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BOUNDS FOR THE NATURAL FREQUENCIES OF A PLATE SUBJECTED TO A THERMAL GRADIENT

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SUMMARY

The effect of a constant thermal gradient on the transverse vibrational frequencies of a simply supported rectangular plate is investigated. Bounds for the eigenfrequencies are obtained for various plate width-to-length ratios as functions of a parameter related to the temperature dependence of the modulus of elasticity of the material. The upper bounds are calculated by the Rayleigh-Ritz method and the lower bounds by the Bazley-Fox Second Projection method. In all instances, the gap between the bounds over their average is less than one half of one per cent.

NOMENCLATURE

Latin Symbols

a	Length of the plate
a_i	Constants of linear combination, equation (III-29)
a_{ij}	Matrix elements, equation (III-18)
b	Width of the plate
b_{ij}	Matrix elements, equation (III-22)
c_i	Constants of linear combination, equation (III-1)
$D(x)$	Flexual rigidity variation, equation (II-4)
D_L	Domain of the operator L
d_{ij}	Matrix elements, equation (III-23)
E	Modulus of elasticity
H	Hilbert space
L	Differential operator, equation (II-14)
L^o	Self-adjoint differential operator, equation (III-8)
L'	Self-adjoint positive definite operator, equation (III-10)
L^k	Intermediate operator, equation (III-17)
$L^{\ell,k}$	Intermediate operator, equation (III-21)
p	Circular frequency of motion, equation (II-7)
p_i	Independent elements of D_L'
r	Parameter, equation (II-11)
T	Temperature excess above a given reference
u	Function
u_i	Function in D_L
$\bar{W}(\bar{x},\bar{y},t)$	Deflection of the plate
x	Non dimensional coordinate
$\bar{x}$	Actual coordinate in the plane of the plate
$\bar{y}$	Coordinate perpendicular to $\bar{x}$ in the plane of the plate

Greek Symbols

α	Parameter, equation (II-3)
β	Slope of the variation of E with T
γ	Positive number, equation (III-21)
δ_{ij}	Kronecker delta
λ	Eigenvalue of the operator L
λ_i^o	Eigenvalue of the operator L^o, equation (III-13)
$\lambda_i^{l,k}$	Eigenvalue of the operator $L^{l,k}$
ϕ	Eigenfunction of the operator L
ϕ_i^o	Eigenfunction of the operator L^o, equation (III-12)
$\phi^{l,k}$	Eigenfunction of the operator $L^{l,k}$
ν	Poisson's ratio

I. INTRODUCTION

The response of structures subjected to thermal environments is affected by the development of thermal stresses and by the deterioration of the materials of construction. The problem of the effect of temperature on the modulus of elasticity is far from being negligible to aircraft and rocket designers, for instance, because for such materials as titanium alloys the modulus may be half of its room temperature value at 1000^{o}F.(1)* Other materials are also affected, and experimental investigations(1,2,3,4,5) have shown that a linear relationship between Young's modulus and temperature provides a good correlation for most engineering materials. Thus, in the presence of steady thermal gradients, the elastic coefficients of homogeneous materials become functions of the space variables. The determination of vibrational characteristics of continuous elastic systems must then be based on non-homogeneous elastic theory.

Although the problem has long been recognized, few solutions appear in the literature. In particular, few attempts have been made at the determination of natural frequencies of plates with variable flexural rigidity. The existing solutions are usually concerned with plates of variable thickness(6-11), but not with material-induced variable rigidities.

The object of this study is to determine the effect of the non-homogeneity caused by a thermal gradient on the natural frequencies of a free, simply supported plate of uniform thickness. A steady temperature distribution is considered in one direction. Any viscoelastic effects are considered negligible in comparison with the induced non-homogeneity

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*Parenthetical references placed superior to the line of the text refer to the bibliography.

effects. Bounds for the eigenfrequencies are computed by the Rayleigh-Ritz method and the Bazley-Fox Second Projection method as functions of a parameter associated with the severity of the thermal gradient and the temperature dependence of the material's modulus of elasticity.

II. FORMULATION OF THE PROBLEM

Consider a rectangular plate of uniform thickness h, length a, and width b, subjected to a steady one-dimensional temperature distribution $T = T_0(1-x)$ where T denotes the temperature excess above the reference temperature at any point x, T_0 denotes the temperature excess above the reference temperature at the end x=a , and $x = \bar{x}/a$.

For most engineering materials, the temperature dependence of the modulus of elasticity is given by a relation of the type

$$E(T) = E_1(1-\beta T) \tag{II-1}$$

where

E_1 is the value of the modulus at some reference temperature

T is the temperature excess above the reference temperature

β is the slope of the variation of E with T .

