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FINAL REPORT

INVESTIGATION TO DEFINE THE PROPAGATION CHARACTERISTICS OF A FINITE AMPLITUDE ACOUSTIC PRESSURE WAVE

October 11, 1966

(Contract No. NAS8-11441)

Prepared by

A. C. Peter
J. W. Cottrell

Approved by

F. C. Hung
Program Manager

NORTH AMERICAN AVIATION, INC.
SPACE and INFORMATION SYSTEMS DIVISION

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FOREWORD

This report presents work performed by North American Aviation, Inc., Space and Information Systems Division in fulfillment of Contract NAS8-11441, "*Investigation to Define the Propagation Characteristics of a Finite Amplitude Pressure Wave.*" This study has been administered by the Unsteady Aerodynamics Branch, National Aeronautics and Space Administration, George C. Marshall Space Flight Center, Huntsville, Alabama.

Dr. F. C. Hung acted as Program Manager for North American Aviation, Inc. Mr. A. C. Peter and J. W. Cottrell were the investigators on the project.

TECHNICAL REPORT INDEX/ABSTRACT

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AN INVESTIGATION TO DEFINE THE PROPAGATION CHARACTERISTICS OF A FINITE AMPLITUDE PRESSURE WAVE	

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ABSTRACT

The contribution of high entropy production regions to the generation characteristics of a finite amplitude pressure wave is considered. The analysis results in a non-linear convected wave equation in terms of entropy and a thermodynamic J-function. It is shown that the equation reduces to the classical form when isentropy is assumed. An acoustical model for non-isentropic noise radiation is derived and comparisons with experiment made.

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SUMMARY

A theoretical a. 'lysis of the propagation characteristics of a finite amplitude pressure wave is presented in the following sections.

The analysis attempts to study the contribution of entropy-producing regions to the mechanism of aerodynamic noise generation. It results in a non-linear convective wave equation in terms of entropy and a thermodynamic J function. A direct analogy between the derived governing equation and those used in classical literature is obtained. An idelization of the processes considered permits the uncoupling of the equations of motion with a consequent construction of an acoustic analogy treating shock wave emission of finite amplitude acoustic waves.

An engineering approach of this analogy is reflected in the concept of an extended plug nozzle whose function it is to facilitate aerodynamic noise attenuation by modifying the entropy-producing regions.

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I. BASIC THEORY

1. INTRODUCTION

This theoretical analysis of the propagation characteristics of a finite amplitude pressure wave has been concerned with the contribution of entropy-producing regions to the mechanism of aerodynamic noise generation in the exhaust of a supersonic nozzle. It is hoped that the present approach will tend to supplement some aspects of the contemporary investigations on the subject of aerodynamic noise generation.

A first attempt to investigate the feasibility of aerodynamic noise attenuation by modifying the geometry of entropy-producing regions (in this particular case, shock waves) resulted in the concept of an extended plug nozzle (Peter and Kamo, 1963). The present analysis attempts to put these early ideas on a more rigorous basis by considering non-isentropic processes coupled with the equations of motion of fluid dynamics.

The analysis results in a nonlinear convective wave equation in terms of entropy and a thermodynamic J-function. A non-isentropic propagation speed α is defined and expressed in terms of the entropy function. This propagation speed reduces to the isentropic speed of sound as the entropy production in the process tends to zero. A proper choice of the thermodynamic J-function allows a direct analogy between the derived governing equation and those used in classical literature (Lighthill, 1952, Phillips, 1960, Ribner, 1954-62) when the entropy terms are assumed small. The equations of motion are simplified and uncoupled by idealizing the process to consist of straight lines in the J-S plane, each referring to a different region in space. This idealization implies a far field noise analysis since near field effects admit of mixed J-S processes taking place.

It is also shown that, within the framework of this analysis, the pressure, density and temperature functions depend on the history of the process rather than the instantaneous states of the fluid particles. It appears, also, that, whereas all functional values of the thermodynamic variables (e.g., p, ρ, T, etc.) are continuous on the boundary of a non-isentropic region (adjoining an isentropic region) their derivatives are not. The non-isentropic propagation speed α differs from its isentropic counterpart, at the end of the non-isentropic process, by a factor which is close, but not equal to, unity. This factor depends directly upon the amount of entropy produced during the process and tends to unity as the entropy production tends to zero. Moreover, it also appears that, even though the entropy variation for weak shocks is of a third order of magnitude, the effects of entropy production cannot be disregarded since, in the governing differential equation, entropy effects are of a first-order magnitude due to an additional factor appearing in the analysis.

2. THE NON-ISENTROPIC CONVECTED WAVE EQUATION

In the proceeding analysis it is assumed that a perfect gas satisfying the equation of state

$$p = \rho R T \tag{2.1}$$

forms a region of entropy production. In a fixed Cartesian Reference Frame the following equations of motion describe the flow characteristics

$$\frac{\partial u_i}{\partial x_i} = -\frac{D}{Dt}\left[\ln\left(\frac{\rho}{\rho_0}\right)\right] \tag{2.2}$$

$$\frac{Du_i}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x_j} = \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\tau_{ij}\right) + F_i \tag{2.3}$$

$$\rho T\frac{DS}{Dt} = \phi + \frac{\partial}{\partial x_j}\left[k\frac{\partial T}{\partial x_j}\right] + Q \tag{2.4}$$

The following notation is used:

ρ = density of the fluid

p = pressure of the fluid

T = temperature of the fluid

u_i = velocity vector

S = entropy

μ = viscosity coefficient

k = heat transfer coefficient

F_i = body forces vector

Q = heat sources

R = gas constant

ϕ = dissipation function

τ_{ij} = viscous part of the stress tensor

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_j\frac{\partial}{\partial x_j}$$

In terms of the above notation the following subsidiary relations are needed:

$$\tau_{ij} = 2\mu\, \varepsilon_{ij} - \frac{2}{3}\mu\, \varepsilon_{kk}\, \delta_{ij}$$

$$\phi = 4\mu\left(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2\right) + \frac{2}{3}\mu\left\{\left(\varepsilon_{11} - \varepsilon_{22}\right)^2 + \left(\varepsilon_{11} - \varepsilon_{33}\right)^2 + \left(\varepsilon_{22} - \varepsilon_{33}\right)^2\right\}$$

with

$$\varepsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$$

δ_{ij} = Kronecker delta

Since the analysis of entropy-producing regions implies a strong coupling between the mechanical conditions (momentum equation) and thermodynamic conditions (energy equation), it will be useful to put the momentum equation in terms of thermodynamic variables. To do so we first take the divergence of the momentum Equation (2.3) to obtain

$$\frac{D}{Dt}\frac{\partial u_i}{\partial x_i} + \frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\frac{\partial p}{\partial x_i}\right] = \frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\tau_{ij}\right)\right] - \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} + \frac{\partial F_i}{\partial x_i} \tag{2.5}$$

Let J be a thermodynamic variable (to be defined at a later stage) and consider the pressure to be a function of density and the variable J. Under these conditions the pressure differential may be written in the form

$$dp = \left(\frac{\partial p}{\partial \rho}\right)_J d\rho + \left(\frac{\partial p}{\partial J}\right)_\rho dJ \tag{2.6}$$

We next define a quantity α having the dimensions of velocity and given by

$$\alpha^2 = \left(\frac{\partial p}{\partial \rho}\right)_J \qquad (2.7)$$

Under these conditions the pressure gradient term in Equation (2.5) may be written as

$$\frac{1}{\rho}\frac{\partial p}{\partial x_i} = \alpha^2 \frac{\partial}{\partial x_i}\left[\ln\left(\frac{\rho}{\rho_0}\right)\right] + \frac{1}{\rho}\left(\frac{\partial p}{\partial J}\right)_\rho \frac{\partial J}{\partial x_i} \qquad (2.8)$$

In view of relations (2.2) and (2.8), Equation (2.5) takes on the following form

$$-\frac{D^2}{Dt^2}\left[\ln\left(\frac{\rho}{\rho_0}\right)\right] + \frac{\partial}{\partial x_i}\left[\alpha^2 \frac{\partial}{\partial x_i}\ln\left(\frac{\rho}{\rho_0}\right)\right] + \frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\left(\frac{\partial p}{\partial J}\right)_\rho \frac{\partial J}{\partial x_i}\right]$$

$$= \frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\frac{\partial}{\partial x_j}(\tau_{ij}) + F_i\right] - \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} \qquad (2.9)$$

Now let us consider the entropy function. To simplify the derivation, a calorically perfect gas will be assumed to yield

$$\ln\left(\frac{\rho}{\rho_0}\right) = \frac{1}{\gamma}\ln\left(\frac{p}{p_0}\right) - \frac{S}{c_p} \qquad (2.10)$$

where c_v and c_p are the specific heats of the gas and γ is their ratio.

A function G will now be defined in such a manner that

$$G = -c_v \ln\left(\frac{p}{p_0}\right) \qquad (2.11)$$

Under these conditions, Relation (2.10) becomes simply:

$$\ln\left(\frac{\rho}{\rho_0}\right) = -\frac{1}{c_p}(S + G) \tag{2.12}$$

In using Relation (2.12) in Equation (2.9) the thermodynamic variables S and J will be regarded as independent variables. As mentioned previously, the thermodynamic variable J will be defined later. Before using Relation (2.12) in the differential equation (2.9), it will be convenient to note the form of the derivatives of Equation (2.12). In accordance with the above hypothesis we have

$$d\left[\ln\left(\frac{\rho}{\rho_0}\right)\right] = -\frac{1}{c_p}\left[dS + dG\right]$$

$$dG = \left(\frac{\partial G}{\partial S}\right)_J dS + \left(\frac{\partial G}{\partial J}\right)_S dJ$$

Hence, taking the first and second differentials of Equation (2.12), the following relations are obtained

$$d\left[\ln\left(\frac{\rho}{\rho_0}\right)\right] = -\frac{1}{c_p}\left\{\left[1 + \left(\frac{\partial G}{\partial S}\right)_J\right]dS + \left(\frac{\partial G}{\partial J}\right)_S dJ\right\} \tag{2.13}$$

$$d^2\left[\ln\left(\frac{\rho}{\rho_0}\right)\right] = -\frac{1}{c_p}\left\{\left[1 + \left(\frac{\partial G}{\partial S}\right)_J\right]d^2S + d\left[1 + \left(\frac{\partial G}{\partial S}\right)_J\right]dS + \right.$$
$$\left. + d\left[\left(\frac{\partial G}{\partial J}\right)_S\right]dJ + \left(\frac{\partial G}{\partial J}\right)_S d^2J\right\} \tag{2.14}$$

Consider next the differential Equation (2.9) and write it in the following form:

$$- \frac{D^2}{Dt^2} \left[\ell n \left(\frac{\rho}{\rho_0} \right) \right] + \alpha^2 \frac{\partial^2}{\partial x_i \partial x_i} \left[\ell n \left(\frac{\rho}{\rho_0} \right) \right] + \frac{\partial \alpha^2}{\partial x_i} \frac{\partial}{\partial x_i} \left[\ell n \left(\frac{\rho}{\rho_0} \right) \right]$$

$$= \frac{\partial}{\partial x_i} \left[\frac{1}{\rho} \frac{\partial}{\partial x_j} (\tau_{ij}) + F_i \right] - \frac{\partial}{\partial x_i} \left[\frac{1}{\rho} \left(\frac{\partial p}{\partial J} \right)_\rho \frac{\partial J}{\partial x_i} \right] - \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} \tag{2.15}$$

From Equations (2.13) and (2.14) the last equation may be written as

$$\frac{1}{c_p} \Bigg\{ \left[1 + \left(\frac{\partial G}{\partial S} \right)_J \right] \frac{D^2 S}{Dt^2} + \frac{D}{Dt} \left[1 + \left(\frac{\partial G}{\partial S} \right)_J \right] \frac{DS}{Dt} + \frac{D}{Dt} \left[\left(\frac{\partial G}{\partial J} \right)_S \right] \frac{DJ}{Dt} +$$

$$+ \left(\frac{\partial G}{\partial J} \right)_S \frac{D^2 J}{Dt^2} - \alpha^2 \left[1 + \left(\frac{\partial G}{\partial S} \right)_J \right] \frac{\partial^2 S}{\partial x_i \partial x_i} - \alpha^2 \frac{\partial}{\partial x_i} \left[1 + \left(\frac{\partial G}{\partial S} \right)_J \right] \frac{\partial S}{\partial x_i}$$

$$- \alpha^2 \frac{\partial}{\partial x_i} \left[\left(\frac{\partial G}{\partial J} \right)_S \right] \frac{\partial J}{\partial x_i} - \alpha^2 \left(\frac{\partial G}{\partial J} \right)_S \frac{\partial^2 J}{\partial x_i \partial x_i} - \frac{\partial \alpha^2}{\partial x_i} \frac{\partial S}{\partial x_i}$$

$$- \frac{\partial \alpha^2}{\partial x_i} \left(\frac{\partial G}{\partial S} \right)_J \frac{\partial S}{\partial x_i} - \frac{\partial \alpha^2}{\partial x_i} \left(\frac{\partial G}{\partial J} \right)_S \frac{\partial J}{\partial x_i} \Bigg\} = \frac{\partial}{\partial x_i} \left[\frac{1}{\rho} \frac{\partial}{\partial x_j} (\tau_{ij}) + F_i \right]$$

$$- \frac{\partial}{\partial x_i} \left[\frac{1}{\rho} \left(\frac{\partial p}{\partial J} \right)_\rho \frac{\partial J}{\partial x_i} \right] - \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j}$$

Collecting terms in the above equation, the following relation is obtained:

$$
\begin{aligned}
&\left[1 + \left(\frac{\partial G}{\partial S}\right)_J\right] \left\{ \frac{D^2 S}{Dt^2} - \alpha^2 \frac{\partial^2 S}{\partial x_i \partial x_i} + \frac{D}{Dt} \left[\ln\left[1 + \left(\frac{\partial G}{\partial S}\right)_J\right]\right] \frac{DS}{Dt} \right. \\
&\quad \left. - \alpha^2 \frac{\partial}{\partial x_i} \left[\ln\left[1 + \left(\frac{\partial G}{\partial S}\right)_J\right]\right] \frac{\partial S}{\partial x_i} - \frac{\partial \alpha^2}{\partial x_i} \frac{\partial S}{\partial x_i} \right\} = \\
&\quad = - \left(\frac{\partial G}{\partial J}\right)_S \left\{ \frac{D^2 J}{Dt^2} - \alpha^2 \frac{\partial^2 J}{\partial x_i \partial x_i} + \frac{D}{Dt} \left[\ln\left(\frac{\partial G}{\partial J}\right)_S\right] \frac{DJ}{Dt} + \right. \\
&\quad \left. - \alpha^2 \frac{\partial}{\partial x_i} \left[\ln\left(\frac{\partial G}{\partial J}\right)_S\right] \frac{\partial J}{\partial x_i} - \frac{\partial \alpha^2}{\partial x_i} \frac{\partial J}{\partial x_i} \right\} + c_p \frac{\partial}{\partial x_i} \left[\frac{1}{\rho} \frac{\partial}{\partial x_j} (\tau_{ij}) + F_i\right] \\
&\quad - c_p \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} - c_p \frac{\partial}{\partial x_i} \left[\frac{1}{\rho}\left(\frac{\partial p}{\partial J}\right)_\rho \frac{\partial J}{\partial x_i}\right]
\end{aligned}
\tag{2.16}
$$

To simplify the notation we introduce two functions H and K given by:

$$
H = \left[1 + \left(\frac{\partial G}{\partial S}\right)_J\right] ; \qquad K = - \left(\frac{\partial G}{\partial J}\right)_S \tag{2.17}
$$

and write the last expression of Equation (2.16) in the form

$$
- c_p \frac{\partial}{\partial x_i}\left[\frac{1}{\rho} \left(\frac{\partial p}{\partial J}\right)_\rho \frac{\partial J}{\partial x_i}\right] = - \frac{c_p}{\rho} \left(\frac{\partial p}{\partial J}\right)_\rho \frac{\partial^2 J}{\partial x_i \partial x_i} - c_p \frac{\partial}{\partial x_i} \left[\frac{1}{\rho}\left(\frac{\partial p}{\partial J}\right)_\rho\right] \frac{\partial J}{\partial x_i} \tag{2.18}
$$

In view of Equations (2.17) and (2.18), Equation (2.16) becomes

$$\frac{D^2S}{Dt^2} - \alpha^2 \frac{\partial^2 S}{\partial x_i \partial x_i} + \frac{D}{Dt}(\ln H)\frac{DS}{Dt} - \alpha^2 \frac{\partial}{\partial x_i}(\ln H + \ln \alpha^2)\frac{\partial S}{\partial x_i} =$$

$$= \frac{K}{H}\left\{ \frac{D^2 J}{Dt^2} - \left[\alpha^2 + \frac{c_p}{K}\frac{1}{\rho}\left(\frac{\partial p}{\partial J}\right)_\rho\right]\frac{\partial^2 J}{\partial x_i \partial x_i} + \frac{D}{Dt}(\ln(-K))\frac{DJ}{Dt} + \right.$$

$$\left. - \left[\alpha^2 \frac{\partial}{\partial x_i}(\ln(-K) + \ln \alpha^2) + \frac{c_p}{K}\frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\left(\frac{\partial p}{\partial J}\right)_\rho\right]\right]\frac{\partial J}{\partial x_i}\right\}$$

$$+ \frac{c_p}{H}\left\{\frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\frac{\partial}{\partial x_j}(\tau_{ij}) + F_i\right] - \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}\right\}$$

Simplifying the above, the following differential equation is obtained:

$$\frac{D^2S}{Dt^2} - \alpha^2\nabla^2 S + \frac{D}{Dt}(\ln H)\frac{DS}{Dt} - \alpha^2 \frac{\partial}{\partial x_i}(\ln \alpha^2 H)\frac{\partial S}{\partial x_i} =$$

$$= \frac{K}{H}\left\{ \frac{D^2 J}{Dt^2} - \left[\alpha^2 + \frac{c_p}{K}\frac{1}{\rho}\left(\frac{\partial p}{\partial J}\right)_\rho\right]\nabla^2 J + \frac{D}{Dt}(\ln(-K))\frac{DJ}{Dt} + \right.$$

$$\left. - \left[\alpha^2 \frac{\partial}{\partial x_i}(\ln(-\alpha^2 K)) + \frac{c_p}{K}\frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\left(\frac{\partial p}{\partial J}\right)_\rho\right]\right]\frac{\partial J}{\partial x_i}\right\} +$$

$$+ \frac{c_p}{H}\left\{\frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\frac{\partial}{\partial x_j}(\tau_{ij}) + F_i\right] - \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}\right\} \tag{2.19}$$

Equation (2.19) represents a nonlinear convective wave equation in terms of the two thermodynamic variables S and J. It implies that, in a non-isentropic region, entropy changes are propagated convectively within the region. The source distribution is represented by the right-hand side of Equation (2.19) and is shown to depend upon the nature of the thermodynamic variable J and also upon the last two terms of the equation representing the effects of viscosity, body forces and velocity fluctuations. The shape of the characteristics is directly a function of the propagation speed

$$\alpha^2 = \left(\frac{\partial p}{\partial \rho}\right)_J \tag{2.20}$$

Formally, this propagation speed depends directly upon the entropy state of the fluid particles and also upon the process J = constant. Through these two thermodynamic variables it is a function of the space variables x_i and the time t. In addition, the form of the equation indicates dissipative processes due to the appearance of first derivatives both in space and time and also due to the highly nonlinear character of the processes involved. This is reflected by the non-linearity of the equation since all coefficients on the left of the above equation are functions of entropy, i.e., of the dependent variable (except the coefficient of the second time derivative which has been incorporated in the forcing function on the right-hand side by suitable division).

3. THE THERMODYNAMIC VARIABLE J

Equation (2.19) governing the flow in non-isentropic regions admits two independent thermodynamic variables, namely, the entropy S and the variable J which, so far, is of an arbitrary character. The remaining variables, whose mathematical forms are sought in the present investigation, e.g., the pressure p, the density ρ, etc., are functions of the space coordinates $x_i (i = 1,2,3)$ and the time t, through these two independent variables. To obtain an analytically meaningful result from the previous derivation, the thermodynamic variable J must be uniquely determined. Now, it appears that the unique determination of this variable cannot be achieved from purely mathematical or thermodynamic considerations, since within the framework of these two sciences the variable J need not be specified to obtain the equation represented by Relation (2.19). This is amply demonstrated by the process of the previous analysis resulting in the actual derivation of the governing equation without recourse to any hypothesis concerning the character of the J function.

To specify the form of the thermodynamic J-function it is necessary to scrutinize some physical aspects of the present derivation. Thus, if $\eta = \eta(S,J)$ is a given thermodynamic variable, its differential may be written in the form:

$$d\eta = \left(\frac{\partial\eta}{\partial J}\right)_S dJ + \left(\frac{\partial\eta}{\partial S}\right)_J dS \tag{3.1}$$

The coefficient of dJ in Equation (3.1) represents an isentropic process with J varying, whereas that of dS reflects entropy variations during a J = constant process. It is apparent that in the case of an entropy-producing region the contribution of the J = constant process must predominate.

The choice of the dimensional units for the J function is completely arbitrary insofar as keeping the process J = constant an invariant. For such a dimensional change may be effectively accomplished by adjoining to the differential dJ arbitrary factors, without affecting the J = constant process. In the present case when non-isentropic pressure variations are considered, the simplest form will be obtained by ascribing to the variable J the dimensions of pressure.

