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AN APPROXIMATE THREE-DIMENSIONAL THEORY OF PLATES WITH APPLICATION TO CRACK PROBLEMS

by
R. J. Hartranft
G. C. Sih

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Department of Mechanical Engineering
and Mechanics
Lehigh University, Bethlehem, Pennsylvania 18015

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R. J. Hartranft[2] and G. C. Sih[3]

Abstract

Using a variational principle, a system of equations is derived for the theory of extension and bending of elastic plates. The equations take into account the stress variations across the thickness of the plate which depend on the characteristic dimensions arising in a given boundary-value problem such as the cavity size, plate thickness, etc. The system of equations are formulated in terms of the generalized transverse displacement and two other functions which represent the distribution of the transverse shear stresses in the plane of the plate. For illustration purposes, the problem of the uniform extension of an infinite plate containing a rectangular crack through the thickness is solved by application of integral representations. Numerical results are displayed graphically and important differences are noted between the solution of the present theory and that obtained by means of the customary equations of generalized plane stress. One of the significant and new features of the solution is that the stress state depends on the plate thickness

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[2]Assistant Professor of Mechanics, Lehigh University, Bethlehem, Pa.

[3]Professor of Mechanics, Lehigh University, Bethlehem, Pa.

to crack length ratio and a dimensionless parameter characterizing the stress distribution across the thickness. It is apparent from the present work that the new theory may also be applied to numerous other problems involving cavities where the deviations from the results of the classical plane extension and plate bending theories are of interest.

Introduction

Since the literature relating to the analysis of plates has been very extensive, a detailed exposition of the subject would be inappropriate here. Within the scope of this paper, however, it should be mentioned that the approximate theories of thin plates are unreliable in the case of plates of considerable thickness, or when the plates are weakened by cavities whose dimensions are of the same order of magnitude as the plate thickness. In such cases, the stress variations in the thickness direction must be accounted for and the problem adopts a three-dimensional character. An obvious recourse is to treat the problem of plates as a three-dimensional problem of elasticity. The stress analysis becomes, consequently, more involved and, up to now, the problem is completely solved only for a few particular cases [1].[4]

The stress distribution in a thick plate containing a smooth circular cavity has been discussed by Sternberg and Sadowsky [2], Green [3], and others. On the basis of the Ritz method, an approximate three-dimensional solution was obtained in [2] by means of the so-called "residual problem of plane stress". An exact formulation of the same problem was presented by Green [3] utilizing series expansions and was solved by Alblas [4]. Their work showed that the thickness of the

[4]Numbers in brackets designate References at end of paper.

plate can exert appreciable influence on the stress concentrations of the circular hole. When the periphery of the cavity contains re-entrant corners or singular points, such as a sharp crack, the problem is considerably more difficult mainly because the conventional mathematical techniques are not suitable for handling three-dimensional problems with geometric singularities. An attempt has been made by Sih et al [5] to investigate the triaxial characteristics of the crack-edge stress field in a thick plate. They made use of the Galerkin biharmonic functions and found the qualitative character of the local stress field <u>interior</u> to the plate. The nature of the stress state in the surface layer, where the crack penetrates through the plate, remains unresolved. In another paper [6], Hartranft and Sih confirmed the results in [5] by a more rigorous method using eigenfunction expansions.

The main concern of the present paper is to develop an approximate theory of plates that can lead to an improved quantitative analysis of the problem of a crack in an elastic plate. The earlier work of Williams [7] and Sih [8] dealt with the stress analysis of the bending of cracked plates in which the fourth-order thin plate theory of Poisson-Kirchhoff [9] was used. Thus, it was inherent in the theory that the physically natural boundary conditions on the edge of the crack can be satisfied only in an approximate manner. Knowles and Wang [10] improved the situation by obtaining a solution to the crack problem using a Reissner's sixth-order theory [11] such that the effect of the transverse shear strains are included. This permits the satisfaction of the three expected boundary conditions on the crack. Their result, however, applies only to the case of a vanishingly thin plate. A further refinement of the bending problem was made by Hartranft and Sih [12] who extended the work in [10] to account for variations in the plate thickness. One of the serious shortcomings, which are common to all of the aforementioned theories, is that in-plane stresses are assumed to vary linearly through the thickness. Recently, Sih [13] has employed the theory of Goldenweiser [14] and obtained the stress distribution around a crack in a bent plate. In his work, the stresses across the plate thickness can be arbitrarily assigned in accordance with experimental evidence and are functions of the plate thickness to crack length ratio.