Taking as the reference temperature, the temperature at the end of the plate x = a, the modulus variation becomes

$$E = E_1\left[1-\alpha(1-x)\right] \tag{II-2}$$

where the parameter α is defined by

$$\alpha = \beta T_0 \qquad (0 \leq \alpha \leq 1) \tag{II-3}$$

The flexural rigidity of the plate can now be written as

$$D(x) = D_0\left[1-\alpha(1-x)\right] \tag{II-4}$$

where

$$D_0 = \frac{E_1 h^3}{12(1-\nu^2)} \tag{II-5}$$

The known differential equation for the deflection $\bar{w}(\bar{x},\bar{y},t)$ of a vibrating plate with flexural rigidity $D(\bar{x},\bar{y})$ is

$$D\nabla^4\bar{w} + 2\frac{\partial D}{\partial \bar{x}}\frac{\partial}{\partial \bar{x}}\nabla^2\bar{w} + 2\frac{\partial D}{\partial \bar{y}}\frac{\partial}{\partial \bar{y}}\nabla^2\bar{w} + \nabla^2 D\,\nabla^2\bar{w} - (1-\nu)\left(\frac{\partial^2 D}{\partial \bar{x}^2}\frac{\partial^2 \bar{w}}{\partial \bar{y}^2} - 2\frac{\partial^2 D}{\partial \bar{x}\partial \bar{y}}\frac{\partial^2 \bar{w}}{\partial \bar{x}\partial \bar{y}} + \frac{\partial^2 D}{\partial \bar{y}^2}\frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\right) = -\rho h\frac{\partial^2 \bar{w}}{\partial t^2} \tag{II-6}$$

For harmonic vibrations, $\bar{w}$ has the form

$$\bar{w}(\bar{x},\bar{y},t) = w(\bar{x},\bar{y})\sin pt \tag{II-7}$$

where p is the circular frequency of the motion.

Separation of the variables $\bar{x}$ and $\bar{y}$, i.e. the search for solutions of the form

$$w(\bar{x},\bar{y}) = \phi(\bar{x})\,\Psi(\bar{y}) \tag{II-8}$$

yields, for simple supports,

$$\Psi_n = A_n \sin\frac{n\pi y}{b} \tag{II-9}$$

and the differential equation for ϕ :

$$\left[1-\alpha(1-x)\right]\left[\frac{d^4\phi}{dx^4} - 2r^2\frac{d^2\phi}{dx^2} + r^4\phi\right] + 2\alpha\left[\frac{d^3\phi}{dx^3} - r^2\frac{d\phi}{dx}\right] = \lambda\phi \tag{II-10}$$

where

$$r = \frac{n\pi a}{b} \tag{II-11}$$

and

$$\lambda = \frac{a^4\rho h p^2}{D_0} \tag{II-12}$$

The boundary conditions for simple supports are

$$\phi = \frac{d^2\phi}{dx^2} = 0 \qquad \text{at} \quad x = 0 \quad \text{and} \quad x = 1 \tag{II-13}$$

The problem of the determination of the plate natural frequencies consists now of the solution of the eigenvalue problem corresponding to equations (II-10) and (II-13).

The differential operator will be represented by L, i.e.

$$L\phi = \lambda\phi \tag{II-13}$$

Consider the Hilbert space H of real square integrable functions defined on $[0,1]$, with the inner product (u,v) defined by

$$(u,v) = \int_0^1 uv\,dx \tag{II-15}$$

for any two functions u and v in H.

The domain of L in H, D_L, consists of the set of functions of class C^4 satisfying the boundary conditions (II-13). As integration by part shows, L is self adjoint over its domain, i.e.

$$(Lu,v) = (u,Lv) \tag{II-16}$$

for every pair u and v in D_L.

Since over D_L, L is positive definite, i.e.

$$(Lu,u) = \int_0^1 \left[1-\alpha(1-x)\right]\left[\left(\frac{d^2u}{dx^2}\right)^2 + 2r^2\left(\frac{du}{dx}\right)^2 + r^4u^2\right]dx \geqslant 0 \tag{II-17}$$

with the equality sign holding only for u = o, the eigenvalues λ are known to be positive and to tend to infinity with no finite limit point. They are assumed to be ordered in increasing order of magnitude

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots \tag{II-18}$$

and the corresponding eigenfunctions ϕ_i are assumed to be normalized to satisfy

$$(\phi_i, \phi_j) = \delta_{ij} \qquad i,j = 1,2,\ldots \tag{II-19}$$

In the following section, the method of solution of the eigenvalue problem is described.