Dimensional considerations indicate that the function:

$$dJ = \beta^2 \left[d\rho - \frac{\rho dp}{\gamma p} \right] - dp \tag{3.2}$$

where β is a constant parameter having the dimensions of velocity fulfills this requirement.* For the present, the form of the J function as given in Equation (3.2) will be regarded as a definition.

It is now necessary to obtain some useful thermodynamic relations satisfied by the thermodynamic variable J. Since the definition of entropy of a perfect gas yields the equation:

$$\frac{dS}{c_v} = \frac{dp}{p} - \gamma \frac{d\rho}{\rho}$$

the combination of Equation (3.2) with the above relation yields:

$$dJ = -\frac{\beta^2}{c_p} \rho \, dS - dp; \qquad dJ = -\left[\frac{\beta^2}{c_p} \rho + \frac{p}{c_v} \right] dS - \frac{\gamma p}{\rho} d\rho \tag{3.3}$$

Hence, we obtain the three relations:

$$\left(\frac{\partial J}{\partial p}\right)_S = -1; \qquad \left(\frac{\partial J}{\partial S}\right)_p = -\frac{\beta^2}{c_p}\rho; \qquad \left(\frac{\partial p}{\partial S}\right)_J = -\frac{\beta^2}{c_p}\rho \tag{3.4}$$

---

* See Concluding Remarks

From the last two equations it follows that:

$$\left(\frac{\partial J}{\partial S}\right)_p = \left(\frac{\partial p}{\partial S}\right)_J \tag{3.5}$$

Likewise, from Relation (3.2) one gets

$$\left(\frac{\partial \rho}{\partial p}\right)_J = \frac{\rho}{\gamma p} + \frac{1}{\beta^2} \tag{3.6}$$

But, from the definition of the non-isentropic propagation speed α (Equation (2.7)) given by

$$\alpha^2 = \left(\frac{\partial p}{\partial \rho}\right)_J$$

it follows, using Equation (3.6) and routine thermodynamic relations, that

$$\frac{1}{\alpha^2} = \frac{\rho}{\gamma p} + \frac{1}{\beta^2} \tag{3.7}$$

for a J = constant process.

Equation (3.7) indicates that the parameter β^2 tends to infinity as the process under consideration approaches an isentropic process in the limit. This property will be confirmed in subsequent derivations by actually defining the parameter β in terms of boundary values.

Again using the definition of the J function in Equation (3.2) the following relation holds

$$\frac{d\rho}{dJ} = \frac{1}{\beta^2} + \left[\frac{\rho}{\gamma p} + \frac{1}{\beta^2}\right]\frac{dp}{dJ}$$

and, by using Equation (3.7), this becomes

$$\frac{d\rho}{dJ} = \frac{1}{\beta^2} + \frac{1}{\alpha^2}\frac{dp}{dJ} \tag{3.8}$$

Hence, using the first of Equation (3.4) it follows that

$$\left(\frac{\partial \rho}{\partial J}\right)_S = \frac{1}{\beta^2} - \frac{1}{\alpha^2}$$

or, using Relation (3.7)

$$\left(\frac{\partial \rho}{\partial J}\right)_S = - \frac{\rho}{\gamma p} \tag{3.9}$$

In a similar manner one deduces from Equation (3.8) that

$$\left(\frac{\partial p}{\partial J}\right)_\rho = - \frac{\alpha^2}{\beta^2} \tag{3.10}$$

Consider the function G and its differential as defined in Equation (2.11):

$$G = - c_v \ln\left(\frac{p}{p_0}\right) ; \qquad dG = \left(\frac{\partial G}{\partial S}\right)_J dS + \left(\frac{\partial G}{\partial J}\right)_S dJ$$

Combining the two it follows that

$$dG = - \frac{c_v}{p} \left(\frac{\partial p}{\partial S}\right)_J dS - \frac{c_v}{p} \left(\frac{\partial p}{\partial J}\right)_S dJ \tag{3.11}$$

From the results obtained in Equation (3.4) this may be written

$$dG = \frac{\beta^2 \rho}{\gamma p} dS + \frac{c_v}{p} dJ \tag{3.12}$$

This last relation implies that

$$\left(\frac{\partial G}{\partial S}\right)_J = \frac{\beta^2 \rho}{\gamma p} ; \qquad \left(\frac{\partial G}{\partial J}\right)_S = \frac{c_v}{p}$$

Consider next the second of Equations (3.3) given by

$$dJ = -\left[\frac{\rho}{\gamma p} + \frac{1}{\beta^2}\right]\frac{\beta^2 p}{c_v}\, dS - \frac{\gamma p}{\rho}\, d\rho$$

Using Equation (3.7) it follows that

$$dJ = -\frac{\beta^2}{\alpha^2}\frac{p}{c_v}\, dS - \frac{\gamma p}{\rho}\, d\rho \tag{3.14}$$

From this result we obtain the relations

$$\left(\frac{\partial J}{\partial \rho}\right)_S = -\frac{\gamma p}{\rho}\, ; \quad \left(\frac{\partial J}{\partial S}\right)_\rho = -\frac{\beta^2}{\alpha^2}\frac{p}{c_v}\, ; \quad \left(\frac{\partial \rho}{\partial S}\right)_J = -\frac{\beta^2}{\alpha^2}\frac{\rho}{c_p} \tag{3.15}$$

It will also be useful to consider the pressure as a function of density and the J-variable and subsequently let the density be a function of J and S [this procedure was used to obtain the governing differential Equation (2.19)]. A routine comparison of coefficients by invoking linear independence yields the relations:

$$\left(\frac{\partial p}{\partial S}\right)_J = \left(\frac{\partial p}{\partial \rho}\right)_J \left(\frac{\partial \rho}{\partial S}\right)_J \tag{3.16a}$$

or

$$\left(\frac{\partial p}{\partial S}\right)_J = \alpha^2 \left(\frac{\partial \rho}{\partial S}\right)_J \tag{3.16b}$$

In a similar manner

$$\left(\frac{\partial p}{\partial J}\right)_S = \left(\frac{\partial p}{\partial \rho}\right)_J \left(\frac{\partial \rho}{\partial J}\right)_S + \left(\frac{\partial p}{\partial J}\right)_\rho \tag{3.17a}$$

or

$$\left(\frac{\partial p}{\partial J}\right)_S = \alpha^2 \left(\frac{\partial \rho}{\partial J}\right)_S + \left(\frac{\partial p}{\partial J}\right)_\rho \tag{3.17b}$$

and these may be easily verified by the use of Table I.

If we now consider the J-function definition and the combined first and second law equation of thermodynamics, the following relation is obtained:

$$\frac{dJ}{\rho} = \left[T - \frac{\beta^2}{c_p}\right] dS - c_p dT \tag{3.18}$$

Equation (3.18) results in the following relations:

$$\left(\frac{\partial J}{\partial T}\right)_S = - c_p \rho \quad \left(\frac{\partial T}{\partial S}\right)_J = \frac{1}{c_p}\left[T - \frac{\beta^2}{c_p}\right] ; \quad \left(\frac{\partial J}{\partial S}\right)_T = \rho\left[T - \frac{\beta^2}{c_p}\right] \tag{3.19}$$

Some of the above relations which may be useful for subsequent derivations are collected in Table I.

TABLE 1

TABLE OF RELATIONS INVOLVING THE THERMODYNAMIC J-FUNCTION

PROCESSES

S = Constant	J = Constant	ρ = Constant	p = Constant	T = Constant
$\left(\frac{\partial J}{\partial p}\right)_S = -1$	$\left(\frac{\partial p}{\partial S}\right)_J = -\frac{\beta^2}{c_p}\rho$	$\left(\frac{\partial p}{\partial J}\right)_\rho = -\frac{\alpha^2}{\beta^2}$	$\left(\frac{\partial J}{\partial S}\right)_p = -\frac{\beta^2}{c_p}\rho$	$\left(\frac{\partial J}{\partial S}\right)_T = \rho\left(T - \frac{\beta^2}{c_p}\right)$
$\left(\frac{\partial \rho}{\partial J}\right)_S = \frac{1}{\beta^2} - \frac{1}{\alpha^2}$	$\left(\frac{\partial \rho}{\partial p}\right)_J = \frac{1}{\alpha^2}$	$\left(\frac{\partial J}{\partial S}\right)_\rho = -\frac{\beta^2}{c_\nu}\frac{p}{\alpha^2}$	$\left(\frac{\partial \rho}{\partial J}\right)_p = \frac{1}{\beta^2}$	
$\left(\frac{\partial G}{\partial J}\right)_S = \frac{c_\nu}{p}$	$\left(\frac{\partial G}{\partial S}\right)_J = \frac{\beta^2\rho}{\gamma p}$			
$\left(\frac{\partial p}{\partial J}\right)_S = \alpha^2\left(\frac{\partial \rho}{\partial J}\right)_S + \left(\frac{\partial p}{\partial J}\right)_\rho$	$\left(\frac{\partial \rho}{\partial S}\right)_J = -\frac{\beta^2}{c_p}\frac{\rho}{\alpha^2}$			
$\left(\frac{\partial J}{\partial T}\right)_S = -c_p\rho$	$\left(\frac{\partial p}{\partial S}\right)_J = \alpha^2\left(\frac{\partial \rho}{\partial S}\right)_J$			
$\left(\frac{\partial J}{\partial \rho}\right)_S = -\frac{\gamma p}{\rho}$	$\left(\frac{\partial T}{\partial S}\right)_J = \frac{1}{c_p}\left(T - \frac{\beta^2}{c_p}\right)$			

4. COMPARISON WITH EXISTING THEORIES

The derivation of the governing equation in terms of two thermodynamic variables S and J and the subsequent definition of the J-function as presented in Sections 2 and 3 respectively, makes it possible to compare the derived equation with those of existing theories and show their equivalence.

Let us consider Equation (2.19) which was shown to be

$$\frac{D^2 S}{Dt^2} - \alpha^2 \nabla^2 S + \frac{D}{Dt}(\ln H)\frac{DS}{Dt} - \alpha^2 \frac{\partial}{\partial x_i}\left(\ln(\alpha^2 H)\right)\frac{\partial S}{\partial x_i}$$

$$= \frac{K}{H}\left\{\frac{D^2 J}{Dt^2} - \left[\alpha^2 + \frac{c_p}{K}\frac{1}{\rho}\left(\frac{\partial p}{\partial J}\right)_\rho\right]\nabla^2 J + \frac{D}{Dt}\left(\ln(-K)\right)\frac{DJ}{Dt} + \right.$$

$$\left. - \left[\alpha^2 \frac{\partial}{\partial x_i}\left(\ln(-\alpha^2 K)\right) + \frac{c_p}{K}\frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\left(\frac{\partial p}{\partial J}\right)_\rho\right]\right]\frac{\partial J}{\partial x_i}\right\} +$$

$$+ \frac{c_p}{H}\left\{\frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\tau_{ij}\right) + F_i\right] - \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}\right\} \tag{4.1}$$

The functions H and K were defined in Equation (2.17) and were given by:

$$H = \left(1 + \left(\frac{\partial G}{\partial S}\right)_J\right) \qquad K = -\left(\frac{\partial G}{\partial J}\right)_S \tag{4.2}$$

However, from the relations involving the J-function in Table 1 we infer that

$$H = \beta^2 \left(\frac{1}{\beta^2} + \frac{\rho}{\gamma p} \right); \qquad K = - \frac{c_v}{p} \tag{4.3}$$

Moreover, from Equation (3.8) these may be written:

$$H = \frac{\beta^2}{\alpha^2}; \qquad K = - \frac{c_v}{p} \tag{4.4}$$

Using these values of H and K in Equation (4.1), the following form is obtained:

$$\frac{D^2 S}{Dt^2} - \alpha^2 \nabla^2 S - \frac{D}{Dt} (\ln \alpha^2) \frac{DS}{Dt} =$$

$$= - \frac{c_v}{\beta^2} \frac{\alpha^2}{p} \left\{ \frac{D^2 J}{Dt^2} - \left[1 + \frac{\gamma p}{\rho \beta^2} \right] \alpha^2 \nabla^2 J - \frac{D}{Dt} (\ln p) \frac{DJ}{Dt} + \right.$$

$$\left. - \left[\alpha^2 \frac{\partial}{\partial x_i} \left[\ln \left(\frac{\alpha^2}{p} \right) + \frac{\gamma p}{\beta^2} \frac{\partial}{\partial x_i} \left(\frac{\alpha^2}{\rho} \right) \right] \right] \frac{\partial J}{\partial x_i} \right\} +$$

$$+ \frac{c_p}{\beta^2} \alpha^2 \left\{ \frac{\partial}{\partial x_i} \left[\frac{1}{\rho} \frac{\partial}{\partial x_j} \left(\tau_{ij} + F_i \right) \right] - \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} \right\} \tag{4.5}$$

We can now divide this Equation by α^2 noting that:

$$\frac{1}{\alpha^2}\frac{D^2S}{Dt^2} - \frac{1}{\alpha^2}\frac{D}{Dt}(\ln\alpha^2)\frac{DS}{Dt} = \frac{D}{Dt}\left[\frac{1}{\alpha^2}\frac{DS}{Dt}\right]$$

$$\frac{1}{p}\frac{D^2J}{Dt^2} - \frac{1}{p}\frac{D}{Dt}(\ln p)\frac{DJ}{Dt} = \frac{D}{Dt}\left(\frac{1}{p}\frac{DJ}{Dt}\right)$$

$$\alpha^2\frac{\partial}{\partial x_i}\left[\ln\frac{\alpha^2}{p}\right] = p\frac{\partial}{\partial x_i}\left(\frac{\alpha^2}{p}\right); \text{ and } \frac{\gamma p}{\rho\beta^2} = \left(\frac{\partial S}{\partial G}\right)_J \tag{4.6}$$

Under these conditions, Equation (4.5) becomes

$$\frac{D}{Dt}\left[\frac{1}{\alpha^2}\frac{DS}{Dt}\right] - \nabla^2 S = -\frac{c_v}{\beta^2}\left\{\frac{D}{Dt}\left(\frac{1}{p}\frac{DJ}{Dt}\right) - \left[1+\left(\frac{\partial S}{\partial G}\right)_J\right]\frac{\alpha^2}{p}\nabla^2 J + \right.$$

$$\left. - \left[\frac{\partial}{\partial x_i}\left(\frac{\alpha^2}{p}\right) + \frac{\gamma}{\beta^2}\frac{\partial}{\partial x_i}\left(\frac{\alpha^2}{\rho}\right)\right]\frac{\partial J}{\partial x_i}\right\} +$$

$$+\frac{c_p}{\beta^2}\left\{\frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\tau_{ij}\right) + F_i\right] - \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right\} \tag{4.7}$$

Finally, the coefficient of the gradient of the J-function may be written:

$$\frac{\partial}{\partial x_i}\left\{\left(\frac{\alpha^2}{p}\right) + \frac{\gamma}{\beta^2}\frac{\partial}{\partial x_i}\left(\frac{\alpha^2}{\rho}\right)\right\} = \frac{\partial}{\partial x_i}\left\{\left[1+\left(\frac{\partial S}{\partial G}\right)_J\right]\frac{\alpha^2}{p}\right\} \tag{4.8}$$

Hence, Equation (4.7) becomes:

$$\frac{\mathcal{D}}{\mathcal{D}t}\left[\frac{1}{\alpha^2}\frac{\mathcal{D}S}{\mathcal{D}t}\right] - \nabla^2 S = -\frac{c_v}{\beta^2}\left\{\frac{\mathcal{D}}{\mathcal{D}t}\left(\frac{1}{p}\frac{\mathcal{D}J}{\mathcal{D}t}\right) - \frac{\partial}{\partial x_i}\left[\left[1+\left(\frac{\partial S}{\partial G}\right)_J\right]\frac{\alpha^2}{p}\frac{\partial J}{\partial x_i}\right]\right\} +$$

$$+\frac{c_p}{\beta^2}\left\{\frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\tau_{ij}\right)+F_i\right] - \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right\} \tag{4.9}$$

Equation (4.9) represents the final form of Equation (2.19), using the thermodynamic J function as defined in Equation (3.2). It may be looked upon as a convected nonlinear wave equation in a non-isentropic region in which the forcing function is represented by the right-hand side of the equation.

Alternately, in a formal manner, one could look upon Equation (4.9) as a convected wave equation in a region in which the J function is propagated, its forcing function being given by entropy generation, viscous effects, body forces and velocity fluctuation. In this case it is convenient to write Equation (4.9) in the alternate form:

$$-\frac{\mathcal{D}}{\mathcal{D}t}\left(\frac{1}{p}\frac{\mathcal{D}J}{\mathcal{D}t}\right) + \frac{\partial}{\partial x_i}\left\{\left[1+\left(\frac{\partial S}{\partial G}\right)_J\right]\frac{\alpha^2}{p}\frac{\partial J}{\partial x_i}\right\} =$$

$$= \frac{\beta^2}{c_v}\left\{\frac{\mathcal{D}}{\mathcal{D}t}\left[\frac{1}{\alpha^2}\frac{\mathcal{D}S}{\mathcal{D}t}\right] - \nabla^2 S\right\} - \gamma\left\{\frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\tau_{ij}\right)+F_i\right] - \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right\} \tag{4.10}$$

The last form of the equation is useful when a non-isentropic region is imbedded within a larger region of quiescent fluid. This is obviously the case when a large amplitude pressure wave is propagated into a quiescent space by highly disturbed non-isentropic flow including moderate shock waves. In this case the right-hand side of Equation (4.10) represents the non-isentropic forcing function.

In the case when the J-wave propagated by Equation (4.10) is generated by isentropic processes, the entropy derivatives vanish from the equation. Also, the J-function varies as the negative pressure function since it was shown that*:

$$\left(\frac{\partial J}{\partial p}\right)_S = -1 \tag{4.11}$$

It will also be shown that, for constant entropy, the propagation speed α^2 becomes the speed of sound a^2 in the limit.

Under these assumptions, Equation (4.10) takes the following form:

$$\frac{D^2}{Dt^2}\left[\ell n\left(\frac{p}{p_0}\right)\right] - \frac{\partial}{\partial x_i}\left\{a^2 \frac{\partial}{\partial x_i}\left[\ell n\left(\frac{p}{p_0}\right)\right]\right\} =$$

$$= \gamma \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} - \gamma \frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\tau_{ij}\right) + F_i\right] \tag{4.12}$$

*Refer to Table 1.

Equation (4.12) is the differential equation derived by Phillips (Phillips, 1960) with the entropy terms deleted. Lighthill's form of Equation (Reference (1)) may be obtained by using isentropic relations between the pressure and density functions (Phillips, 1960, also Ribner, 1962).

This derivation indicates the equivalence of Equation (2.19) to the forms used in classical investigations of the aerodynamic noise problem. It should be noted, however, that this equivalence holds true only when isentropic pressure variations are postulated, as it is in the case of most classical investigations. The present analysis which is aimed at an investigation of entropy-producing regions with the inclusion of moderate shock waves and their contribution to aerodynamic sound generation, puts the emphasis on the dissipative terms of the equation. It assumes those terms to be the main contribution to the forcing function of the wave equation causing the propagation of a finite amplitude pressure wave.

5. SOME ADDITIONAL ASPECTS OF THE GOVERNING EQUATION

The form of Equation *(4.10)* obtained in the preceding section makes it apparent that one of the focal points of this analysis is the choice of the form of the thermodynamic J-function, which allows one to draw an analogy between the general form of the derived equation with those used in classical studies of the aerodynamic noise problem. Moreover, it has been shown this analogy exists only when isentropic variations of the J-function are allowed.

A more general approach to the problem, in view of the form of the derived equation, seems to indicate that, in a region admitting an arbitrary thermodynamic process having the form

$$J = J(S) \tag{5.1}$$

the propagation of J and S waves are mutually dependent so that entropy generation causes a J-wave emission (and vice-versa) with additional contributions to those phenomena being made by the viscous effects, body forces and velocity fluctuations. In addition, the obtained Equation *(4.10)* is still coupled to the remaining equations of motion. It would thus be useful to consider the possibilities of uncoupling these equations by idealizing the process to straight lines in the J-S plane, a procedure which would be particularly suitable for the application of this analysis to the propagation characteristics of a finite amplitude pressure wave. For it is feasible to consider the two regions, a highly perturbed region with a possible appearance of moderate shock waves which is imbedded in a region of quiescent fluid, this second quiescent region admitting of finite amplitude acoustic waves caused by the perturbed region.

This idealization implies, basically, a far field noise analysis since near field effects must admit a mixed J-S process taking place as represented by Equation (5.1).

It is apparent from the previous analysis and the obtained results that entropy changes would predominate in the perturbed region due to the highly dissipative character of these processes (e.g., shock wave appearances). In such a case the pressure and density functions would be primarily dependent upon the entropy states of the fluid particles and the thermodynamic process may conceivably be approximated by J=constant and S varying. In terms of the previous analysis the differential equation governing such a dissipative region would be given by

$$\frac{\beta^2}{c_v}\left\{\frac{\mathcal{D}}{\mathcal{D}t}\left(\frac{1}{\alpha^2}\frac{\mathcal{D}S}{\mathcal{D}t}\right) - \nabla^2 S\right\} = \gamma\,\frac{\partial u_i}{\partial x_j}\,\frac{\partial u_j}{\partial x_i}$$
$$-\gamma\left\{\frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\tau_{ij}\right) + F_i\right]\right\} \qquad (5.2)$$

In other words, in a region in which velocity fluctuations, viscous effects and non-conservative body forces predominate, an S-wave is generated whose propagation characteristics are largely determined by the non-isentropic speed α. Moreover, in the presence of sudden discontinuities in the region due to instantaneous velocity or viscous changes, entropy effects will also be generated and must be considered as indicated by the form of Equation (5.2).