The most popular theory for problems of plane extension is that of generalized plane stress. Loosely speaking, the approximate nature of this theory lies in the assumption that the in-plane stresses are uniform in the average sense through the thickness and the remaining stress components are sufficiently small in magnitude so that they may be neglected. Despite the above oversimplifications, the foundation of the current theories of crack extension in plates [15] rests basically upon the generalized plane stress solution of Inglis [16] who published his work on the elliptical cavity over fifty years ago. In reality, the stress distribution in a cracked plate of finite thickness is neither in a state of plane strain nor generalized plane stress. Hence, the need for a more refined solution of the basic crack problem which accounts for some of the three-dimensional effects is apparent.

The proposed theory is constructed with the objective of reducing the analytical difficulties associated with the three-dimensional equations of elasticity while the essential features of the three-dimensional characteristics of the solution are retained. This is guided by the solution in [5,6] from which an understanding of what is disregarded and what is retained in the approximate theory can be gained. The method of derivation is somewhat related to one of Reissner's theories [17] and the resulting equations meet all of the requirements of the three-dimensional theory of elasticity except for the stress-displacement relations which are satisfied approximately. Depending upon the evenness and oddness of the function that describes the stress distribution in the thickness direction, the newly developed theory can be applied to problems of either bending or extension of plates.

As an example, the problem of a through crack in an infinite plate stretched uniformly at infinity is considered. With the aid of Fourier transforms, the boundary conditions of the crack problem lead to a set of dual integral equations which can in turn be reduced to the solution of a single Fredholm equation of the second kind. Asymptotic expansions of the stresses near the end points of the crack are carried out and reveal that the qualitative character of the singular solution interior to the plate coincides with the exact solution found in [5,6]. The thickness dependence of the local solution is reflected through a function that can be determined from the plane strain condition as suggested in [5,6]. Although the stresses in a layer close to the plate surface are not known, they can be chosen in such a way so that the traction free condition on the plate surface and the condition of continuity with the interior solution are satisfied. Finally, the results are compared with those obtained from the generalized plane stress theory discussed in connection with the possibility of generalizing the current theories of fracture to include the effect of plate thickness.

The Equilibrium of an Elastic Plate

Consider the equilibrium of a homogeneous, isotropic, elastic plate, i.e., of an elastic medium bounded by two parallel planes. The medium occupies the region $|z|<h/2$ and the middle surface of the plate coincides with the xy-plane as shown in Fig. 1. The plate is subjected to certain load conditions at infinity, leaving the faces of the plate at $z = \pm h/2$ free of tractions, i.e.,

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0 \qquad \text{for} \quad z = \pm h/2 \tag{1}$$

Without undue complication, the loadings are assumed to be symmetric about the xy-plane so that only extensional deformation is produced with no bending. The bending problem follows in the same way simply by taking the loads to be skew-symmetric with respect to the mid-plane of the plate. In order to reduce the three-dimensional problem to manageable proportions, the following form of stress state is assumed:

$$\begin{aligned} [\sigma_x, \sigma_y, \tau_{xy}] &= \frac{4}{h^2} f''(2z/h)[S_x, S_y, T_{xy}] \\ [\tau_{xz}, \tau_{yz}] &= -\frac{2}{h} f'(2z/h)[Z_x, Z_y] \\ \sigma_z &= f(2z/h)Z_z \end{aligned} \tag{2}$$

in which the functions S_x, S_y, T_{xy}, Z_x, Z_y and Z_z depend on x and y only and the function f, which is to be determined subsequently, depends on z only. Inserting eqs. (2) into the stress equations of equilibrium yields

$$Z_x = \frac{\partial S_x}{\partial x} + \frac{\partial T_{xy}}{\partial y} , \quad Z_y = \frac{\partial T_{xy}}{\partial x} + \frac{\partial S_y}{\partial y}$$

$$Z_z = \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} \tag{3}$$

The differential equations and boundary conditions of the theory in terms of the functions S_x, S_y, etc. can be best determined by requiring the complementary energy of the system to be as small as is possible with the admitted equilibrium state of stress in eqs. (2). This enables the reduction of the three-dimensional equations of elasticity to a system of equations involving only two variables, namely x and y. The mode of stress distribution in the z-direction as governed by the function f will be found separately with the condition that

$$f(\pm 1) = f'(\pm 1) = 0 \tag{4}$$

which corresponds to satisfying the free-surface requirements stated in eq. (1).

Application of the Calculus of Variations

The application of the minimum energy principle for deriving equations in the theory of plates or shells has been well explored in the past. Thus, only a brief statement of the principle will be made and most of the mathematical details leading up to the governing differential equations will be omitted.