III. METHOD OF SOLUTION

Since exact solutions to equation (II-10) are not known, approximate solutions must therefore be sought. The number of techniques to estimate eigenvalues of self-adjoint operators that have appeared in the literature is enormous. (See for instance references 12,13,14,15). In this study, the Rayleigh-Ritz method was used to calculate upper bounds to the desired eigenvalues, and the Bazley-Fox Second Projection method to compute lower bounds. The theoretical foundations for these methods have appeared in the literature, and consequently only the details needed for their application to the problem on hand are presented. For mathematical proofs, the reader is referred to references 12 through 16.

a. Upper Bounds

The basic idea of the Rayleigh-Ritz method consists in determining the stationary values of the Rayleigh quotient, $(Lu,u)/(u,u)$, not over all admissible functions u, but only over the linear manifold spanned by n linearly independent functions u_i satisfying the prescribed boundary conditions of the operator L. The problem then consists in finding the functions u of the form

$$u = \sum_{i=1}^{n} c_i u_i \tag{III-1}$$

i.e. in finding the constants c_i, making the Rayleigh quotient stationary. The result is the general matrix eigenvalue problem

$$\left[(u_i, Lu_j)\right]\left[c_j\right] = \bar{\lambda}\left[(u_i, u_j)\right]\left[c_j\right] \tag{III-2}$$

Since the class of admissible functions is restricted to the linear manifold, it follows that the eigenvalues $\bar{\lambda}_j$ are upper bounds for those of L, i.e.

$$\lambda_j \leq \bar{\lambda}_j \qquad j = 1, 2, \ldots n \tag{III-3}$$

Furthermore, it follows that as n increases the upper bounds are improved, or at least not worsened.

For the problem under consideration, these functions were chosen to be

$$u_i = \sqrt{2} \sin i\pi x \tag{III-4}$$

which satisfy the boundary conditions (II-13), and are such that

$$(u_i, u_j) = \delta_{ij} \tag{III-5}$$

Evaluation of the inner products needed in equation (III-2) yields

$$\begin{aligned} (u_i, Lu_j) &= 0 && \text{for } i \neq j, (i \pm j) \text{ even} \\ &= \frac{-8\alpha(ij)\left[\pi^4 (ij)^2 + r^2\pi^2(i^2+j^2) + r^4\right]}{\pi^2 (i^2-j^2)^2} && \text{for } i \neq j, (i \pm j) \text{ odd} \\ &= \left(1 - \frac{\alpha}{2}\right)\left[(i\pi)^2 + r^2\right]^2 && \text{for } i = j \end{aligned} \tag{III-6}$$

These expressions were substituted in equation (III-2), and the resulting matrix eigenvalue problem was solved numerically for various values of the parameter α and of the length-to-width ratio, a/b, for 15 x 15 matrix sizes. The improvement in the upper bounds with increasing matrix size is illustrated in Table 1 where the results correspond to the value of $\alpha = 0.50$.

b. Lower Bounds

The computation of lower bounds presents considerably more difficulties. The outline of the Bazley-Fox Second Projection method is presented here. For details of proofs or construction of the needed operators, the reader is referred to reference 16.

Consider the operator L as the sum of two operators L^0 and L' such that

$$L = L^0 + L' \tag{III-7}$$

where L^0 is the self-adjoint operator

$$L^0 u = (1-\alpha)\left[\frac{d^4 u}{dx^4} - 2r^2 \frac{d^2 u}{dx^2} + r^4 u\right] \tag{III-8}$$

with boundary conditions

$$u = \frac{d^2 u}{dx^2} = 0 \qquad \text{at} \quad x=0 \ \text{and} \ x=1 \tag{III-9}$$

and L' is the operator

$$L' u = \alpha \frac{d^2}{dx^2}\left[x \frac{d^2 u}{dx^2}\right] - 2\alpha r^2 \frac{d}{dx}\left[x \frac{du}{dx}\right] + \alpha r^4 x u \tag{III-10}$$

with the boundary conditions given by equation (III-9). L' can be easily shown to be self-adjoint and positive definite.

The eigenvalue problem for L^0, namely

$$L^0 \phi^0 = \lambda^0 \phi^0 \tag{III-11}$$

is readily solvable. The normalized eigenfunctions ϕ_i^0 are

$$\phi_i^0 = \sqrt{2} \sin i\pi x \tag{III-12}$$

and the corresponding eigenvalues are

$$\lambda_i^0 = (1-\alpha)\left[(i\pi)^2 + r^2\right]^2. \tag{III-13}$$

Comparison of the Rayleigh quotients for L and L^0 indicates that

$$(L^0 u, u) \leq (Lu, u) \tag{III-14}$$

for all functions u in the coinciding domains of L and L^0.