On the other hand, when the quiescent region is considered, the assumption of far field effects would also call for the additional stipulation of isentropy. In this case, excluding the perturbed region from consideration, the governing Equation (4.10) reduces simply to:

$$\frac{\mathcal{D}^2}{\mathcal{D}t^2}\left[\ln\left(\frac{p}{p_0}\right)\right] - \frac{\partial}{\partial x_i}\left\{\alpha^2 \frac{\partial}{\partial x_i}\left[\ln\left(\frac{p}{p_0}\right)\right]\right\} = 0 \tag{5.3}$$

Here, the functional value of α^2 tends to the isentropic propagation speed a^2, as entropy changes approach zero.*

Finally, when both regions are considered simultaneously, with the dissipative region playing the role of a forcing function propagating acoustic waves into the quiescent region, the governing equation (4.10) takes on the following form:

$$\frac{\mathcal{D}^2}{\mathcal{D}t^2}\left[\ln\left(\frac{p}{p_0}\right)\right] - \frac{\partial}{\partial x_i}\left\{\alpha^2 \frac{\partial}{\partial x_i}\left[\ln\left(\frac{p}{p_0}\right)\right]\right\} = \frac{\beta^2}{c_v}\left\{\frac{\mathcal{D}}{\mathcal{D}t}\left[\frac{1}{\alpha^2}\frac{\mathcal{D}S}{\mathcal{D}t}\right] - \nabla^2 S\right\} +$$

$$- \gamma\left\{\frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\tau_{ij}\right) + F_i\right]\right\} + \gamma\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} \tag{5.4}$$

It should be noted, finally, that in the more general case where no simple assumption of two distinct thermodynamic regions can be made and where the respective paths of the thermodynamic processes in the J-S plane do not follow straight lines, as is conceivable in these extreme physical conditions, the analogy between Equation (4.10) and that derived above cannot be drawn *a priori*.

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*See Section VI.

6. THE PROCESS J = CONSTANT

In accordance with the preceding derivation, the region in which dissipative motion predominates will be approximated by the thermodynamic process J=constant. This region, in which entropy variations predominate, permits thermodynamic variables which are, by hypothesis, functions of the space coordinates and time through the medium of entropy.

Using Equation (3.6) it is inferred that

$$\left(\frac{\partial \rho}{\partial p}\right)_J - \frac{\rho}{\gamma p} = \frac{1}{\beta^2} \tag{6.1}$$

Integration of this equation yields the pressure density relation for the J=constant process:

$$\rho = \frac{\gamma}{\gamma-1}\,\frac{1}{\beta^2}\,p + A\,p^{\frac{1}{\gamma}} \tag{6.2}$$

with A being the integration constant (Figure 1).

Coupling Equation (6.2) with the entropy equation for a perfect gas

$$\rho = p^{\frac{1}{\gamma}} B \exp\left(-\frac{S-S_0}{c_p}\right) \tag{6.3}$$

where B is a constant, yields the pressure as a function of entropy for the process:

$$p = \left\{\frac{\gamma-1}{\gamma} B\,\beta^2\left[\exp\left(-\frac{S-S_0}{c_p}\right) - \frac{A}{B}\right]\right\}^{\frac{\gamma}{\gamma-1}} \tag{6.4}$$

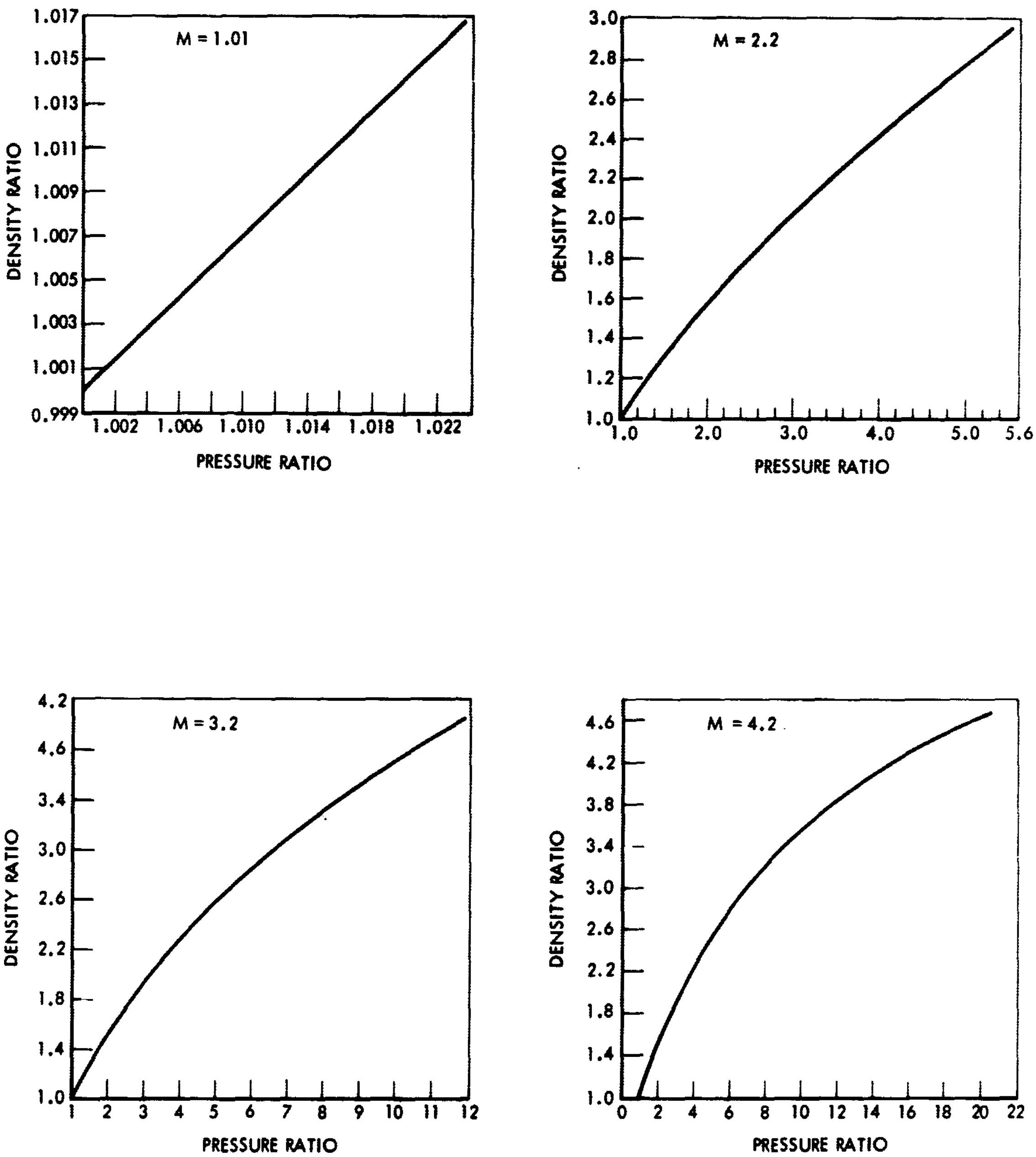

Figure 1. Pressure vs Density for J=Constant Process Applied to Shock Layer Transition Flow

Let p_i and ρ_i refer to the isentropic values of these functions when $S = S_i$ $(i = 0,1)$ respectively. Under these conditions the constant B is determined in terms of either initial or end conditions (Equation (6.3)).

From the condition that

$$p = p_0 \qquad \text{when} \qquad S = S_0 \tag{6.5}$$

the constant ratio $\frac{A}{B}$ in Equation (6.4) is also determined. The resulting expression for the pressure function takes on the following form:

$$p = p_0 \left\{ (\gamma-1)\frac{\beta^2}{a_0^2}\left[\exp\left(-\frac{S-S_0}{c_p}\right)-1\right]+1\right\}^{\frac{\gamma}{\gamma-1}} \tag{6.6}$$

Here a_0^2 is the isentropic propagation speed when $S=S_0$. The constant parameter β^2 is now evaluated from the condition that $p=p_1$ when $S=S_1$ at the end of the process

$$\beta^2 = \frac{\frac{a_0^2}{\gamma-1}\left[\left(\frac{p_1}{p_0}\right)^{\frac{\gamma-1}{\gamma}}-1\right]}{\left[\exp\left(-\frac{S_1-S_0}{c_p}\right)-1\right]} \tag{6.7}$$

It may be shown in a specific application* that, when no entropy pro-

*For moderate shock layers, the numerator and denominator of Equation (6.7) are expressed in terms of the upstream Mach number M_0^2. Setting $M_0^2 = 1+\varepsilon$ it is found that, as $\varepsilon \to 0$, $\lim_{\varepsilon \to 0} \beta^2 = O\left(-\frac{1}{\varepsilon^2}\right)$; i.e., as $S_1 \to S_0$, $\beta^2 \to -\infty$ (see Section 7).

duction occurs, i.e., when $S_1 \to S_0$ in the limit, the value of the parameter β^2 tends to minus infinity in the limit. This property makes Equation (6.1) isentropic when $S_1 \to S_0$, as expected.

On the other hand, once a non-isentropic, J=constant process takes place, i.e., $S_1 > S_0$, the param β^2 becomes finite. Under these conditions its existence and constancy serve to underline both entropy production and irreversibility of the process. For once the parameter β^2 becomes finite it reflects the process of entropy production and its constancy determines the irreversible character of the event.

To evaluate the density ρ as a function of entropy for the process J = constant, Equations (6.3) and (6.6) are used to yield:

$$\rho = \rho_0 \left\{ (\gamma-1) \frac{\beta^2}{a_0^2} \left[exp\left(- \frac{S-S_0}{c_p} \right) - 1 \right] + 1 \right\}^{\frac{1}{\gamma-1}} exp\left(- \frac{S-S_0}{c_p} \right) \qquad (6.8)$$

Equation (2.1) combined with Equations (6.6)and (6.8) yields the temperature T as a function of entropy for the process J=constant.

$$T = T_0 \left\{ (\gamma-1) \frac{\beta^2}{a_0^2} \left[exp\left(- \frac{S-S_0}{c_p} \right) - 1 \right] + 1 \right\} exp\left(\frac{S-S_0}{c_p} \right) \qquad (6.9)$$

It is apparent from Equation (6.9) that, as the entropy S tends to S_0, T tends to T_0, which is the isentropic value of the temperature function.* In a similar manner, when S tends to S_1, T tends to T_1. These can be readily verified by using Equations (6.3), (6.7) and (6.9)

*Note that, as $S-S_0 \to \varepsilon$, $\beta^2 = O\left(\frac{1}{\varepsilon^2}\right)$, and $\left[exp\left(- \frac{S_1-S_0}{c_p} \right) - 1 \right] = O(\varepsilon^3)$ (See Section 7).

The preceding derivations allow the determination of the non-isentropic propagation speed α^2 as a function of entropy. Using Equations (3.8), (6.6) and (6.8) it is apparent that:

$$\alpha^2 = \frac{(\gamma-1)\ \beta^2\left[\exp\left(-\frac{S-S_0}{c_p}\right)-1\right]+a_0^2}{\gamma\left[\exp\left(-\frac{S-S_0}{c_p}\right)-1\right]+\frac{a_0^2}{\beta^2}+1} \qquad (6.10)$$

The non-isentropic propagation speed α^2 is plotted for different entropy production values in Figure 2.

In the limiting case when S tends to S_0, α^2 tends to the isentropic propagation speed a_0^2, as indicated by the form of Equation (6.10).* Here it is of interest, however, to consider the value of the propagation speed $\alpha = \alpha_1$ when $S = S_1$, i.e., at the termination of the non-isentropic J=constant process. To this end, Equation (6.10) is written in the form

$$\alpha_1^2 = \frac{(\gamma-1)\ \beta^2\left[\exp\left(-\frac{S_1-S_0}{c_p}\right)-1\right]+a_0^2}{(\gamma-1)\left[\exp\left(-\frac{S_1-S_0}{c_p}\right)-1\right]+\left[\exp\left(-\frac{S_1-S_0}{c_p}\right)+\frac{a_0^2}{\beta^2}\right]} \qquad (6.11)$$

Equations (6.3) and (6.7) are then used to obtain the following substitutions:

$$\left(\frac{p_1}{p_0}\right)^{1-\frac{1}{\gamma}} = \frac{a_1^2}{a_0^2}\left[\exp\left(-\frac{S_1-S_0}{c_p}\right)\right] \qquad (6.12a)$$

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*When no J =constant process occured, i.e., when $|\beta^2| \to \infty$

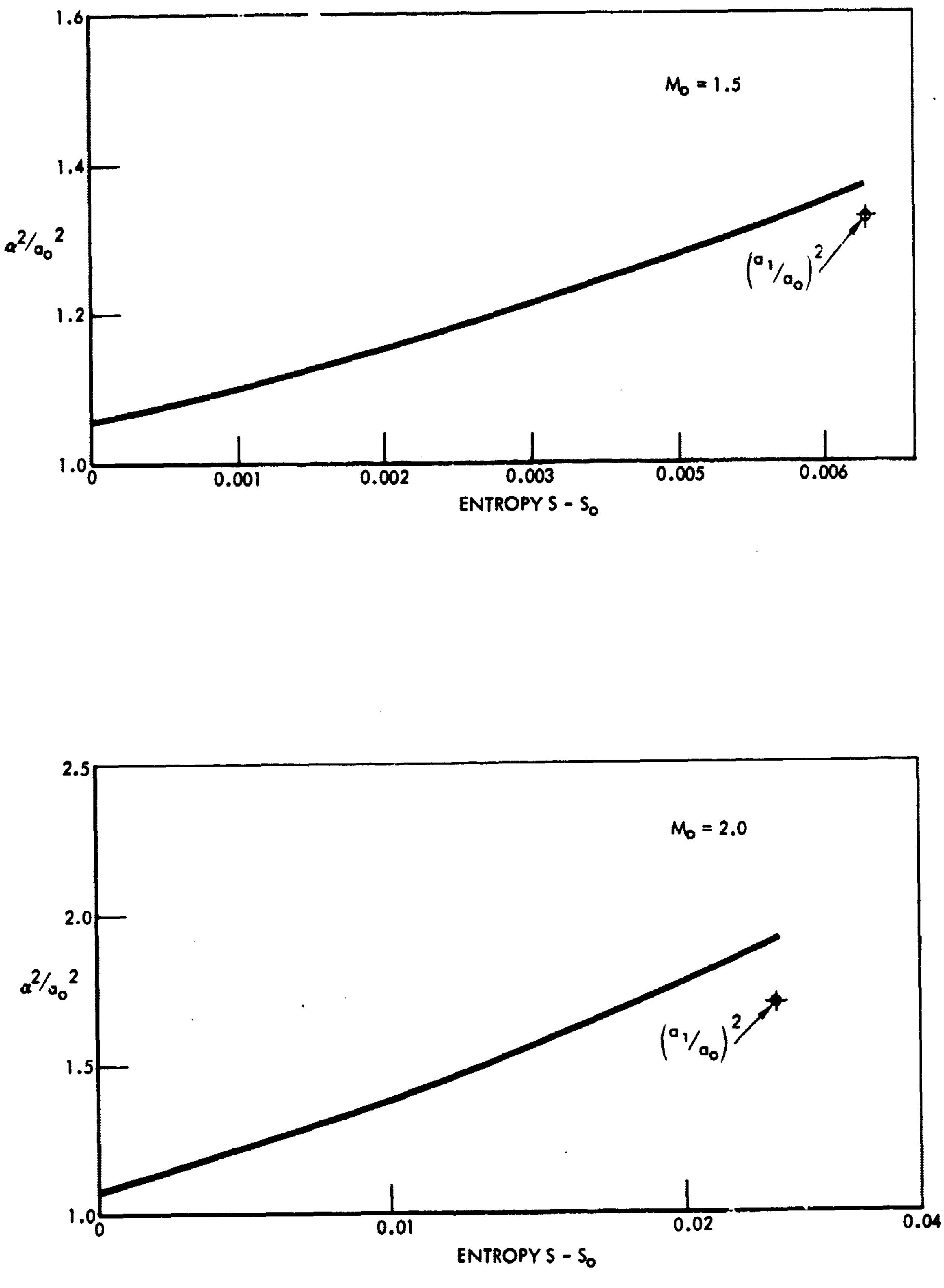

Figure 2. The non-isentronic Propagation Speed a vs Entropy Applied to Shock Laver Transition Flow

and

$$(\gamma-1)\left[\exp\left(-\frac{S_1-S_0}{c_p}\right)-1\right] = \frac{a_0^2}{\beta^2}\left\{\left(\frac{p_1}{p_0}\right)^{1-\frac{1}{\gamma}}-1\right\} \tag{6.12b}$$

Under these conditions Equation (6.11) simplifies to

$$\alpha_1^2 = \frac{a_1^2}{1+\dfrac{a_1^2}{\beta^2}} \tag{6.13}$$

In this case, however, the parameter β^2 is finite, since a non-isentropic process took place. Thus the propagation speed at the end of the process does not reduce to its corresponding isentropic speed, but is modified by a factor

$$\frac{1}{\sqrt{1+\dfrac{a_1^2}{\beta^2}}}$$

It will subsequently be shown that, for small entropy changes, this factor is close to unity.*

It appears from the above considerations that when a process J=constant occurs in which entropy changes predominate, the thermodynamic variables (p, ρ, T, α^2, etc.) are functions of the history of the process rather than the instantaneous values of the state of the fluid particles (note the reference to S_0 as the initial state to which all values of the

*For large entropy variations ($M > 5.0$) the above statement does not hold.

variables are referred). Moreover, whereas all functional values of the pressure, density and temperature are continuous at the boundary of the region in which the process takes place, this is not so for the derivatives of these functions.

For it is noted that all functions take on their isentropic values on the boundary of the region, excepting the propagation speed α, which differs from its isentropic counterpart at the termination of the process by a factor which is directly dependent upon the amount of entropy production during the process and tends to unity as the entropy production tends to zero.

It then formally appears that a non-isentropic region may have a discontinuous boundary due to a difference in the propagation speeds when it adjoins an isentropic region. However, the functional values of the thermodynamic variables are not discontinuous.

7. THE CASE OF MODERATE SHOCK WAVES

The preceding analysis lends itself to simplification when the entropy production region under consideration is the region of a moderate shock wave.

To evaluate the functional variations of the thermodynamic variables in the shock layer as a function of entropy production, it is convenient to evaluate initially certain recurring expressions appearing in the preceding analysis and given by

$$\left[\left(\frac{p_1}{p_0}\right)^{1-\frac{1}{\gamma}} - 1\right]; \quad \left\{exp\left(-\frac{S_1-S_0}{c_p}\right) - 1\right\}; \quad \beta^2; \text{ etc.} \tag{7.1}$$

Here state zero denotes the value of the variables in front of the shock layer, and state one refers to their value behind the shock layer. From the shock transition equations it is evident that

$$\left(\frac{p_1}{p_0}\right)^{\frac{\gamma-1}{\gamma}} = \left\{\frac{2\gamma M_0^2 \sin^2\theta - (\gamma-1)}{\gamma+1}\right\}^{\frac{\gamma-1}{\gamma}} \tag{7.2}$$

$$\exp\left(-\frac{S_1-S_0}{c_p}\right) = \frac{(\gamma-1)M_0^2\sin^2\theta}{(\gamma-1)M_0^2\sin^2\theta+2}\left\{\frac{\gamma+1}{2\gamma M_0^2\sin^2\theta-(\gamma-1)}\right\}^{\frac{1}{\gamma}} \tag{7.3}$$

$$\frac{\rho_1}{\rho_0} = \left\{\frac{(\gamma+1)M_0^2\sin^2\theta}{(\gamma-1)M_0^2\sin^2\theta+2}\right\} \tag{7.4}$$

Let us consider the value of the parameter β^2. It is apparent from Relation (6.7) that

$$\beta^2 = \frac{\frac{a_0^2}{\gamma-1}\left\{\left(\frac{2\gamma M_0^2\sin^2\theta-(\gamma-1)}{\gamma+1}\right)^{\frac{\gamma-1}{\gamma}}-1\right\}}{\left\{\left(\frac{(\gamma+1)M_0^2\sin^2\theta}{(\gamma-1)M_0^2\sin^2\theta+2}\right)\left(\frac{\gamma+1}{2\gamma M_0^2\sin^2\theta-(\gamma-1)}\right)^{\frac{1}{\gamma}}-1\right\}} \tag{7.5}$$

In the limit, when S_1 tends to S_0, the normal component of the Mach number upstream of the shock layer tends to unity. Consequently, Equation (7.5) may be evaluated in the limit by setting

$$M_0^2\sin^2\theta = 1+\varepsilon \tag{7.6}$$

Following this procedure it can readily be shown that:

$$\left\{\left(\frac{p_1}{p_0}\right)^{1-\frac{1}{\gamma}} - 1\right\} = 2\left(\frac{\gamma-1}{\gamma+1}\right)\varepsilon - 2\left(\frac{\gamma-1}{\gamma}\right)\varepsilon^2 + \ldots \quad (7.7a)$$

$$\left\{\exp\left(-\frac{S_1-S_0}{c_p}\right) - 1\right\} = -\frac{2}{3}\frac{(\gamma-1)}{(\gamma+1)^2}\varepsilon^3 + \ldots \quad (7.7b)$$

and

$$\frac{\rho_1}{\rho_0} = 1 - \frac{2}{\gamma+1}\varepsilon + \frac{2}{\gamma+1}\varepsilon^2 - \frac{4}{3}\frac{(1+2\gamma)}{(1+\gamma)^2}\varepsilon^3 + \ldots \quad (7.7c)$$

Hence, as $S_1 \rightarrow S_0$, i.e., as ε approaches zero (to the first approximation)

$$\lim_{S_1 \rightarrow S_0} \beta^2 = \lim_{\varepsilon \rightarrow 0}\left\{-3a_0^2\left[\frac{\gamma+1}{\gamma-1}\right]\frac{1}{\varepsilon^2}\right\} = -\infty \quad (7.8)$$

This is the condition which makes the J=constant process isentropic in the limit, in accordance with previous derivations [see Relations (3.8) and (6.7)].