The principle of minimum complementary energy states that, among all possible states of stress which satisfy equilibrium in the interior of the body and the prescribed surface tractions, the actual state of stress makes Π a minimum. Here, Π stands for the complementary energy

$$\Pi = \frac{1}{2E} \iiint_V [\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x)$$

$$+2(1+\nu)(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)]\, dxdydz \tag{5}$$

$$-\iint_{S_1} [\sigma_n\overline{U}_n + \tau_{ns}\overline{U}_s + \tau_{nz}\overline{U}_z]\, dsdz$$

where E is Young's modulus and ν is Poisson's ratio. In eq. (5), V is the volume occupied by the elastic body and S_1 is the portion of the surface of V on which the displacements $\overline{U}_n$, $\overline{U}_s$ and $\overline{U}_z$ are prescribed. The displacement components $\overline{U}_n$ and $\overline{U}_s$ are defined in the directions normal and tangential to S and $\overline{U}_z$ is the displacement component in the direction perpendicular to the plate.

According to the minimum principle, the variation of Π in eq. (5) is made equal to zero in such a way that the constraints shown by eqs. (3) remain satisfied. To this end, the Lagrange multipliers u_x, u_y and u_z are introduced and the desired variation becomes

$$\delta\{\Pi - \iint_R [u_x(Z_x - \frac{\partial S_x}{\partial x} - \frac{\partial T_{xy}}{\partial y}) \tag{6}$$

$$+u_y(Z_y - \frac{\partial T_{xy}}{\partial x} - \frac{\partial S_y}{\partial y}) + u_z(Z_z - \frac{\partial Z_x}{\partial x} - \frac{\partial Z_y}{\partial y})]dxdy\} = 0$$

where R represents the area of the plate. Making use of the stresses in eqs. (2) and carrying out the integration by parts, it can be shown that

$$\iint_R \{[\frac{\partial u_x}{\partial x} - \frac{1}{(1-\nu^2)D}(S_x-\nu S_y + \nu\alpha^2 Z_z)]\delta S_x$$

$$+[\frac{\partial u_y}{\partial y} - \frac{1}{(1-\nu^2)D}(S_y-\nu S_x + \nu\alpha^2 Z_z)]\delta S_y$$

$$+[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} - \frac{2}{(1-\nu)D} T_{xy}]\delta T_{xy} \tag{7}$$

$$+[u_x + \frac{\partial u_z}{\partial x} - \frac{2\alpha^2}{(1-\nu)D} Z_x]\delta Z_x + [u_y + \frac{\partial u_z}{\partial y} - \frac{2\alpha^2}{(1-\nu)D} Z_y]\delta Z_y$$

$$+ \frac{1}{(1-\nu^2)D}[(1-\nu^2)D\, u_z - \alpha^4(\beta^2+1)Z_z + \nu\alpha^2(S_x+S_y)]\delta Z_z\}dxdy$$

$$-\int_{L_1} [(u_n-\bar{u}_n)\delta S_n + (u_s-\bar{u}_s)\delta T_{ns} + (u_z-\bar{u}_z)\delta Z_n]dS = 0$$

and L_1 is that part of the boundary on which the surface tractions are not prescribed. The parameters α,β and D are defined as

$$\alpha^2 = \frac{1}{6}h^2 I_2, \quad \beta^2 = \frac{3}{2}(I_1/I_2^2)-1, \quad D = \frac{Eh^3}{12(1-\nu^2)} \tag{8}$$

in which I_1 and I_2 stand for

$$I_1 = \int_{-1}^{1} [f(\zeta)]^2 d\zeta \ , \quad I_2 = \int_{-1}^{1} [f'(\zeta)]^2 d\zeta$$

with ζ being the normalized thickness coordinate 2z/h. The generalized displacements u_x, u_y, u_z in eq. (6) are weighted averages of the displacement components U_x, U_y, U_z through the thickness and they are given by

$$u_x(x,y) = \frac{2}{h}\int_{-1}^{1} f''(\zeta)\, U_x(x,y,\tfrac{h}{2}\zeta)\, d\zeta$$

$$u_y(x,y) = \frac{2}{h}\int_{-1}^{1} f''(\zeta) U_y(x,y,\tfrac{h}{2}\zeta)\, d\zeta \tag{9}$$

$$u_z(x,y) = -\int_{-1}^{1} f'(\zeta) U_z(x,y,\tfrac{h}{2}\zeta)\, d\zeta$$

Without loss in generality, the choice

$$\int_{-1}^{1} [f''(\zeta)]^2 d\zeta = 3/2$$

has been made arbitrarily for the sake of convenience. Eqs. (7) provide the differential equations of the theory and also the natural boundary conditions of the problem, which allow, under the assumptions made, either stress or displacement to be specified, i.e.,

$$\begin{aligned} S_n &= \overline{S}_n \quad &\text{or}\quad u_n &= \overline{u}_n \\ T_{ns} &= \overline{T}_{ns} \quad &\text{or}\quad u_s &= \overline{u}_s \qquad \text{on } L \\ Z_n &= \overline{Z}_n \quad &\text{or}\quad u_z &= \overline{u}_z \end{aligned} \tag{10}$$