Consequently, their ordered eigenvalues satisfy the inequalities

$$\lambda_i^0 \leq \lambda_i \qquad , \quad i = 1, 2, \ldots \tag{III-15}$$

The lower bounds λ_i^0 are, however, quite remote from the desired eigenvalues. To improve them, intermediate operators L^k are constructed so that they have the same domains as L^0 and their eigenvalues satisfy the inequalities

$$\lambda_i^0 \leq \lambda_i^k \leq \lambda_i^{k+1} \leq \lambda_i. \qquad , \quad i = 1, 2, \ldots \tag{III-16}$$

Considering k linearly independent vectors belonging to the domain of L' , the k-th intermediate operator is given by

$$L^k u = L^0 u + \sum_{i=1}^{k} \sum_{j=1}^{k} a_{ij}\,(u, L' p_j)\, L' p_i \tag{III-17}$$

where the elements a_{ij} are the entries of the following matrix

$$[a_{ij}] = \left[(L' p_i, p_j)\right]^{-1} \tag{III-18}$$

Having chosen the linearly independent vectors as

$$p_i = \phi_i^0 = \sqrt{2}\,\sin i\pi x \tag{III-19}$$

the inner products take the forms

$$(L'p_i, p_j) = (Lu_i, u_j) \quad \text{for } i \neq j$$
$$= \frac{\alpha}{2}\left[(i\pi)^2 + r^2\right]^2 \quad \text{for } i = j \qquad \text{(III-20)}$$

where the expressions for (Lu_i, u_j) are given by equation (III-6).

To avoid the difficulties associated with the solution of the eigenvalue problem for L^k, Bazley and Fox[16] have introduced smaller operators for which the problem reduces to a finite linear algebraic problem. These operators are denoted by $L^{l,k}$. They are given by

$$L^{l,k}u = \left[A^0 - \gamma\right]u + \sum_{i=1}^{l}\sum_{j=1}^{l} b_{ij}(u, \phi_j^0)\,\phi_i^0 \qquad \text{(III-21)}$$

where γ is a positive number whose optimum value for the determination of $\lambda_j^{l,k}$ is such that $\lambda_{l+1}^0 - \gamma = \lambda_j^{l,k}$. Since $\lambda_j^{l,k}$ is not known, γ must be chosen from an estimate of it, which for this problem was taken as $\bar{\lambda}_j$. This choice, therefore, tends to give a lower bound for λ_j slightly smaller than the value obtainable from the optimum γ.

The matrix $[b_{ij}]$ is the inverse of the matrix with elements

$$[b_{ij}]^{-1} = \frac{1}{\gamma}\left[\delta_{ij} - \sum_{m=1}^{k}\sum_{n=1}^{k}(\phi_i^0, L'p_m)\,d_{mn}\,(L'p_n, \phi_j^0)\right] \qquad \text{(III-22)}$$

where $[d_{mn}]$ is the matrix inverse to that with elements

$$[d_{mn}]^{-1} = \left[(L'p_m, L'p_n) + \gamma(L'p_m, p_n)\right] \qquad \text{(III-23)}.$$

Its evaluation requires the following inner products, in addition to those given in equation (III-20):

$$\left(L' p_i, L' p_j\right) = \frac{(-1)^{i+j} 4 i j \alpha^2 \sqrt{\lambda_i^0 \lambda_j^0}}{(1-\alpha)(i^2-j^2)} \left\{ \frac{2\sqrt{\lambda_i^0 \lambda_j^0}}{\pi^2 (1-\alpha)(i^2-j^2)} + \frac{\sqrt{\lambda_i^0} - \sqrt{\lambda_j^0}}{\sqrt{1-\alpha}} \right\} \quad \text{for } i \neq j$$

$$= \frac{2\alpha^2 \lambda_i^0}{(1-\alpha)} \left\{ \frac{\lambda_i^0 \left[2(i\pi)^2 - 3\right]}{12 (i\pi)^2 (1-\alpha)} + \frac{\sqrt{\lambda_i^0}}{2\sqrt{1-\alpha}\, i\pi} + 2 i^2 \pi^2 \right\} \quad \text{for } i = j \qquad \text{(III-24)}$$

The eigenvalues of $L^{l,k}$ can be shown to satisfy the inequalities

$$\lambda_i^{l,k} \leq \lambda_i^{l+1,k} \leq \lambda_i^{k} \leq \lambda_i \qquad \text{(III-25)}$$

The solution of the eigenvalue problem

$$L^{l,k} \phi^{l,k} = \lambda^{l,k} \phi^{l,k} \qquad \text{(III-26)}$$

is now accomplished as follows:

1. If $\phi^{l,k}$ is orthogonal to the span of $\{\phi_i^0\}_{i=1}^{l}$, then the problem reduces to

$$\left[L^0 - \gamma\right] \phi^{l,k} = \lambda^{l,k} \phi^{l,k} \qquad \text{(III-27)}$$

which has solution

$$\left.\begin{aligned} \lambda_j^{l,k} &= \lambda_j^0 - \gamma \\ \phi_j^{l,k} &= \phi_j^0 \end{aligned}\right\} \quad \text{for } j > l \qquad \text{(III-28)}$$

2. If $\phi^{l,k}$ is in the span of $\{\phi_i^o\}_{i=1}^{l}$, it can be written as

$$\phi^{l,k} = \sum_{m=1}^{l} a_m \phi_m^o \qquad \text{(III-29)}$$

which upon substitution in equation (III-26) and consideration of the linear independence of the eigenfunctions of L^o yields the matrix eigenvalue problem

$$\left[(\lambda_i^o - \gamma)\,\delta_{ij} + b_{ij} - \lambda^{l,k}\delta_{ij}\right]\left[a_i\right] = \left[0\right] \quad , \; i,j = 1,2 \ldots l \qquad \text{(III-30)}$$

The eigenvalues of $L^{l,k}$ are then obtained by ordering the results of equations (III-28) and (III-30).