The parameter β^2 is plotted as a function of Mach number for a given shock transition process in Figure 3.

In a similar manner, using the shock transition relations, the values of the functions p and ρ versus entropy are plotted in Figures 4 and 5.

It should also be noted that most of the derived relations for pressure, density, etc., contain the factor

$$E = \beta^2 \left\{ exp\left(- \frac{S_1 - S_0}{c_p}\right) - 1 \right\} \tag{7.9}$$

Now in the limit as S_1 tends to S_0, the parameter β^2 tends to negative infinity as $\left(- \frac{1}{\epsilon^2}\right)$ [see Equation (7.8)], whereas the second factor in Equation (7.9) tends to zero as $(-\epsilon^3)$ [see Equation (7.7)]. Hence, the total expression in Equation (7.9) tends to zero when $S_1 \to S_0$ (and $S \to S_1$) as expected.

The following important consequence of the above derivation should be noted. Even when a small entropy change occurs during any arbitrary process, the non-isentropic effects cannot be arbitrarily disregarded, for it could be reasoned that, even for the case of weak shocks, the entropy variation is of a third-order magnitude, and therefore its effects could conceivably be disregarded. On the other hand, the present theoretical considerations point to the fact that this reasoning may not be justified. This is due to the fact that, even though the entropy variation is of a third-order magnitude, its product with the process parameter β^2 varies as a first-order magnitude [Equation (7.9)]. Thus, all the functional values of pressure, density, temperature, etc., in the non-isentropic region also vary as a first-order magnitude. Moreover, the differential equation governing the aerodynamic sound generation, as derived in Equation (5.4) has also first-order variation of the non-isentropic terms. Consequently, the contribution of the non-isentropic terms cannot be disregarded within the framework of this analysis and the entropy production terms must be taken into account even for small variations of the entropy functions in the process considered (e.g., weak shocks).

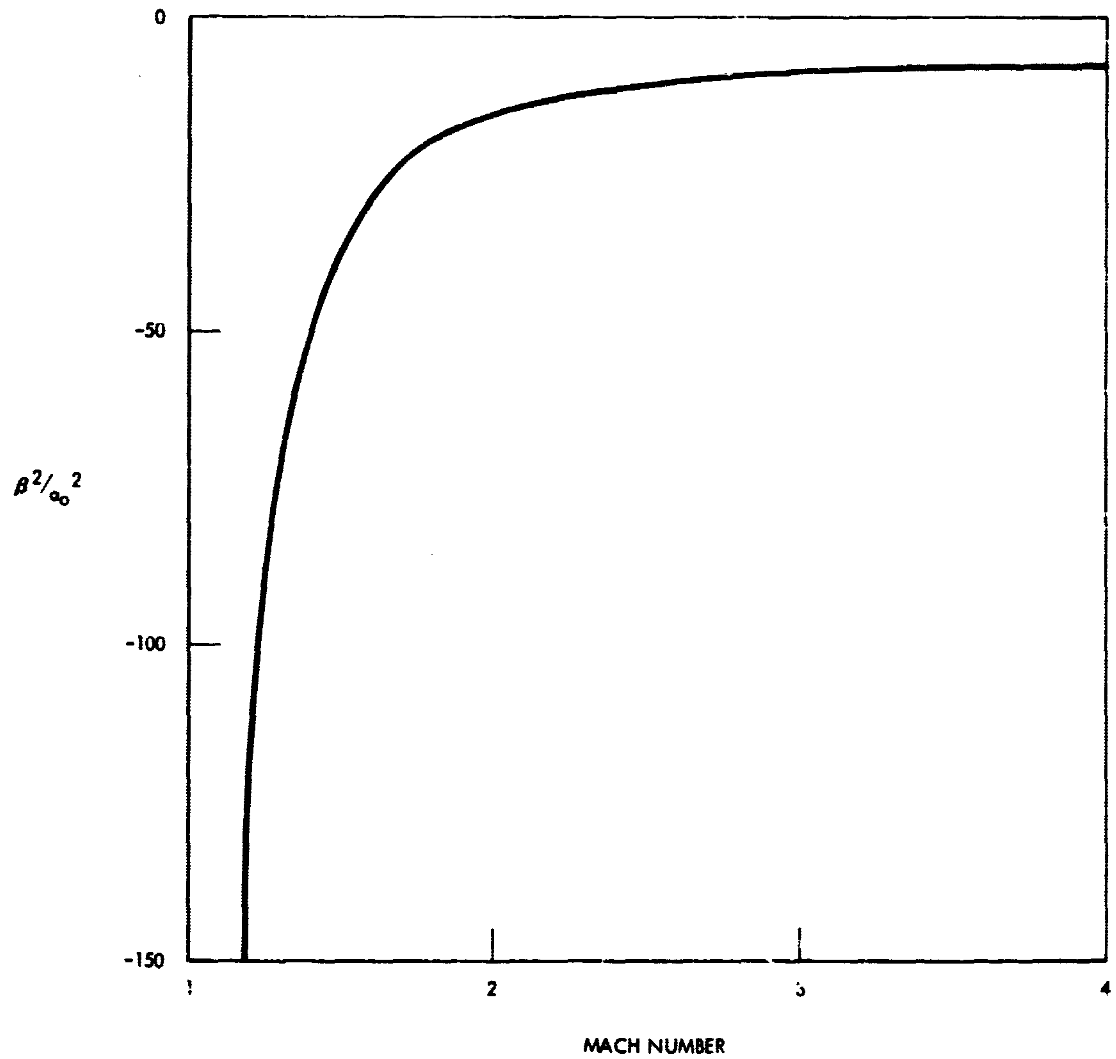

Figure 3. The parameter β^2/a_0^2 vs Upstream Mach Number for J=Constant Process Applied to a Normal Shock Transition Flow.

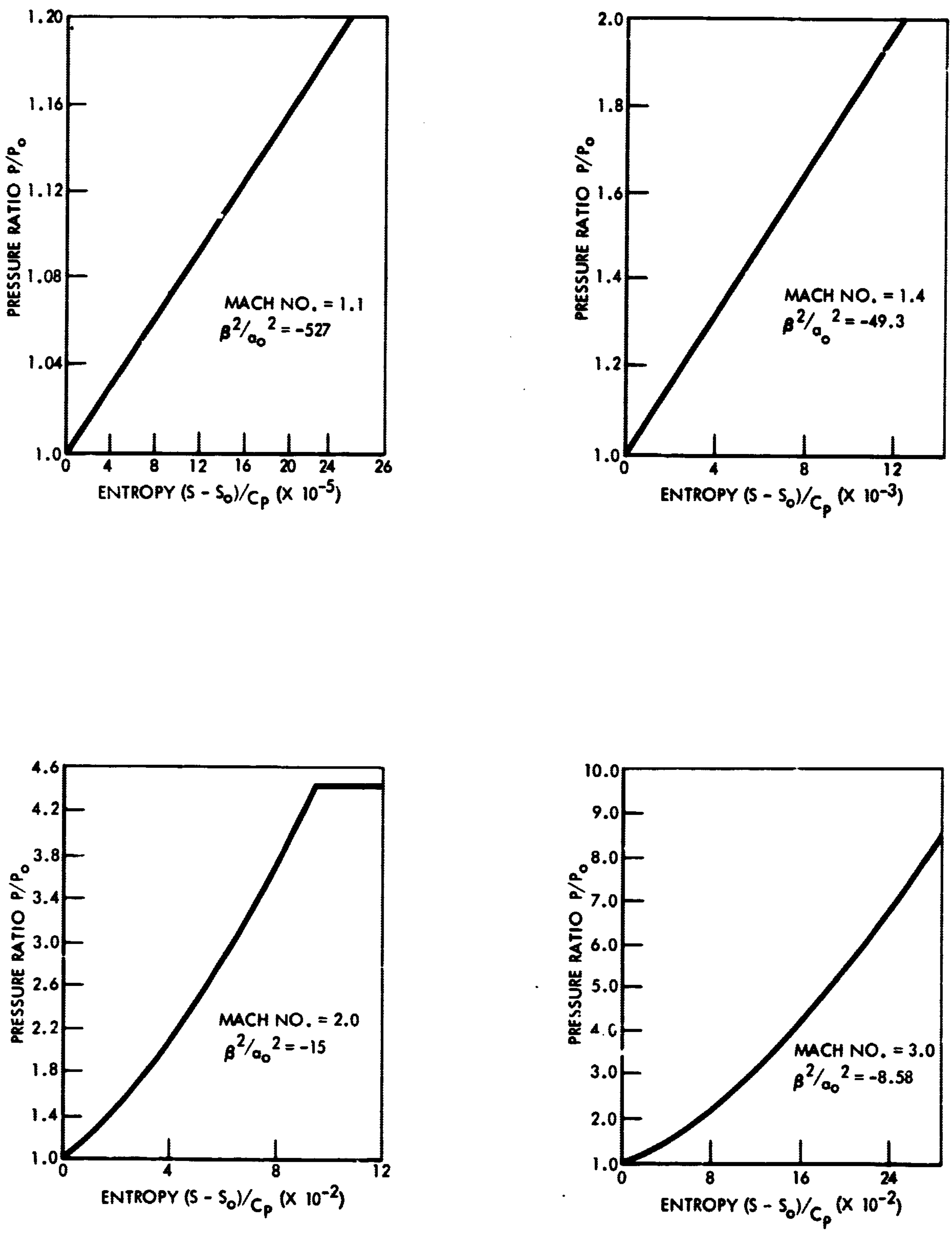

Figure 4. Pressure vs. Entropy for J=Constant Process Applied to Shock Layer Transition Flow.

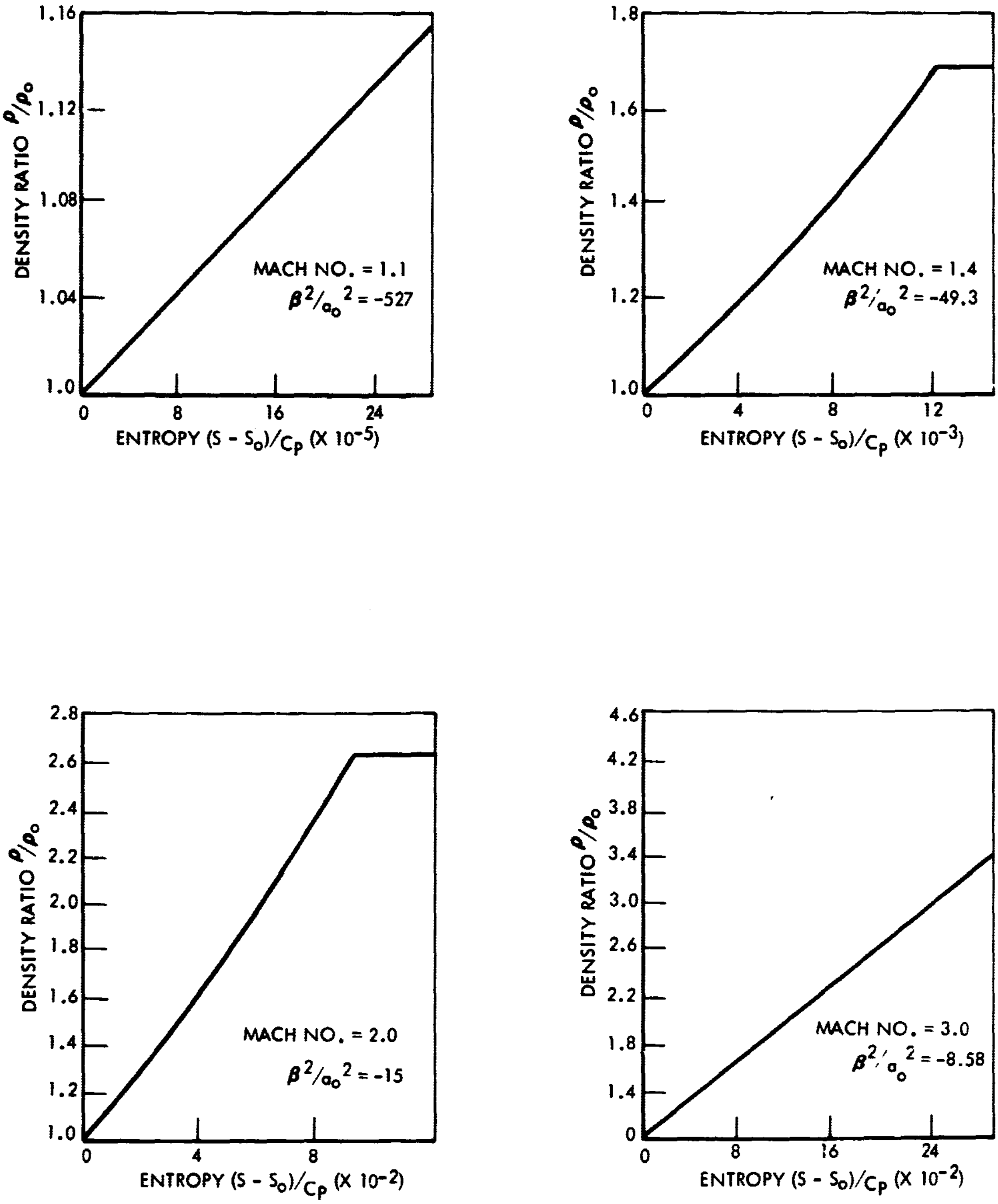

Figure 5. Density vs. Entropy for J=Constant Process Applied to Shock Layer Transition Flow.

II. ENGINEERING APPLICATION

8. INTRODUCTION

A simplified engineering model of the propagation characteristics of a finite amplitude pressure wave in high-speed flow with moderate shock wave formation is presented in the subsequent sections. The actual computations of the results were based initially on a previous model (Peter and Li, 1965) in which only non-isentropic changes of the pressure function were considered, since at that time some of the results of the present analysis could not be incorporated into the computer programs.* Nevertheless, the basic characteristics of the derivations are similar in both cases and serve to illustrate some physical aspects of non-isentropic forcing functions in aerodynamic noise generation.

In accordance with the theoretical aspects of the preceding analysis an entropy-producing region imbedded in an isentropic medium will produce acoustic excitations in the far field. In the near field a mixed J-S process takes place** with a resulting entropy wave emission. The near field mixed process will not be considered here. Emphasis is placed mainly on the effect of finite amplitude wave propagation into a quiescent isentropic region.

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* The only difference consists in the Mach number dependence of the wave amplitude. This adds a constant factor to the decibel count of the resulting intensity.

** See Section 5.

The governing equations for the propagation phenomena have been derived in Equations (5.2) and (5.4). Equation (5.2) indicates that non-conservative effects of fluid motion, i.e., rotational velocity fluctuations, viscous effects and dissipative body forces cause an entropy wave propagation in the perturbed region. Equation (5.4) indicates that, when the perturbed region is imbedded in an isentropic quiescent medium, an acoustic excitation of the quiescent region also takes place.

In the present engineering approach the thickness of the transition layer is disregarded and a discontinuous jump in the value of the forcing function is assumed. The time dependence of the forcing function enters through a series of disturbances caused by random distribution of impulses imparted to the shock layer. An order of magnitude analysis indicates that entropy contributions to the forcing function predominate while velocity fluctuations are of second order magnitude even for the case of weak shock waves. Their effect on the forcing function seems to be indirect by imparting initial excitations to the entropy function and thus triggering the mechanism of emission.

9. ENGINEERING REPRESENTATION

In accordance with the preceding discussion, the governing differential equation will now be applied to analyze the propagation characteristics of moderate shock layers in supersonic nozzle flow.

It was shown in Section 5 that the equation representing non-isentropic wave propagation into a quiescent region is given by:

$$\frac{D^2}{Dt^2}\left[\ell n\left(\frac{p}{p_0}\right)\right] - \frac{\partial}{\partial x_i}\left[a_0^2 \frac{\partial}{\partial x_i} \ell n\left(\frac{p}{p_0}\right)\right] =$$

$$= \frac{1}{c_v}\left[\frac{D}{Dt}\left(\frac{\beta^2}{\alpha^2}\frac{DS}{Dt}\right) - \beta^2\nabla^2 S\right] - \gamma\frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\tau_{ij}\right) + F_i\right]$$

$$+ \gamma \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} \qquad (9.1$$

The case of a shock layer is now considered. To do this it is necessary to specify the right-hand side of Equation (9.1), particularly the entropy function and its gradients. A rigorous functional evaluation of these variables cannot be accomplished, however, without solving simultaneous the remaining equations of motions [see Equations (2.1) to (2.4)] with which Equation (9.1) is still coupled. To circumvent this difficulty it will be necessary to represent the J-constant process as if it is taking place in a region of zero thickness, so that a sudden discontinuous jump in the dependent variables across the shock layer occurs. This representation ascribes to the remaining equations of motion a contribution whose mean square value is given by the Rankine-Hugoniot relations. However, the formal derivation of the governing equations has implied the existence and continuity of the dependent variables up to and including second derivatives. This restriction limits the analytical representation of the discontinuous character of the dependent variables to those functions whose derivatives admit analytical characteristics. For this reason generalized functions have been employed to represent the formalistic behavior of the jump conditions during transition. Their gradients become Dirac Delta functions whose successive derivatives admit analytical interpretations.

It will be shown subsequently that this discontinuous character of the dependent variables during the process of transition results in a radiating acoustic dipole field.* The radiating dipoles are distributed over the shock layer and their local strength is proportional in the mean to the Rankine-Hugoniot transition values.

A notable simplification of the mathematical derivations may be achieved by imposing upon the radiating field the condition** that the time dependence of the forcing function will enter through a series of disturbances caused by a random distribution of impulses imparted to the shock layer. The time derivatives of these impulse functions are multiplied by mean transition values which determine the amplitude of the propagated wave and which may be regarded as slowly varying continuous time functions with negligible gradients. As a consequence, the time derivatives of the impulse functions may be neglected in the first approximation in view of the fact that no time integration appears in the solution to the wave equation. Physically, the assumption implies that mean transition values of the forcing function determine the amplitude of the emitted waves, whereas the random time impulses supply the mechanism triggering the emission.

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* Section 10.

** This condition is similar to that imposed on the forcing function used in computations (Peter and Li, 1965, also Section 8).

Under these conditions, Equation (9.1) becomes

$$\frac{D^2}{Dt^2}\left[\ln\left(\frac{p}{p_0}\right)\right] - \frac{\partial}{\partial x_i}\left[a_0^2 \frac{\partial}{\partial x_i} \ln\left(\frac{p}{p_0}\right)\right] =$$

$$= \frac{1}{c_v}\left\{ u_i \frac{\partial}{\partial x_i}\left(\frac{\beta^2}{\alpha^2} u_j \frac{\partial S}{\partial x_j}\right) - \beta^2 \nabla^2 S \right\} + \gamma \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j}$$

$$- \gamma \frac{\partial}{\partial x_i}\left[\frac{1}{\rho} \frac{\partial}{\partial x_j}(\tau_{ij}) + F_i\right] \qquad (9.2)$$

An order-of-magnitude analysis of the forcing function of Equation (9.2) will now be conducted. In the first place, since the region of the shock in which the J=constant process takes place is represented by a layer of vanishingly small thickness, the viscous term in the equation may be accounted for by the jump across th layer to which the action of the viscous forces is confined. (It can be shown that the viscous terms in such a discontinuous representation of the forcing function generate a quadrupole radiation field which is of the next order of magnitude). In addition, the body forces do not appear under normal flow conditions and may be disregarded.

For the sake of convenience we will designate the remaining terms of the function by $W_i (i = 1,2,3)$

$$W_1 = u_i \left\{\frac{\partial}{\partial x_i}\left(\frac{\beta^2}{\alpha^2} u_j \frac{\partial S}{\partial x_j}\right)\right\}$$

$$W_2 = - \beta^2 \nabla^2 S$$

$$W_3 = \gamma \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} \qquad (9.3)$$

It is evident that the two functions, $\mathcal{W}_1$ and $\mathcal{W}_2$ represent the contribution of entropy production to the forcing function, whereas $\mathcal{W}_3$ designates the interaction of velocity gradients (fluctuations) and represents their contribution to sound generation. These mechanical velocity fluctuations will now be shown to constitute a second-order magnitude when compared with the entropy terms of the forcing functions, even for the case of weak shock layers.

