom. Gilbert chose to work with the ID system because it is decoupled.

The canonical form to which the ID system can be transformed is now discussed. Only the case of controllable systems (Kalman, 1960) is treated although Gilbert considered both controllable and uncontrollable systems. Almost all practical systems are controllable (Schultz and Melsa, 1967), and state variable feedback never causes loss of controllability (Brockett, 1965); these two facts justify the decision to omit the discussion of uncontrollable systems.

There are two parts to Gilbert's canonical form. First, the A, B, and C matrices must have the form shown in Figure 3.3(a); the system is now said to be in Standard Decoupled (SD) form. The diagonal nature of the upper portion of the A and B matrices and the left half of C permit the identification of m decoupled subsystems, and it is shown later that the submatrix A^c_{m+1} is important in accounting for the peculiarities of the type discussed in Example 3.2. If the subsystem exhibits the additional internal structure of Figure 3.3(b), it is said to be Canonically Decoupled (CD).

The ID system and the CD system are related by a linear change of variables. Let the respective state variables be x and $\hat{x}$. These variables are related by the nonsingular matrix Q, as

$$\hat{x} = Qx \qquad 3.61$$

The transformation matrix Q is found as follows.

Let $\mathcal{Q}$ denote the n-dimensional space of n-element row matrices, and for i = 1, ..., m, define

$$\mathcal{Q}_1 = [\ \eta\ ;\ \eta \in \mathcal{Q};\ \eta A^j B_k = 0 \text{ for } k = 1, \ldots, m,\ k \neq i \text{ and } j = 0, \ldots, n-1] \qquad 3.62$$

$$A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 & 0 \\ 0 & A_{22} & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & A_{mm} & 0 \\ A_1^c & A_2^c & \cdots & A_m^c & A_{m+1}^c \end{bmatrix} \quad \begin{array}{l} A_{ii} \text{ is } n_i \times n_i \\ A_i^c \text{ is } n_{m+1} \times n_i \end{array}$$

$$B = \begin{bmatrix} B_{11} & 0 & \cdots & 0 \\ 0 & B_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & B_{mm} \\ B_1^c & B_2^c & \cdots & B_m^c \end{bmatrix} \quad \begin{array}{l} B_{ii} \text{ is } n_i \times 1 \\ B_i^c \text{ is } n_{m+1} \times 1 \end{array}$$

$$C = \begin{bmatrix} C_{11} & 0 & \cdots & 0 & 0 \\ 0 & C_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & C_{mm} & 0 \end{bmatrix} \quad C_{ii} \text{ is } 1 \times n_i$$

(a) Standard Decoupled Form

Figure 3.3 Canonically Decoupled System Representation

$$A_{ii} = \begin{bmatrix} \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} & 0 \\ \Upsilon_i & \Phi_i \end{bmatrix}$$

I is $d_i \times d_i$

Υ_i is $l_i \times (d_i+1)$

Φ_i is $l_i \times l_i$

$$B_{ii} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \gamma_i \\ \beta_i \end{bmatrix}$$

β_i is $r_i \times 1$

$$C_{ii} = [1 \quad 0 \quad \ldots \quad 0]$$

(b) Canonically Decoupled Form

Figure 3.3 Canonically Decoupled System Representation (Continued) where B_k is the kth column of B. The numbers n_i and l_i, which are needed in the definition of the CD representation, are defined by

$$n_i = \dim \mathcal{C}_i \qquad 3.63$$

$$i = 1, \ldots, m$$

$$l_i = n_i - d_i - 1 \qquad 3.64$$

The number n_i is the number of poles of subsystem i which can be controlled by state variable feedback; l_i is the number of fixed zeroes of the subsystem.

The transformation matrix Q is written as

$$Q = \begin{bmatrix} Q_1 \\ \vdots \\ Q_{m+1} \end{bmatrix} \qquad 3.65$$

where the rows of the $n_i \times n$ matrix Q_i, $i = 1, \ldots, m$, are a basis for $\mathcal{C}_i$. The first d_i+1 rows of each of these Q_i are chosen to be C_i, C_iA, $\ldots$, $C_iA^{d_i}$ and the last $n - d_i - 1$ rows are any other linearly independent row vectors which make the set of rows of Q_i a basis for $\mathcal{C}_i$. For Q_{m+1}, the rows are chosen so that the collection of rows of Q form a basis for $\mathcal{C}$.

The state equations for the ID system are

$$\dot{x} = (A + BF^*)x + BG^*u \qquad 3.66$$

$$y = Cx \qquad 3.67$$

After substituting $\hat{x} = Qx$, these equations become

$$\dot{\hat{x}} = Q(A + BF^*)Q^{-1}x + QBG^*u \qquad 3.68$$

$$y = CQ^{-1}\hat{x} \qquad 3.69$$

Gilbert proved that the matrices $Q(A + BF^*)Q^{-1}$, QBG^*, and CQ^{-1} have the structure required for the CD system representation.

Some familiarity with the transformation can be gained by finding the CD representation for Example 3.1.

Example 3.3

The state variable representation is shown in Figure 3.1(b) and the F* and G* appear in Figure 3.1(d). These equations and matrices are needed to calculate the ID representation

$$\dot{x} = (A + BF^*)x + BG^*u \qquad 3.70$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ \frac{50}{7} & -1 & 0 \\ -\frac{50}{7} & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ \frac{10}{7} & 0 \\ -\frac{10}{7} & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} x \qquad 3.71$$

Once the ID representation has been found, the calculation of the transformation matrix Q can proceed. The first step in calculating Q is the characterization of the subspaces of $\mathcal{Q}$. For $\mathcal{Q}_1$ Equation 3.62 becomes

$$\mathcal{Q}_1 = [\eta ; \eta A^j B_2 = 0, \; j = 0, 1, 2] \qquad 3.72$$

Let the row vector η be (η_1, η_2, η_3). Then

$$\eta B_2 = \eta_3 = 0$$
$$\eta AB_2 = 0$$
$$\eta A^2 B_2 = 0 \qquad 3.73$$

Thus

$$\mathcal{Q}_1 = [\eta : \eta = (\eta_1, \eta_2, 0)] \qquad 3.74$$

By definition the rank of $\mathcal{Q}_1$ is n_1, or, for this example, 2. The same procedure can be carried out for the subspace $\mathcal{Q}_2$, to yield

$$\mathcal{Q}_2 = [\eta : \eta = (0, \eta_2, \eta_2)] \qquad 3.75$$

The rank of $\mathcal{Q}_2$ is 1; hence n_2 is 1. For this example the dimensions of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ add up to the dimension of $\mathcal{Q}$, and $\mathcal{Q}_3$ is not needed.

The matrix Q is composed of Q_1 and Q_2. Since d_1 and d_2 are both 0, the first row of Q_1 is C_1, the first row of the output matrix C shown in Figure 3.1(b), and the first row of Q_2 is C_2. The second row of Q_1 can be any row vector which is independent of C_1 and which belongs to the subspace $\mathcal{Q}_1$. Choose the row vector (0, 1, 0). Thus Q is completely defined, as

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad 3.76$$

and Q^{-1} is found to be

$$Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad 3.77$$

The system matrices in CD form are found by using the system matrices for the ID system and the above transformation matrix, and performing the computations required in Equations 3.68 and 3.69. The results are shown below.

$$\hat{A} = Q(A + BF^*)Q^{-1} = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ \frac{50}{7} & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \qquad 3.78$$

$$\hat{B} = QBG^* = \left[\begin{array}{c|c} 1 & 0 \\ \frac{10}{7} & 0 \\ \hline 0 & 1 \end{array}\right] \qquad 3.79$$

$$\hat{C} = CQ^{-1} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \qquad 3.80$$

Note that these matrices satisfy the requirements for the CD representation given in Figure 3.3(a) and (b).

The partitioning in Equations 3.78 - 3.80 is used in illustrating what matrices are associated with each subsystem. The two distinct subsystems for this example have the state equations

$$\dot{\hat{x}}^1 = \begin{bmatrix} 0 & 0 \\ \frac{50}{7} & -1 \end{bmatrix} \hat{x}^1 + \begin{bmatrix} 1 \\ \frac{10}{7} \end{bmatrix} u^1 \qquad 3.81$$

$$y^1 = [1 \quad 0]\hat{x}^1 \qquad 3.82$$

$$\dot{\hat{x}}^2 = [0]\hat{x}^2 + [1]u^2 \qquad 3.83$$

$$y^2 = [1]\hat{x}^2 \qquad 3.84$$

The superscripts do not represent powers but are used to indicate a partitioning of $\hat{x}$, y, and u into two disjoint parts, each of which is associated with one of the systems.

The CD representation is the means by which each subsystem is identified and isolated in a manner that permits the application of the results of state variable feedback as it is formulated for single-input, single-output systems. This is shown clearly in the previous example. The identification of a set of open-loop transfer functions of the ID system is now possible. In terms of the CD representation of the ID system these transfer functions are

$$p_{ii}(s,F^*,G^*)(s) = C_{ii}(sI - A_{ii})^{-1}B_{ii} \qquad 3.85$$

where $p_{ii}(s,F^*,G^*)$ is the ith diagonal element of the nonsingular, diagonal transfer matrix $P(s,F^*,G^*)$, relating the output y to the control input u of the ID system. It may be helpful to recall that P relates y to u and H relates y to r.

These open-loop systems are to be compensated by the control law $[\hat{F},\hat{G}]$. Gilbert shows that $\hat{F}$ and $\hat{G}$ preserve noninteraction if and only if they have the forms

$$\hat{F} = \begin{bmatrix} \theta_1 & 0 & \cdots & 0 & 0 \\ 0 & \theta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & \theta_m & 0 \end{bmatrix} \qquad 3.86$$

where each θ_i is a $1 \times n_i$ matrix, and

$$\hat{G} = \Lambda = \mathrm{diag}[\lambda_{11}, \ldots, \lambda_{mm}] \qquad 3.87$$

The important point is that when compensating the multivariable plant, one need only integrator decouple the system and then change variables to get the canonically decoupled form; from then on each subsystem can be treated separately. The transfer function for the ith subsystem is given by Equation 3.85. In keeping with the results of state variable feedback for single-input, single-output systems (Schultz and Melsa, 1967), all n_i of the subsystem poles can be arbitrarily placed, but the l_i zeroes of the open-loop system--given by the numerator of Equation 3.85--are not affected by state variable feedback.

Gilbert shows that the zeroes of the ith subsystem are the zeroes of the equation

$$\det(sI - \Phi_i) = 0 \qquad 3.88$$

where, according to Figure 3.3(a) and (b), Φ_i is an $l_i \times l_i$ submatrix of A_{ii}, the system matrix for the ith subsystem. The poles of the subsystem are deduced by using the previously noted fact that the ith subsystem of an ID system has d_i+1 poles at the origin and its remaining poles at the locations of the subsystem zeroes. Let $p^i(s,F^*,G^*)$ be the characteristic polynomial of the ith subsystem of the ID system; then

$$p^i(s,F^*,G^*) = \det(sI - A_{ii}) = s^{d_i+1}\det(sI - \Phi_i) \quad i = 1, \ldots, m \qquad 3.89$$

When the control law [F,G] is used for compensating the ID system, all n_i poles of each subsystem can be moved from the locations determined from Equation 3.89 to arbitrary locations.

If $\sum_{i=1}^{m} n_i < n$, there are additional roots of the characteristic polynomial which are not accounted for by Equation 3.89. These roots are the zeroes of the polynomial $\det(sI - A_{m+1,m+1})$; they are not affected by state variable feedback. Thus, q(s,F,G), the characteristic polynomial of the compensated ID system has $\sum_{i=1}^{m} n_i$ roots which are controlled by F and G; and, in addition, the factor $\det(sI - A_{m+1,m+1})$ is present. The presence of the additional factor in q(s,F,G) is detected during the completion of the procedure for finding the transformation matrix Q because if $\sum_{i=1}^{m} n_i < n$, then $n_{m+1} \neq 0$, $\alpha_{m+1} \neq 0$, and $A_{m+1,m+1} \neq 0$. Since the factor is not affected by $[\hat{F},\hat{G}]$, the system is unstable if any of the roots of $\det(sI - A_{m+1,m+1})$ are in the right half plane.

This is an appropriate place to state Gilbert's design procedure. First, the given multivariable system is tested for decoupling by applying Falb and Wolovich's criterion that B* be nonsingular. If the test is successful then the system is integrator decoupled by using F* and G*. Next the matrix Q is found and used to change variables and transform the system to Gilbert's canonically decoupled form. Now there are m subsystems, the ith subsystem has l_i fixed zeroes and n_i poles which can be arbitrarily placed by state variable feedback. The characteristic polynomial of the system consists of the product of the characteristic polynomials of each of the subsystems and, in addition, the factor $\det(sI - A_{m+1,m+1})$ is present. Once the individual subsystems have been compensated, the matrices $\hat{F}$ and $\hat{G}$ are completely determined. The corresponding compensation matrices in terms of the original system variables are

$$F = F^* + B^{*-1}\hat{F}Q \qquad 3.90$$

$$G = B^{*-1}\hat{G} \qquad 3.91$$

These formulas are the result of the application of the transformation matrix Q and Equations 3.59 and 3.60.

Example 3.4

The system of Example 3.1 again provides a convenient means for illustrating the design procedure. The CD representation is given in Equations 3.78 - 3.80.

The two subsystems for this example are found by using Equation 3.85. They are

$$p_{11}(s,F^*,G^*) = \frac{s+1}{s(s+1)} \qquad 3.92$$

$$p_{22}(s,F^*,G^*) = \frac{1}{s} \qquad 3.93$$

Subsystem 1 has a fixed zero at s = -1 and two poles which can be arbitrarily placed. Subsystem 2 has one arbitrary pole. Suppose it is desired to achieve the following transfer functions by the proper choice of $\hat{F}$ and $\hat{G}$.

$$h_{11}(s,\hat{F},\hat{G}) = \frac{2(s+1)}{s^2+2s+2} \qquad 3.94$$

$$h_{22}(s,\hat{F},\hat{G}) = \frac{1}{s+1} \qquad 3.95$$

Techniques for calculating the $\hat{F}$ and $\hat{G}$ that result in a given $H(s,\hat{F},\hat{G})$ are discussed in Chapter 5. Here the results for this example are merely written down because this particular system is treated again in Chapter 5 as Example 5.1. The compensation matrices for the ID system in terms of the state variables of the CD representation are

$$\hat{F} = \begin{bmatrix} -\frac{3}{4} & -\frac{7}{40} & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \hat{G} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad 3.96$$

The corresponding matrices for the original system, found by substituting in Equations 3.90 and 3.91, are

$$F = \begin{bmatrix} \frac{17}{28} & -\frac{1}{40} & 0 \\ -\frac{85}{14} & \frac{1}{4} & 1 \end{bmatrix} \qquad G = \begin{bmatrix} \frac{2}{7} & 0 \\ -\frac{20}{7} & 1 \end{bmatrix} \qquad 3.97$$

A block diagram of the compensated system appears in Figure 3.4.

The procedure for characterizing the class of all F matrices which decouple is now discussed and illustrated for Example 3.4.

When the system is in CD form, the only F matrix which decouples is $\hat{F}$, given by Equation 3.86. For the current example, $n_1 = 2$ and $n_2 = 1$, so that

$$\hat{F} = \begin{bmatrix} \theta_{11} & \theta_{12} & 0 \\ 0 & 0 & \theta_{23} \end{bmatrix} \qquad 3.98$$

The corresponding F for the original system is found by using Equation 3.90. After the matrix calculations have been performed there results

$$F = \begin{bmatrix} \frac{5}{7}+\frac{1}{7}\theta_{11} & \frac{1}{7}\theta_{12} & 0 \\ -10(\frac{5}{7}+\frac{1}{7}\theta_{11}) & 1-\frac{10}{7}\theta_{12}+\theta_{23} & 2+\theta_{23} \end{bmatrix} \qquad 3.99$$

The work of Gilbert described in this section is notable for its completeness. If the multivariable system can be decoupled, then his work provides a means for determining the form of each component subsystem. In particular, the number of subsystem zeroes and the number of poles are known; and, just as in single-input, single-output systems, state variable feedback allows the arbitrary placement of all the subsystem poles, but the

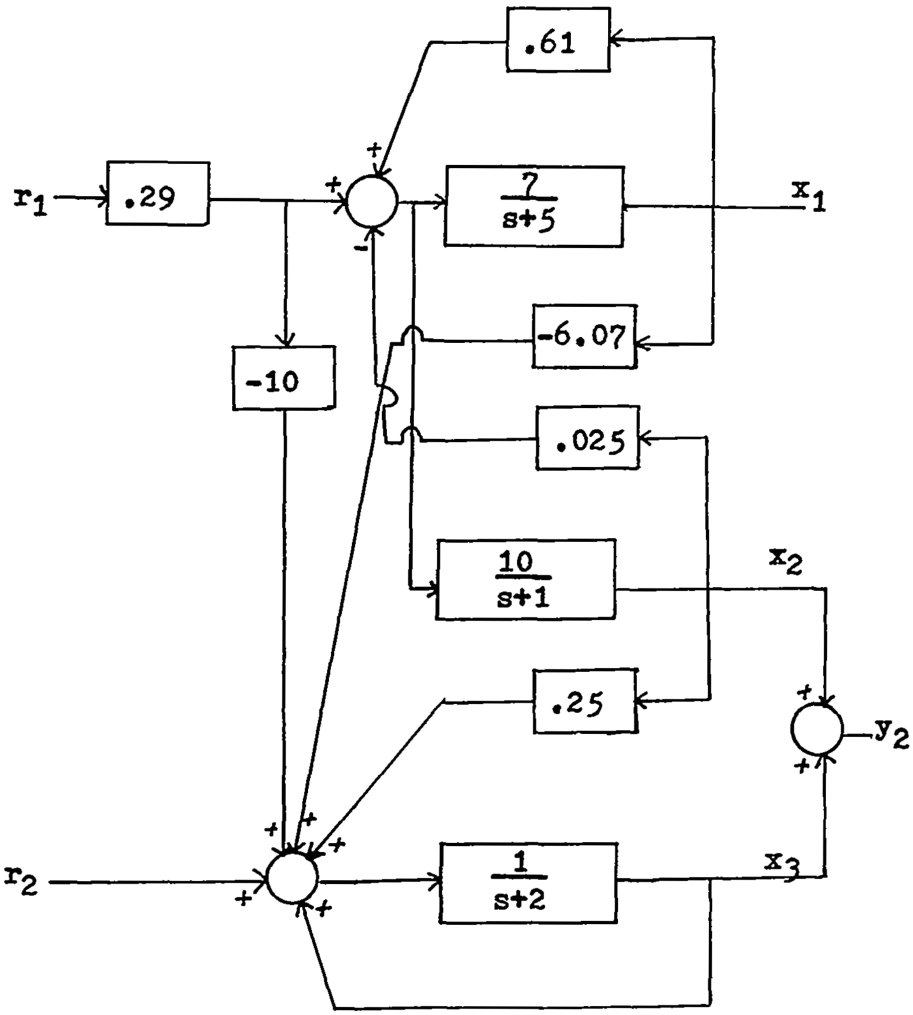


Figure 3.4 Compensated System of Example 3.4

zeroes remain unchanged. A step-by-step design procedure is given in Chapter 5.