Computations were performed for various values of the parameter α and of the length-to-width ratio, a/b, for $k = l = 15$. The results presented in Table 2 illustrate the improvement in the lower bounds as the sizes of the intermediate operators are increased. The results shown correspond to $\alpha = 0.50$.

IV. RESULTS AND DISCUSSION

The following procedure was used in the calculation of the eigenfrequencies of the plate subjected to a thermal gradient. For a given a/b ratio, and for a given value of α , the bounds for the eigenvalues λ in equation (II-10) were computed for fixed values of the parameter r defined in equation (II-11). They were obtained by letting the order n take the discrete values 1 through 10 successively. The upper bounds were computed using 15 x 15 matrix sizes and the lower bounds were calculated for intermediate operators with $k = l = 15$. Consequently, a sequence of eigenvalue problems (one for each value of r) were solved to take into account the possible combinations of mode shapes in the x- and y-directions. For a given value of the a/b ratio and a given value of α , the results were arranged in two matrices, one for the upper bounds and one for the lower bounds, giving the approximations to the eigenvalues of the plate for the order of the mode shapes in the x- and y-directions. It was then possible to order by inspection the eigenfrequencies in ascending orders of magnitude.

In all instances, the Rayleigh-Ritz method coupled with the Bazley-Fox Second Projection method yielded excellent results since the gaps between the bounds over their average remained less than one half of one per cent.

The effect of the temperature gradient on the first three eigenvalues of the plates is illustrated in Figures 1 through 3 for values of the a/b ratio equal to 0.5, 1.0, 1.5, and 2.0. The ratio of each eigenvalue to the corresponding eigenvalue of the plate at the reference temperature is presented as a function of the a/b ratio and of the parameter α which represents the severity of the thermal gradient and of the temperature dependence of the plate modulus of elasticity. Comparison between the variations of the three eigenvalues indicates that the effect of the thermal gradient depresses the higher harmonics faster than the fundamental. This conclusion appears to be valid for harmonics of higher orders too.

Of particular interest is the phenomenon illustrated in Figure 3 by the variation of the third eigenvalue for the square plate. As shown on the figure, the third eigenvalue for a/b = 1.0 is much lower than the third eigenvalue corresponding to the other length-to-width ratios. Furthermore, the point $\lambda_3 / \lambda_{3_{\alpha=0}}$ = 1.00 is a point of discontinuity. The explanation for this phenomenon is as follows: for the plate at the reference temperature, i.e. for α = 0.0, the second eigenvalue corresponds indifferently to the combination of the first mode in the x-direction and the second mode in the y-direction, or to the combination of the second mode in the x-direction and the first mode in the y-direction. For instance, for α = 0.0 and a/b = 1.0, λ_2 = 2435.227. The third eigenvalue in this case corresponds to the combination of the second modes in the x- and y-directions. It has the value λ_3 = 6234.181. In the presence of the thermal gradient, however, the rigidity of the plate in the x-direction is weakened. Consequently, the second eigenvalue of the reference plate "branches off" to yield two frequencies: the combination of the first mode in the x-direction with the second mode in the y-direction yields λ_2 , while the combination of the second mode in the x-direction with the first mode in the y-direction yields λ_3 . For instance, for a/b = 1.0, α = 0.1,

$\lambda_2 = 2312.092$ and $\lambda_3 = 2312.318$. Hence the third eigenvalue is quite remote from the third eigenvalue of the reference plate. This phenomenon is also exhibited by plates with other length-to-width ratios for harmonics of orders higher than three.

The results presented here apply only to a linear temperature distribution. In themselves, they should be of value to designers of equipment subjected to thermal gradients. Furthermore, since in many instances the temperature distribution can be approximated by a linear variation, they can be useful in estimating the effect of other gradients on the natural frequencies of simply supported plates.

The method of solution can be used for other temperature variations. The main difficulty in the lower bounds method resides in splitting the differential operator so that a convenient base problem results.