To that end, consider the case of one-dimensional flow and transform the term $\mathcal{W}_3$ by the use of Equations (2.2), (2.13) and (4.4) to obtain (4.4) to obtain

$$\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} = \left(\frac{\beta^2}{\alpha^2}\frac{DS}{Dt}\right)\left(\frac{\beta^2}{\alpha^2}\frac{DS}{Dt}\right) \qquad (9.4)$$

where $[u_i = U(x),\ 0,\ 0\ ]$.

Equation (9.4) reduces to

$$\frac{\partial u_i}{\partial x_j}\ \frac{\partial u_j}{\partial x_i} \sim \left(U\frac{\beta^2}{\alpha^2}\ \frac{\partial S}{\partial x}\right)^2 \qquad (9.5)$$

when mean intensity values are considered.

In the entropy-producing region, i.e., in the shock layer represented by the process J=constant, the propagation speed α is of the order of the isentropic sound speed a_0 for very weak shock layers.* Likewise, the

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* Section 7.

velocity U also approaches sonic speed* and is of the same order of magnitude.

Referring now to previous derivations, it was shown** that

$$\beta^2 \nabla S \sim O(a_0^2 \varepsilon) \tag{9.6a}$$

$$\frac{\beta^2}{\alpha^2} \nabla S \sim O(\varepsilon) \tag{9.6b}$$

with $\varepsilon = (M_0^2 - 1)$ and such** that $\varepsilon << 1$. It then follows that

$$U \frac{\beta^2}{\alpha^2} S \sim O(a_0 \varepsilon) \tag{9.7}$$

Using these estimates in the term W_i of the forcing function, as in Equation (9.3), it is inferred that

$$W_1 \sim O(a_0^2 \varepsilon)$$

$$W_2 \sim O(a_0^2 \varepsilon)$$

$$W_3 \sim O(a_0^2 \varepsilon^2) \tag{9.8}$$

Thus, the present theory indicates that velocity fluctuations (mechanical effects) in a non-isentropic region are of a second-order magnitude when compared to the entropy terms even for the case of weak shocks when entropy variations, *per se*, are very small.

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* From above

** Section 7.

It should be noted, however, that such velocity fluctuations may affect the non-isentropic sound generation indirectly by imparting initial excitations to the entropy function, thus providing the triggering mechanism necessary for emission.*

10. AN ACOUSTIC ANALOGY

An acoustic analogy is now constructed in order to evaluate the propagation characteristics of a stationary shock layer forming in the exhaust of a fixed supersonic nozzle. The shock layer is regarded as a stationary region from which acoustic excitations propagate into the surrounding quiescent medium. The analogy disregards convective and dispersive effects of the emission mechanism. On the other hand, the local entropy gradients which predominate in this analogy were shown** to be of a comparable order of magnitude and represent a substantial part of the overall problem.

The differential equation governing this emission process may be put, in view of Equation (9.2) and the subsequent discussions, in the following form:

$$\frac{\partial^2}{\partial t^2}\left[\ln\left(\frac{p}{p_0}\right)\right] - a_0^2\nabla^2\left[\ln\left(\frac{p}{p_0}\right)\right] = - \frac{\beta^2}{c_v}\nabla^2\left[A(\bar{x})H(z-h)\delta(t)\right] \tag{10.1}$$

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* See Equation (5.2)

** See Equation (9.8)

Here the entropy function S has been written in the form discussed previously

$$S = A(\bar{x})H(z-h)\delta(t) \tag{10.2}$$

It is noted that the spatial dependence of the entropy function is represented by a product of an amplitude function $A(\bar{x})$ and the Heavyside step function. The amplitude function is proportional in the mean to the transition values of entropy. Its dependence upon spatial coordinates is introduced here to take into account thermodynamic non-equilibrium flows in the region (e.g., curved shock layers).* For constant curvature shocks the amplitude function depends upon the constant normal component of the upstream Mach number. The present investigation will concentrate largely on the latter case.

The shock surface is represented by the equation

$$z = h(x,y) \tag{10.3}$$

The spatial dependence in Equation (10.3) will imply axial symmetry.

In actuality the shock layer structure in underexpanded supersonic nozzles admits, in certain cases, of a slight curvature before the onset of the Mach disc (Peter and Kamo, 1963). However, in the present formulation, the curved effects will introduce unwarranted complications due to the non-equilibrium character of the flow behind the shock layer at the end of the non-isentropic process (i.e., spatial entropy gradients). For in this case the process parameter β^2 will not retain its constant character stipulated

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* It can be shown that such non-equilibrium flows make the thermodynamic parameter β^2 space-dependent.

here unless its average value is used. Moreover, conical structures with a possible Mach disc superposition should afford a good approximation, which becomes exact in many cases.

Under these conditions, the total amplitude of the emission is seen to depend upon the product $(\beta^2 A)$ independent of spatial variation, with the upstream normal Mach number appearing as a flow parameter.

The time variation of the entropy function $\delta(t)$, representing an impulse-disturbance imparted to the J=constant region, will be kept in a general form

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma(\omega)\, e^{-i\omega t} d\omega \tag{10.4}$$

where $\Gamma(\omega)$ is the Fourier transform of $\delta(t)$.

The solution of Equation (10.1) may now be written

$$\ell n\left(\frac{p}{p_0}\right) = -\frac{A}{8\pi^2}\,\frac{\beta^2}{a_0^2}\int_{-\infty}^{\infty} \Gamma(\omega)\, e^{-i\omega t} d\omega$$

$$\times \iiint \frac{e^{i\frac{\omega}{a_0}r}}{r}\,\nabla^2[H(z-h)]\,dv \tag{10.5}$$

The integration is taken over the region of the disturbance.*

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* Here the radius r is defined in the usual manner $r = |\vec{\xi} - \vec{x}|$; $\vec{\xi}$ = observation point coordinates; $\vec{x}$ = perturbed region coordinates.

Let us consider first the simplest case of a normal shock layer located at some point $z = h_0$, where h_0 = constant. Under these conditions the term $\nabla^2 H$ becomes a derivative of the delta function $\delta(z-h_0)$ with respect to z. As a consequence, Equation (10.5) may be written(using the properties of the delta function):

$$\ln\left(\frac{p}{p_0}\right) = \frac{A}{8\pi^2}\frac{\beta^2}{a_0^2}\int_{-\infty}^{\infty}\Gamma(\omega)\, e^{-i\omega t} d\omega \iint \frac{\partial}{\partial z}\left[\frac{e^{i\frac{\omega}{a_0}r}}{r}\right]_{z=h_0} d\sigma \qquad (10.6)$$

The integral is taken over the surface σ over which the disturbance takes place.

It is evident that the obtained solution represents a radiating dipole field located at $z = h_0$. The radiating dipoles are distributed over the shock layer emitting pressure waves whose amplitude is proportional to the transition values of the product $(A\beta^2)$. The product determines to the first approximation the dependence of the acoustic intensity of radiation upon the upstream Mach number. From previous considerations it is readily deduced that:

$$A\beta^2 = \frac{\dfrac{a_0^2}{\gamma - 1}\left\{\left[\dfrac{2\gamma M_0^2 \sin^2\theta - (\gamma-1)}{\gamma + 1}\right]^{1-\frac{1}{\gamma}} - 1\right\}}{\left\{\left[\dfrac{(\gamma+1)M_0^2\sin^2\theta}{(\gamma-1)M_0^2\sin^2\theta + 2}\right]\left[\dfrac{\gamma + 1}{2\gamma M_0^2\sin^2\theta - (\gamma-1)}\right]^{\frac{1}{\gamma}} - 1\right\}} \times$$

$$\times\ \ln\left\{\left[\frac{2\gamma M_0^2\sin^2\theta - (\gamma-1)}{\gamma + 1}\right]\left[\frac{(\gamma+1)M_0^2\sin^2\theta}{(\gamma-1)M_0^2\sin^2\theta + 2}\right]^{-\gamma}\right\} \qquad (10.7)$$

Here, $M_0\sin\theta$ denotes the normal component of the Mach number to a shock layer inclined at an angle θ=constant.

Equation (10.5) will now be generalized to include shock layers of an arbitrary shape. Thus we let the shock layer surface be given by the equation

$$F_0^1(x,y,z) = \psi_0^1 \ * \qquad (10.8a)$$

where ψ^1 denotes a family of given surfaces and ψ_0^1=constant is that particular surface which defines the shock layer.

This equation may also be written $z - f(x,y) = 0$, on the shock; $z - f$ =constant, otherwise.

For convenience we define two mutually orthogonal families of surfaces by

$$F^2(x,y,z) = \psi^2 \qquad\qquad F^3(x,y,z) = \psi^3 \tag{10.8b}$$

which are orthogonal to $F^1 = \psi^1$

Equation (10.5) is now transformed to the curvilinear coordinates ψ^i [i=1,2,3]. It should be noted that these coordinates refer only to the perturbed region defined by the vector $\vec{x}$ and not to the point of observation which is denoted by the vector $\vec{\xi}$. Equation (10.5) is now written in the form:

$$\ln\left(\frac{p}{p_0}\right) = -\frac{A}{8\pi^2}\frac{\beta^2}{a_0^2}\int_{-\infty}^{\infty}\Gamma(\omega)e^{-i\omega t}d\omega \times$$

$$\times \iiint G(\vec{\xi},\vec{\psi})\nabla^2\left[H\left(\psi^1-\psi_0^1\right)\right]h_1h_2h_3\ d\psi^1 d\psi^2 d\psi^3 \tag{10.9}$$

where

$$G(\vec{\xi},\vec{\psi}) = \left[\frac{e^{i\frac{\omega}{a_0}r}}{r}\right] \tag{10.10}$$

is a function of $\vec{\xi}$ and the ψ^i's.

$$h_1h_2h_3 = \frac{\partial(x,y,z)}{\partial(\psi^1,\psi^2,\psi^3)} \tag{10.11}$$

represents the Jacobian of transformation. The h_i's ($i = 1,2,3$) represent the metric coefficients of the curvilinear system.

In terms of the coordinates ψ^i it is evident that

$$\nabla^2 H = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial \psi^1} \left[\left(\frac{h_2 h_3}{h_1} \right) \frac{\partial H}{\partial \psi^1} \right] \right\} \tag{10.12}$$

since $H = H(\psi^1)$ only. Thus, Equation (10.9) becomes

$$\ln\left(\frac{p}{p_0}\right) = - \frac{A}{8\pi^2} \frac{\beta^2}{a_0^2} \int_{-\infty}^{\infty} \Gamma(\omega) e^{-i\omega t} d\omega \times$$

$$\times \iiint G(\vec{\xi}, \vec{\psi}) \left\{ \frac{\partial}{\partial \psi^1} \left[\frac{h_2 h_3}{h_1} \frac{\partial H}{\partial \psi^1} \right] \right\} d\psi^1 d\psi^2 d\psi^3 \tag{10.13}$$

Noting that

$$\frac{\partial}{\partial \psi^1} \left[H\left(\psi^1 - \psi_0^1\right) \right] = \delta\left(\psi^1 - \psi_0^1\right) \tag{10.14}$$

and using the properties of the delta function, Equation (10.13) takes on the following form:

$$\ell n\left(\frac{p}{p_0}\right) = \frac{A}{8\pi^2}\,\frac{\beta^2}{a_0^2}\int_{-\infty}^{\infty}\Gamma(\omega)e^{-i\omega t}d\omega \;\times$$

$$\times \iint \frac{\partial}{\partial\psi^1}\left[\frac{e^{i\frac{\omega}{a_0}r}}{r}\right]\frac{h_2h_3}{h_1}\,d\psi^2 d\psi^3 \qquad (10.15)$$

If we denote the linear increments along the curvilinear coordinate lines by $d\ell_i (i = 1,2,3)$ so that

$$d\ell_i = h_i d\psi^i$$

Equation (10.15) can conveniently be written in the form

$$\ell n\left(\frac{p}{p_0}\right) = \frac{A}{8\pi^2}\,\frac{\beta^2}{a_0^2}\int_{-\infty}^{\infty}\Gamma(\omega)e^{-i\omega t}d\omega \;\times$$

$$\times \iint \frac{\partial}{\partial\ell_i}\left[\frac{e^{i\frac{\omega}{a_0}r}}{r}\right]_{\psi^1=\psi_0^1} d\ell_2 d\ell_3 \qquad (10.17$$

This last equation clearly defines a dipole radiation field generated by a shock layer whose form is represented by the surface ψ_0^1. It is noted that the dipole axis pointing in the direction of ℓ_1 is normal to the shock surface. It is then of immediate consequence that the maximum intensity of radiation of a supersonic nozzle in the presence of shock waves will tend, in the light of the present theory, to be normal to the shock surfaces. Thus a conical shock whose angle with respect to the jet axis is 45° will radiate sound whose intensity will tend to be maximum along the 45° axis of the jet. Since most shock layers in underexpanded supersonic nozzles lie in the range of 30° to 60° inclination, it appears that the maximum intensity of radiation will tend to follow this pattern.*

It should be noted that the resultant field of radiation is symmetric with respect to the plane passing through the origin and which is normal to the axis of the jet. This is due to the fact that the supersonic state of the conditions prevailing upstream of the shock wave were disregarded in this analogy. As a matter of fact, backward radiation will be impeded by the upstream conditions to an extent which cannot be envisaged from the present development. A more accurate picture of these effects could, it is hoped, be obtained from the effects of the terms disregarded here but appearing in the governing differential Equation (9.1).

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*See Section 11 for a more complete discussion of conical shock propagation characteristics.

It should also be noted that additional sources of radiation appear in actual supersonic nozzle flow which could be, but are not, treated by the present approach. For instance, at the termination of the shock at the boundary of the nozzle flow a ring of radiating dipoles appears whose axis is directed along the shock. Consequently, additional radiating sources appear due to the tangential entropy difference between the terminating shock and the surrounding medium. These effects, which must be treated separately, will not change the present directionality and propagation characteristics to a large extent due to the orders of magnitude of the respective surface areas.

The appearance of Mach discs in the flow could be treated by superposition but it is doubtful if more useful information could be obtained without a rigorous analysis of the complete non-isentropic phenomena of high speed flows. This is due to the highly idealized nature of the present analogy. For instance, it is idealized here that the shock transition takes place locally. In actuality, a succession of waves appears in the flow, but due to their geometrical similarity of shape, which appears to be of a predominant importance in the light of this analysis, the total effect would not be materially different.

This hypothesis does not seem to hold, however, when Mach discs appear at higher Mach numbers. In this case, the successive shock formation does not possess the geometrical similarity characteristics of shock layer shapes. On the contrary, the successive shock waves which are distinguished by their highly unstable characteristics, take the form of a fluctuating cone. For this reason it is feasible that the idealization of conical shapes could be extended to the propagation characteristics of transition layers, even for higher Mach numbers, in spite of Mach disc appearance. In any case, a separate basic investigation of these phenomena seems desirable.

11. THE CONICAL SHOCK LAYER

It was shown in the previous section that the radiation characteristics of an arbitrarily shaped shock layer were described by the forcing function:

$$\ell n \left[\frac{p}{p_0}\right] = \frac{A}{8\pi^2} \frac{\beta^2}{a_0^2} \int \Gamma(\omega) e^{-i\omega t} d\omega \iint \frac{\partial}{\partial \ell_i} \left(\frac{e^{ikr}}{r}\right) d\ell_j d\ell_k \tag{11.1}$$

where $k = \frac{\omega}{a_0}$ is the wave number and the integral is evaluated at the surface $\psi^i = \psi_0^i$. The subscripts i, j and k can take on any one of the values between 1 and 3, but it is self-evident that $i \neq j \neq k$. This notation is more flexible since it allows the choice of any coordinate of the curvilinear triplet as the shock surface.

In the particular case of conical shocks it is convenient to choose the spherical coordinate system:

$$\psi^1 = \tilde{\rho} \qquad \psi^2 = \theta \qquad \psi^3 = \phi \tag{11.2}$$

so that the conical surface is represented by the coordinate $\psi_0^2 = \theta_0$, where θ_0 is the value of the conical angle formed by the shock. Thus, referring to the notation in Equation (11.1), it is inferred that $i = 2$, $j = 1$, and $k = 3$.

In view of the above definitions the shock region is represented by the coordinates:

$$x = \tilde{\rho}\sin\theta\cos\phi \qquad y = \tilde{\rho}\sin\theta\sin\phi \qquad z = \tilde{\rho}\cos\theta \tag{11.3}$$

Using the above notation, Equation (11.1) takes on the following form

$$\ell n \left[\frac{p}{p_0}\right] = \frac{A}{8\pi^2} \frac{\beta^2}{a_0^2} \int \Gamma(\omega) e^{-i\omega t} d\omega \iint \frac{\partial}{\partial\theta} \left[\frac{e^{ikr}}{r}\right]_{\theta=\theta_0} \sin\theta_0 \, d\tilde{\rho} d\phi \tag{11.4}$$

Let the observation point $\vec{\xi} = [\xi, \eta, \zeta]$ also be defined in terms of spherical system of coordinates

$$\xi = R\sin\gamma\cos\nu$$
$$\eta = R\sin\gamma\sin\nu$$
$$\zeta = R\cos\gamma \qquad (11.5)$$

Under these conditions the quantity $r = |\vec{\xi} - \vec{x}|$ may be written

$$r = \left(R^2 + \tilde{\rho}^2 - 2\tilde{\rho}R\cos\sigma\right)^{\frac{1}{2}} \qquad (11.6)$$

where

$$\cos\sigma = \sin\theta\sin\gamma\cos(\phi - \nu) + \cos\theta\cos\gamma \qquad (11.6a)$$

It is now apparent that in the far field analysis the linear dimensions of the shock layer are much smaller than the distance R from the origin to the observation point. Thus Equation (11.6) may be expanded in terms of the ratio $\tilde{\rho}/R$

$$r = R\left(1 - \frac{\tilde{\rho}}{R}\cos\sigma + \ldots\right) \qquad (11.7)$$

Consider next the integrand in Equation (11.4). The quantity r in the denominator affects the magnitude of the radiated field but it has no effect on the phase of the emitted wave, which is modified by the wave number k. Thus, in the denominator, little error will be affected by replacing r with R.

We will write Equation (11.4) in the following form:

$$\ln\frac{p}{p_0} = \frac{A\beta^2\sin\theta_0}{8\pi^2 a_0^2 R}\int_{\infty}^{\infty}\Gamma(\omega)e^{i(kR-\omega t)}d\omega\left(\frac{\partial}{\partial\theta}\iint e^{-ik\tilde{\rho}\cos\sigma}\,d\tilde{\rho}d\phi\right)_{\theta=\theta_0} \qquad (11.8)$$

and consider the expression in the bracket:

$$I = \frac{\partial}{\partial\theta}\iint e^{-ik\tilde{\rho}\cos\sigma}\,d\tilde{\rho}d\phi \qquad (11.9)$$

which should be evaluated at $\theta = \theta_0$ after the differentiation. Using Relation (11.6a) for $\cos\sigma$ this may be written as

$$I = \frac{\partial}{\partial\theta} \int_0^L e^{-ik\tilde{\rho}\cos\theta\cos\gamma}\, d\tilde{\rho} \int_0^{2\pi} e^{-ik\tilde{\rho}\sin\theta\sin\gamma\cos(\phi-\nu)}\, d\phi \qquad (11.10)$$

Employing the integral form of the Bessel function representation, the above expression takes on the following form:

$$I = 2\pi \frac{\partial}{\partial\theta} \int_0^L e^{ik\tilde{\rho}\cos\theta\cos\gamma}\, J_0(k\tilde{\rho}\sin\theta\sin\gamma)\, d\tilde{\rho} \qquad (11.11)$$

Performing the differentiation and setting $\theta = \theta_0$ one gets

$$I = 2\pi ik\sin\theta_0\cos\gamma \int_0^L e^{-ik\tilde{\rho}\cos\theta_0\cos\gamma}\, J_0(k\tilde{\rho}\sin\theta_0\sin\gamma)\tilde{\rho}\, d\tilde{\rho} +$$

$$-2\pi k\cos\theta_0\sin\gamma \int_0^L e^{-ik\tilde{\rho}\cos\theta_0\cos\gamma}\, J_1(k\tilde{\rho}\sin\theta_0\sin\gamma)\tilde{\rho}\, d\tilde{\rho} \qquad (11.12)$$

Using these results in Equation (11.8) the following expression for the forcing function of the conical shock results:

$$\ell n\left(\frac{p}{p_0}\right) = \frac{i\, A\beta^2\sin^2\theta_0\cos\gamma}{4\pi a_0^2 R} \int_{-\infty}^{\infty} \Gamma(\omega)k\, e^{i(kR-\omega t)}\, d\omega \int_0^L e^{-ik\tilde{\rho}\cos\theta_0\cos\gamma} \times$$

$$\times J_0(k\tilde{\rho}\sin\theta_0\sin\gamma)\tilde{\rho}\, d\tilde{\rho} - \frac{A\beta^2\sin 2\theta_0\cos\gamma}{8\pi a_0^2 R} \int_{-\infty}^{\infty} \Gamma(\omega)k\, e^{i(kR-\omega t)}\, d\omega \times$$

$$\times \int_0^L e^{-ik\tilde{\rho}\cos\theta_0\cos\gamma}\, J_1(k\tilde{\rho}\sin\theta_0\sin\gamma)\tilde{\rho}\, d\tilde{\rho} \qquad (11.13)$$

Equation (11.13) represents the forcing function of a conical shock layer radiating acoustic pressure waves into the surrounding quiescent region subject to the simplifying assumptions of the present analogy. The angle of the cone with the z-axis of the coordinates is represented by $\theta = \theta_0$. The length of the cone is represented by the radial coordinate $\tilde{\rho}$ such that:

$$0 \leq \tilde{\rho} \leq L$$

where L is the total length of the layer measured from the origin along the conical surface. The diameter of the shock is given by:

$$d = 2\tilde{\rho}\sin\theta_0 \qquad \text{thus} \quad D = 2L\sin\theta_0$$

D being the maximum shock diameter which is assumed to be of the order of the nozzle diameter unless more exact measurements are obtainable from experiments.