The computations required in finding the ID system and the CD representation are tedious if performed by hand. Thus the digital computer is an indispensable aid. More is said about the computational aspects in Chapter 5. Here, it is sufficient to note that explicit formulas are characteristic of this section, rather than iterative methods. This means that the programming job is simplified since much of the task consists merely of re-coding formulas in a form that is acceptable to the computer.

Summary

In a sense, the state variable feedback technique has been described from start to finish. The original work on the design technique is described in the early sections of the chapter. The research described subsequently has caused the original work to be relegated to a position of historical value only. Somewhat the same remark applies to some of Falb and Wolovich's work. However, their test for decoupling is highly useful and the matrices F* and G* are basic in Gilbert's work.

Gilbert provides a complete theoretical treatment of the state variable feedback problem. His theory describes the limitations of state variable feedback by providing explicit formulas for determining the specific form of each subsystem of the compensated system. He also gives an analytical explanation of the situation where state variable feedback alone does not provide control over all the roots of the characteristic polynomial.

At least two problems yet remain. The first problem arises when state variable feedback alone does not allow the designer enough freedom to achieve the required system response. For example, suppose a third-order subsystem with one zero is necessary for a satisfactory response, but the system is capable of only a second-order response. If the system has one input and one output, the solution to this problem would be to add a lead-lag compensator. For the multivariable system a similar technique is applicable. The second problem is the computational problem. For a given choice for the transfer matrix of the compensated system, how does one carry out the computations required to find F and G? Both computational aspects and additional compensation are discussed in the next two chapters.

CHAPTER 4

SERIES COMPENSATION AND STATE VARIABLE FEEDBACK

In the previous chapter the limitations of state variable feedback are discussed. The use of Gilbert's canonically decoupled representation of the integrator decoupled multivariable system makes available the option of treating the system as a collection of m single-input, single-output subsystems. The ith subsystem of the integrator decoupled system has l_i zeroes and n_i poles; by applying state variable feedback to the multivariable system all the subsystem poles can be placed arbitrarily, but the subsystem zeroes remain fixed.

There are design problems in which state variable feedback alone does not offer enough flexibility to meet the performance specifications. Usually these situations arise when zeroes are required in the transfer functions of one or more of the subsystems of the final, compensated system. The primary reason for wanting zeroes in a closed-loop transfer function of a single-input, single-output system is that their presence makes it possible to achieve an infinite velocity-error coefficient, or zero position error for ramp inputs (Truxal, 1955; Schultz and Melsa, 1967). Zeroes which are an inherent part of the plant, or fixed portion of the system, are usually at undesirable locations, and so additional zeroes must be added. For a single-input, single-output system, series compensation is added to realize the required zeroes; similar techniques are developed in this chapter for multivariable systems.

Three methods are discussed for inserting additional dynamics into the subsystems of the multivariable system. The first technique, Method A, is directly analogous to the procedure used in the single-input, single-output case. Basically, it requires that the multivariable plant be augmented by inserting compensation networks in the control input channels and that all the states of the resulting augmented plant be fed back. An example is used to show that Method A does not apply in many design problems because it is not always possible to decouple the augmented plant. Another example shows that even when Method A is applicable, there are serious problems associated with its use.

In Method B the plant is first decoupled, and then the series compensation is added in the control input channels of the decoupled plant. No additional feedback is needed from any of the states of the resulting augmented plant. In particular, the compensator states are not fed back so that parts of the final, compensated system have no feedback at all; and noise and sensitivity problems may be present. In both the matters of utility and the amount of attention given in this chapter, Methods A and B do not deserve equal ranking with Method C; rather, they are best considered as steps along the path to the most general technique, Method C.

Method C is similar to Method B in that the plant is first decoupled before the series compensation is added in the control input channels of the resulting decoupled plant. But now, state variable feedback is used again; and this time all the states of the augmented plant, including compensator states, are fed back around the augmented plant as the final step in the design. This method represents the ultimate in state variable feedback design because through its use the designer has the greatest freedom in achieving the response required for each of the subsystems of the final, compensated multivariable system.

It is proved in this chapter that, when the zeroes and poles are added in the manner prescribed by Method C, the zeroes appear unchanged in the proper compensated subsystem transfer functions. Further, all the poles of the augmented plant are arbitrarily positioned by the final application of state variable feedback. The proof of this central result requires the formulation and proof of several intermediate results. Some of these intermediate steps are generalizations of theorems and lemmas proved by Gilbert (1968).

Gilbert mentions the problem of augmenting the multivariable plant in such a manner that it could be decoupled, but no work has been reported on the specific problem of augmenting the multivariable system for improvement in response. Except where explicitly stated to the contrary, the procedures, theorems, and discussions presented in this chapter are new.

Methods for Series Compensation

In this section three methods are introduced for providing more flexibility in system design than that available through the use of state variable feedback alone. Methods A and B are discussed in greater detail than Method C because this section is the only one in which the former are considered; Method C is discussed in full detail in the succeeding sections of this chapter.

Method A

Method A consists of three steps. First, the subsystem transfer functions of the integrator decoupled (ID) system are identified with the aid of Gilbert's canonically decoupled (CD) representation. At the completion of this step, one learns the location of the fixed zeroes and the number of subsystem poles. If the design criteria can be met by merely adjusting the gains and the poles of each subsystem, then state variable feedback alone can be used to design the system. If the form of any of the subsystem transfer functions does not permit the system response specifications to be achieved, then series compensation is needed; and it is necessary to proceed to the second step of Method A.

The second step is the insertion of compensation networks into the control input channels of the multivariable plant. The question of exactly what form of compensation to insert cannot be answered a priori; there is no guarantee that in the final, compensated system zeroes inserted in the control input channel will appear in the proper subsystem transfer function, or even at all. An even more serious difficulty is that the type of compensation being described could lead to the loss of the ability of the system to be decoupled by state variable feedback. However, series compensation networks with the desired zeroes are added in each of the input channels, as needed. The assumptions are made that decoupling is still possible and that the zeroes which have been added will appear in the proper subsystem transfer function after state variable feedback.

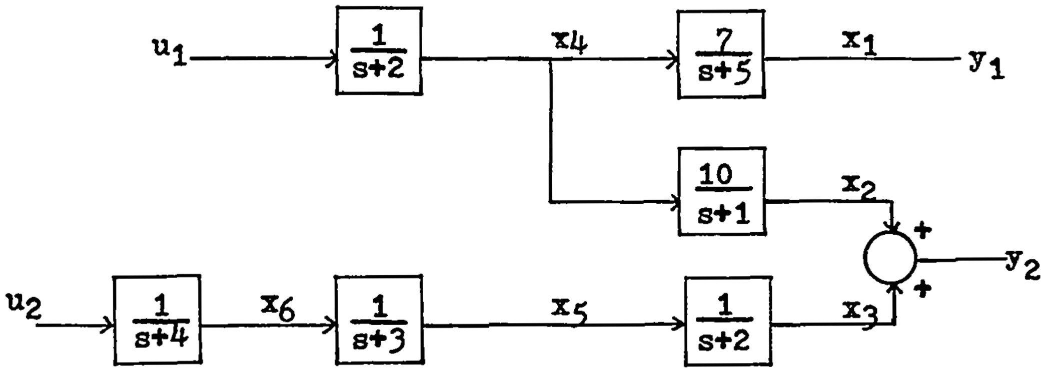
The final step of the design procedure requires that the two assumptions made above be tested. First, can the system be decoupled, or is B^* nonsingular for the augmented plant? Second, do the subsystems contain the desired zeroes that are to appear in the closed-loop system? If these assumptions are indeed valid, then state variable feedback is used to alter the subsystem poles and to introduce gain to meet the specifications. The following two examples are attempts to apply Method A. Both attempts fail and by doing so illustrate the two basic deficiencies of Method A.

Example 4.1. The plant of Examples 3.1, 3.3, and 3.4 is again used. It is already known from Equations 3.92 and 3.93 that subsystem 1 has a zero at $s = -1$ and two arbitrary poles, and that subsystem 2 has no zeroes and one arbitrary pole. Assume that the specifications require that in the final design both subsystems have third-order responses. For simplicity, no attempt is made to add any zeroes.

An appropriate choice for augmenting the plant is shown in the block diagram of Figure 4.1(a) (the block diagram before augmentation appears in 3.1(a)), and the state equations are shown in Figure 4.1(b). A straightforward calculation shows that d_1 and d_2 are both 0 and that B^* is given by

$$B^* = \begin{bmatrix} 7 & 0 \\ 10 & 0 \end{bmatrix} \qquad 4.1$$

Clearly B^* is a singular matrix, and so the multivariable system cannot be decoupled by state variable feedback. This example demonstrates that situations may arise where the addition of series compensation violates the assumption of decoupling.



(a) Block Diagram of the Augmented System

$$\dot{x} = \begin{bmatrix} -5 & 0 & 0 & 7 & 0 & 0 \\ 0 & -1 & 0 & 10 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} x$$

(b) State Equations

Figure 4.1 Example 4.1

Example 4.2. Consider the augmented plant whose block diagram and state equations are shown in Figure 4.2(a) and (b), respectively. In this example an attempt is being made to introduce a zero at s = -3 in subsystem 1. In order to find out whether the zero does appear in the subsystem, the CD representation for the integrator decoupled augmented system must be found. Then the transfer functions for its two subsystems can be found. The subsystem transfer functions for the CD system are

$$p_{11}(s,F^*,G^*) = \frac{s+1}{s^2(s+1)} \qquad 4.2$$

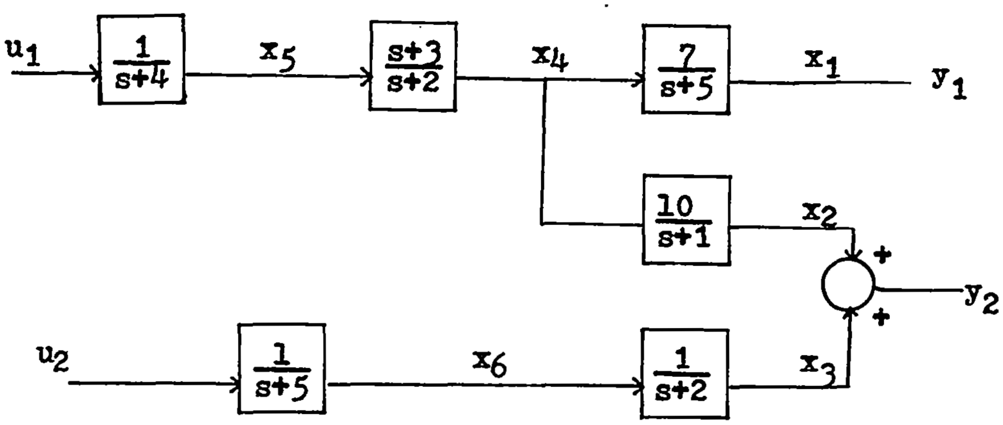
$$p_{22}(s,F^*,G^*) = \frac{1}{s^2} \qquad 4.3$$

The characteristic polynomial for the system is

$$p(s,F^*,G^*) = s^4(s+1)(s+3) \qquad 4.4$$

The only place the factor (s+3) appears is in $p(s,F^*,G^*)$. As Equations 4.2 and 4.3 show, neither subsystem can ever have a zero at s = -3, and so the attempt to add a zero has failed.

Examples 4.1 and 4.2 illustrate the two deficiencies of Method A. In Example 4.1 it is shown that augmenting the plant in the prescribed manner could lead to loss of coupling. Example 4.2 shows that even when the augmented system can be decoupled, there is no way, except trial-and-error, of knowing what to insert in the input channels in order to make the subsystem have the required form.



(a) Block Diagram of the Augmented System

$$\dot{x} = \begin{bmatrix} -5 & 0 & 0 & 7 & 0 & 0 \\ 0 & -1 & 0 & 20 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} x$$

(b) State Equations

Figure 4.2 Example 4.2

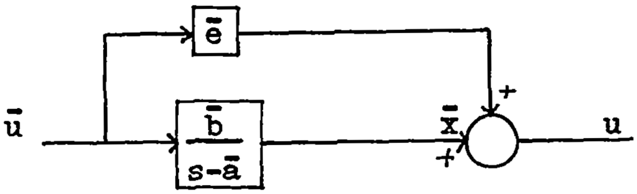
Apparently some types of transfer functions can be inserted in the control input channels without losing the ability to decouple, while others cannot. The case where first-order series compensators are added is now discussed. This case is important because it is frequently desired to insert a first-order compensator containing one pole and one zero in one of the subsystems so that the zero can be used to increase the velocity-error coefficient of the subsystem.

Consider the most general first-order series compensator shown in Figure 4.3(a). If both $\bar{e}$ and $\bar{b}$ are non-zero, then

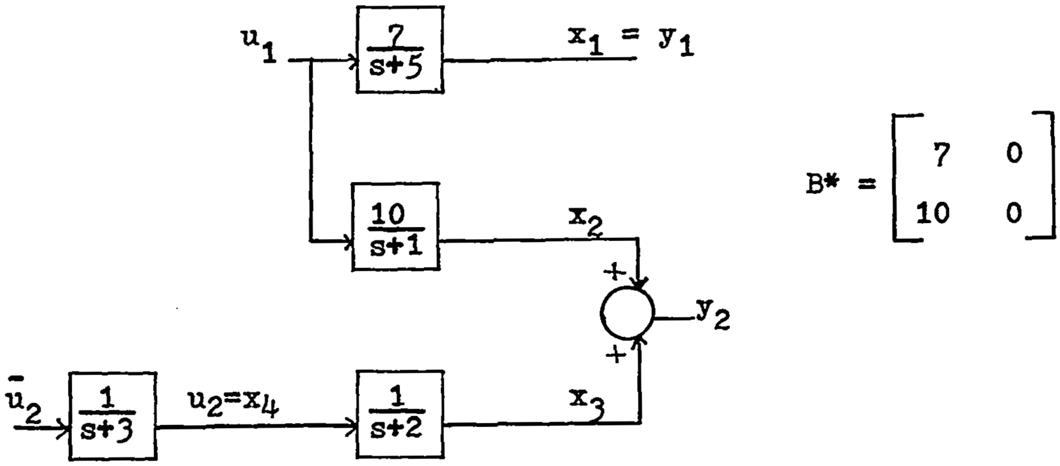
$$\frac{u}{\bar{u}} = \frac{\bar{e}s - \bar{e}\bar{a} + \bar{b}}{s - \bar{a}} \qquad 4.5$$

and the transfer function has a zero at $\frac{\bar{e}\bar{a} - \bar{b}}{\bar{e}}$. If $\bar{e}$ is 0, a pole is being added in the control input channel; if $\bar{b}$ is 0, gain is being added.

Figure 4.3(b) and (c) show two examples where first-order compensation has been added to the control input channels of the plant of Example 4.1. In both cases, B* is singular, and the ability to decouple has been lost. These examples show that even when first-order series compensation is required, the use of Method A could lead to loss of coupling.

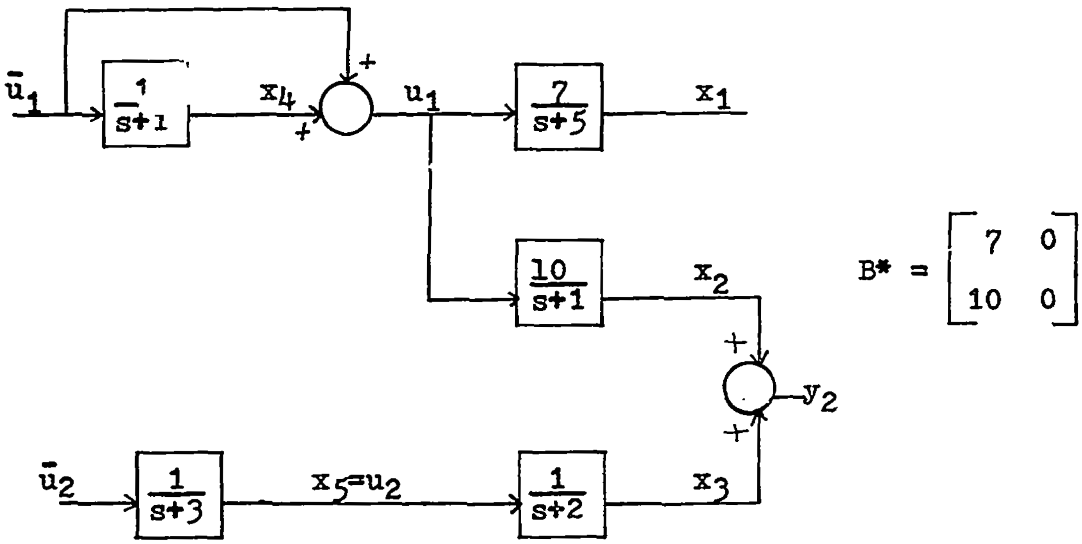


(a) General First-Order Series Compensator



(b) System Which Cannot Be Decoupled

Figure 4.3 First-Order Series Compensation



(c) System Which Cannot Be Decoupled

Figure 4.3 First-Order Series Compensation (Continued)

There are two special cases of first-order series compensation which preserve decoupling; these are embodied in the following theorem.