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APPENDIX A

TABLE 1

UPPER BOUNDS

$\alpha = 0.50$

a/b	Order	Matrix Size		
		3 x 3	5 x 5	15 x 15
0.5	1	112.318	112.300	112.298
	2	287.121	287.073	287.068
	3	756.137	755.994	755.980
1.0	1	308.669	287.073	287.068
	2	1921.999	1782.059	1782.021
	3	1926.745	1789.367	1789.268
1.5	1	813.863	755.994	755.980
	2	3011.857	2797.900	2797.736
	3	9904.087	7043.085	7042.871
2.0	1	1921.999	1782.059	1782.021
	2	4938.209	4589.557	4589.263
	3	13,245.020	12,095.120	12,089.490

TABLE 2

LOWER BOUNDS

$\alpha = 0.50$

a/b	Order	Matrix Size		
		3 x 3	5 x 5	15 x 15
0.5	1	112.102	112.139	112.146
	2	286.498	286.577	286.622
	3	755.104	755.289	755.291
1.0	1	286.498	286.577	286.622
	2	1781.365	1781.780	1781.820
	3	1785.879	1788.649	1789.088
1.5	1	755.104	755.289	755.291
	2	2793.130	2797.098	2797.660
	3	8843.354	9041.179	9047.130
2.0	1	1781.365	1781.780	1781.820
	2	4582.040	4588.270	4589.148
	3	11,767.570	12,081.850	12,089.250