Equation (11.13) is plotted in Fig. 6 and 7 showing constant intensity lines for a number of frequencies.

In addition, simplified computer runs for the case of paraboloidal shock layers were performed using average constant values for the process parameter β^2. These are shown in Fig. 8.

The obtained results indicate that, for low frequencies, the directionality tends to align itself with the jet axis, whereas the higher frequency contributions swing away from the axis and tend to be normal to it (e.g., shock layer with $\theta_0 = 25°$. see Figure 6). A similar trend is noted in the case of $\theta_0 = 45°$, even though, in this case, the maximum high frequency directionality tends to but does not reach 90° (see Figure 7). The same trends hold true for other shock shapes (Figure 8).

As mentioned previously,* the overall intensity of radiation tends to a maximum in the directions normal to the shock layer. Noting that for the relevant range of Mach numbers the shock angles vary between 30° to 60° approximately (excluding Mach discs), it is to be expected, in the light of the present theory, that the maximum intensity will tend to occur in the same range with respect to the jet axis.

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* Section 10

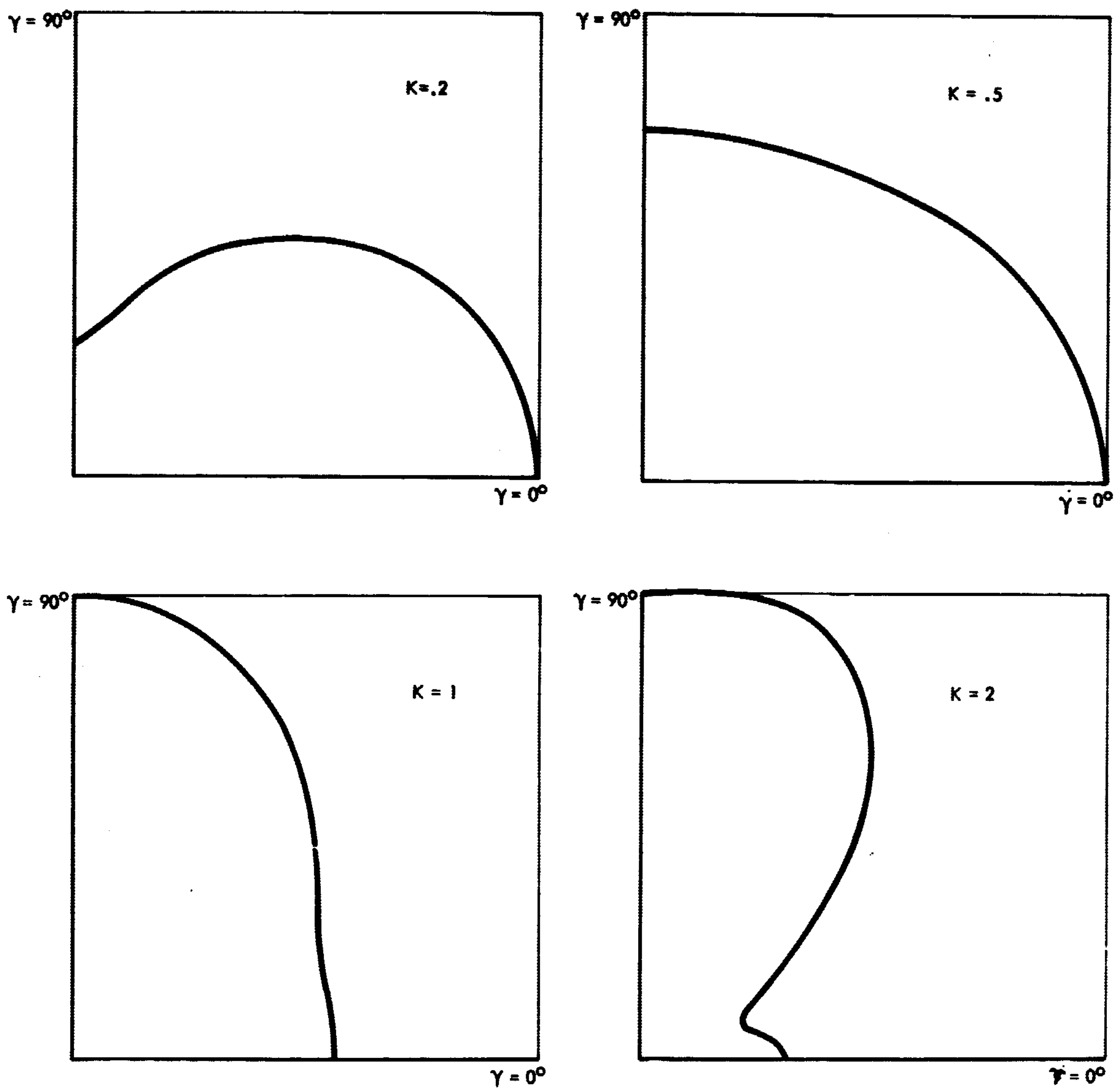

Figure 6a. Polar graphs of lines of constant intensity in the noise field of a 25° conical shock showing the frequency dependent directionality.

$$k = \frac{\omega}{a_0} \text{ varies between .2 and 2 (cgs units)}$$

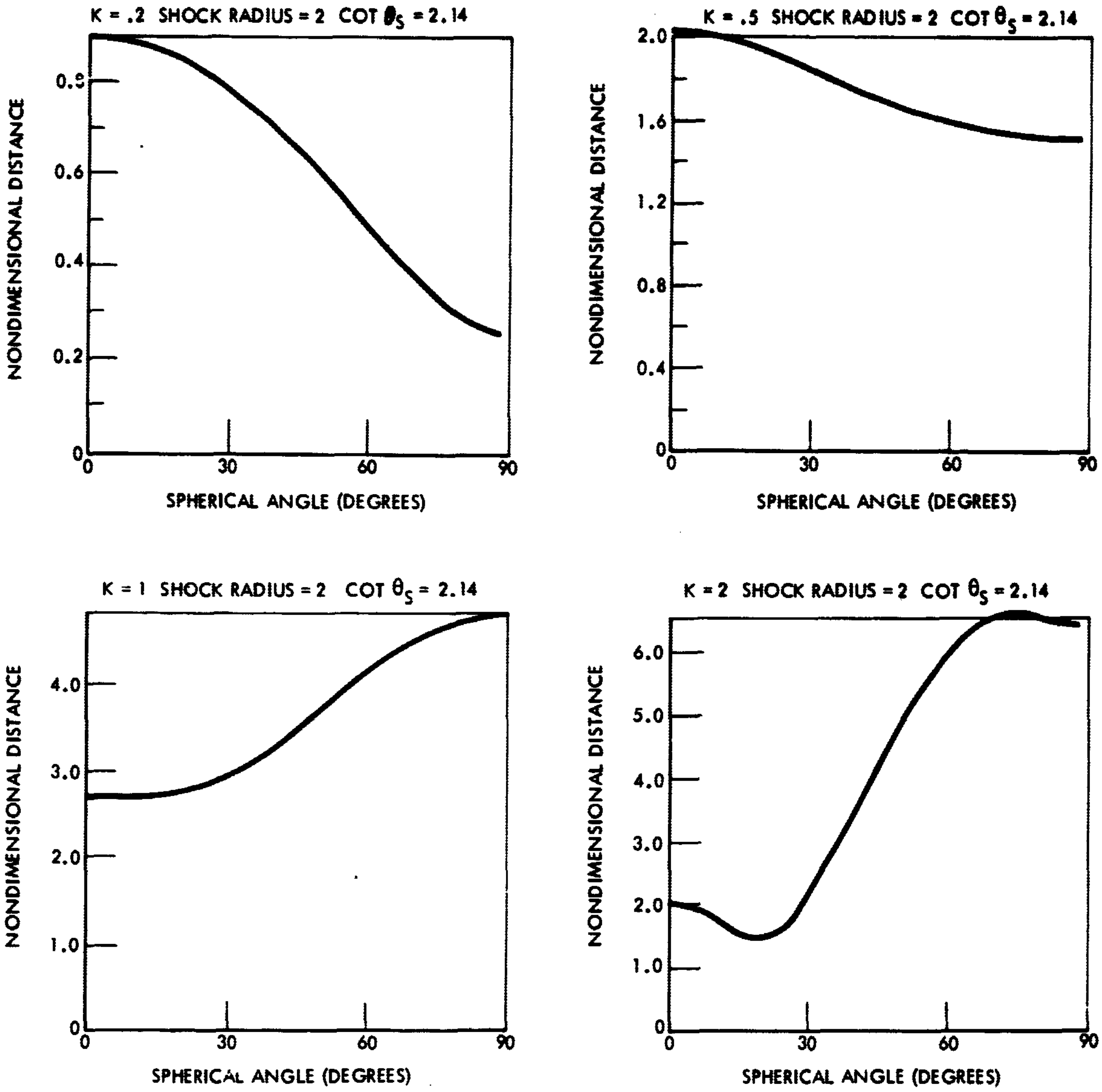

Figure 6b. Distance from shock versus spherical angle for lines of constant intensity in the field of a 25° conical shock.

$k = \frac{\omega}{a_0}$ varies between .2 and 2 (cgs units)

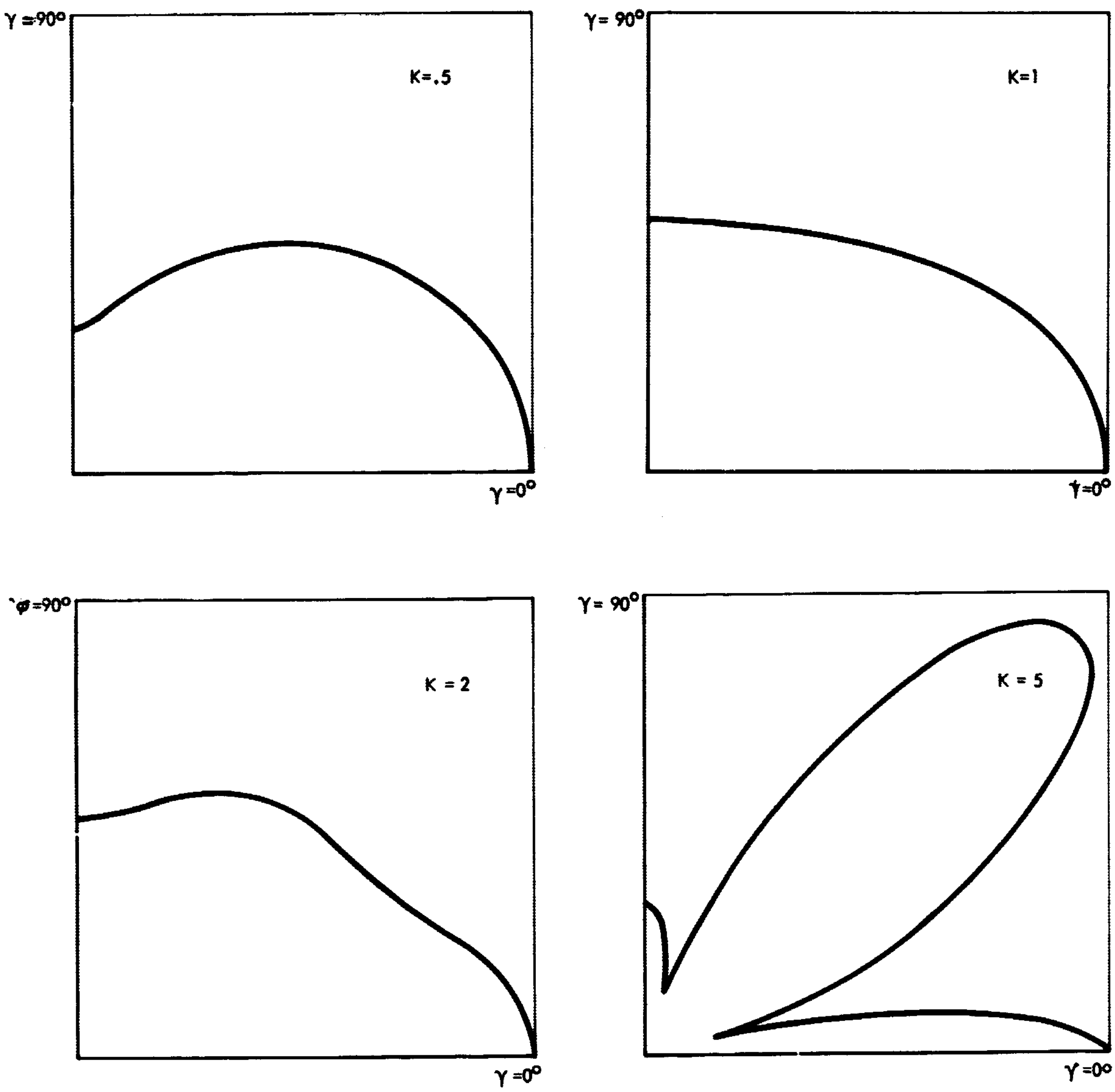

Figure 7a. Polar graphs of lines of constant intensity in the noise field of a 45° conical shock showing the frequency dependent directionality.

$k = \frac{\omega}{a_0}$ varies between .5 and 5 (cgs units)

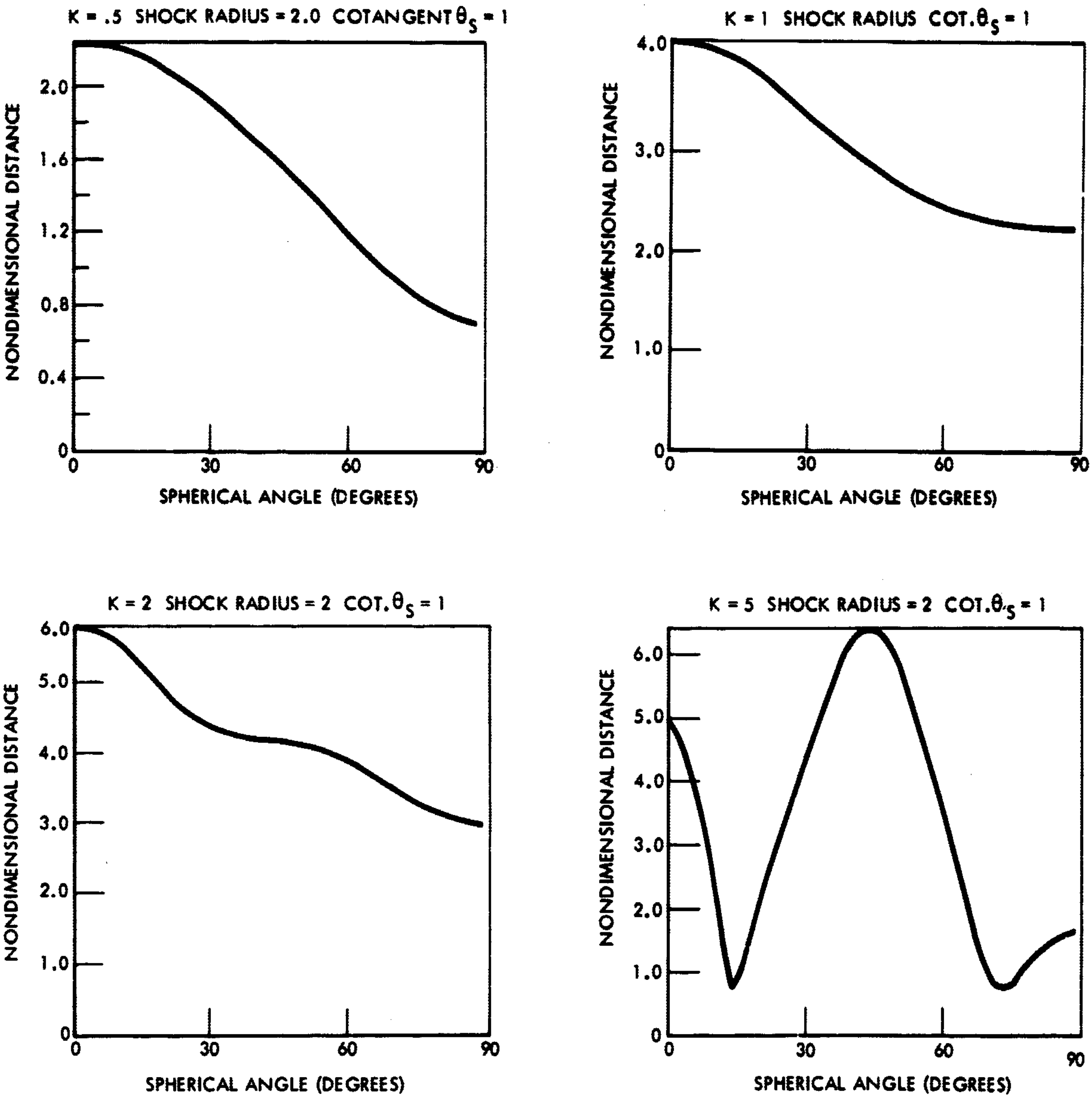

Figure 7b. Distance from shock versus spherical angle for lines of constant intensity in the field of a 45° conical shock. $k = \frac{\omega}{a_0}$ varies between .5 and 5 (cgs units).

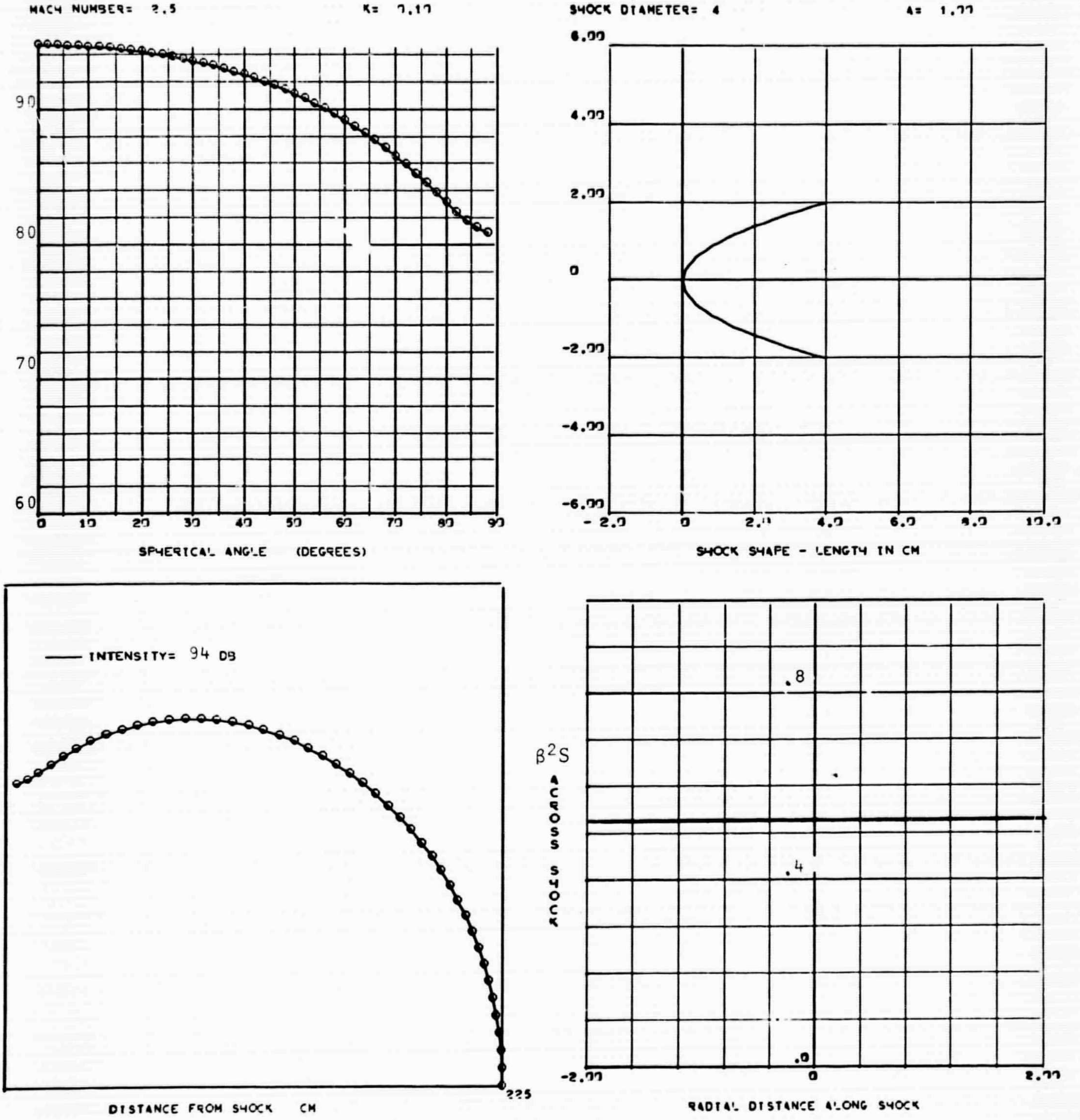

Figure 8a.

Directionality field of a Parabaloidal Shock at $k = \omega/a_0 = .10$ (cgs units) Upper left gives intensity vs. spherical angle. Lower left shows a polar plot of a line of constant intensity.