Theorem 4.1 Let the multivariable plant

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

be compensated by the first-order series compensation

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u}$$

$$u = \bar{x} + \bar{E}\bar{u}$$

where $\bar{A}$, $\bar{B}$, and $\bar{E}$ are m x m diagonal matrices having the respective diagonal elements $\bar{a}_{ii}$, $\bar{b}_{ii}$, and $\bar{e}_{ii}$, $i = 1, \ldots, m$. In addition, assume that $\bar{B}$ is nonsingular. Then, provided that the original plant can be decoupled, the resulting augmented multivariable plant can be decoupled if

(i) $\bar{E}$ is 0

or (ii) $\bar{E}$ is nonsingular

Proof. The theorem is proved by finding the matrix B* for case (i) and then for case (ii); for each case it is shown to be nonsingular.

case (i) $\bar{E} = 0$

Let $\tilde{x} = \begin{bmatrix} x \\ \bar{x} \end{bmatrix}$ be the state variables for the system augmented by first-order series compensation. It is easily shown that the state equations for the augmented system are

$$\dot{\tilde{x}} = \begin{bmatrix} A & B \\ 0 & \bar{A} \end{bmatrix} \tilde{x} + \begin{bmatrix} B\bar{E} \\ \bar{B} \end{bmatrix} \bar{u} \qquad 4.6$$

$$y = [C \quad 0]\tilde{x} \qquad 4.7$$

In order to find $\tilde{B}^*$ the sequence of row vectors $\tilde{C}_i$, $\tilde{C}_i\tilde{B}$, ..., $\tilde{C}_i\tilde{A}^{d_i}\tilde{B}$ is formed. Now since $\bar{E}$ is 0, $\tilde{B}$ is just $\begin{bmatrix} 0 \\ \bar{B} \end{bmatrix}$ and

$$\tilde{C}_i\tilde{B} = [C_i \quad 0]\begin{bmatrix} 0 \\ \bar{B} \end{bmatrix} = 0$$

$$\tilde{C}_i\tilde{A}\tilde{B} = [C_i \quad 0]\begin{bmatrix} A & B \\ 0 & \bar{A} \end{bmatrix}\begin{bmatrix} 0 \\ \bar{B} \end{bmatrix} = C_iB\bar{B}$$

$$\vdots$$

$$\tilde{C}_i\tilde{A}^{d_i+1}\tilde{B} = (C_iA^{d_i}B + C_iA^{d_i-1}B + \ldots + C_iB)\bar{B}$$

By assumption, $\bar{B}$ is nonsingular and diagonal; also, by the definition of d_i, the terms C_iB, C_iAB, ..., $C_iA^{d_i-1}B$ are zero. Thus in the sequence C_iB, C_iAB, ..., $C_iA^{d_i}B$, only $C_iA^{d_i}B$ is nonzero; in fact,

$$\tilde{C}_i\tilde{A}^{d_i+1}\tilde{B} = C_iA^{d_i}B\bar{B}$$

$$= B_i^*\bar{B} \qquad i = 1, \ldots, m \qquad 4.8$$

or

$$\tilde{B}^* = B^*\bar{B} \qquad 4.9$$

The matrix $\tilde{B}^*$ is nonsingular because both B^* and $\bar{B}$ are nonsingular, and the augmented system can be decoupled.

case (ii) $\bar{\bar{E}}$ nonsingular

Consider the following sequence

$$\tilde{C}_i\tilde{B} = [C_i \quad 0] \begin{bmatrix} B\bar{\bar{E}} \\ \bar{B} \end{bmatrix} = C_iB\bar{\bar{E}}$$

$$\tilde{C}_i\tilde{A}\tilde{B} = [C_i \quad 0] \begin{bmatrix} A & B \\ 0 & \bar{A} \end{bmatrix} \begin{bmatrix} B\bar{\bar{E}} \\ \bar{B} \end{bmatrix} = C_iAB\bar{\bar{E}} + C_iB\bar{B}$$

$$\vdots$$

$$\tilde{C}_i\tilde{A}^{d_i}\tilde{B} = C_iA^{d_i}B\bar{\bar{E}} + (C_iA^{d_i-1}B + \dots + C_iB)\bar{B}$$

Again the terms C_iB, C_iAB, ..., $C_iA^{d_i-1}B$ are zero and so

$$\tilde{C}_i\tilde{A}^{d_i}\tilde{B} = C_iA^{d_i}B\bar{\bar{E}}$$

$$= B_i^*\bar{\bar{E}} \qquad i = 1, \dots, m \qquad 4.10$$

or

$$\tilde{B}^* = B^*\bar{\bar{E}} \qquad 4.11$$

The matrix $\tilde{B}^*$ is nonsingular because both B^* and $\bar{\bar{E}}$ are nonsingular, and the theorem is proved.

By virtue of Theorem 4.1 the only sure way of adding first-order compensators by Method A requires that every control input channel contains both a pole and a zero, or that every control input channel contains only a pole. If this consistency in the choice of first-order compensators is abandoned, then loss of decoupling could result, as shown in Figure 4.3 (b) and (c). Even when the consistency is maintained and decoupling is assured, there is a danger of "losing" the compensator zero. This was observed in Example 4.2 where the zero inserted in the control input channel for u_1 did not appear in the transfer function for subsystem 1.

Method A is best understood as a trial-and-error approach. The outstanding feature of Method A is simplicity, and for some problems it may prove to be satisfactory. When it does not yield satisfactory results, Method B or Method C should be used.

Method B

In Method B the multivariable plant is decoupled before additional compensation is added in the control input channels. The steps in the design procedure are

(1) The CD representation of the ID plant is found and used in determining the locations of the fixed zeroes and the number of poles for each subsystem.

(2) State variable feedback is used to decouple the plant so that the resulting transfer function for each subsystem is by itself a factor of the desired transfer function of that subsystem.

(3) Compensator networks are inserted in the control input channels of the decoupled plant. The transfer function for each series compensator is selected so that its product with the transfer function

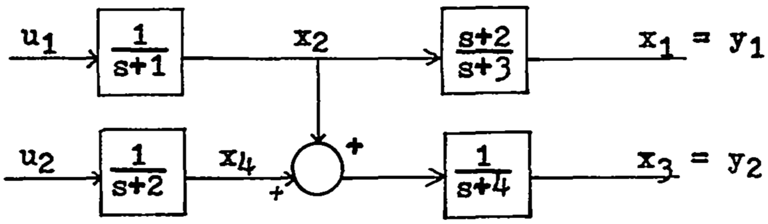
of the corresponding subsystem of the decoupled plant is equal to the desired transfer function for that subsystem.

Here the problems associated with Method A are no longer present because the plant is decoupled before compensation is added and because there is no feedback around the compensation. In fact, the chief disadvantage of the method is that no feedback is used around the series compensation. As a result, the augmented system is sensitive to changes in the parameters in the compensation, and the system is likely to perform poorly in the presence of noise. Both of these considerations are discussed in the following example.

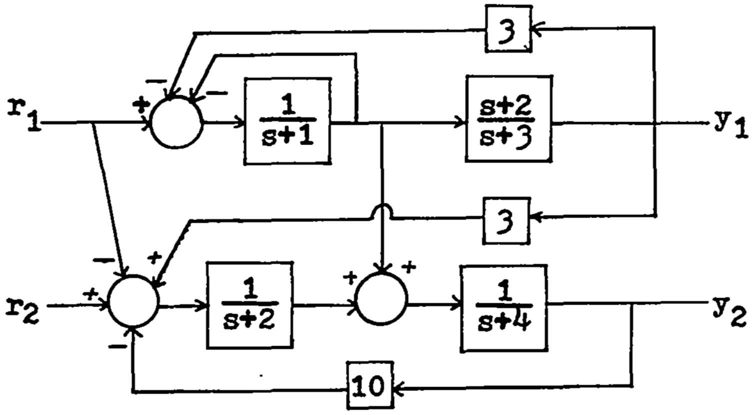
Example 4.3. The block diagram for the given plant is shown in Figure 4.4(a). After completing step (1) it is learned that subsystem 1 has a zero at s = -2 and two poles which can be controlled by state variable feedback; subsystem 2 has two arbitrary poles. Suppose it is desired to achieve the transfer matrix

$$\tilde{H}(s,F,G) = \begin{bmatrix} \frac{12(s+5)}{(s+10)(s+6)} & 0 \\ 0 & \frac{18}{(s+3)^2+9} \end{bmatrix} \qquad 4.12$$

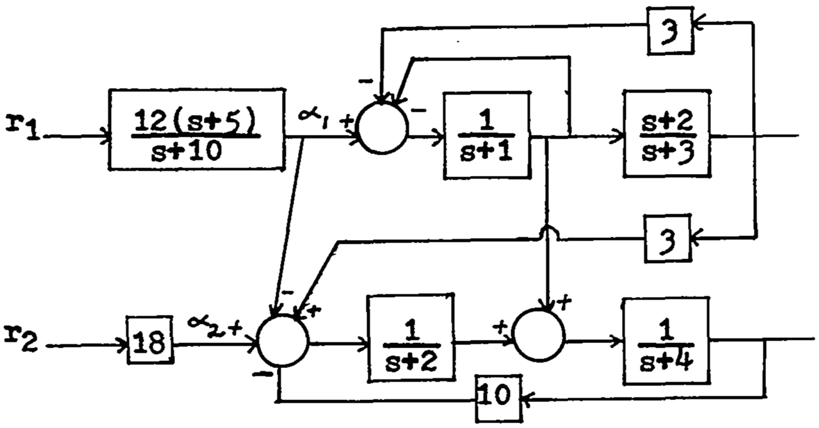
where $\tilde{H}(s,F,G)$ is the transmission matrix from r to y of the final, compensated system, and F and G are the compensation matrices for that system.



(a) Block Diagram



(b) Intermediate Step



(c) Final Design

Figure 4.4 Example 4.3

The first step toward this end is the decoupling of the given plant. Let the plant be compensated so that as much as possible of the desired transmission matrix is achieved, or let

$$P(s,F,G) = \begin{bmatrix} \frac{1}{s+6} & 0 \\ 0 & \frac{1}{(s+3)^2+9} \end{bmatrix} \qquad 4.13$$

(Here, P is used instead of H because P(s,F,G) relates y and u rather than y and r). The compensation matrices that result in the above P(s,F,G) are

$$F = \begin{bmatrix} -3 & -1 & 0 & 0 \\ 3 & 0 & -10 & 0 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

and the block diagram for this intermediate form of the system is shown in Figure 4.4(b). Now compare the diagonal elements of the matrices in Equations 4.12 and 4.13. For subsystem 1 the additional factor $\frac{12(s+5)}{(s+10)}$ is needed to realize the desired transfer function; for subsystem 2 a gain of 18 is necessary. The additional compensation is added as shown in Figure 4.4(c).

The deficiencies of Method B are clearly evident from a study of the block diagram of Figure 4.4(c). Any noise occurring at the points labelled α_1 and α_2 on that diagram passes through the subsystems and appears unattenuated at the outputs. Furthermore, there is no feedback around the series compensation networks to reduce the effects of parameter changes.

There is a variation of the procedure of Method B that is similar to Configuration II design, discussed in Chapter 2. State variable feedback and series compensation are used to develop a set of subsystems such that when the ith output is fed directly back to the ith input, the ith subsystem exhibits the required response. The similarity to Configuration II lies in the fact that in both cases the open-loop subsystem must be altered in a manner such that the closed-loop system meets the design specifications.

The advantage of using this variation of Method B is that feeding back the outputs insures that there is feedback around both the decoupled plant and the series compensation. On the other hand, since the plant states have already been measured and since the compensator states are presumably easy to measure, why not feedback all the states instead of merely the outputs? This is exactly what is done in Method C.

Method C

The steps in the third design procedure are

(1) The CD representation of the ID plant is found and used in determining the locations of the fixed zeroes and the number of poles for each subsystem.

(2) State variable feedback is used to decouple the plant.

(3) Based on the knowledge gained in step (1) and the design specifications, appropriate compensator networks are inserted in the control input channels of the decoupled plant.

(4) State variable feedback, including feedback of the compensator states, is used to add gain and put the subsystem poles in the required locations.

The following questions arise. Does the plant remain decoupled when series compensation is added in the control input channels and all the states are fed back? If so, what can be said about the form of the subsystems of the augmented system in terms of the known structure of the original decoupled plant and the added compensation?

The answer to the first question is reasonably obvious. There are two ways of determining whether the augmented system can be decoupled. First, the matrix B^* can be found for the augmented system and tested for non-singularity; or second, a particular F and G can be found which decouples the augmented system. The second method is by far the easier one to apply in this case. Before adding the compensation the plant is decoupled, so that the matrix $P(s)$ relating y to u is diagonal. Now after the compensation is added in the control input channels, the control input u_i, $i = 1, \ldots, m$, is effectively changed to $\bar{p}_{ii}(s)\bar{u}_i$ where $\bar{p}_{ii}(s)$ is the transfer function of the ith compensator. Thus $P(s)$ is replaced by $P(s)\bar{P}(s)$ and is still diagonal. Since the new plant matrix is diagonal, the matrices $F = 0$ and $G = I$ are a suitable choice for decoupling the system and so the augmented plant can indeed be decoupled.

The answer to the second question agrees with one's expectations. Unlike Method A, in Method C zeroes added in the compensator networks always appear in the proper subsystem transfer functions after all the states are fed back. If the ith subsystem of the decoupled plant has l_i zeroes and n_i poles and if $\bar{l}_i$ zeroes and $\bar{n}_i$ poles are contained in the series compensator added in the ith control input channel, then in the final, compensated system, the $l_i + \bar{l}_i$ zeroes appear unchanged in the ith subsystem transfer function, and the $n_i + \bar{n}_i$ poles of that transfer function are controlled by the state variable feedback. The proof of this central result is the main contribution of this dissertation.

The proof of the central result stated above requires several steps. In the following section additional notation and a precise formulation of the state equations are given; then in the next section the central result is proved.

Decoupled Compensation

It is now necessary to be more precise than previously in describing the compensation to be placed in the control input channels of the multivariable system. State variable equations are written to describe the compensation for each channel and then these sets of equations are combined to give a single set of system equations to describe all the compensation. Coupling between the channels is purposely omitted and so the term <u>decoupled compensation</u> is appropriate for describing the compensation added to the control input channels of the plant.

The state equations for the decoupled compensation take the form

$$\dot{\bar{\bar{x}}} = \bar{\bar{A}}\bar{\bar{x}} + \bar{\bar{B}}\bar{\bar{u}} \qquad 4.14$$

$$u = \bar{\bar{C}}\bar{\bar{x}} + \bar{\bar{E}}\bar{\bar{u}} \qquad 4.15$$

where the structure of $\bar{\bar{x}}$ and the various matrices is given in Figure 4.5. The matrix $\bar{\bar{E}}$ is needed when a first-order compensator having both a pole and a zero is to be added to the system because in such a situation the control input is fed directly forward to the subsystem output; see Figure 4.3(c) for an example where $\bar{\bar{E}}$ is needed.

Recall that the system equations for the original system are

$$\dot{x} = Ax + Bu \qquad 4.16$$

$$y = Cx \qquad 4.17$$

$$\bar{\bar{x}} = \begin{bmatrix} \bar{\bar{x}}^1 \\ \bar{\bar{x}}^2 \\ \circ \\ \circ \\ \circ \\ \bar{\bar{x}}^m \end{bmatrix}$$

$\bar{x}$ is $\bar{n} \times 1$

$\bar{x}^1$ is $\bar{n}_1 \times 1$

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & 0 & \cdots & 0 \\ 0 & \bar{A}_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \bar{A}_{mm} \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} \bar{B}_{11} & 0 & \cdots & 0 \\ 0 & \bar{B}_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \bar{B}_{mm} \end{bmatrix}$$

$\bar{A}$ is $\bar{n} \times \bar{n}$

$\bar{A}_{ii}$ is $\bar{n}_i \times \bar{n}_i$

$\bar{B}$ is $\bar{n} \times m$

$\bar{B}_{ii}$ is $\bar{n}_i \times 1$

$$\bar{C} = \begin{bmatrix} \bar{C}_{11} & 0 & \cdots & 0 \\ 0 & \bar{C}_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \bar{C}_{mm} \end{bmatrix} \qquad \bar{E} = \begin{bmatrix} \bar{e}_{11} & 0 & \cdots & 0 \\ 0 & \bar{e}_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \bar{e}_{mm} \end{bmatrix}$$

$\bar{C}$ is $m \times \bar{n}$

$\bar{C}_{ii}$ is $1 \times \bar{n}_i$

$\bar{E}$ is $m \times m$

$\bar{e}_{ii}$ is 1×1

Figure 4.5 Structure of the Decoupled Compensation

Let the states of the original system and the decoupled compensation be combined into the single vector $\tilde{x}$, with

$$\tilde{x} = \begin{bmatrix} x \\ \bar{x} \end{bmatrix} \qquad 4.18$$

Then the state equations for the augmented multivariable system are written

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\bar{u} \qquad 4.19$$

$$y = \tilde{C}\tilde{x} \qquad 4.20$$

In terms of the matrices in Equations 4.14 - 4.17, these system equations have the form

$$\dot{\tilde{x}} = \begin{bmatrix} A & B\bar{C} \\ 0 & \bar{A} \end{bmatrix}\tilde{x} + \begin{bmatrix} B\bar{E} \\ \bar{B} \end{bmatrix}\bar{u} \qquad 4.21$$

$$y = [C \quad 0]\tilde{x} \qquad 4.22$$

The control input for the augmented system is

$$\bar{u} = \tilde{F}\tilde{x} + \tilde{G}r \qquad 4.23$$

Equations 4.18 - 4.22 provide an exact description of the multivariable system in terms of the original plant matrices and the matrices of the decoupled compensation.

Proof of the Central Result for Method C

The purpose of this section is to present the proof of the central result for Method C. When Method A is used to add dynamics to the multivariable system, there is a danger that decoupling is lost and that the compensator zeroes do not appear in the transfer functions of the compensated augmented system. For Method C it is already clear that decoupling is never lost by the addition of decoupled compensation. The structure of the augmented subsystems is now discussed and proved to be valid.

The zeroes and the number of poles of the subsystem transfer functions of the decoupled plant are presumed known. This implies that the designer has obtained the CD representation for the original, coupled plant and from it has determined the number of fixed subsystem zeroes and subsystem poles; then he has used state variable feedback to decouple the plant and move the subsystem poles to some known locations. The form of the subsystems in the decoupled compensation are certainly known because they are added to the system by the designer. The central result (Theorem 4.3) to be shown is that the subsystems of the augmented plant can be treated individually, each having an "open-loop" transfer function whose zeroes are the zeroes of the added subsystem compensation and the zeroes present in the subsystem before compensation, and whose poles are under the control of state variable feedback. The number of subsystem poles for the ith subsystem is $\bar{\bar{n}}_i + n_i$ and the number of fixed zeroes is $\bar{\bar{l}}_i + l_i$, where n_i and $\bar{n}_i$ and l_i and $\bar{\bar{l}}_i$ are, respectively, the number of zeroes and poles of the component parts of the ith open loop subsystem.

The demonstration of the central result requires several steps, so an outline of the proof is helpful. The steps are

(1) Show that a decoupled multivariable plant can always be put in standard form by a linear change of variables (Theorem 4.2).

(2) Show that the central result is true when a system in standard form is augmented with decoupled compensation. (Lemma 3)

(3) Show that the system described in (2) and the original augmented system are related by a linear change of variables.

(4) Show that the two systems are Control Law Equivalent and thus prove the central result (Theorem 4.3).

Before the theorem accompanying step (1) (Theorem 4.2) is proved, two subsidiary results are needed. The required results are properties of decoupled systems; they are important because in forming the augmented plant, decoupled compensation is added to a <u>decoupled</u> plant.

Lemma 1 For a decoupled multivariable system

$$C_i A^j B = \gamma_{ij} I_i \quad i = 1, \ldots, m; \; j = 0, \ldots, n-1 \qquad 4.24$$

where I_i is the ith row of the m x m identity matrix and at least one of the m(m x n) numbers

γ_{ij} is nonzero. Furthermore, the matrix B^* is diagonal and nonsingular.