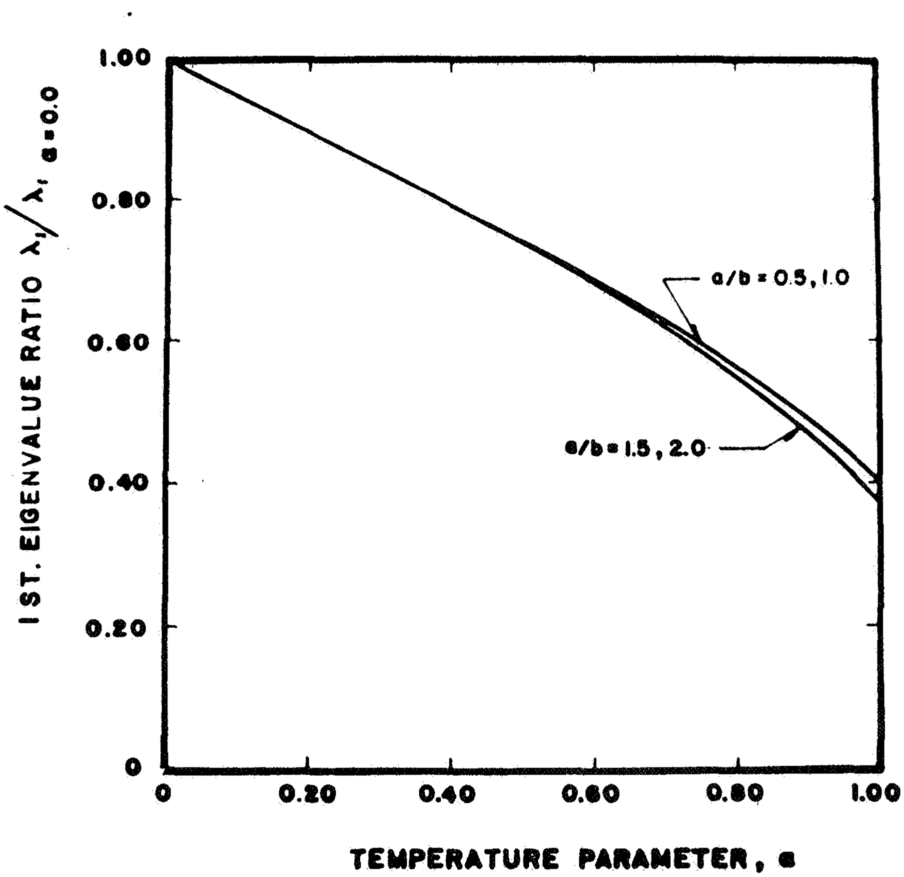

Figure 1

Effect of Thermal Gradient on Fundamental Eigenvalue Ratio

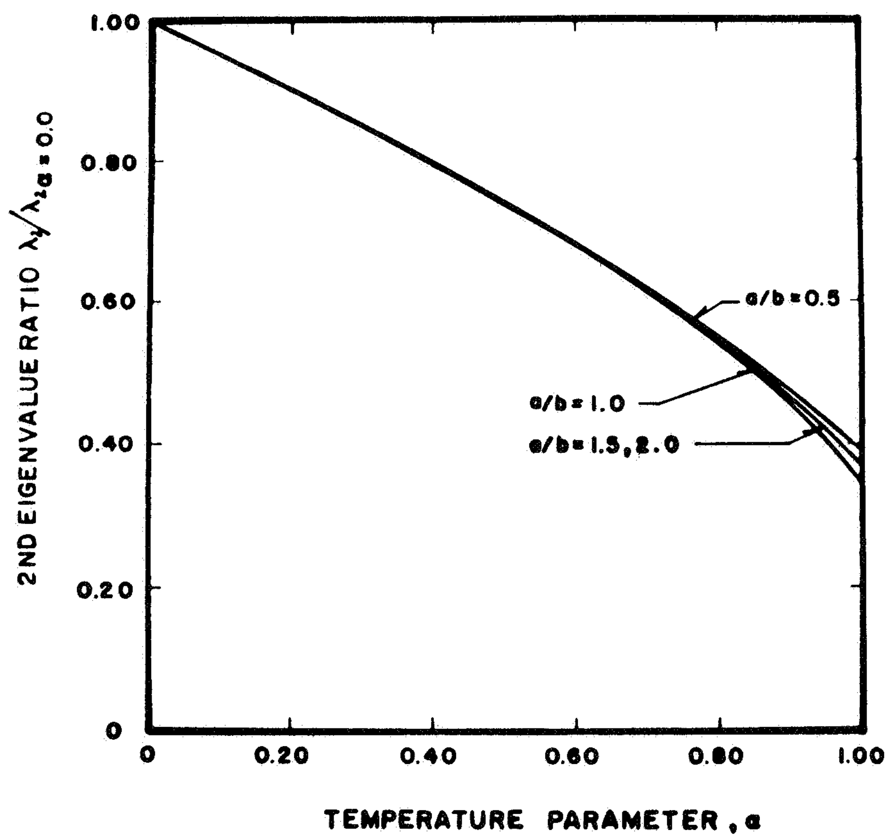

Figure 2

Effect of Thermal Gradient on Second Eigenvalue Ratio

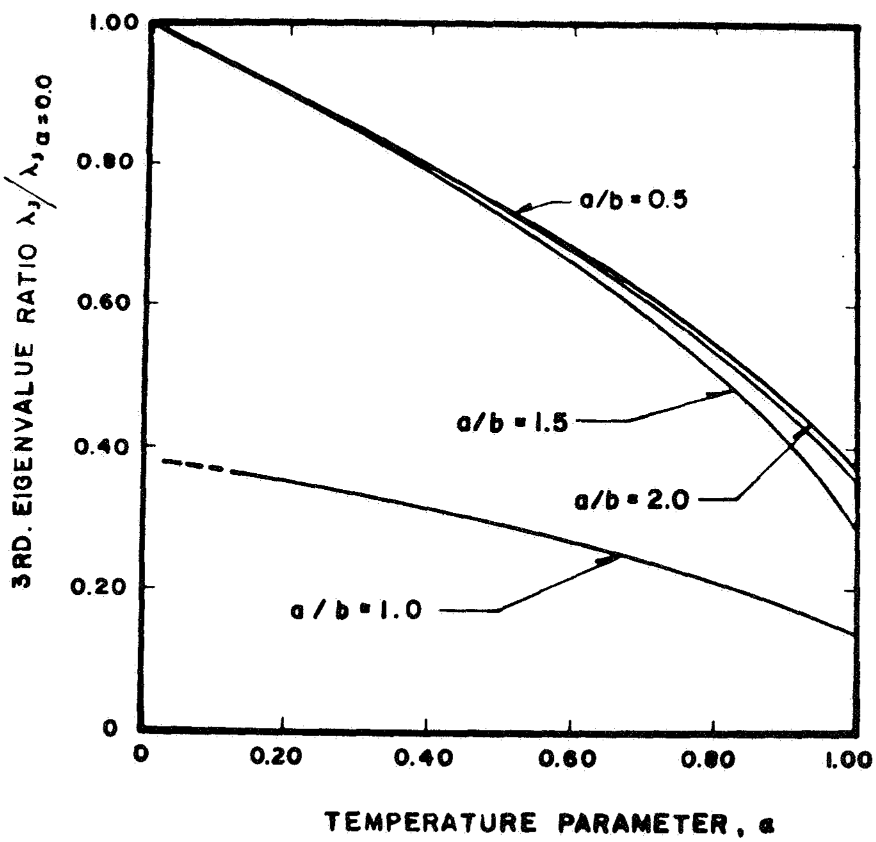

Figure 3

Effect of Thermal Gradient on Third Eigenvalue Ratio

APPENDIX B

Table 3 - Upper and Lower Bounds for the Eigenvalues of a Rectangular Plate with Linear Modulus Variation

$\alpha = 0.00$

a/b	Order	Upper Bounds	Lower Bounds
	1	152.20	152.20
0.5	2	389.64	389.64
	3	1028.88	1028.88
	1	389.64	389.64
1.0	2	2435.23	2435.23
	3	6234.18	6234.18
	1	1028.88	1028.88
1.5	2	3805.04	3805.04
	3	9740.91	9740.91
	1	2435.23	2435.23
2.0	2	6234.18	6234.18
	3	16,462.13	16,462.13