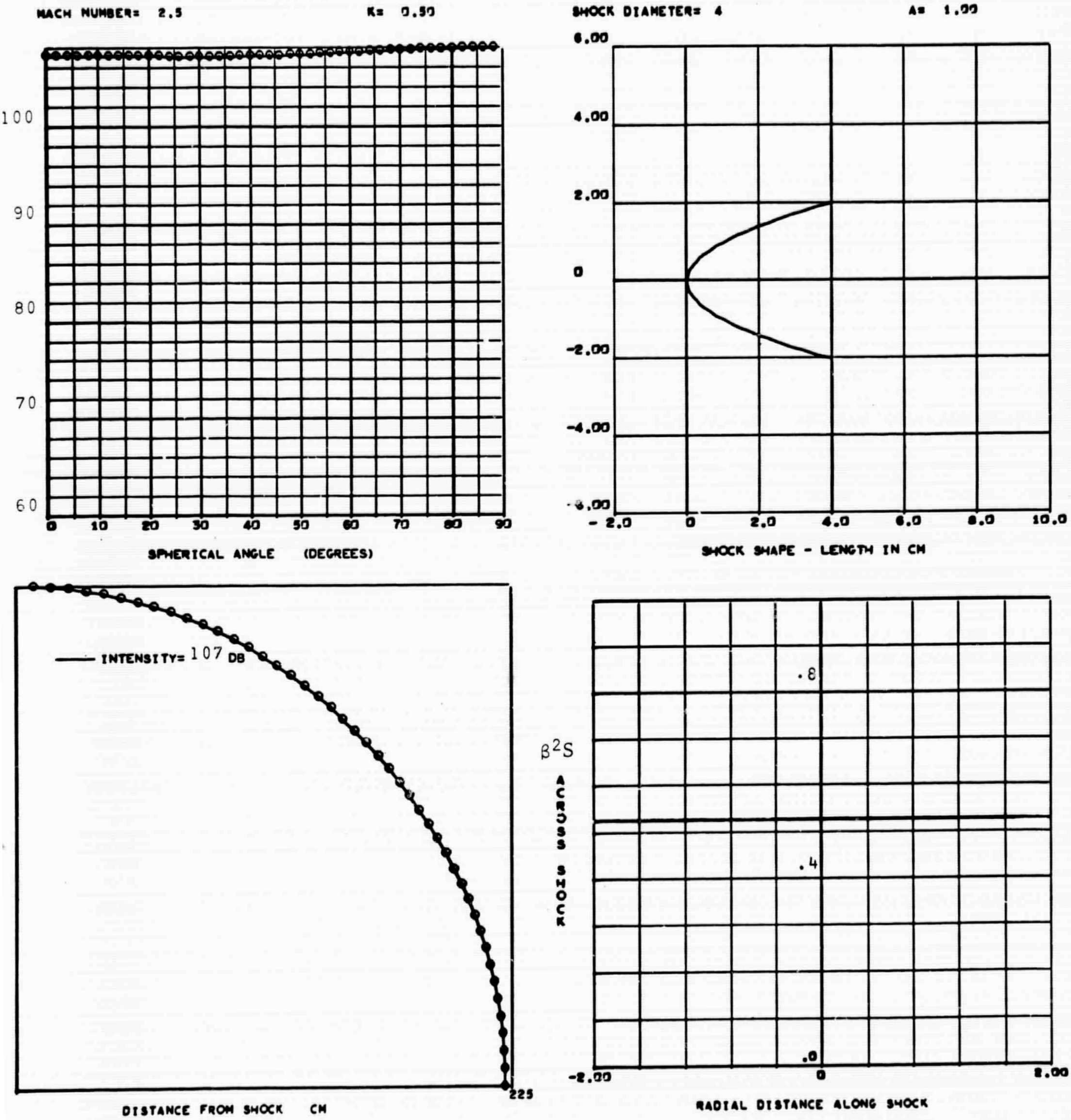

Figure 8b.

Directionality Field of a Parabaloidal Shock at $k = \omega/a_0 = .50$ (cgs units)

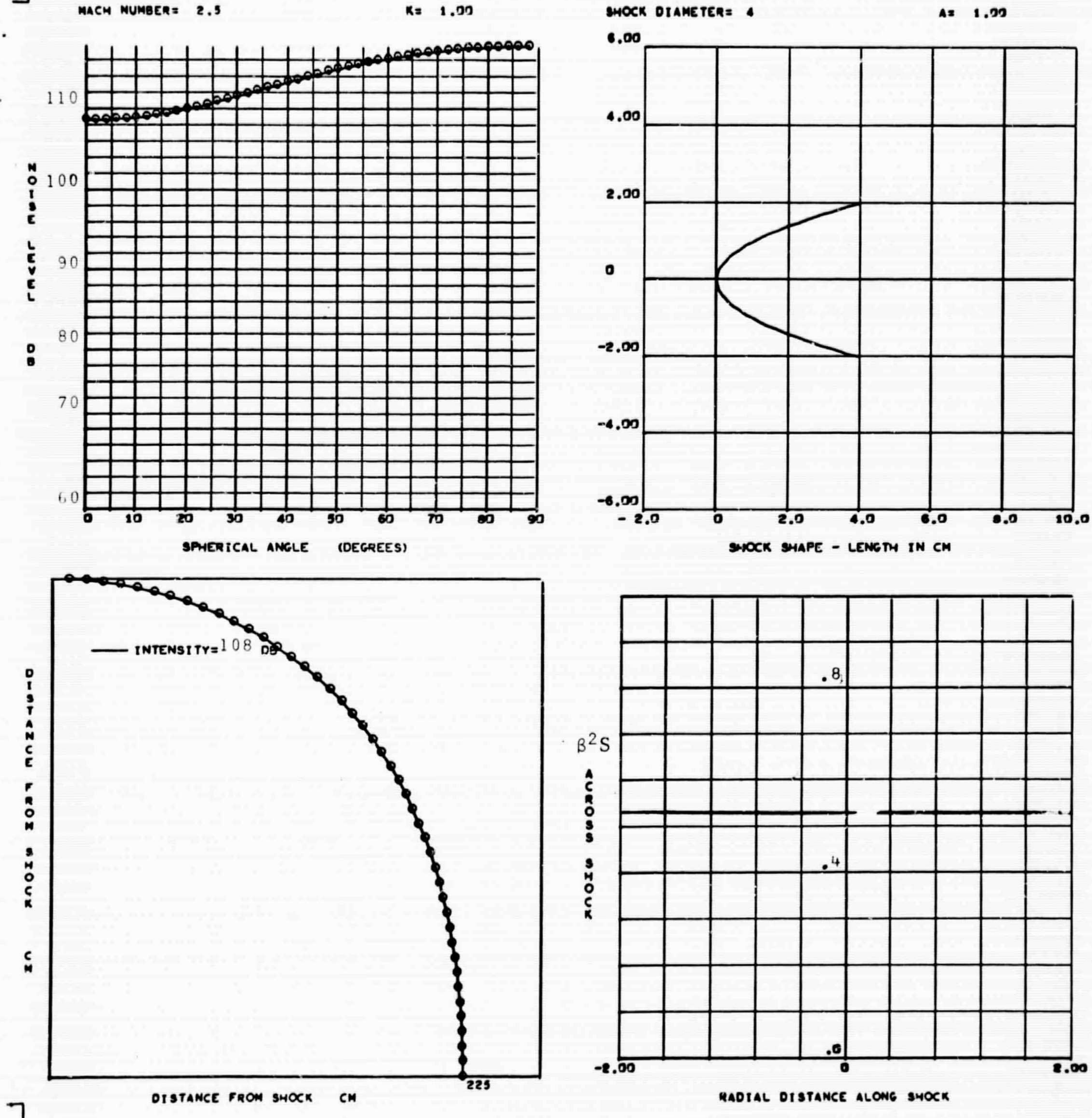

Figure 8c.

Directionality Field of a Parabaloidal Shock at $k = \omega/a_0 = 1.00$ (cgs units)

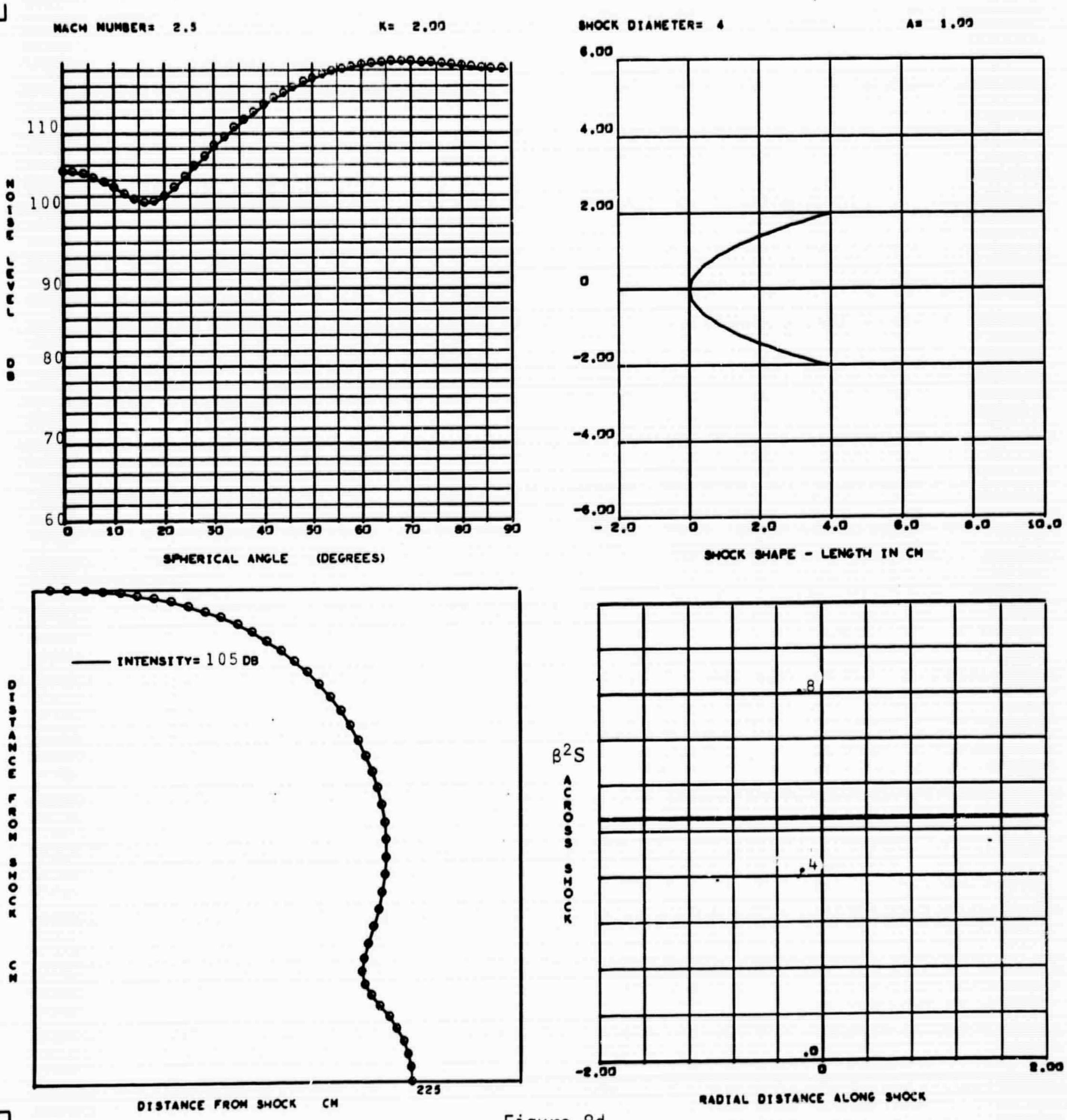

Figure 8d.

Directionality Field of a Parabaloidal Shock at $k = \omega/a_0 = 2.00$ (cgs units)

12. THE TRIGGERING MECHANISM

The attempted theoretical treatment of non-isentropic flow characteristics makes it feasible to offer some initial speculations concerning the causes and effects of the mechanism triggering wave emission from the entropy-producing regions. In accordance with the present derivation, an entropy-producing region is governed by a non self-adjoint (i.e., non-conservative) hyperbolic partial differential equation. As a consequence, an entropy wave is propagated into the dissipative region whenever a perturbation appears on its boundary or within the region itself (non-homogereous case). The stipulated growth of the propagated wave is assured by the second law of thermodynamics which determines the direction of the process. Incidentally, it may be readily verified that, when a non-distorted propagated wave pattern is assumed, calling for linearization of - and the vanishing of first-order derivatives from - the equation (self-adjoint form), isentropic flow conditions are implied, for in this case the propagation speed $\alpha = \alpha(S)$ must be constant.

It is now apparent that, in the light of the present theory, the behavior of such entropy-producing regions will be determined by the dissipative effects within this region and the excitations imparted to it at the boundary. In the present approach of shock layer application when the thickness of the region tends to zero the boundary effects would tend to be predominant. As a consequence of these effects, there seems to be a marked distinction between shock layers (regarded here as entropy-producing regions) formed by a supersonic motion of a solid in an unbounded medium and those formed in the exhaust of a supersonic nozzle. For in the first case when almost uniform conditions prevail upstream and downstream of the layers, their infinite extent in the remaining directions tends to suppress any possible excitations from these sources. On the other hand, shock regions formed in the exhaust of supersonic nozzles have to contend with the conditions of the free boundary at the termination of the layer in addition to its upstream and downstream perturbations. Thus, for supersonic-exhaust conditions potential instabilities are formed in the presence of any dissipative effects (e.g., shear layers, velocity fluctuations, vorticity, etc.), but the shock layers represent a first-order magnitude in comparison with other sources, which in turn enhance their instability by appearing at the free boundary where these shock layers terminate. It should also be noted that the same unstable characteristics occur in a moving rigid body when boundary-layer shock interactions take place, since the effect of rigid boundaries at the extremities of the shock are effectively counteracted by the fluctuating medium.

In the present engineering approach, the characteristic input of the boundary excitations is represented by a series of randomly distributed impulses imparted to the shock layer. The actual physical nature of these impulses is both space- and time-dependent, but their random characteristic fosters the belief that ergodic theory application would be a justification to represent them in terms of any one of these independent variables. As a consequence, a time-dependent behavior is used in the present approach.

Intuitively one could expect the Fourier transforms of these impulse functions to reflect different probability characteristics for different frequency bands into which these impulses are decomposed. It is thus feasible that the characteristic Fourier transform should also be a statistical aggregate of random impulse functions acting upon the shock layer. Accordingly this characteristic Fourier transform could hopefully be expressed as a frequency dependent probability distribution. From heuristic considerations such a distribution should tend to zero for large requency values to effectively ensure the existence of the Fourier Integral.

It is felt that a separate, more rigorous investigation of these initial conditions is needed in future developments of the subject matter. However, for the present application, a simple representation of the characteristic impulse function will be employed. Thus, the respective distribution has been assumed for

$$0 \leq \omega \leq \omega_0$$

to be equal to unity. This implies that the function $\Gamma(\omega)$, in terms of which the time-dependence of the forcing function has been written [Equation (10.4)] is equal to unity. Thereafter the transform is chosen to fall off as $\frac{\omega_0}{\omega}$ as ω grows large. This choice effectively prevents the divergence of the Fourier integral for large values of frequency, even though it is by no means unique.

Based upon the above model, it was found from the computational efforts that the overall behavior of the resultant theoretical curves could be correlated with experimental data in a straightforward manner with a single exception. Namely, in all cases the frequency spectrum of the theoretical results had to be shifted by a constant factor. This could be expected due to the employment of the present acoustical analogy in which convective effects and the Doppler shift were not included.*

Employment of the above model for the disturbances triggering non-isentropic wave emission completes the overall theoretical representation of the forcing function governing the acoustic propagation of shock layers as derived in Section 10. The computation of power spectral densities of the resulting noise field were accomplished by the application of random noise theory.** The choice of the Poisson distribution as a representative probability function for the successive occurrences of the emission phenomena was decided upon.** A typical CRT plot of spectrum for the initial conditions stipulated above is presented in Figure 9. From the results of the computation it can readily be seen that empirical results also indicate the necessity of the distribution to tend to zero for large values of frequency [see Figure 9 and Figure 11]. The plot represents a simulated run based upon Smith's Report (E. B. Smith, 1966) for $\gamma = 50°$.

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* Ribner, 1962.

** Rice, 1941, pp. 294-325.

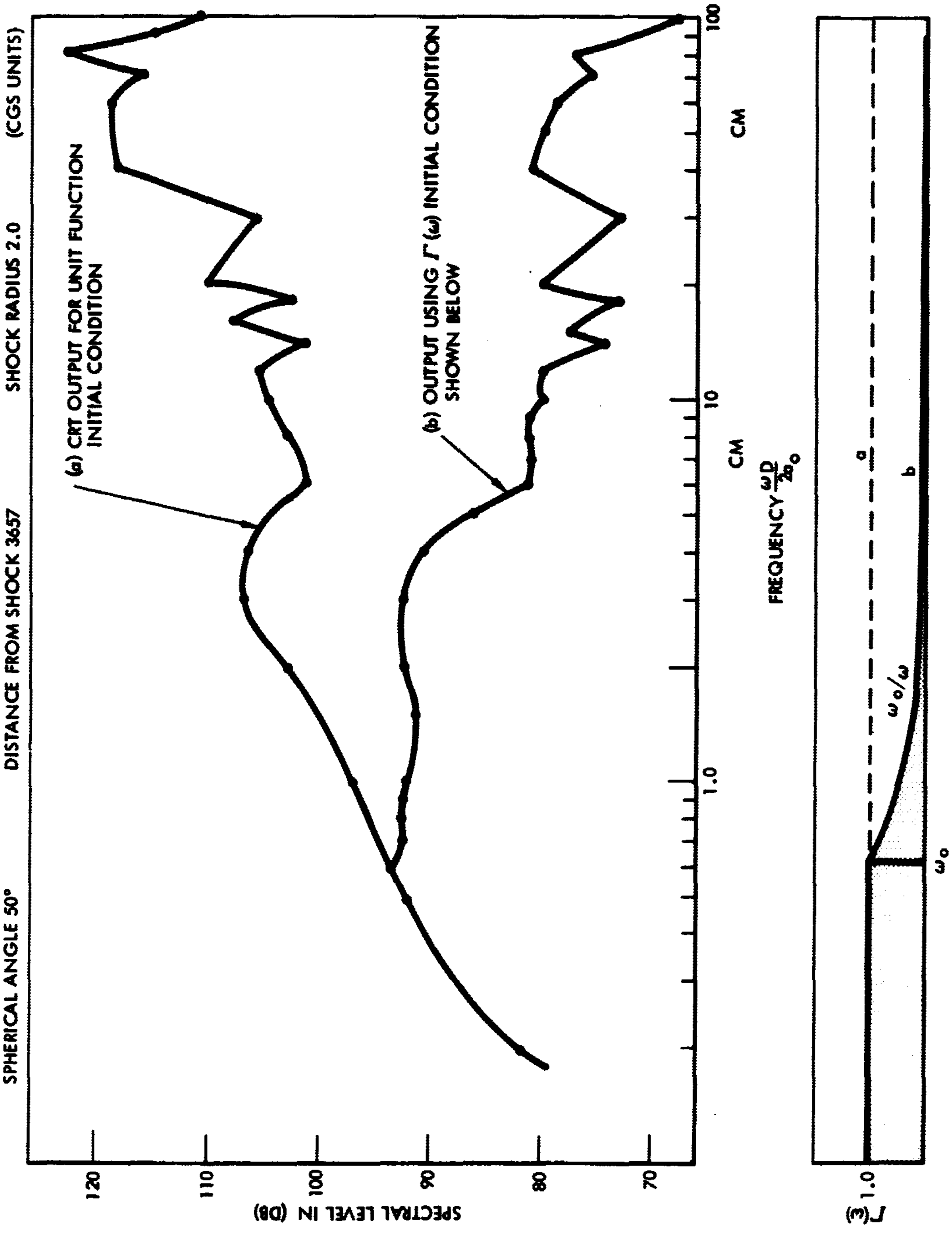

Figure 9. A Characteristic CRT Plot of Spectrum for Two Typical Time Input Functions (Simulation of Smith's Experiment; γ = 50°.

13. EXPERIMENTAL VERIFICATION

Correlation of the obtained theoretical results with experiments was attempted by actual simulation of given nozzle and flow characteristics. In certain cases conjecture had to be used concerning the shock structures of the nozzle under consideration, but whenever possible these were obtained from personal contacts.* Thus, actual simulation runs were carried out using appropriate values for shock layer angle, its diameter, upstream Mach number, etc. The various flow parameters used in the simulation of Smith's experiments are presented in Table II. The results of these theoretical and experimental correlations are shown in Figures 10 to 12.

Two shortcomings of the present acoustical analogy became apparent as soon as comparison with the available data for a given simulation run were made. In the first place, a shift in the frequency spectra of the theoretical computations was noted, its value being an absolute constant for each nozzle simulated. Since the present analogy did not account for convective and Doppler shift effects (Section 10) it appears that such behavior of the computed results could be avoided by taking these into account.

Secondly, it was found that the acoustic contribution of the theoretical model tends to be overly exaggerated in the neighborhood of the jet axis. The reason for this behavior may again be found in the

* e.g., Martin Co. report, Courtesy E. B. Smith.

simplified character of the present analogy, since the attenuative nature of this region is probably due to the effects of refraction (Ribner, 1966) which were disregarded in the engineering computations.