Proof. Let P(s) be the diagonal matrix relating y and u and let $P_i(s)$ be the ith row of P(s). Let

$$p(s) = \det(sI - A)$$
$$= s^n - p_1 s^{n-1} - \dots - p_n \qquad 4.25$$

The expression for $P_i(s)$ is

$$P_i(s) = C_i(sI - A)^{-1}B \qquad 4.26$$

and the following formula, taken from Gantmacher (1959) and used by Morgan (1963) and Gilbert (1968) can be used to calculate $P_i(s)$.

$$P_i(s) = [p(s)]^{-1}(C_i Bs^{n-1} + C_i R_1 Bs^{n-2} + \dots + C_i R_{n-1}B) \quad 4.27$$

where

$$R_1 = A - p_1 I$$
$$R_2 = A^2 - p_1 A - p_2 I$$
$$\vdots$$
$$R_{n-1} = A^{n-1} - p_1 A^{n-2} - \dots - p_{n-1}I \qquad 4.28$$

Substituting the expressions for the R_j into Equation 4.27 gives

$$P_i(s) = [p(s)]^{-1}[C_i Bs^{n-1} + C_i(A - p_1 I)Bs^{n-2} + C_i(A^2 - p_1 A - p_2 I)Bs^{n-3} + \dots + C_i(A^{n-1} - p_1 A^{n-2} - \dots - p_{n-1}I)B] \qquad 4.29$$

Since the system is decoupled, P(s) is diagonal and nonsingular, and

$$P_i(s) = p_{ii}(s)I_i \qquad 4.30$$

where $p_{ii}(s)$ is the ith diagonal element of P(s). In order that Equations 4.29 and 4.30 be compatible, the following relationships must hold:

$$
\begin{aligned}
C_i B &= \psi_{i0} I_i \\
C_i AB &= \psi_{i1} I_i \\
&\vdots \\
C_i A^{n-1} B &= \psi_{i,n-1} I_i
\end{aligned}
\tag{4.31}
$$

At least one of the ψ_{ij}, j = 0, 1, ..., n-1, is nonzero because otherwise $P_i(s)$ would be 0 and P(s) would be singular. In addition, recall that the matrix B* is defined to be

$$
B^* = \begin{bmatrix} C_1 A^{d_1} B \\ \vdots \\ C_m A^{d_m} B \end{bmatrix} \tag{4.32}
$$

so that B* is diagonal and nonsingular, and the proof is complete.

Another property of decoupled systems is given in the following lemma.

Lemma 2 For a decoupled, controllable system the following conditions are satisfied. ($\mathcal{Q}_i$ is defined by Equation 3.62)

(1) $\mathcal{Q}_i$ is a row invariant subspace of A; i.e., $\eta \in \mathcal{Q}_i$ implies $\eta A \in \mathcal{Q}_i$.

(2) $\mathcal{Q}'_i \cap \mathcal{Q}'_j = 0$ for $i = 1, \ldots, m$, $i \neq j = 1, \ldots, m$

Proof of (1). It is given that $\eta A^j B_k = 0$, $k = 1, \ldots, m$, $k \neq i$, $j = 0, \ldots, n-1$; it must be shown that $\eta A A^j B_k = 0$ for the same set of i, k, and j. Except for $j = n-1$ the proof follows immediately. From the Hamilton-Cayley theorem (Nering, 1963) $A^n = q_1 A^{n-1} + \ldots q_n I$. Premultiply A^n by η and post multiply by B_k; $\eta A^n B_k = q_1 \eta A^{n-1} B_k + \ldots + q_n \eta B_k = 0$. Therefore, (1) is true for all $i = 0, 1, \ldots, m$.

Proof of (2). Let $\eta \in \mathcal{Q}'_i \cap \mathcal{Q}'_j$. Then since η is in both $\mathcal{Q}'_i$ and $\mathcal{Q}'_j$, $\eta A^j B_k = 0$, $j = 0, \ldots, n-1$, $k = 1, \ldots, m$. Because of controllability n of the vectors from the set $A^j B_k$, $j = 0, \ldots, n-1$, $k = 1, \ldots, m$ are linearly independent. Let the columns of the matrix Z be those vectors. Then $\eta Z = 0$, or $\eta = 0$ because Z is nonsingular.

Lemma 1 and Lemma 2 are needed in proving Theorem 4.2--the first step on the path to establishing the central result for Method C.

Theorem 4.2 Any controllable, decoupled multivariable system can be put in standard form by a linear change of state variables. (Standard form is defined in Figure 3.3(a).)

Proof. The matrix Q is defined by $\hat{x} = Qx$ where $\hat{x}$ are the new state variables, in terms of which the matrices $\hat{A} = QAQ^{-1}$, $\hat{B} = QB$, and $\hat{C} = CQ^{-1}$ are to be in standard form. The structure of Q is the following

$$Q = \begin{bmatrix} Q_1 \\ \cdot \\ \cdot \\ \cdot \\ Q_{m+1} \end{bmatrix} \qquad Q_i = \begin{bmatrix} \mathscr{g}_1^i \\ \cdot \\ \cdot \\ \cdot \\ \mathscr{g}_{n_i}^i \end{bmatrix}$$

where the $\mathscr{g}_j^i$, $j = 1, \ldots, n_i$, are rows of the $n_i \times n$ matrix Q_i and are a basis for $\mathscr{Q}_i$. The first row of Q_i, $i = 1, \ldots, m$, is always chosen to be C_i, the ith row of the C matrix. This row vector is a member of the invariant subspace $\mathscr{Q}_i$ because of Lemma 1. The matrices $\hat{A}$, $\hat{B}$, and $\hat{C}$ are considered in turn.

(a) To show that $\hat{A}$ has the required form. The matrix A is assumed to represent a linear transformation σ. Let the rows of A be the n-tuples which represent the images of the basis vectors of $\mathscr{Q}$, under σ. Let the rows of Q, the $\mathscr{g}_j^i$ as defined above, be the sets of n-tuples representing the new basis of $\mathscr{Q}$ in terms of the original basis.

In order to find $\hat{A}$, the image of each $\mathscr{g}_j^i$ in terms of the set of all $\mathscr{g}_l^k$ (in terms of $\{\mathscr{g}_l^k\}$) must be found. Consider $\mathscr{g}_1^1$. Since the vectors in $\mathscr{Q}_1$ are transformed by σ back into the same subspace ($\mathscr{Q}_1$ is an invariant subspace by Lemma 2), the first n_1 rows of $\hat{A}$ must have the form $[A_{11} \quad 0 \quad \ldots \quad 0]$, where A_{11} is an $n_1 \times n_1$ matrix. The second group of rows of $\hat{A}$ must take the form $[0 \quad A_{22} \quad 0 \quad \ldots \quad 0]$ and so on for all but the last group of rows. Nothing is known about how the vectors in

$\mathscr{C}_{m+1}$ are transformed by σ, so that no special structure can be ascribed to the last n_{m+1} rows of $\hat{A}$. When the m+1 groups of rows are put together to form $\hat{A}$, the matrix is found to have the structure required for the standard form representation.

(b) To show that $\hat{B}$ has the required form. Recall that $\hat{B} = QB$. The first row of $\hat{B}$ is $\wp_1^1 B$ or $\sum_{k=1}^{m} \wp_1^1 B_k$ where is the kth column of B. From the definition of $\mathscr{C}_1$, $\wp_1^1 B_k = 0$ for $k \neq 1$ so that the first row of $\hat{B}$ has the required form, shown in Figure 3.3(a). In the same way, the remaining rows of $\hat{B}$ are found to have the required form.

(c) To show that $\hat{C}$ has the required form. Recall that $\hat{C} = CQ^{-1}$ or $\hat{C}Q = C$. Since the $\wp_1^i$, $i = 1, \ldots, m$, were chosen to be the C_i, $i = 1, \ldots, m$; $\hat{C}$ must satisfy the requirements for the output matrix of the standard form representation in order to satisfy the last equation. The proof of Theorem 4.2 is now complete.

The second step leading to the central result for Method C is now considered; the requirements of step (2) are embodied in Lemma 3.

Lemma 3 Augmenting a decoupled multivariable plant which is in standard form leads to a design problem in which there are m open-loop subsystems, each having a transfer function which is the product of the transfer function for the ith subsystem and the transfer function introduced into ith control input channel. When state variable feedback is applied, the subsystem zeroes remain fixed and the subsystem poles can be placed arbitrarily.

Proof. The required result is demonstrated for the case of two inputs and two outputs. Figure 4.6 (a), (b), and (c) shows the state equations for the original plant, the decoupled compensation, and the augmented plant, respectively. Change variables by defining $z^1 = x^1$, $z^2 = \bar{x}^1$, $z^3 = x^2$, $z^4 = \bar{x}^2$, and $z^5 = x^3$. In terms of the z variables the system equations are

$$\dot{z} = \begin{bmatrix} A_{11} & B_{11}\bar{C}_{11} & 0 & 0 & 0 \\ 0 & \bar{A}_{11} & 0 & 0 & 0 \\ 0 & 0 & A_{22} & B_{22}\bar{C}_{22} & 0 \\ 0 & 0 & 0 & \bar{A}_{22} & 0 \\ A_1^c & B_1^c\bar{C}_{11} & A_2^c & B_2^c\bar{C}_{22} & A_3^c \end{bmatrix} z + \begin{bmatrix} B_{11}\bar{e}_{11} & 0 \\ \bar{B}_{11} & 0 \\ 0 & \bar{B}_{22}\bar{e}_{22} \\ 0 & \bar{B}_{22} \\ B_1^c\bar{e}_{11} & B_2^c\bar{e}_{22} \end{bmatrix} \bar{n} \qquad 4.33$$

$$y = \begin{bmatrix} C_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{22} & 0 & 0 \end{bmatrix} \qquad 4.34$$

$$\dot{x} = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ A_1^c & A_2^c & A_3^c \end{bmatrix} x + \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \\ B_1^c & B_2^c \end{bmatrix} u \qquad y = \begin{bmatrix} C_{11} & 0 & 0 \\ 0 & C_{22} & 0 \end{bmatrix} x$$

(a) Original System in Standard Form

$$\dot{\bar{\bar{x}}} = \begin{bmatrix} \bar{\bar{A}}_{11} & 0 \\ 0 & \bar{\bar{A}}_{22} \end{bmatrix} \bar{\bar{x}} + \begin{bmatrix} \bar{\bar{B}}_{11} & 0 \\ 0 & \bar{\bar{B}}_{22} \end{bmatrix} \bar{\bar{u}}$$

$$y = \begin{bmatrix} \bar{\bar{C}}_{11} & 0 \\ 0 & \bar{\bar{C}}_{22} \end{bmatrix} \bar{\bar{x}} + \begin{bmatrix} \bar{\bar{e}}_{11} & 0 \\ 0 & \bar{\bar{e}}_{22} \end{bmatrix} \bar{\bar{u}}$$

(b) Decoupled Compensation

$$\dot{x} = \begin{bmatrix} A_{11} & 0 & 0 & B_{11}\bar{\bar{C}}_{11} & 0 \\ 0 & A_{22} & 0 & 0 & B_{22}\bar{\bar{C}}_{22} \\ A_1^c & A_2^c & A_3^c & B_1^c\bar{\bar{C}}_{11} & B_2^c\bar{\bar{C}}_{22} \\ 0 & 0 & 0 & \bar{\bar{A}}_{11} & 0 \\ 0 & 0 & 0 & 0 & \bar{\bar{A}}_{22} \end{bmatrix} x + \begin{bmatrix} B_{11}\bar{\bar{e}}_{11} & 0 \\ 0 & B_{22}\bar{\bar{e}}_{22} \\ B_1^c\bar{\bar{e}}_{11} & B_2^c\bar{\bar{e}}_{22} \\ \bar{\bar{B}}_{11} & 0 \\ 0 & \bar{\bar{B}}_{22} \end{bmatrix} \bar{\bar{u}}$$

$$y = \begin{bmatrix} C_{11} & 0 & 0 & 0 & 0 \\ 0 & C_{22} & 0 & 0 & 0 \end{bmatrix} x$$

(c) Augmented System

Figure 4.6 Two Input, Two Output System

The equations for subsystem 1 are enclosed in boxes in the matrices above. Let $\tilde{p}_{11}(s)$ be the transfer function for subsystem 1 which relates $\bar{\bar{u}}_1$ and y_1. Then

$$\tilde{p}_{11}(s) = [C_{11} \quad 0] \begin{bmatrix} sI-A_{11} & -B_{11}\bar{\bar{C}}_{11} \\ 0 & sI-\bar{\bar{A}}_{11} \end{bmatrix}^{-1} \begin{bmatrix} B_{11}\bar{\bar{e}}_{11} \\ \bar{\bar{B}}_{11} \end{bmatrix}$$

$$= C_{11}(sI-A_{11})^{-1}B_{11}[\bar{\bar{e}}_{11}+\bar{\bar{C}}_{11}(sI-\bar{\bar{A}}_{11})^{-1}\bar{\bar{B}}_{11}]$$

$$= p_{11}(s)\bar{\bar{p}}_{11}(s) \qquad 4.35$$

where $p_{11}(s)$ is the transfer function which relates u_1 and y_1 and $\bar{\bar{p}}_{11}(s)$ is the transfer function from $\bar{\bar{u}}_1$ to u_1. It is known from single-input, single-output theory that the use of state variable feedback allows all $n_1 + \bar{\bar{n}}_1$ poles of the system to be placed at arbitrary positions and that the zeroes of the compensated system are the zeroes of $p_{11}(s)\bar{\bar{p}}_{11}(s)$. Thus the lemma is proved for subsystem 1 and, in the same manner, for subsystem 2. The proof for $m > 2$ is straightforward.

In step (2) the original system is assumed to be in standard form. For step (3) it is shown that the original system need only be decoupled. The augmented systems in both cases are related by a change of variables. Let $\tilde{x}$ be the state variables for the augmented plant in which the original plant is decoupled, but is not necessarily in standard form; thus, $\tilde{x} = \begin{bmatrix} x \\ \bar{\bar{x}} \end{bmatrix}$. The state equations for this system are given in Equations 4.21 and 4.22 and are symbolized by $\tilde{S}[\tilde{A},\tilde{B},\tilde{C}]$. Let $\underline{\tilde{x}}$ be the state variables for the augmented plant in which the original plant is in standard form; thus $\underline{\tilde{x}} = \begin{bmatrix} \hat{x} \\ \bar{x} \end{bmatrix}$ and the system is represented by $\underline{\tilde{S}}[\underline{\tilde{A}},\underline{\tilde{B}},\underline{\tilde{C}}]$.

Now change variables in the system $\tilde{S}[\tilde{A},\tilde{B},\tilde{C}]$. Let $z = T\tilde{x}$ where

$$T = \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \qquad 4.36$$

In terms of the z variables, the system is represented by $\tilde{S}[T\tilde{A}T^{-1},\ T\tilde{B},\ \tilde{C}T^{-1}]$. But

$$T\tilde{A}T^{-1} = \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B\bar{\bar{C}} \\ 0 & \bar{A} \end{bmatrix} \begin{bmatrix} Q^{-1} & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} QAQ^{-1} & QB\bar{\bar{C}} \\ 0 & \bar{A} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{A} & \hat{B}\bar{\bar{C}} \\ 0 & \bar{A} \end{bmatrix} = \underline{\tilde{A}} \qquad 4.37$$

Similarly,

$$T\tilde{B} = \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B\bar{\bar{E}} \\ \bar{B} \end{bmatrix} = \begin{bmatrix} QB\bar{\bar{E}} \\ \bar{B} \end{bmatrix} = \underline{\tilde{B}} \qquad 4.38$$

and

$$\tilde{C}T^{-1} = [C \quad 0] \begin{bmatrix} Q^{-1} & 0 \\ 0 & I \end{bmatrix} = [\hat{C} \quad 0] = \underline{\tilde{C}} \qquad 4.39$$

This development shows that $\tilde{S}[T\tilde{A}T^{-1}, T\tilde{B}, \tilde{C}T^{-1}]$ is identical with $\underline{\tilde{S}}[\underline{\tilde{A}}, \underline{\tilde{B}}, \underline{\tilde{C}}]$, or that the two augmented plants are related by a linear change of state variables, and the plants are similar.

The final step, step (4), leading to Theorem 4.3 is the demonstration that similar systems are control law equivalent. This particular result is due to Gilbert (1968); the proof is repeated here for the sake of completeness.

Let $S[A,B,C]$ and $S_1[A_1,B_1,C_1]$ be similar systems; i.e., the state variables x of the system $S[A,B,C]$ are related to the state variables v of $S_1[A_1,B_1,C_1]$ by a non-singular transformation matrix T_1, as $v = T_1x$. For $S[A,B,C]$ the transmission matrix after compensation by state variable feedback is

$$H(s,F,G) = C(sI - A - BF)^{-1}BG \qquad 4.40$$

Control law equivalency between $S[A,B,C]$ and $S_1[A_1,B_1,C_1]$ will hold if an F_1 and a G_1 can be found such that $H_1(s,F_1,G_1)$ is identical to $H(s,F,G)$. Since $v = T_1x$, $A_1 = T_1AT_1^{-1}$, $B_1 = T_1B$, and $C_1 = CT_1^{-1}$. Thus,

$$\begin{aligned} H_1(s,F_1,G_1) &= C_1(sI - A_1 - B_1F_1)^{-1}B_1G_1 \\ &= CT_1^{-1}(sI - T_1AT_1^{-1} - T_1BF_1)^{-1}T_1BG_1 \\ &= C(sI - A - B\hat{F}_1T_1)^{-1}BG_1 \qquad 4.41 \end{aligned}$$

Choosing $F_1 = FT_1^{-1}$ and $G_1 = G$ causes $H_1(s, F_1, G_1)$ to be equal to $H(s, F, G)$, the desired conclusion.

The goal of this development is the proof of the central result that the use of Method C permits the multivariable system to be treated as m single-input, single-output systems. In step (1) it is shown that any decoupled plant is similar to a decoupled plant in standard form. This fact and the simplicity of the standard form representation are the motivation for considering the augmentation of a plant in standard form with decoupled compensation.

In step (2) it is shown that such a configuration leads to m subsystems each of whose forms is completely determined by a knowledge of the structure of the original plant in standard form and the structure of the decoupled compensation. This is the result which is being sought for the more general case of the compensation of decoupled plant with decoupled compensation, and the keys to its proof are given in steps (3) and (4). Steps (3) and (4) establish the conclusion that the two augmented plants can be made to have the same transfer matrices by the proper choices of the respective compensation matrices; thus the result proved for the augmented plant whose original system is in standard form is also true for the augmented plant in which the plant is decoupled but not necessarily in standard form.

By this line of reasoning the following theorem has been proved.

Theorem 4.3 Let decoupled compensation be used to augment a decoupled multivariable plant. Then for the purpose of compensating the resulting augmented plant, the system consists of m decoupled subsystems; the ith subsystem has $n_i + \bar{n}_i$ poles which can be arbitrarily placed by state variable feedback, and $l_i + \bar{l}_i$ zeroes which are not affected by the feedback.

The choice of Method C avoids the problem of loss of coupling which plagues Method A. The arbitrariness of Method A is also eliminated; one can be certain, for example, that if 1 zero and 1 pole are needed in subsystem i, then the insertion of the corresponding compensation network in channel i leads to the appearance of the zero and an arbitrarily positioned pole in the transfer function for $\frac{y_i(s)}{r_i(s)}$. This simple illustration is the essence of Theorem 4.3, even though the proof of the theorem is quite abstract and requires the introduction of a formal representation for the two parts of the augmented plant.

The first step in the application of Method C requires that the fixed plant be decoupled. The matrices F* and G* always decouple the plant, but the subsystems resulting from these compensation matrices have poles at the origin and at the locations of the zeroes of the plant. A slight change in the system parameters may cause the subsystem poles to move into the right half plane. A further disadvantage is that the subsystem gain is reduced to unity.