$\alpha = 0.10$

a/b	Order	Upper Bounds	Lower Bounds
	1	144.53	144.38
0.5	2	369.99	369.34
	3	976.95	976.14
	1	369.99	369.34
1.0	2	2312.09	2311.81
	3	2312.32	2312.10
	1	976.95	976.14
1.5	2	3613.06	3613.02
	3	9245.68	9244.75
	1	2312.09	2311.81
2.0	2	5919.81	5919.59
	3	15,631.10	15,630.95

$\alpha = 0.20$

a/b	Order	Upper Bounds	Lower Bounds
0.5	1	136.74	136.50
	2	349.99	349.45
	3	923.95	923.20
1.0	1	349.99	349.45
	2	2185.90	2185.63
	3	2186.85	2186.59
1.5	1	923.94	923.20
	2	3417.23	3417.12
	3	8732.24	8731.36
2.0	1	2185.90	2185.63
	2	5599.50	5599.36
	3	14,782.41	14,782.23

$\alpha = 0.30$

a/b	Order	Upper Bounds	Lower Bounds
0.5	1	128.79	128.67
	2	329.58	329.07
	3	869.65	868.91
1.0	1	329.58	329.07
	2	2056.67	2055.81
	3	2058.35	2058.09
1.5	1	869.65	868.91
	2	3216.80	3216.70
	3	8197.19	8196.31
2.0	1	2056.07	2055.81
	2	5272.12	5271.94
	3	13,912.65	13,912.51

$\alpha = 0.40$

a/b	Order	Upper Bounds	Lower Bounds
0.5	1	120.66	120.52
	2	308.64	308.15
	3	813.79	813.06
1.0	1	308.64	308.15
	2	1921.81	1921.59
	3	1926.13	1925.91
1.5	1	813.79	813.06
	2	3010.77	3010.73
	3	7636.10	7635.33
2.0	1	1921.81	1921.59
	2	4936.10	4935.88
	3	13,017.19	13,016.99

$\alpha = 0.50$

a/b	Order	Upper Bounds	Lower Bounds
0.5	1	112.30	112.15
	2	287.07	286.62
	3	755.98	755.29
1.0	1	287.07	286.62
	2	1782.02	1781.82
	3	1789.27	1789.09
1.5	1	755.98	755.29
	2	2797.73	2797.66
	3	7042.87	7042.21
2.0	1	1782.02	1781.82
	2	4589.26	4589.15
	3	12,089.49	12,089.25

$\alpha = 0.60$

a/b	Order	Upper Bounds	Lower Bounds
0.5	1	103.64	103.50
	2	264.66	264.20
	3	695.64	695.02
1.0	1	264.66	264.20
	2	1635.08	1634.92
	3	1646.36	1646.18
1.5	1	695.64	695.02
	2	2575.56	2575.47
	3	6408.79	6408.13
2.0	1	1635.08	1634.92
	2	4228.28	4228.08
	3	11,119.66	11,119.29

$\alpha = 0.70$

a/b	Order	Upper Bounds	Lower Bounds
0.5	1	94.57	94.46
	2	241.13	240.69
	3	631.85	631.27
1.0	1	241.13	240.69
	2	1478.40	1478.23
	3	1495.15	1494.91
1.5	1	631.85	631.27
	2	2340.80	2340.67
	3	5720.39	5719.75
2.0	1	1478.40	1478.23
	2	3847.77	3847.54
	3	10,091.52	10,090.82

$\alpha = 0.80$

a/b	Order	Upper Bounds	Lower Bounds
0.5	1	84.92	84.78
	2	215.93	215.52
	3	562.96	562.42
1.0	1	215.93	215.52
	2	1307.29	1307.11
	3	1331.47	1331.19
1.5	1	562.96	562.42
	2	2087.13	2086.89
	3	4954.18	4953.50
2.0	1	1307.29	1307.11
	2	3437.71	3437.35
	3	8975.01	8973.34

$\alpha = 0.90$

a/b	Order	Upper Bounds	Lower Bounds
0.5	1	74.26	74.13
	2	187.90	187.52
	3	485.32	484.78
1.0	1	187.90	187.52
	2	1111.40	1111.06
	3	1145.89	1145.16
1.5	1	485.32	484.78
	2	1799.96	1799.05
	3	4058.98	4057.33
2.0	1	1111.40	1111.06
	2	2974.86	2973.43
	3	7699.54	7691.90

$\alpha = 1.00$

a/b	Order	Upper Bounds	Lower Bounds
0.5	1	61.11	60.95
	2	152.73	152.33
	3	385.11	384.51
1.0	1	152.73	152.33
	2	850.67	850.25
	3	900.23	899.62
1.5	1	385.11	384.51
	2	1419.57	1418.62
	3	2833.56	2381.78
2.0	1	850.67	850.25
	2	2361.99	2360.38
	3	5969.26	5961.03