On the other hand, the overall theoretical results and the characteristic trends of jet noise phenomena seem to be closely following the general pattern of experimental findings. From the simulation runs of Smith's report the following facts may be noted.

a) In both the theory and experiment the directivity of higher frequency contribution shifts toward 90° from the jet axis (Figure 10).

b) The computed results show a directivity shift back towards a 65° angle from the axis for frequency increase above 3200 cps (Figure 6b, K=2.0 plot). The experimental data of the report indicate an identical trend, a 2.5 db intensity drop occurring between 70° and 90° from the axis for center band frequency of 10^4 cps. [Smith, page B-3, Program Firing #1].

c) The total relative theoretical intensity change (as a function of the spherical angle γ) is computed to be of the order of 12 db. The above characteristic and the actual decibel count are well substantiated by the experimental findings (Figure 10).

d) The experimental measurements of density spectra exhibit a sufficient number of data points to highlight characteristic fluctuations of the curve in the higher frequency regions. The computed values seem to analytically confirm this behavior indicating it to be a characteristic Bessel function variation (Figure 11).*

e) The computed spectral characteristics show a relative increase of about 20 db/decade in the low frequency ranges. The measured experimental values confirm the validity of the above result (Figure 11).

f) The overall directionality characteristics of the theoretical model show a marked correlation with the experimental measurements (Figure 12).

A point of experimental controversy arises when the changes of intensity with frequency are considered as a function of the spherical angle γ. In Smith's report the directionality curves have the same bell-shaped character, the maximum intensity varying with the frequency. On the other hand, both the General Electric report (Lee, Smith, et al. 1961) and the F-1 engine data** indicate an exchange of energy takes place in the range

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* In correlating the spectral densities, a conversion from the 1/3 octave band analysis used in experiments has been effected.

** Courtesy G. Wilhold, Unsteady Aerodynamics Group, NASA, MSFC.

$40° \leq \gamma \leq 90°$. Thus, as the frequency increases, the maximum intensity at about 40° decreases and that about 90° increases, the curve pivoting about an almost stationary maximum value. The theoretical results of the present model tend to support the latter but not the former trend

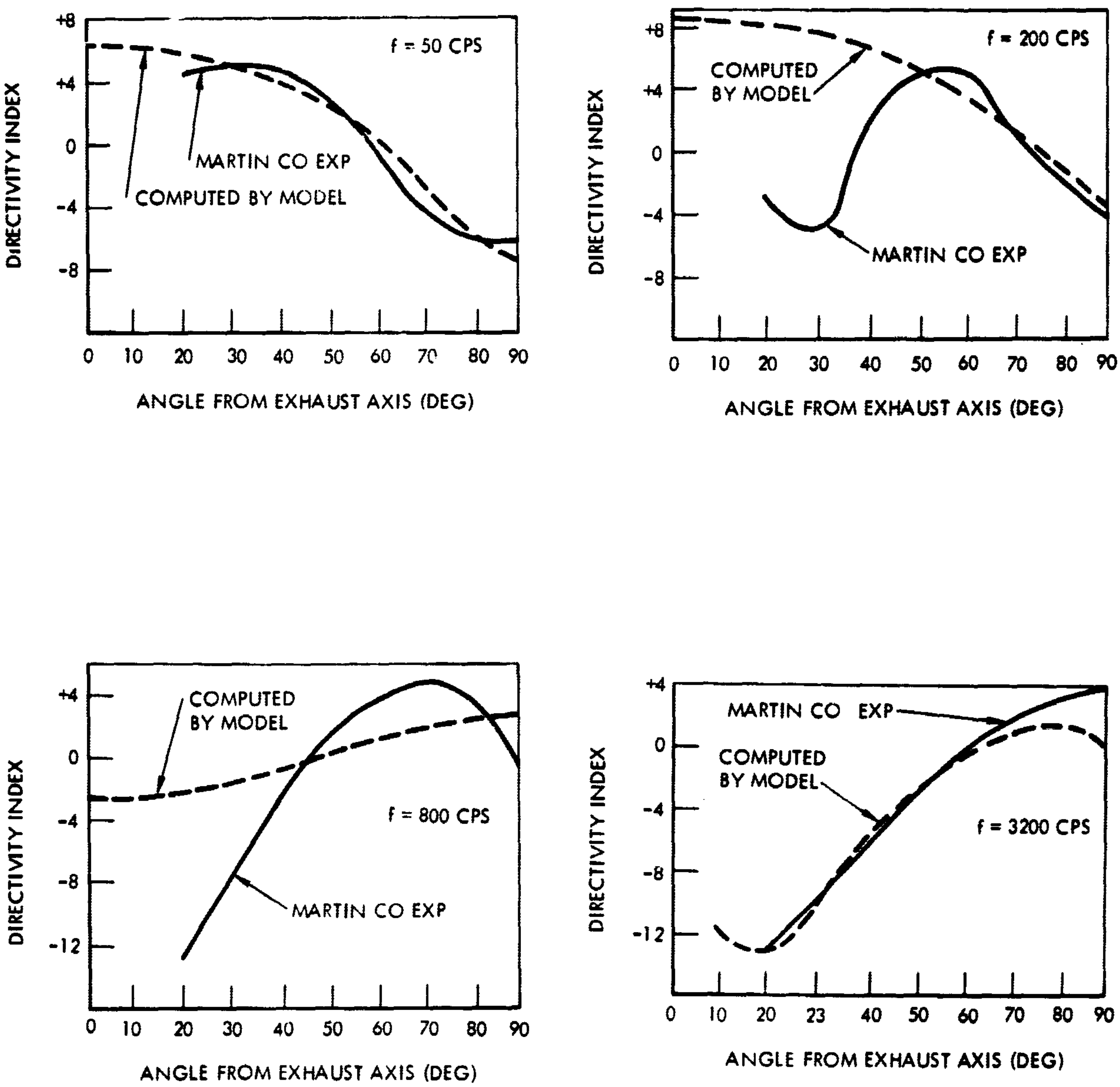

Figure 10. Comparison of Frequency Directivity Indices as Computed by Model and as Determined Experimentally by Smith.

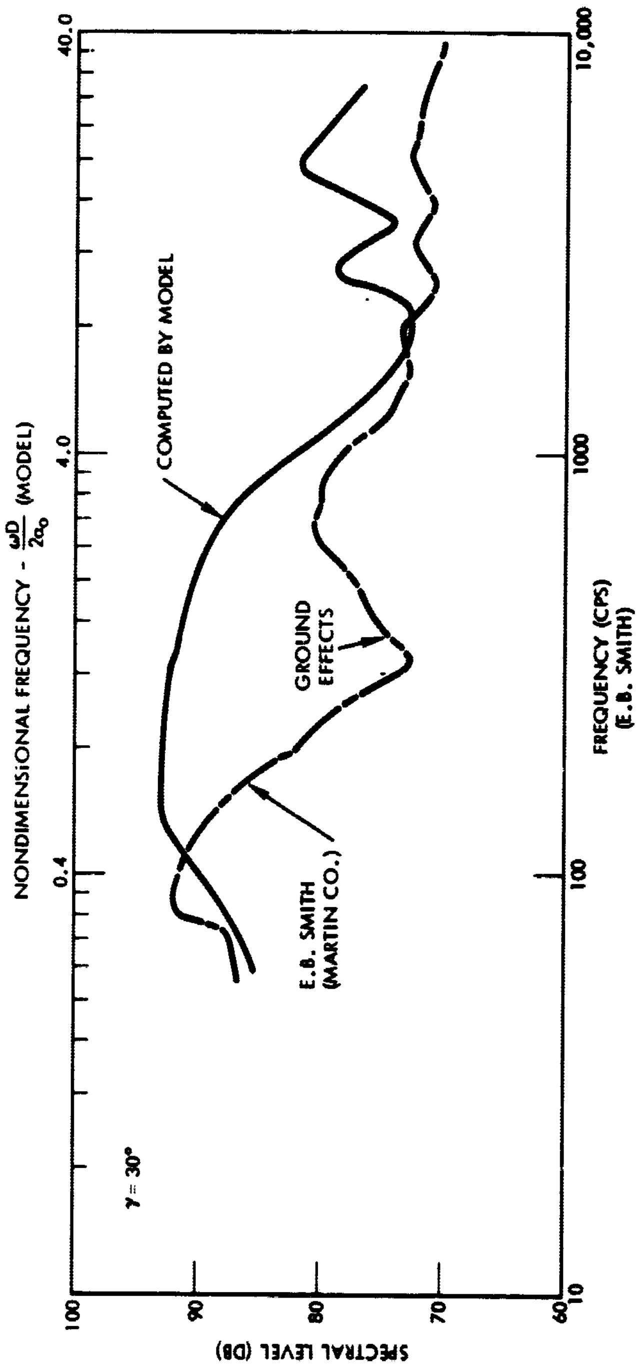

Figure 11a. Comparison of Spectra Computed by Model with Those Obtained Experimentally by Smith (Program Firing No. 1)

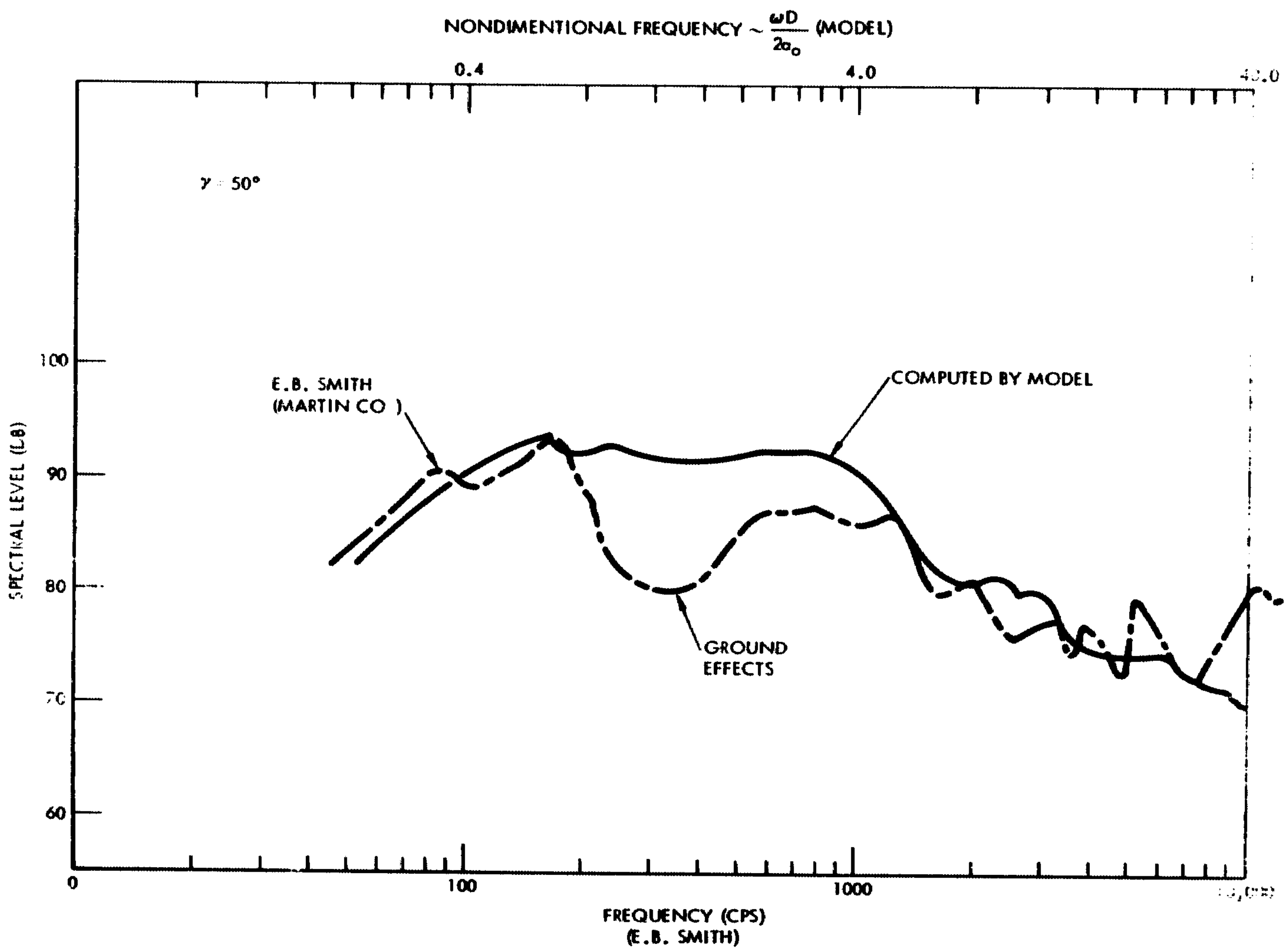

Figure 11b. Comparison of Spectra Computed by Model with Those Obtained Experimentally by Smith (Program Firing No. 1)

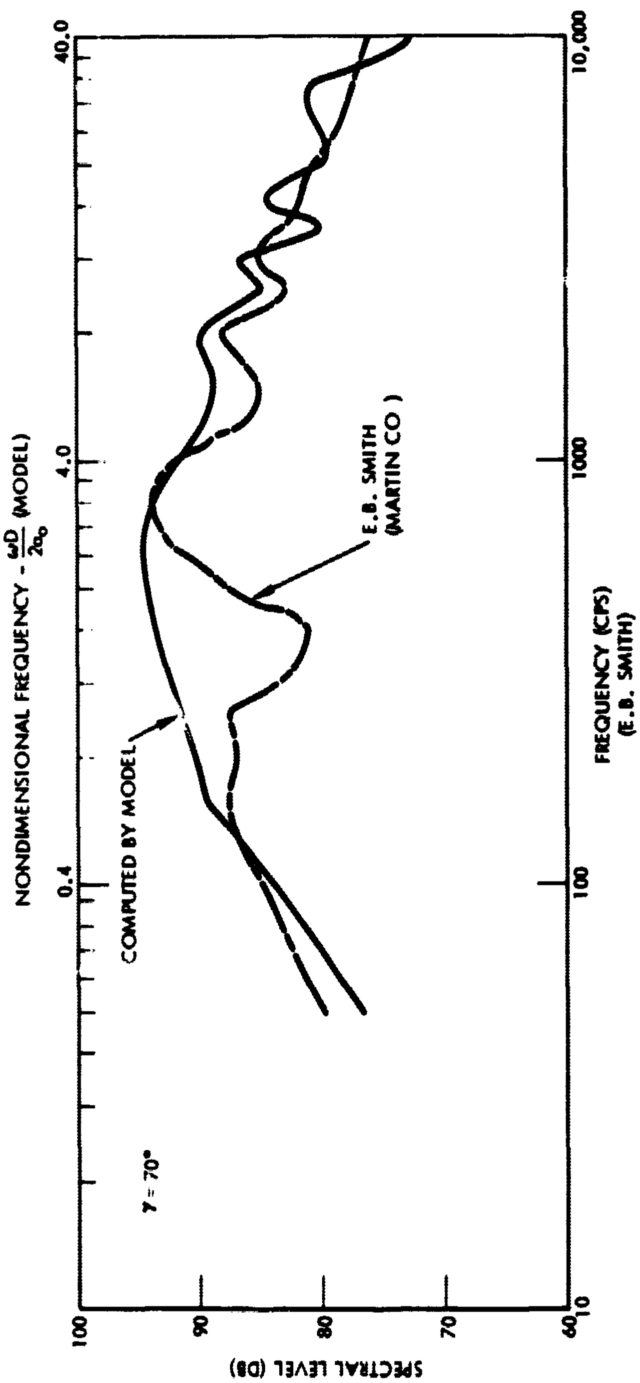

Figure 11c. Comparison of Spectra Computed by Model with Those Obtained Experimentally by Smith (Program Firing No. 1)

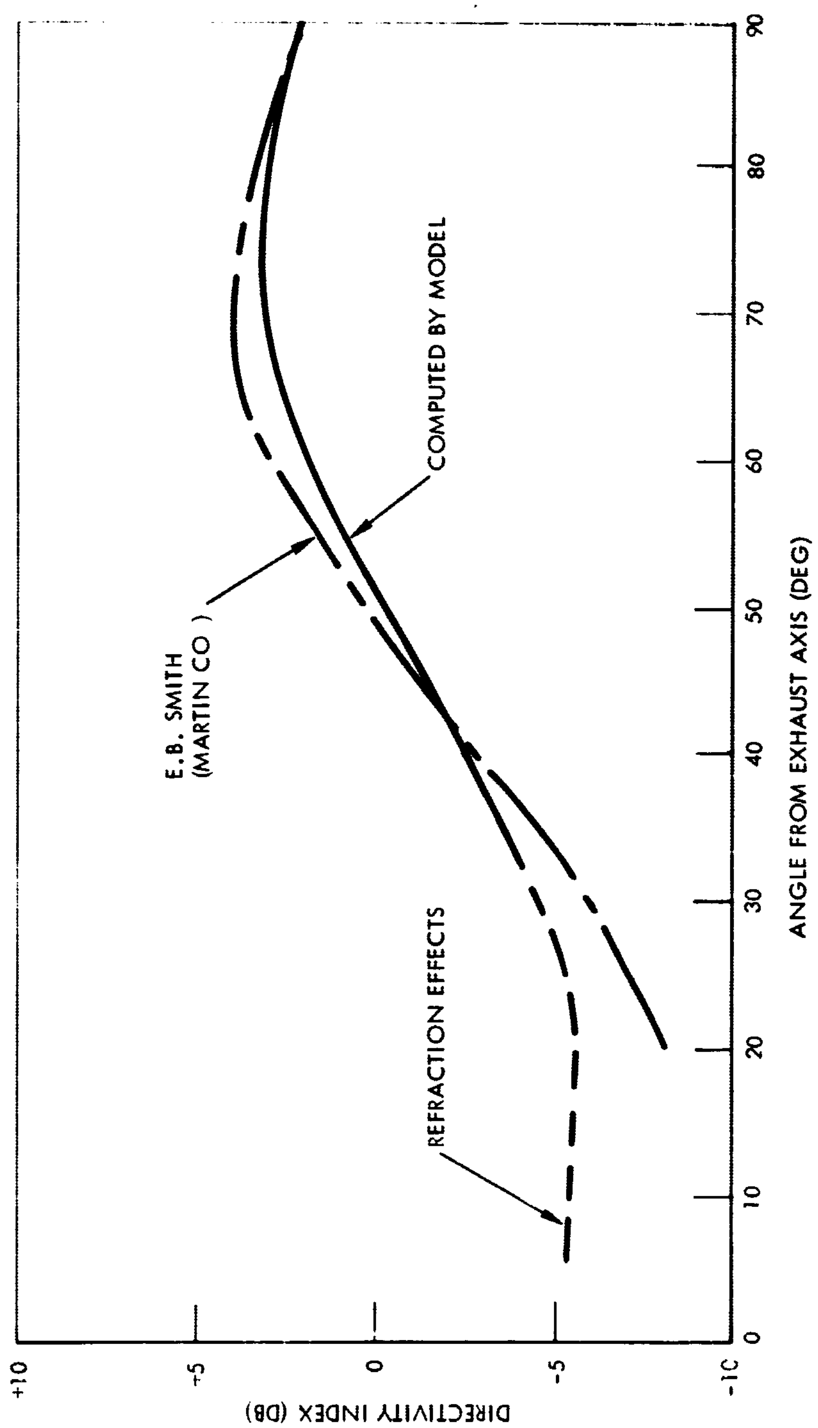

Figure 12. Overall Directivity Simulation of Smith's Experiments (Program Firing No. 1)

TABLE II

Smith's Simulation

Shock Diameter (D)	4 cm
Shock angle (θ)	25°
Distance from Shock R	3756 cm
Exit Mach Number	3.5
Non-dimensional cutoff frequency, $\frac{\omega D}{2a_0}$	.6

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CONCLUDING REMARKS

The preceding investigation of non-isentropic propagation phenomena was motivated by the desirability to represent in an analytical manner the contribution of dissipative regions to aerodynamic noise generation. An engineering application of this analysis is reflected in the concept of an EPN (extended plug nozzle) device. Its function is to modify entropy-producing regions (shock waves) in order to attenuate noise generation of of high-speed nozzle exhausts. In the present study a mathematical model is formulated and analyzed for some specific cases to determine the noise generation characteristics of supersonic nozzles. For these specific cases existing test data appear to correlate well with the theoretical results.

As it is in most cases, the present approach is based upon a rather simple basic idea which may have a tendency to become lost in the technicalities of the derivation. Its essence is the choice of independent thermodynamic variables which would depict non-isentropic pressure fluctuations.

It may be readily seen that if, for instance, the pressure and entropy be chosen as independent variables, all pressure gradients become isentropic and, as a consequence, so do pressure fluctuations. This fact seemed an unwarranted restriction upon the physical phenomena taking place in entropy-producing regions.

It should also be remarked that the choice of the thermodynamic J-function was based upon physical aspects of the preceding analysis coupled with dimensional considerations. The actual derivation of the J process was obtained, however, from an analysis of convected entropy changes in equilibrium flows. This latter alternate approach has not been presented here due to the initial stages of its development and also to avoid excessive complications of the present analysis. It is felt, moreover, that the scope of such a derivation warrants additional analytical efforts in a subject which may be incidental to the propagation characteristics of a finite amplitude pressure wave.

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