As far as the theory is concerned it makes no difference how the system is decoupled or what the gain or subsystem poles are made. A practical method for determining the compensation needed to decouple the plant involves the characterization of the class of all F and G matrices which decouple the system. According to Theorem 3.1, all G matrices which decouple are of the form

$$G = B^{*-1}\Lambda \qquad 4.42$$

where Λ is diagonal and nonsingular. This equation shows that the diagonal elements of G can always be chosen to be 1 and that the elements of the columns of G are multiples of the diagonal element contained in the column. The choice of 1 is recommended for the diagonal elements of G because this choice assures that no system gain is being deliberately canceled.

The class of F matrices which decouple is given by Equation 3.90, repeated below

$$F = F^{*} + B^{*-1}\hat{F}Q \qquad 4.43$$

The easiest compensation matrices to implement are those with the maximum number of elements which are 0. This criterion and Equation 4.43 form the basis for the selection of F, as shown in Example 4.4 below.

Example 4.4. Consider the application of Method C to the multivariable plant considered in the first two examples of this chapter. In Example 4.1 the addition of decoupled compensation leads to loss of coupling, and in Example 4.2 the required zero at $s = -3$ could not be obtained. In the present case Theorem 4.3 guarantees the required results.

The block diagram for the plant is shown in Figure 4.7(a). This particular system was discussed in Chapter 3, in Examples 3.1, 3.3, and 3.4. The class of G matrices which decouple is defined by Equation 4.42 which, in this instance, is

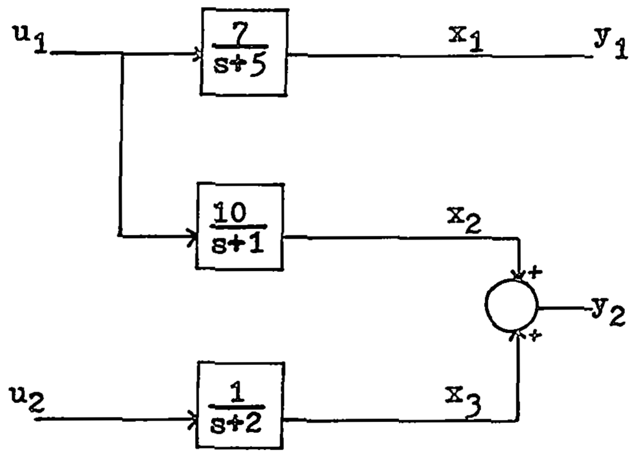
$$G = \begin{bmatrix} \frac{1}{7}\lambda_{11} & 0 \\ \frac{10}{7}\lambda_{11} & \lambda_{22} \end{bmatrix} \qquad 4.44$$

Let each of the diagonal elements of G be 1, so that $\lambda_{11} = 7$, $\lambda_{22} = 1$, and

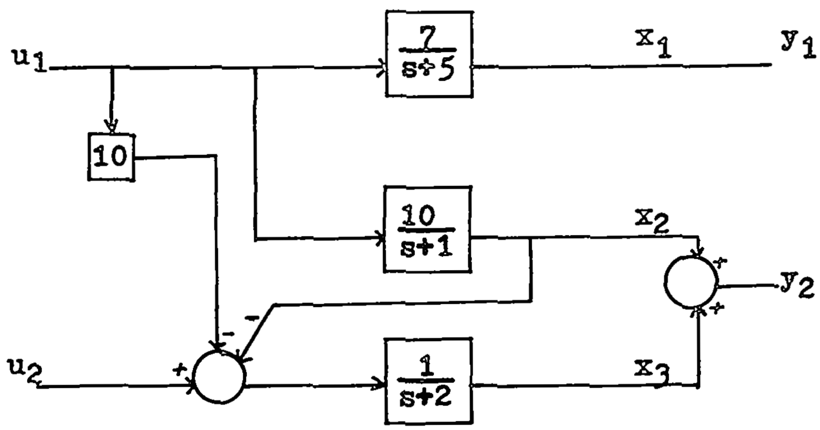
$$G = \begin{bmatrix} 1 & 0 \\ -10 & 1 \end{bmatrix} \qquad 4.45$$

The class of F matrices for this example is discussed in Chapter 3, Equation 3.99, repeated for convenience as

$$F = \begin{bmatrix} \frac{5}{7}+\frac{1}{7}\theta_{11} & \frac{1}{7}\theta_{12} & 0 \\ -10(\frac{5}{7}+\frac{1}{7}\theta_{11}) & 1-\frac{10}{7}\theta_{12}+\theta_{23} & 2+\theta_{23} \end{bmatrix} \qquad 4.46$$

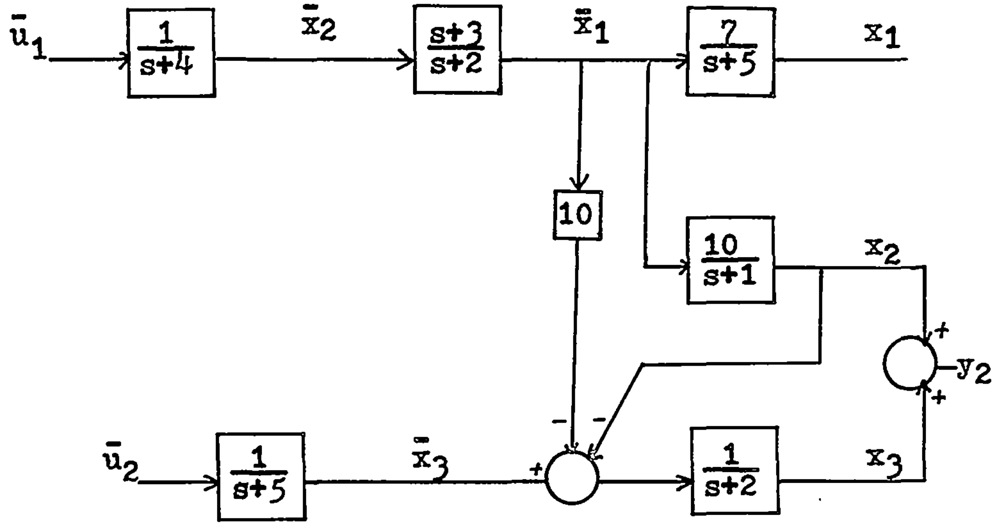


(a) Fixed Plant



(b) Decoupled Plant

Figure 4.7 Example 4.4



(c) Augmented System

Figure 4.7 Example 4.4 (Continued)

For $\theta_{11} = -5$, $\theta_{12} = 0$, and $\theta_{23} = -2$, F takes the simple form

$$F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad 4.47$$

The decoupled plant is shown in Figure 4.7(b). By inspection, the transfer matrix for the decoupled plant is

$$p(s,F,G) = \begin{bmatrix} \frac{7(s+1)}{(s+5)(s+1)} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix} \quad 4.48$$

Let the desired transfer matrix be

$$\tilde{H}(s,\tilde{F},\tilde{G}) = \begin{bmatrix} \frac{7(s+3)(s+1)}{(s^2+4s+8)(s+4)(s+1)} & 0 \\ 0 & \frac{1}{s^2+6s+18} \end{bmatrix} \quad 4.49$$

The form of the decoupled compensation required for the system is found by comparing Equations 4.48 and 4.49. For subsystem 1 a second-order compensation network with a zero at s = -3 is needed and for subsystem 2 a first-order network must be added to the decoupled plant. All the poles of the compensation network are arbitrary. One possible choice for the decoupled compensation is shown in Figure 4.7(c). The design is complete when the compensation matrices F and G are found for the decoupled plant. Methods for finding F and G are the subject of Chapter 5. For this example,

$$F = \begin{bmatrix} -.93 & 0 & 0 & -.50 & 3.50 & 0 \\ 0 & -10 & -10 & 0 & 0 & 1 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.50}$$

The block diagram for the final design appears in Figure 4.8.

Summary

The need for the techniques of this chapter is present whenever state variable feedback by itself does not provide enough flexibility for meeting the design specifications. Method A, in which compensation is added to the control input channels of the plant, does not appear to be widely applicable because in this procedure the addition of decoupled compensation may lead to loss of the ability of state variable feedback to decouple the augmented system. Also, the task of choosing the compensation is complicated by the uncertainty of the form of the structure of the subsystems of the augmented system; Examples 4.1, 4.2, and 4.3 illustrate these aspects of the method.

Method B does not suffer from either of the two disadvantages of the previous method because the decoupled compensation is added after the fixed plant has been decoupled. Its chief drawback is that the states of the decoupled compensation appear unchanged in the transfer functions of the compensated system, and there is no feedback around the decoupled compensation. Thus the

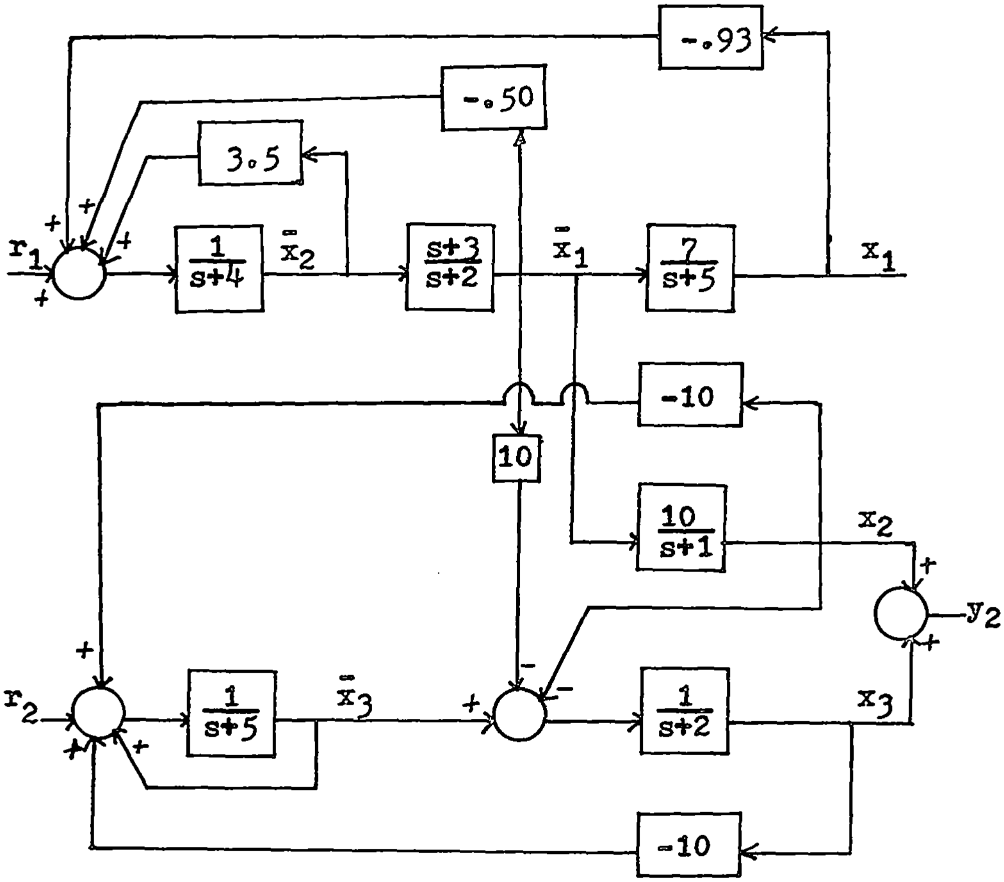


Figure 4.8 Final Design for Example 4.4

method is similar to the much maligned open loop design technique in which the compensation is used to cancel the system dynamics and to insert the required dynamics in their place.

In Method C the plant is decoupled before decoupled compensation is added, and then all states, including compensator states, are fed back. The method is amenable to a rather abstract analysis which culminates in Theorem 4.3. The significance of Theorem 4.3 is that it opens up the field of multivariable systems design to those engineers who are familiar with only single-input, single-output systems. All the techniques used in the design of single-input, single-output systems are now applicable to the multivariable system design problem. In particular, the state variable feedback technique is applicable, and this is one which is emphasized.

The first step in Method C is the decoupling of the fixed plant. In Example 4.4 the criterion used in deciding how the step should be carried out was the simplicity of the matrix of feedback coefficients. Other criteria such as system sensitivity, gain requirements, and other effects of the relative positions of the poles of the augmented system and the compensated system could be considered.

The development of design procedures has now reached the stopping point for this dissertation. Other developments may follow but the remainder of this work is devoted to the development of design procedures for implementing the techniques already known and to solving a practical problem. Chapters 5 and 6 are reserved for these purposes.

CHAPTER 5

COMPUTATIONAL PROCEDURES

The theory needed for the design of multivariable systems by state variable feedback is discussed in Chapters 3 and 4. The canonically decoupled (CD) representation of the integrator decoupled (ID) system is the means by which the subsystems are isolated and their structure is identified. If the form of the subsystems is unsatisfactory, decoupled compensation is used to change the structure to one for which state variable feedback design permits the design specifications to be met.

In this chapter procedures are discussed for the calculation of the numerical values of the compensation matrices F and G. The first part of the chapter contains a step-by-step design procedure which applies in design problems where the addition of decoupled compensation is not needed. The relevant formulas from Chapter 3 are repeated and used in describing all but one of the steps in detail. The step, given a brief treatment at this stage, is that of calculating $\hat{F}$ and $\hat{G}$, the compensation matrices for the canonically decoupled representation of the integrator decoupled plant.

The second chapter section presents the well-known phase-variable transformation, discussed by Johnson and Wonham (1966), and explains its use in an easily programmed technique for calculating $\hat{F}$ and $\hat{G}$. The utility of the phase-variable transformation in the state variable feedback design of single-input, single-output systems has been recognized by Morgan (1963) and Melsa (1967), among others; the results presented here are an implicit part of a lemma quoted by Gilbert (1968).

The third section adapts the above computational procedure to fit the case where decoupled compensation must be added to the decoupled plant by using Method C of Chapter 4. For the case where $\mathscr{C}_{m+1} = 0$, or where the sum of the orders of the subsystems is n, the means is discussed for avoiding the intermediate step of finding the CD representation for the augmented system. When $\mathscr{C}_{m+1} \neq 0$, there is no choice except to find the CD representation.

No new theory is developed here, but the results are new to the extent that previously known computational procedures are adapted to fit the new design technique. In particular, the concept of a multivariable, phase-variable transformation and the accompanying algorithm for finding the corresponding transformation matrix are new. An attempt to maintain unity of presentation is made by taking the two examples of this chapter from Chapters 3 and 4.

Step-by-Step Design Procedure

The presentation of the computational techniques is simplified by making reference to the following design procedure. This design procedure is applicable for design problems in which decoupled compensation is not needed.

(1) Find the matrix B*; if it is nonsingular, the multivariable system can be decoupled.

(2) Calculate F* and G*, the compensation matrices which put the system in integrator decoupled form.

(3) Calculate the matrix Q and use it to change variables and find the canonically decoupled representation of the integrator decoupled system.

(4) Identify subsystems and note the fixed zeroes and number of poles for each.

(5) Select the desired transmission matrix and compensate the canonically decoupled system by finding the numerical $\hat{F}$ and $\hat{G}$ matrices which cause the resulting compensated subsystems to have transfer functions that meet the design requirements.

(6) Calculate F and G, the matrices which are used for compensating the original system so that it exhibits the response achieved in step (5).

The first two steps are straightforward and are easily programmed on a digital computer. The multivariable system is described by the equations

$$\dot{x} = Ax + Bu \qquad 5.1$$

$$y = Cx \qquad 5.2$$

and B* is defined by

$$B^* = \begin{bmatrix} C_1A^{d_1}B \\ \vdots \\ C_mA^{d_m}B \end{bmatrix} \qquad 5.3$$

where d_i, $i = 1, \ldots, m$, is the smallest nonnegative integer for which the row matrix $C_iA^{d_i}B \neq 0$. In addition, F* and G* are

$$F^* = -B^{*-1}A^* = -B^{*-1}\begin{bmatrix} C_1A^{d_1+1} \\ \vdots \\ C_mA^{d_m+1} \end{bmatrix} \qquad 5.4$$

$$G^* = B^{*-1} \qquad 5.5$$

and the ID system representation is

$$\dot{x} = (A + BF^*)x + BG^*u \qquad 5.6$$

$$y = Cx \qquad 5.7$$

In step (3) the matrix Q is needed in finding the CD representation of the integrator decoupled system. The rows of Q are grouped together in m+1 blocks labelled Q_i and having n_i, i = 1, ..., m+1, rows. For Q_i, i = 1, ..., m, the first d_i+1 rows are C_i, C_iA,[1] ..., $C_iA^{d_i}$ and the last n_i-d_i-1 rows are any row vectors which, together with the first d_i+1 rows, form a basis for the (row) vector space $\mathcal{Q}_i$, defined by

$$\mathcal{Q}_i = \{ \eta ; \eta A^k B_j = 0, \ k = 0, 1, \ldots n-1, \ j = 1, \ldots, m, \ j \neq i \quad i = 1, \ldots, m\} \qquad 5.8$$

For each i, i = 1, ..., m, Equation 5.8 defines a set of n linear algebraic equations whose unknowns are the components of the row vector η. The solutions of the equations form the vector space of $\mathcal{Q}_i$ of dimension n_i and the row vectors C_i, C_iA, ..., $C_iA^{d_i}$ are linearly independent members of $\mathcal{Q}_i$.

Once n_i, the number of poles of subsystem i, is known, the number of zeroes of the subsystem, l_i, can be calculated. The relevant equation, as discussed in Chapter 3, is

$$l_i = n_i - d_i - 1 \quad i = 1, \ldots, m \qquad 5.9$$

The problem of extending the row vectors C_i, C_iA, ..., $C_iA^{d_i}$ to form a basis for $\mathcal{Q}_i$ is simplified by the use of the Hermite normal form (Nering, 1963). The Hermite normal form is defined and its existence is

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1. The matrix A is the system matrix for the ID system.

assured by Theorem 5.1 below. This theorem is, by coincidence, Theorem 5.1 in Nering (1963).

Theorem 5.1 Given any m x n matrix D of rank α, then by a sequence of elementary row operations on D a matrix D' can be formed, where D' has the following structure:

(1) There is at least one non-zero element in each of the first α rows of D', and the elements in all remaining rows are zero.

(2) The first non-zero element appearing in row i $(i \leq \alpha)$ is a 1 appearing in column k_i, where $k_1 < k_2 < \dots < k_\alpha$.

(3) In column k_i, the only non-zero element is 1 in row i.

The form of D' is uniquely determined by D. Thus the matrix D' has the form

$$\begin{array}{c} \\ \\ \end{array}\begin{bmatrix} 0 & \cdots & 0 & \overset{\text{column } k_1}{1} & d'_{1,k_1+1} & \cdots & \overset{\text{column } k_2}{0} & d'_{1,k_2+1} & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & d'_{2,k_2+1} & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \end{bmatrix}$$

In the present discussion D is the coefficient array for the set of linear algebraic equations found from

Equation 5.8. The matrix D' is found from D (with the aid of a digital computer) and then used in obtaining a standard basis for $\mathscr{C}_i$. Each of the vectors in the standard basis is checked for linear dependence on the set $C_i, C_iA, \ldots, C_iA^{d_i}$. If the vector is linearly independent of that set, it is added to $C_i, C_iA, \ldots, C_iA^{d_i}$; if if is dependent, the vector is discarded. This procedure is always successful (Theorem 3.6 in Nering (1963)) and in addition is easily programmed.

If $\sum_{i=1}^{m} n_i < n$ the basis vectors for $\mathscr{C}_i$, i = 1, ..., m, are not sufficient to span the n dimensional space of row vectors. The remaining rows of Q are then found by choosing as rows the n tuples representing the vectors which are needed to form a basis for $\mathscr{C}$; these vectors are not unique. After Q is found, the CD representation can be computed, as

$$\dot{\hat{x}} = Q(A + BF^*)Q^{-1}\hat{x} + QBG^*u \qquad 5.10$$

$$y = CQ^{-1}\hat{x} \qquad 5.11$$

The special structure of the matrices of the CD representation is used in step (4) in identifying the form of the subsystems. According to Figure 3.3(a) the ith subsystem has the state equations

$$\dot{\hat{x}}^i = A_{ii}\hat{x}^i + B_{ii}u^i \qquad 5.12$$

$$y^i = \hat{x}^i_1 \qquad 5.13$$

The transfer function $p_{ii}(s,F^*,G^*)$, relating the output y_i to the control input u_i has d_i+1 poles at the origin and l_i poles at the same locations as the zeroes of the ith subsystem. The subsystem zeroes are the zeroes of the characteristic equation of Φ_i.

$$\det(sI - \Phi_i) = 0 \qquad 5.14$$

where Φ_i is a submatrix of the matrix A_{ii} and is defined in Figure 3.3(b). An efficient computer program for finding and factoring the characteristic equation of a matrix is available in the report by Melsa (1967).

In step (5) the CD representation is used, and the matrices $\hat{F}$ and $\hat{G}$ are found which cause the compensated system to exhibit a response which satisfies the design requirements. This step is best carried out with the aid of a phase-variable transformation and is discussed in detail in the next section.

The final step in the design procedure is the calculation of the matrices F and G to be used in compensating the original system. The matrices $\hat{F}$ and $\hat{G}$ are the known compensation matrices for the CD system. Now

$$u = \hat{F}\hat{x} + \hat{G}r \qquad 5.15$$

But since $\hat{x} = Qx$, Equation 5.15 can be written

$$u = \hat{F}Qx + \hat{G}r \qquad 5.16$$

which shows that the compensation matrices F_1 and G_1 for the ID system are $\hat{F}Q$ and $\hat{G}$, respectively. The ID system and the original system are control law equivalent, so that the matrices F and G can be found which apply to the original system. The required formulas relating F to F_1 and B*, and G to G_1 and B* appear in Equations 3.59 and 3.60. For the F_1 and G_1 found above, these equations give

$$F = F^* + B^{*-1}\hat{F}Q \qquad 5.17$$

$$G = B^{*-1}\hat{G} \qquad 5.18$$

The remaining topic to be discussed is that of finding $\hat{F}$ and $\hat{G}$. Once this has been done, F and G are calculated from Equations 5.17 and 5.18, and the design is ready to be implemented on the physical system.

Use of the Phase-Variable Transformation for Compensation

Consider a controllable, single-input, single-output system described in the state equations

$$\dot{x} = Ax + bu \qquad 5.19$$

$$y = cx \qquad 5.20$$

where x is an n-vector, u and y are now scalars instead of vectors, b is an n-vector, and c is a 1 x n row vector. The transfer function p(s) relating y and u is

$$p(s) = \frac{y}{u} = \frac{k(s^l + a_1 s^{l-1} + \ldots + a_l)}{s^n - p_1 s^{n-1} - p_2 s^{n-2} \ldots - p_n} \qquad 5.21$$

A well-known system representation, the phase-variable representation, takes a form in which the n coefficients

p_i and the l coefficients a_j appear directly. If x^o denotes the phase variables, then the system equations are

$$\dot{x}^o = A^o x^o + b^o u$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ p_n & p_{n-1} & & \cdots & p_1 \end{bmatrix} x^o + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u \qquad 5.22$$

$$y = c^o x^o$$

$$= k[a_l \quad a_{l-1} \quad \cdots \quad a_1 \quad 0 \quad \cdots \quad 0] x^o \qquad 5.23$$

Let state variable feedback be used for compensating the system in phase-variable form, as

$$u = f^o x^o + g^o r \qquad 5.24$$

where f^o is a 1 x n row vector having elements f_i^o, and g^o and r are scalars. Then Equation 5.22 becomes

$$\dot{x}^o = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ p_n+f_1^o & p_{n-1}+f_2^o & \cdots & p_1+f_n^o \end{bmatrix} x^o + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g^o \end{bmatrix} r \qquad 5.25$$

and Equation 5.22 is unchanged. The transfer function relating y and r for the compensated system $h(s,f^o,g^o)$ is given by

$$h(s,f^o,g^o) = \frac{g^o k(s^l + a_1 s^{l-1} + \cdots + a_l)}{s^n - (p_1+f_n^o)s^{n-1} - \cdots - (p_n+f_1^o)} \qquad 5.26$$

If the desired characteristic polynomial is

$$q(s) = s^n - q_1 s^{n-1} - q_2 s^{n-2} - \dots - q_n \qquad 5.27$$

then the elements of f' should be chosen as

$$f'_1 = q_n - p_n$$
$$f'_2 = q_{n-1} - p_{n-1}$$
$$\vdots$$
$$f'_n = q_1 - p_1 \qquad 5.28$$

The variables x and x' are related by the nonsingular matrix T, as

$$x = Tx' \qquad 5.29$$

and so the matrix f for the original system which corresponds to f' is

$$f = f'T^{-1} \qquad 5.30$$

The change of variables is concerned with the state variables x and x', and not the input variable r; thus, g and g' are related directly, as

$$g = g' \qquad 5.31$$

The input gain g' is selected from the requirement that the factor g'k in Equation 5.26 be equal to the required subsystem gain.

An algorithm for calculating T is given by Johnson and Wonham (1966). Let

$$T = [T_1 \quad T_2 \quad \dots \quad T_n] \qquad 5.32$$

where T_i are n x 1 column matrices. The algorithm is

$$T_n = b$$
$$T_{n-1} = AT_n - p_1T_n$$
$$T_{n-2} = AT_{n-1} - p_2T_n$$
$$\vdots$$
$$T_1 = AT_2 - p_{n-1}T_n \qquad 5.33$$

Note that the coefficients of the characteristic polynomial p(s) are required in using the algorithm. The discussion for single-input, single-output systems is now complete. Next, the multivariable system is considered.

Once step (4) of the design procedure is accomplished, m decoupled subsystems are in evidence. Each subsystem can be treated as a single-input, single-output system. For the ith subsystem, the state equations of the subsystem, Equations 5.12 and 5.13, replace Equations 5.19 and 5.20, and the transformation,

$$\hat{x}^i = T^i x'^i \quad i = 1, \ldots, m \qquad 5.34$$

where the subscripts are used to designate the subsystem, replaces Equation 5.29. Since the system is integrator decoupled, the ith subsystem has d_i+1 poles at the origin and l_i poles at the locations of the zeroes of that subsystem, as found from Equation 5.14. Therefore, the characteristic polynomial for the ith subsystem, $p^i(s,F^*,G^*)$, is

$$p^i(s,F^*,G^*) = s^{d_i+1}\det(sI - \Phi_i) \qquad 5.35$$

The coefficients of $p^i(s,F^*,G^*)$ are needed in the calculation of T^i.

There are now m separate design problems; in each one the fixed zeroes and the number of poles are known. After the desired gain and characteristic polynomial are selected for a subsystem, the required input gain and feedback coefficients are calculated. For the ith subsystem the input gain g'_{ii} is found by setting it equal to the required system gain stipulated by the design specifications because the ID subsystems have unity gain. In accordance with Equation 5.31 the corresponding input gain $\hat{g}_{ii}$ for the CD rather than the phase-variable representation is just g'_{ii}. The matrices G' and $\hat{G}$ are thus identical; they are diagonal matrices because the multivariable system is decoupled when it is put in integrator decoupled form.

The row vector of feedback coefficients for the ith subsystem is labelled f'^i; it is found from $p^i(s,F^*,G^*)$ and $q^i(s,F,G)$, the desired subsystem characteristic polynomial. Equation 5.28 is used for this purpose. After f'^i is found, the corresponding row vector for the CD representation is calculated. Recall from Chapter 3 that $\hat{F}$ must be of the form

$$\hat{F} = \begin{bmatrix} \theta_1 & 0 & \cdots & 0 & 0 \\ 0 & \theta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & \theta_m & 0 \end{bmatrix} \qquad 5.36$$

where each θ_i, $i = 1, \ldots, m$, is a $1 \times n_i$ matrix. Each θ_i is found from f'^i and T^i by using the equation corresponding to Equation 5.30; namely,

$$\theta_i = f'^i (T^i)^{-1} \qquad 5.37$$

At the completion of the design of all m subsystems, all m rows of $\hat{F}$ and all diagonal elements of $\hat{G}$ are known. The corresponding matrices for the original system are then found from Equations 5.17 and 5.18; these last computations complete step (6) of the design procedure. The design procedure is now illustrated by Example 5.1.

Example 5.1

As an example of the step-by-step design procedure consider the system used for Examples 3.1, 3.3, and 3.4. Steps (1), (2), and (3) have already been carried out in Chapter 3; for convenience the results of these steps are shown in Figure 5.1(a), (b), and (c). In step (3) only one row of Q_1, namely $C_1 = [1 \quad 0 \quad 0]$ is known directly because $d_1 = 0$. The set of linear equations associated

$$d_1 = 0 \qquad B^* = \begin{bmatrix} 7 & 0 \\ 10 & 1 \end{bmatrix} \qquad B^{*-1} = \begin{bmatrix} \frac{1}{7} & 0 \\ -\frac{10}{7} & 1 \end{bmatrix}$$

$$d_2 = 0$$

(a) Step (1), Test for Decoupling

$$F^* = \begin{bmatrix} \frac{5}{7} & 0 & 0 \\ -\frac{50}{7} & 1 & 2 \end{bmatrix} \qquad G^* = \begin{bmatrix} \frac{1}{7} & 0 \\ -\frac{10}{7} & 1 \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{50}{7} & -1 & 0 \\ -\frac{50}{7} & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ \frac{10}{7} & 0 \\ -\frac{10}{7} & 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} x$$

(b) Step (2), Integrator Decoupled System

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$Q_2 = [0 \quad 1 \quad 1]$$

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{50}{7} & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 1 & 0 \\ \frac{10}{7} & 0 \\ 0 & 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \hat{x}$$

(c) Step (3), Canonically Decoupled System

Figure 5.1 Example 5.1

with $\mathcal{C}_1$ is found from Equation 5.8 with $i = 1$. These equations are

$$\eta B_2 = \eta_3 = 0$$
$$\eta A B_2 = 0$$
$$\eta A^2 B_2 = 0 \qquad 5.38$$

so that a suitable basis for $\mathcal{C}_1$ is $\{[1, 0, 0], [0, 1, 0\}$.

The dimension of $\mathcal{C}_1$ is 2, indicating that another row vector is needed in forming the matrix Q_1. A suitable choice for this vector is [0, 1, 0], the basis vector which is linearly independent of C_1. In this example the coefficient array for Equation 5.38 is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is already in Hermite normal form. For more complicated examples it is necessary to reduce the coefficient array to Hermite normal as an aid in finding a basis for the subspace.

Step (4) has also been carried out for this example in Chapter 3, and the subsystem transfer functions were found to be

$$p_{11}(s, F^*, G^*) = \frac{s+1}{s(s+1)} \qquad 5.39$$

$$p_{22}(s, F^*, G^*) = \frac{1}{s} \qquad 5.40$$

Again, subsystem 1 has a fixed zero at $s = -1$ and two arbitrary poles; subsystem 2 has one arbitrary pole.

Choose as the transfer functions relating the inputs to the outputs

$$h_{11}(s,F,G) = \frac{2(s+1)}{s^2+2s+2} \qquad 5.41$$

$$h_{22}(s,F,G) = \frac{1}{s+1} \qquad 5.42$$

These are the same choices as in Example 3.4 of Chapter 3.

Step (5) is the calculation of the numerical matrices $\hat{F}$ and $\hat{G}$. Consider subsystem 1; comparing $p_{11}(s,F^*,G^*)$ in Equation 5.39 with the transfer function of Equation 5.21 and the phase-variable representation of Equations 5.22 and 5.23, one arrives at the following phase-variable representation

$$\dot{x}'^1 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x'^1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad 5.43$$

$$y = [1 \quad 1]x'^1 \qquad 5.44$$

For convenience the superscript 1 is dropped from y and u.

The compensation matrix f'^1 and the scalar g'_{11} needed to achieve the response required by Equation 5.41 are found by using Equation 5.28; $f'^1_1 = -2$, $f'^1_2 = -1$, or $f'^1 = [-2 \quad -1]$. The scalar g'_{11} must supply the required gain; thus $g'_{11} = 2$.

In order to find θ_1, the compensation matrix corresponding to f'^1, but which applies to the CD system representation rather than the phase-variable representation, the matrix T^1 relating $\hat{x}^1$ and x'^1 must be found by the algorithm of Equation 5.33. The required calculations are

$$T_2^1 = \begin{bmatrix} 1 \\ \frac{10}{7} \end{bmatrix} \qquad T_1^1 = AT_2^1 + T_2^1 = \begin{bmatrix} 1 \\ \frac{50}{7} \end{bmatrix}$$

or

$$T^1 = \begin{bmatrix} 1 & 1 \\ \frac{10}{7} & \frac{50}{7} \end{bmatrix} \qquad 5.45$$

Now step (5) is completed for subsystem 1 by calculating θ_1 from f'^1 and T^1 using Equation 5.37, as

$$\theta_1 = f'^1(T^1)^{-1}$$

$$= [-2 \quad -1] \begin{bmatrix} -\frac{1}{4} & \frac{7}{40} \\ \frac{5}{4} & -\frac{7}{40} \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} & -\frac{7}{40} \end{bmatrix} \qquad 5.46$$

Subsystem 2 is already in phase-variable form. Its pole is required to be placed at $s = -1$; the required value of the feedback coefficient is found from Equation 5.28 to be $\theta_2 = -1$. The gain for subsystem 2 is unity, or $\hat{g}_{22} = 1$. The complete compensation matrices $\hat{F}$ and $\hat{G}$ are found by putting together the rows found in the design of the individual subsystems, in conformity with Equation 5.36; they are

$$\hat{F} = \begin{bmatrix} -\frac{3}{4} & -\frac{7}{40} & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad G = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad 5.47$$

The final design step, step (6), is the calculation of F and G from Equations 5.17 and 5.18, as

$$F = F^* + B^{*-1}\hat{F}Q$$

$$= \begin{bmatrix} \frac{5}{7} & 0 & 0 \\ -\frac{50}{7} & 1 & 2 \end{bmatrix} + \begin{bmatrix} \frac{1}{7} & 0 \\ -\frac{10}{7} & 1 \end{bmatrix} \begin{bmatrix} -\frac{3}{4} & -\frac{7}{40} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{17}{28} & -\frac{1}{40} & 0 \\ -\frac{85}{14} & \frac{1}{4} & 1 \end{bmatrix} \qquad 5.48$$

$$G = B^{*-1}\hat{G}$$

$$= \begin{bmatrix} \frac{1}{7} & 0 \\ -\frac{10}{7} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & 0 \\ -\frac{20}{7} & 1 \end{bmatrix} \qquad 5.49$$

Parts of the design procedure which have not been discussed are design specifications and the selection of the transfer functions to meet the specifications. These subjects have been treated by many authors (Truxal, 1955; Bower and Schultheiss, 1959). Except for this task, the design procedure is amenable to digital computer computation; in connection with this dissertation working programs have been written and used to check the numerical examples. The computer programs are not included in the text because further usage is needed to be sure that the programs are reliable. An excellent program for the state variable feedback design of single-input, single-output systems is described by Melsa (1967); this program has definite utility in the multivariable system design.

Procedure Applicable to the Augmented System

In this section the modifications of the design procedure and the associated computational procedures are presented for the case where series compensation is added to the multivariable plant. Methods A, B, and C are used when additional dynamics are needed.

Both Method A and Method B are similar to design by state variable feedback alone as far as the computational requirements are concerned. In Method A steps (1) - (4) are completed for the given plant, and the subsystems are identified. Then series compensators are placed in the control inputs of the plant, and steps (1) - (4) are repeated. The repetition of steps (1) - (4) is necessary because there is no guarantee that the augmented plant can be decoupled or that zeroes added in the compensation will appear in the appropriate subsystems. If the augmented plant can be decoupled and the forms of the subsystems of the augmented plant are satisfactory, then steps (5) and (6) of the design procedure are completed.

For Method B steps (1) - (4) are completed for the given plant, and the subsystems are identified--just as in Method A. But now the plant is compensated by state variable feedback in order to realize as much of the transfer matrix as possible. Steps (5) and (6) are required to calculate the compensation matrices at this stage. The design is completed by inserting the series compensators in the new, decoupled plant which cause the system specifications to be met.

The computational procedures for Method C are now discussed in detail. The first four steps of the previous design procedures (checking decoupling, finding the ID system and the CD representation, and identifying the plant subsystems) apply in the present case. After the subsystems have been identified, the designer has two new tasks. He must decide what additional compensator networks are needed to meet the design requirements; commonly, lead-lag networks are necessary for increasing the velocity-error coefficients or one or more poles are used to cancel unwanted plant zeroes. It must also be decided how the plant is to be decoupled.

One suggestion for deciding how to proceed in decoupling the plant is discussed in Chapter 4. The criterion used for determining F is cost, and the lowest-cost design is assumed to be the one in which the largest number of entries of F are zero. The use of this scheme requires that the class of all decoupling F matrices be found from the equation

$$F = F^* + B^{*-1}\hat{F}Q \qquad 5.50$$

Here, F^*, B^{*-1}, and Q are numerical matrices and $\hat{F}$ has the form

$$\hat{F} = \begin{bmatrix} \theta_1 & 0 & \cdots & 0 & 0 \\ 0 & \theta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \theta_m & 0 \end{bmatrix} \qquad 5.51$$

where each θ_i, $i = 1, \ldots, m$, is a $1 \times n_i$ matrix. After F has been found in terms of the elements of the θ_i, as many of its elements are made zero as possible.

The recommended G is the matrix in which the diagonal elements are 1, and the off-diagonal elements satisfy the equation

$$G = B^{*-1}\Lambda \qquad 5.52$$

where Λ is a diagonal matrix whose diagonal elements are nonzero. The above G matrix is simple and requires a minimum amount of gain.

The equations describing the decoupled compensation are

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u} \qquad 5.53$$

$$u = \bar{C}\bar{x} + \bar{E}\bar{u} \qquad 5.54$$

and the augmented system has the equations

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\bar{u}$$

$$= \begin{bmatrix} A & B\bar{C} \\ 0 & \bar{A} \end{bmatrix} \tilde{x} + \begin{bmatrix} B\bar{E} \\ \bar{B} \end{bmatrix} \bar{u} \qquad 5.55$$

$$y = \tilde{C}\tilde{x}$$

$$= [C \quad 0]\tilde{x} \qquad 5.56$$

The CD representation for the integrator decoupled form of the original system is no longer applicable because the plant is now described by Equations 5.55 and 5.56, rather than Equations 5.1 and 5.2. Therefore, the completion of step (5) of the design procedure (calculating $\tilde{F}$ and $\tilde{G}$), in its present form, requires finding the ID plant and the CD representation for the augmented system. Except for this change, steps (5) and (6) are exactly the same as in the case where no decoupled compensation is needed.

If the designer proceeds in the manner described above, the CD representation must be found both for the plant and the augmented plant. The CD representation of the plant is needed to determine the plant's structure, and the CD representation of the augmented plant is used in steps (5) and (6) for finding $\hat{F}$ and $\hat{G}$ and then F and G.

There is a special case in which the second CD transformation is not needed. By Theorem 4.3 the form of the augmented system is known from the form of the decoupled plant and the decoupled compensation. Once the augmented plant is integrator decoupled, the ith subsystem has $\tilde{d}_i+1$ poles at the origin and $\tilde{l}_i$ known zeroes which are canceled by poles. The important conclusion is that the characteristic polynomial for the subsystems are known because all of their pole locations are known.

The special case referred to above is the one in which the matrix $A_{m+1,m+1}$ is not needed in the CD representation of the ID plant. For the special case, $A_{m+1,m+1}$ is also unnecessary in the CD representation of the ID augmented plant because of Theorem 4.3. Thus each of the poles of the ID augmented plant is associated with one and only one subsystem. This fact makes possible a change of state variables for which the A, B, and C matrices have the form shown in Figure 5.2(a) and (b).

The structure in part (a) of the figure can be achieved by a linear change in the state variables of the ID augmented plant because that system is decoupled and because $A_{m+1,m+1}$ is assumed not to be needed in the CD representation. The structure of part (b) indicates m subsystems each of which is in phase variable form. The fact that this structure can be achieved is proved by giving a procedure for constructing the required transformation matrix, but first the representation is defined formally, as follows:

Definition 5.1 A decoupled multivariable system is in multivariable phase variable form if the matrices in the equations

$$\dot{x}' = A'x + B'u$$

$$y = C'x'$$

take the form shown in Figure 5.2(a) and (b).

$$A' = \begin{bmatrix} A'_{11} & 0 & \cdots & 0 \\ 0 & A'_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & A'_{mm} \end{bmatrix} \quad \begin{array}{l} A'_{ii} \text{ is } n_i \times n_i \\ i = 1, \ldots, m \end{array}$$

$$B' = \begin{bmatrix} B'_{11} & 0 & \cdots & 0 \\ 0 & B'_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & B'_{mm} \end{bmatrix} \quad \begin{array}{l} B'_{ii} \text{ is } n_i \times 1 \\ i = 1, \ldots, m \end{array}$$

$$C' = \begin{bmatrix} C'_{11} & 0 & \cdots & 0 \\ 0 & C'_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & C'_{mm} \end{bmatrix} \quad \begin{array}{l} C'_{ii} \text{ is } 1 \times n_i \\ i = 1, \ldots, m \end{array}$$

(a) Structure of A', B', and C'

Figure 5.2 Multivariable Phase Variable Form

$$A'_{ii} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 1 \\ p^i_{n_i} & p^i_{n_i-1} & p^i_{n_i-2} & \cdots & p^i_1 \end{bmatrix}$$

$$B'_{ii} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C'_{ii} = [a^i_{l_i} \quad a^i_{l_i-1} \quad \cdots \quad a^i_1 \quad 0 \quad \cdots \quad 0]$$

(b) Structure of A'_{ii}, B'_{ii}, and C'_{ii}

Figure 5.2 Multivariable Phase Variable Form (Continued)

In the present case the numbers p_j^i, $j = 1, \ldots, n_i$, are the coefficients of the characteristic polynomial of subsystem i of the augmented ID plant; i.e.,

$$\tilde{p}^i(s,F^*,G^*) = s^{n_i} - p_1^i s^{n_i-1} - \ldots - p_{n_i}^i \qquad 5.57$$

Similarly, the numbers a_j^i, $j = 1, \ldots, l_i$, are the coefficients of the numerator polynomial of subsystem i. Both sets of numbers are known because the structure of the augmented plant is known from the structures of the decoupled plant and the decoupled compensation.

An algorithm is now developed for finding the multivariable phase variable representation from the integrator decoupled, augmented plant representation. The intermediate step of finding the CD representation is being by-passed.

Let the augmented plant be integrator decoupled so that the equations for the ID augmented plant are

$$\dot{\tilde{x}} = (\tilde{A} + \tilde{B}\tilde{F}^*)\tilde{x} + \tilde{B}\tilde{G}^*\bar{u} \qquad 5.58$$

$$y = \tilde{C}\tilde{x} \qquad 5.59$$

Define x' as the state vector for the multivariable phase variable representation for the system of Equations 5.58 and 5.59. The state vector $\tilde{x}$ is related to x' by the nonsingular matrix T, as

$$\tilde{x} = Tx' \qquad 5.60$$

In terms of x' the state equations are

$$\dot{x}' = A'x' + B'u \qquad 5.61$$

$$y = C'x' \qquad 5.62$$

and on substitution of Equation 5.60 these equations become

$$\dot{\tilde{x}} = TA'T^{-1}\tilde{x} + TB'u \qquad 5.63$$

$$y = C'T^{-1}\tilde{x} \qquad 5.64$$

A comparison of Equations 5.63 and 5.64 with Equations 5.58 and 5.59 reveals the following

$$TA' = (\tilde{A} + \tilde{B}\tilde{F}^*)T \qquad 5.65$$

$$TB' = \tilde{B}\tilde{G}^* \qquad 5.66$$

$$C'T^{-1} = \tilde{C} \qquad 5.67$$

The unknown in these equations is the transformation matrix T. An algorithm analogous to that of Equation 5.33 is being sought, but in the present case the change of variables is being made for the entire system rather than for each subsystem considered separately. This is necessary because the matrices $\tilde{A} + \tilde{B}\tilde{F}^*$, $\tilde{B}\tilde{G}^*$, and $\tilde{C}$ have no special structure when considered separately.

Let T be partitioned into n columns in the following way

$$T = [T_1^1 T_2^1 \ldots T_{n_1}^1 T_1^2 T_2^2 \ldots T_{n_2}^2 \ldots T_1^m T_2^m \ldots T_{n_m}^m] \qquad 5.68$$

where each T_j^i is an n x 1 column matrix.

Consider Equation 5.66. Because of the special structure of B' shown in Figure 5.2(a) and (b), the left-hand side of Equation 5.66 is

$$TB' = [T_{n_1}^1 T_{n_2}^2 \ldots T_{n_m}^m] \qquad 5.69$$

Substituting this expression for TB in Equation 5.66 gives an expression for m of the n columns of T, as

$$T^{i}_{n_i} = (\tilde{B}\tilde{G}^*)_i \qquad i = 1, \ldots, m \tag{5.70}$$

where $(\tilde{B}\tilde{G}^*)_i$ is the ith column of $\tilde{B}\tilde{G}^*$. Next, consider Equation 5.65. The matrix TA' has a special form because A' has a special form. For simplicity consider just the first n_1 columns of TA'; in order, they are

$$p^1_{n_1}T^1_{n_1},\ T^1_1 + p^1_{n_1-1}T^1_{n_1},\ T^1_2 + p^1_{n_1-2}T^1_{n_1},\ \ldots,\ T^1_{n_1-1} + p^1_1 T^1_{n_1}$$

The use of these expressions in Equation 5.65 gives

$$\begin{aligned} p^1_{n_1}T^1_{n_1} &= (\tilde{A} + \tilde{B}\tilde{F}^*)T^1_1 \\ T^1_1 + p^1_{n_1-1}T^1_{n_1} &+ (\tilde{A} + \tilde{B}\tilde{F}^*)T^1_2 \\ &\vdots \\ T^1_{n_1-1} + p^1_1 T^1_{n_1} &= (\tilde{A} + \tilde{B}\tilde{F}^*)T^1_{n_1} \end{aligned} \tag{5.71}$$

When the rest of the columns of T are considered, equations of the form of Equations 5.71 result, with the superscript 1 replaced by i, i = 2, ..., m.

The recursion relationship for the columns of T is found from Equations 5.70 and 5.71. In Equation 5.71 the last equation is solved for $T^1_{n_1-1}$ in terms of $T^1_{n_1}$ and then the next equation is solved for $T^1_{n_1-2}$ in terms of $T^1_{n_1-1}$ and $T^1_{n_1}$, and so on. In compact form the resulting algorithm is

$$T^i_{n_i} = (BG^*)_i$$

$$i = 1, \ldots, m, \; j = 1, \ldots, n_i - 1$$

$$T^i_{n_i - j} = (A + BF^*)T^i_{n_i - j + 1} - p^i_j T^i_{n_i} \qquad 5.72$$

These equations are easily programmed on a digital computer.

Once the ID augmented system has been put into the multivariable phase variable form, step (5) (calculating F' and G') is easily carried out. The matrix F' has the following form

$$F' = \begin{bmatrix} f'^1 & 0 & \cdots & 0 \\ 0 & f'^2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & f'^m \end{bmatrix} \qquad 5.73$$

where each f'^i, $i = 1, \ldots, m$ is a $1 \times n_i$ matrix. For the ith subsystem the characteristic polynomial is $p^i(s)$ and the desired characteristic polynomial is $q^i(s)$. The elements of f'^i are given by the equations corresponding to Equation 5.28; namely,

$$f'^i_{n_1} = q^i_{n_i} - p^i_{n_i}$$

$$\vdots$$

$$f'^i_{n_i} = q^i_1 - p^i_1 \qquad 5.74$$

The matrix G' is diagonal and its ith diagonal element is equal to the required gain of the ith subsystem of the final, compensated system.

For the augmented system expressed in terms of its physical variables, the corresponding $\tilde{F}$ and $\tilde{G}$ matrices (step 6) are

$$\tilde{F} = \tilde{F}^* + \tilde{B}^{*-1}F'\tilde{T}^{-1} \qquad 5.75$$

$$\tilde{G} = \tilde{B}^{*-1}G' \qquad 5.76$$

The discussion of the computational procedures for design problems which require the addition of dynamics is now complete. The following example illustrates the application of the particular procedure which is given the most attention in this section; namely, the one in which Method C is needed and in which the matrix $A_{m+1,m+1}$ does not appear in the CD representation of the integrator decoupled plant.

Example 5.2

Consider the example which is used to illustrate Method C in Chapter 4; namely, Example 4.4. Steps (1) - (4) have already been carried out and the augmented system is shown in Figure 4.7(c). Only steps (5) and (6) remain in the design.

The state equations for the augmented plant are

$$\dot{\tilde{x}} = \begin{bmatrix} -5 & 0 & 0 & 7 & 0 & 0 \\ 0 & -1 & 0 & 10 & 0 & 0 \\ 0 & -1 & -2 & -10 & 0 & 1 \\ 0 & 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \bar{u} \qquad 5.77$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \tilde{x} \qquad 5.78$$

With the aid of a digital computer B*, F*, and G*, are found to be

$$B^* = \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix} \quad F^* = \begin{bmatrix} -3.571 & 0 & 0 & 7 & 1 & 0 \\ 0 & -4 & -4 & 0 & 0 & 7 \end{bmatrix}$$

$$G^* = \begin{bmatrix} .1429 & 0 \\ 0 & 1 \end{bmatrix} \qquad 5.79$$

and the state equations for the integrator decoupled, augmented system are

$$\dot{\tilde{x}} = (\tilde{A} + \tilde{B}\tilde{F}^*)x + \tilde{B}\tilde{G}^*\bar{u}$$

$$= \begin{bmatrix} -5 & 0 & 0 & 7 & 0 & 0 \\ 0 & -1 & 0 & 10 & 0 & 0 \\ 0 & -1 & -2 & -10 & 0 & 1 \\ -3.571 & 0 & 0 & 5 & 0 & 0 \\ -3.571 & 0 & 0 & 7 & -3 & 0 \\ 0 & -4 & -4 & 0 & 0 & 2 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ .1429 & 0 \\ .1429 & 0 \\ 0 & 1 \end{bmatrix} \bar{u} \qquad 5.80$$

$$y = \tilde{C}\tilde{x}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \tilde{x} \qquad 5.81$$

Subsystem 1 has two zeroes, one at $s = -3$ and one at $s = -1$. In the ID augmented plant two of the four poles are used for canceling the zeroes and the remaining two are at the origin. Thus the characteristic polynomial is

$$\tilde{p}^1(s,\tilde{F}^*,\tilde{G}^*) = s^2(s+1)(s+3)$$

$$= s^4 + 4s^3 + 3s^2 \qquad 5.82$$

In a similar fashion $\tilde{p}^2(s)$ is found to be

$$\tilde{p}^2(s,\tilde{F}^*,\tilde{G}^*) = s^2 \qquad 5.83$$

Enough information has been given so that T can be calculated from Equation 5.72, as

$$T = \begin{bmatrix} 3.00 & 4.00 & 1.00 & 0 & 0 & 0 \\ 21.43 & 11.43 & 1.43 & 0 & 0 & 0 \\ -21.43 & -11.43 & -1.43 & 0 & 1.00 & 0 \\ 2.14 & 3.29 & 1.29 & 0.14 & 0 & 0 \\ 1.43 & 2.43 & 1.14 & 0.14 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.00 & 1.00 \end{bmatrix} \qquad 5.84$$

The desired transfer matrix for the compensated augmented system is given by Equation 4.49. From this equation the characteristic polynomials of the compensated system are identified as

$$\tilde{q}^1(s) = (s^2+4s+8)(s+4)(s+1)$$

$$= s^4 + 9s^3 + 32s^2 + 56s + 32 \qquad 5.85$$

$$\tilde{q}^2(s) = s^2 + 6s + 18 \qquad 5.86$$

The characteristic polynomials of the subsystems of both the ID augmented plant and the final, compensated system are now known. The coefficients of these polynomials are used to calculate the rows of the compensation matrix F'. For this purpose, Equation 5.74 is used, and the resulting F' is

$$F' = \begin{bmatrix} -32 & -56 & -29 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -18 & -6 \end{bmatrix} \qquad 5.87$$

The compensation matrix $\tilde{F}$, which applies to the augmented system expressed in terms of its physical variables, is found from Equation 5.75 to be

$$\tilde{F} = \begin{bmatrix} -.93 & 0 & 0 & -.50 & 3.50 & 0 \\ 0 & -10 & -10 & 0 & 0 & 1 \end{bmatrix} \qquad 5.88$$

According to the design specifications embodied in Equation 4.49, no additional gain beyond that already present in the plant is required. The gain of the ID system has been made unity so that the plant gain (represented by the diagonal elements of B* in Equation 5.79) must be restored by the matrix G'. Thus,

$$G' = \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix} \qquad 5.89$$

and G is calculated from Equation 5.76, as

$$G = B^{*-1}G'$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad 5.90$$

The design is now complete. A block diagram for the designed system appears in Chapter 4 as Figure 4.8.

Summary

There are two parts to this chapter. In the first part complete computational procedures are described for performing the calculations required in the design of a multivariable system by state variable feedback. The relevant design formulas from Chapter 3 are organized as part of an orderly design procedure, and the phase variable transformation is introduced and utilized for the calculation of the compensation matrix F. The discussion is complete in the sense that the designer can use the design procedure in going from the start (the system equations) to the finish (the compensation matrices F and G). For all but simple examples the use of the design procedure requires the aid of a digital computer. In fact, the procedure is formulated with this requirement in mind.

The second part of the chapter extends the first part to cover the case where series compensation is needed in order to meet the design specifications.

Methods A, B, and C, presented in the previous chapter, are now discussed from a computational point of view. The first two methods are given a brief treatment because their computational aspects are similar to those already described. Method C is treated in more detail.

Two applications of state variable feedback are needed in the design of control systems by Method C. The calculation of the feedback coefficients for each application could require a separate transformation to the CD representation. For the special case where each of the plant poles is assigned to one and only one subsystem a technique is given for avoiding one of the transformations to the CD representation. The technique employs the multivariable phase variable representation, a concept which is introduced in the chapter.

The comments on the computational procedure of the first part of the chapter apply to the second part. Again, the procedures are complete and especially tailored for digital computer usage.

CHAPTER 6

A PRACTICAL EXAMPLE

The theory and design procedures for the state variable feedback design of multivariable systems is presented in the preceding chapters. The present chapter is concerned with the application of the state variable feedback technique to a practical example. The physical system chosen is a coupled-core nuclear reactor (Weaver, 1968). The inputs to the system are the reactivities for each core, as determined by the positions of the core control rods, and the system outputs are the power levels of the individual cores. The total power for the system is obtained by adding the powers for each of the cores. Coupling between the cores exists because of neutron leakage between the cores. Thus, if the reactivity input to one core is changed, then the power levels of all the cores are affected.

The mode of operation desired is that in which all cores are given the same input and are required to respond in an identical manner. This goal is achieved by using state variable feedback to decouple the system and to cause each subsystem of the compensated system to exhibit the same response as the other subsystems. The advantage of this mode of operation is that all cores share equally the task of providing the power output.

Without the addition of series compensation, the desired subsystem responses cannot be realized. One of the methods of Chapter 4 is needed to supply the additional dynamics required. Method A is well suited for this example because the addition of a single pole to each subsystem allows the design specifications to be met. Theorem 4.1 assures that decoupling is not lost by the addition of the series compensation; the loss of zeroes is not a concern (as it turns out) since no zeroes are being added in the compensation.

The values of the parameters used in the description of the physical system are the same as those used by Weaver and Vanasse (1967). Three cores are assumed, and so the multivariable system has three inputs and three outputs.

Coupled-Core Reactor Design

A coupled-core nuclear reactor is a critical reactor consisting of two or more subcritical cores (Weaver, 1968). There is a mutual exchange of neutrons among the cores due to the neutron leakage of the cores. It is this neutron leakage between cores which makes the entire system capable of sustaining a nuclear chain reaction. Because of the neutron leakage, the behavior of each core is influenced by the behavior of all other cores; in other words, the system is coupled.

The specific case of three coupled cores is considered. In the plant equations, the effect of delayed neutrons and controller dynamics are excluded, and the cores are assumed to be identical with the same neutron coupling coefficient. Even so, the equations are still nonlinear and must be linearized about the steady-state reactivity and power levels. These matters are discussed fully in the reference cited above. Here the linearized equations are assumed to be given, as

$$\dot{x}_1 = -\frac{D}{\tau}x_1 - \frac{\alpha n_0}{\tau}x_2 + \frac{D}{\tau}x_3 + \frac{D}{\tau}x_5 + \frac{n_0}{\tau}u_1$$

$$\dot{x}_2 = K_1x_1 - ax_2$$

$$\dot{x}_3 = \frac{D}{\tau}x_1 - \frac{D}{\tau}x_3 - \frac{\alpha n_0}{\tau}x_4 + \frac{D}{\tau}x_5 + \frac{n_0}{\tau}u_2$$

$$\dot{x}_4 = K_1x_3 - ax_4$$

$$\dot{x}_5 = \frac{D}{\tau}x_1 + \frac{D}{\tau}x_3 - \frac{D}{\tau}x_5 - \frac{\alpha n_0}{\tau}x_6 + \frac{n_0}{\tau}u_3$$

$$\dot{x}_6 = K_1x_5 - ax_6$$

$$y_1 = x_1$$

$$y_2 = x_3$$

$$y_3 = x_5 \qquad 6.1$$

where x_1 = neutron density or power in core 1

x_2 = temperature in core 1

x_3 = neutron density or power in core 2

x_4 = temperature in core 2

x_5 = neutron density or power in core 3

x_6 = temperature in core 3

u_1 = reactivity input from controller 1

u_2 = reactivity input from controller 2

u_3 = reactivity input from controller 3

y_1 = total neutron density or power of core 1

y_2 = total neutron density or power of core 2

y_3 = total neutron density or power of core 3

Assume the following values for the system parameters (Weaver and Vanasse, 1967)

$n_0 = 10^5$ watts $\quad K_1 = 10^{-5}$ degree/watt-sec.

$a = 10^{-2}$ sec^{-1} $\quad \tau = 0.1$ sec

$\alpha = 10^{-3}$ degree^{-1} $\quad D = 0.1$

Then the state equations, in matrix notation, are

$$\dot{x} = \begin{bmatrix} -1 & -10^3 & 1 & 0 & 1 & 0 \\ 10^{-5} & -10^{-2} & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -10^3 & 1 & 0 \\ 0 & 0 & 10^{-5} & -10^{-2} & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & -10^3 \\ 0 & 0 & 0 & 0 & 10^{-5} & -10^{-2} \end{bmatrix} x + \begin{bmatrix} 10^6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 10^6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 10^6 \\ 0 & 0 & 0 \end{bmatrix} u \tag{6.2}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x \tag{6.3}$$

The plant equations are now known, and the step-by-step design procedure of the first section of Chapter 5 is applicable. The first step in the design procedure is the test for decoupling. To carry out the test, the matrix B* must be formed and checked for nonsingularity. For the present example, B* is easily formed because each row matrix C_iB, i = 1, 2, 3, is non-zero. Thus each d_i, i = 1, 2, 3, is 0, and, in accordance with Equation 5.3,

$$B^* = \begin{bmatrix} C_1B \\ C_2B \\ C_3B \end{bmatrix} = \begin{bmatrix} 10^6 & 0 & 0 \\ 0 & 10^6 & 0 \\ 0 & 0 & 10^6 \end{bmatrix} \qquad 6.4$$

Clearly, B* is nonsingular, and the system can be decoupled by state variable feedback.

In step (2) of the design procedure F* and G*, the compensation matrices which put the system in integrator decoupled form, are calculated by using Equations 5.4 and 5.5. They are

$$F^* = \begin{bmatrix} 10^{-6} & 10^{-3} & -10^{-6} & 0 & -10^{-6} & 0 \\ -10^{-6} & 0 & 10^{-6} & 10^{-3} & -10^{-6} & 0 \\ -10^{-6} & 0 & -10^{-6} & 0 & 10^{-6} & 10^{-3} \end{bmatrix} \qquad 6.5$$

$$G^* = \begin{bmatrix} 10^{-6} & 0 & 0 \\ 0 & 10^{-6} & 0 \\ 0 & 0 & 10^{-6} \end{bmatrix} \qquad 6.6$$

The state equations for the ID plant are

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 10^{-5} & -10^{-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10^{-5} & -10^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10^{-5} & -10^{-2} \end{bmatrix} x + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} u \qquad 6.7$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x \qquad 6.8$$

In step (3) the matrix Q is needed in finding the canonically decoupled representation of the ID plant. Here, the first n_1 rows of Q are discussed in detail. In order to find these rows the subspace $\mathscr{Q}_1$ is considered. The vector space $\mathscr{Q}_1$ is the set of all row vectors η which satisfy the relation

$$\mathscr{Q}_1 = \{\eta;\ \eta A^j B_k = 0,\ j = 0, 1, \ldots, 5,\ k = 2, 3\} \qquad 6.9$$

where A and B are the matrices in Equation 6.7. As usual, the row vector η is written $[\eta_1\ \ \eta_2\ \ \ldots\ \ \eta_n]$, and the coefficient array for the equations resulting from Equation 6.8 is formed, as

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10^{-5} & 0 & 0 \\ 0 & 0 & 0 & -10^{-7} & 0 & 0 \\ 0 & 0 & 0 & 10^{-9} & 0 & 0 \\ 0 & 0 & 0 & -10^{-11} & 0 & 0 \\ 0 & 0 & 0 & 10^{-13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10^{-5} \\ 0 & 0 & 0 & 0 & 0 & -10^{-7} \\ 0 & 0 & 0 & 0 & 0 & 10^{-9} \\ 0 & 0 & 0 & 0 & 0 & -10^{-11} \\ 0 & 0 & 0 & 0 & 0 & 10^{-13} \end{bmatrix} \qquad 6.10$$

With the aid of a digital computer, the Hermite normal form of the above array is found to be

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad 6.11$$

where the all-zero rows are deleted. The 4 x 6 array yields the following relationships among the elements of η:

$$\eta_3 = \eta_4 = \eta_5 = \eta_6 = 0 \qquad 6.12$$

and so a suitable basis for $\mathcal{Q}_1$ is $\{[1 \; 0 \; 0 \; 0 \; 0 \; 0],$ $[0 \; 1 \; 0 \; 0 \; 0 \; 0]\}$. The rank of $\mathcal{Q}_1$ is n_1 or 2, and the number of zeroes of subsystem 1 is $n_1 - d_1 + 1$ or 1.

The first row of Q_1 is C_1, which is also a member of the above basis, and the second row is taken as the remaining basis vector. Proceeding in a similar manner for Q_1 and Q_2, one finds that

$$Q = I \qquad 6.13$$

where I is the 6 x 6 identity matrix. The fact that Q is the identity matrix indicates that the ID plant is already in canonically decoupled form, and no change of variables is needed. Equations 6.7 and 6.8 apply to both the ID plant and the CD representation of the ID plant.

The subsystem matrices for the ID plant are outlined in the matrices of Equations 6.7 and 6.8. As expected, the subsystem equations are identical because the cores are assumed to be identical. A comparison of the matrices of Equation 6.7 with those in Figures 3.3(a) and (b) reveals that

$$\Phi_i = [-10^{-2}] \qquad i = 1, 2, 3 \qquad 6.14$$

and so each subsystem has a fixed zero at s = -.01 and two poles which are under the control of state variable feedback. Step (4) is now complete.

The remaining steps of the design procedure require that a suitable response be selected for each subsystem and that the compensation matrices be found that give the desired response. Suppose that the desired dynamics of each subsystem are embodied in the following transfer functions (Weaver and Vanasse, 1967):

$$\frac{y_i}{r_i} = \frac{10^6}{s^2 + 2s + 2} \qquad i = 1, 2, 3 \qquad 6.15$$

The presence of the fixed zero close to the origin must be taken into account. Only two poles are present in each subsystem so that if one of them is used for cancellation, a first-order response results. Apparently, one additional pole is needed in each subsystem; then one pole can be used for cancellation and two poles are left to achieve the second-order subsystem response. This technique is the one used below.

It is desired to add one pole to each subsystem. Method A applies, and by Theorem 4.1 decoupling is not lost. Let the three, identical series compensators each have unity gain and one pole at s = -1. The equations for the compensation are

$$\dot{\bar{x}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{u} \qquad 6.16$$

$$u = \bar{x}$$

Using Equations 5.55 and 5.56 the state equations for the augmented plant are found to be

$$\dot{x} = \begin{bmatrix} -1 & -10^3 & 1 & 0 & 1 & 0 & 10^6 & 0 & 0 \\ 10^{-5} & -10^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -10^3 & 1 & 0 & 0 & 10^6 & 0 \\ 0 & 0 & 10^{-5} & -10^{-2} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & -10^3 & 0 & 0 & 10^6 \\ 0 & 0 & 0 & 0 & 10^{-5} & -10^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{u} \qquad 6.18$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} x \qquad 6.19$$

The basic design procedure is now applied to the augmented plant. The compensation matrices F* and G* required to form the ID augmented plant are

$$F^* = \begin{bmatrix} -2.99\times10^{-6} & -1.01\times10^{-3} & 10^{-6} & 10^{-3} & 10^{-6} & 10^{-3} & 2 & -1 & -1 \\ 10^{-6} & 10^{-3} & -2.99\times10^{-6} & -1.01\times10^{-3} & 10^{-6} & 10^{-3} & -1 & 2 & -1 \\ 10^{-6} & 10^{-3} & 10^{-6} & 10^{-3} & -2.99\times10^{-6} & -1.01\times10^{-3} & -1 & -1 & 2 \end{bmatrix} \qquad 6.20$$

$$G^* = \begin{bmatrix} 10^{-6} & 0 & 0 \\ 0 & 10^{-6} & 0 \\ 0 & 0 & 10^{-6} \end{bmatrix} \qquad 6.21$$

and the system matrices for the ID augmented plant are found by forming $A + BF^*$ and BG^*.

Each subsystem has three poles and one fixed zero, or $d_i = 1$, for $i = 1, 2, 3$. This means that the matrix Q, which is needed to find the CD representation, has C_i and C_iA, $i = 1, 2, 3$ as rows. A simpler Q is obtained by using the standard bases for Q_1, Q_2, and Q_3 found by using the Hermite normal form. The Q resulting from this procedure is

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10^{-6} & 0 & 10^{-6} & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 10^{-6} & 0 & 0 & 0 & 10^{-6} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 10^{-6} & 0 & 10^{-6} & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad 6.22$$

In terms of the new state variables for the ID augmented system the system matrices are in block diagonal form with the following matrices along the diagonals

$$A_{ii} = \begin{bmatrix} -1 & -10^{3} & 10^{6} \\ 10^{-5} & -10^{-2} & 0 \\ -9.9\text{x}10^{-7} & -1.01\text{x}10^{-3} & 1 \end{bmatrix} \qquad 6.23$$

$$B_{ii} = \begin{bmatrix} 0 \\ 0 \\ 10^{-6} \end{bmatrix} \qquad 6.24$$

$$C_{ii} = [1 \quad 0 \quad 0] \qquad 6.25$$

For the ith subsystem the desired transfer function which takes into account the fixed zero is

$$\frac{y_i}{r_i} = \frac{(s+.01)10^6}{(s+.01)(s^2+2s+2)} \qquad 6.26$$

The set of feedback coefficients θ_1 and the gain g_{11} must be found to realize the above subsystem response. This task was accomplished by using the computer program of Melsa (1967), a program which uses the phase variable transformation discussed in Chapter 5. The results are shown in the following compensation matrices which apply to the system expressed in terms of the state variables corresponding to the matrices of Equations 6.23 - 6.25.

$$F = \begin{bmatrix} 0 & 2x10^{-3} & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2x10^{-3} & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2x10^{-3} & -2 \end{bmatrix} \qquad 6.27$$

$$G = \begin{bmatrix} 10^6 & 0 & 0 \\ 0 & 10^6 & 0 \\ 0 & 0 & 10^6 \end{bmatrix} \qquad 6.28$$

In terms of the state variables for the original augmented plant, the compensation matrices F and G are obtained from Equations 5.17 and 5.18, repeated as

$$F = F^* + B^{*-1}FQ \qquad 6.29$$

$$G = B^{*-1}G \qquad 6.30$$

All of the quantities on the right-hand sides of Equations 6.29 and 6.30 have already been calculated (B^{*-1} is just G^*). Performing the required matrix multiplications and addition yields

$$F = \begin{bmatrix} -2.99\times10^{-6} & 9.9\times10^{-4} & -10^{-6} & 10^{-3} & -10^{-6} & 10^{-3} & 0 & -1 & -1 \\ -10^{-6} & 10^{-3} & -2.99\times10^{-6} & 9.9\times10^{-4} & -10^{-6} & 10^{-3} & -1 & 0 & -1 \\ -10^{-6} & 10^{-3} & -10^{-6} & 10^{-3} & 2.99\times10^{-6} & 9.9\times10^{-4} & -1 & -1 & 0 \end{bmatrix} \qquad 6.31$$

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad 6.32$$

With these compensation matrices the multivariable system is decoupled into 3 noninteracting subsystems, each of which has the transfer function of Equation 6.26.

As a practical matter it is noted that, because a simple reactor model is being used, all the states can be measured. As a result the design, though complicated, can be physically implemented.

This example has been worked for the case where the following subsystem transfer function is desired

$$\frac{y_1}{r_1} = \frac{10^6(s+.01)}{s^2 + 20s + 200} \qquad 6.33$$

In this instance the form of the response is the same as in the previous case, but the system bandwidth has been increased by a factor of 10. The corresponding F and G matrices are

$$F = \begin{bmatrix} -1.83\text{x}10^{-4} & 1.899\text{x}10^{-2} & -1.9\text{x}10^{-5} & 10^{-3} & -1.9\text{x}10^{-5} & 10^{-3} & -18 & -1 & -1 \\ -1.9\text{x}10^{-5} & 10^{-3} & -1.83\text{x}10^{-4} & 1.899\text{x}10^{-2} & -1.9\text{x}10^{-5} & 10^{-3} & -1 & -18 & -1 \\ -1.9\text{x}10^{-5} & 10^{-3} & -1.9\text{x}10^{-5} & 10^{-3} & -1.83\text{x}10^{-4} & 1.899\text{x}10^{-2} & -1 & -1 & -18 \end{bmatrix} \quad 6.34$$

$$G = I \quad 6.35$$

The first stage of the design process is now complete. Still needed before the process is finished are simulation studies to verify noninteraction and the subsystem responses and a sensitivity investigation. One would be especially interested in determining what effect changes in or the removal of some of the feedback coefficients has on the response. These studies are best carried out by those who are directly responsible for the design of the physical system.

Summary

The example of this chapter is taken from a recently published textbook on reactor dynamics and control (Weaver, 1968). It is a problem that has some engineering significance. Although the design has not been carried to completion (physical implementation), the results which are given indicate that the design techniques presented in this dissertation should be considered when designing multivariable systems.

CHAPTER 7

CONCLUSIONS

In this chapter all previous results are summarized, and suggestions are given for further research.

Summary

The study of design techniques for multivariable systems is the topic of this report. Both conventional, frequency-domain techniques and modern, combined frequency-domain, time-domain procedures are considered. Noninteraction is taken as one of the two basic design requirements; the other is that specified subsystem transfer functions be achieved. Conventional methods are quickly shown to have the disadvantage of complexity--both in carrying out the design calculations and in the physical implementation of the compensation. There are, however, some problems for which the conventional methods yield satisfactory designs, and research continues in this area (Chen, 1968 a, b).

The bulk of the attention to design is given to the state variable feedback design of multivariable systems. After its introduction by Morgan in 1963, several authors studied the technique, with the most recent and complete treatment being given by Gilbert (1968). Gilbert's results make possible the identification of the fixed zeroes of the subsystems of the multivariable system and the number of subsystem poles which are controlled by state variable feedback. By treating each subsystem individually, the designer can apply some of the previously developed knowledge of state variable feedback design of single-input, single-output systems.

A topic which has not been previously studied is the addition of dynamics to the multivariable system before state variable feedback is applied, for the purpose of improving the system response. Three methods are proposed and analyzed in Chapter 4 for adding dynamics. The first method, Method A, requires that the compensation be placed in the control-input channels of the multivariable plant and that all the states of the augmented system be fed back. This method is the preferred one when it works, because of its simplicity. However, its use could lead to loss of coupling or loss of zeroes. An alternate approach, Method B, is shown to have serious practical limitations.

Method C applies in every case in which the multivariable plant can be decoupled. According to Theorem 4.3, the use of Method C makes it possible to apply the same techniques for the multivariable plant as are applied in single-input, single-output design.

In particular, zeroes and poles can be added with the assurance that decoupling is not lost, the added zeroes and plant zeroes appear unchanged in the proper subsystem transfer functions, and both the added poles and the plant poles can be arbitrarily positioned by state variable feedback.

Chapter 5 is intended to serve as a clear outline of what must be done to apply the state variable feedback design techniques of Chapters 3 and 4. The presentation is oriented toward digital computer usage because practical multivariable design problems are frequently of high order and require tedious calculations that are most accurately performed by the computer. In the case where dynamics are added to the decoupled multivariable plant, a short-cut is given to cut down on computer time.

The practical application of Chapter 6 shows that the design techniques of the previous chapter do indeed have value in control system design.

Further Research

Although the design techniques presented here are sufficiently complete to be used in practical design problems, there are several topics which merit further research. Among these are

1. The decoupling of multivariable systems for which B^* is singular.

2. Further study of Method A.

3. Further study of the considerations involved in the initial decoupling step of Method C.

4. The application of the techniques of gain-insensitivity to multivariable systems.

5. The relationship of the design methods to those involving integral performance indices.

6. Multivariable system design by state variable feedback where noninteraction is not required.

7. State Estimation in multivariable systems.

Each of these topics is now discussed briefly.

For topic 1, Gilbert (1968) mentions that as long as the plant matrix P(s) is nonsingular, dynamics can be added to the multivariable system so that the resulting augmented system can be decoupled by state variable feedback. The practical implications of this procedure have not been reported. In particular, one needs to know how to find the added compensation and whether it is physically realizable. In the present study, dynamics are added to make it possible to meet the design specifications. In problems which cannot be decoupled by state variable feedback unless dynamics are added, it would be desirable to be able to choose the dynamics which permitted decoupling and also contributed to a good design.

For topic 2, more work is needed to find out when series compensation causes loss of coupling and loss of zeroes. Theorem 4.1 provides answers for the simplest form of series compensation, but other situations have yet to be considered.

For topic 3, the best way of decoupling the plant before adding decoupled compensation is not known, nor is it even known what criteria for defining the best way should be used. Perhaps sensitivity theory could be of value here.

Topic 4 appears to be related to the previous topic because, according to Herring (1967), systems are made gain-insensitive by conditioning the plant before the final application of state variable feedback. Herring's results apply to single-input, single-output systems; the multivariable case has yet to be studied.

In topic 5 performance indices are mentioned as an alternate means for specifying the desired system response. In fact, the idea of using state variable feedback originated in connection with design for minimizing a particular integral performance index (Schultz and Melsa, 1967). This dissertation uses desired transfer matrices as the performance specification. Relations between the designs resulting from the two different types of specifications are known for single-input, single-output systems, but not for multivariable systems. Here the constraint of noninteraction should prove useful.

For topic 6, study of the design situation in which noninteraction is not a requirement needs to be conducted. In an aircraft, for example, the plane rolls when making turns, so that changes in yaw are accompanied by changes in roll, and these changes are tolerated. One would like to be able to choose a specific, non-zero transfer function between r_i and y_j ($i \neq j$), and realize it by state variable feedback. At present, no results are available in this area.

The final topic is concerned with the very important practical problem of estimating state variables which cannot be measured directly. Due to the large number of state variables in a typical multivariable system, the need for estimating states is great. For the case where no noise is present the work of Luenberger (1964, 1966, 1967) and others (Singer, 1968) should be investigated as a basis for developing the theory for the case where decoupled multivariable systems are being designed.

With the increasing complexity of the design problems being considered by control engineers, the continued development of multivariable system theory seems assured. State variable feedback design should share in this development.

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