Referring to Fig. 1 in which ϕ is used to denote the angle between the x-axis and the normal direction, the relationships

$$S_n\cos\phi - T_{ns}\sin\phi = S_x\cos\phi + T_{xy}\sin\phi$$

$$S_n \sin\phi + T_{ns}\cos\phi = T_{xy}\cos\phi + S_y \sin\phi$$
$$Z_n = Z_x \cos\phi + Z_y \sin\phi \qquad (11)$$

and

$$u_n = u_x \cos\phi + u_y \sin\phi \ , \quad u_s = - u_x \sin\phi + u_y \cos\phi \qquad (12)$$

can be easily established from the transformation properties of the stress tensor and displacement vector.

Now, since the variations of the six quantities S_x, S_y, etc. in the double integral of eq. (6) are independent and arbitrary, there results a system of six equations which when coupled with the three equations of equilibrium in eqs. (3) determine the nine unknowns u_x, u_y, u_z, S_x, S_y, etc. After some algebra, these equations may be re-arranged to form a system of three simultaneous equations in the unknown u_z, Z_x and Z_y as given by

$$\alpha^4[1+\beta^2/(1-\nu^2)]\ \nabla^4 u_z - 2\alpha^2\nabla^2 u_z + u_z = 0$$
$$Z_x - \alpha^2\nabla^2 Z_x = \frac{1}{\alpha^2}\frac{\partial N}{\partial x} \qquad (13)$$
$$Z_y - \alpha^2\nabla^2 Z_y = \frac{1}{\alpha^2}\frac{\partial N}{\partial y}$$

provided that

$$\alpha^4(1+\beta^2\kappa^2)\left(\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y}\right) = (1-\nu)D[(1-\nu)u_z + \nu\alpha^2\nabla^2 u_z] \qquad (14)$$

and

$$N = \frac{D}{1+\beta^2\kappa^2}\ [(1-\nu)(u_z-\alpha^2\nabla^2 u_z) + \alpha^2\beta^2\kappa^2\nabla^2 u_z]$$

where

$$\kappa^2 = \frac{1-\nu}{1+\nu}$$

The symbol ∇^2 denotes the Laplacian operator in x and y. The remaining six unknowns can then be found directly from u_z, Z_x and Z_y, i.e.,

$$u_x = -\frac{\partial u_z}{\partial x} + \frac{2\alpha^2}{(1-\nu)D} Z_x, \quad u_y = -\frac{\partial u_z}{\partial y} + \frac{2\alpha^2}{(1-\nu)D} Z_y \tag{15}$$

and

$$\begin{aligned} S_x &= -D\left(\frac{\partial^2 u_z}{\partial x^2} + \nu \frac{\partial^2 u_z}{\partial y^2}\right) + \frac{\alpha^2}{1-\nu}\left[(2-\nu)\frac{\partial Z_x}{\partial x} + \nu \frac{\partial Z_y}{\partial y}\right] \\ S_y &= -D\left(\frac{\partial^2 u_z}{\partial y^2} + \nu \frac{\partial^2 u_z}{\partial x^2}\right) + \frac{\alpha^2}{1-\nu}\left[(2-\nu)\frac{\partial Z_y}{\partial y} + \nu \frac{\partial Z_x}{\partial x}\right] \\ T_{xy} &= -D(1-\nu)\frac{\partial^2 u_z}{\partial x \partial y} + \alpha^2\left(\frac{\partial Z_x}{\partial y} + \frac{\partial Z_y}{\partial x}\right) \\ Z_z &= \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} \end{aligned} \tag{16}$$

In some ways, the newly developed equations (13), (15) and (16) resemble those found by Reissner [11] for the bending of elastic plates and they are considered to be an improvement over the equations of generalized plane stress. Similar expressions in terms of circular or elliptical coordinates can also be written down which will be more suited for studying the stress concentration around circular or elliptical cavities.

A Pressurized Rectangular Crack

Let a plate of thickness h be cut along the x-axis from $x = -a$ to $x = a$ resulting in a rectangular crack through the thickness as illustrated in Fig. 2. The surfaces of the crack are pressurized such that the plate is stretched symmetrically about the xz-and yz-planes. Thus, it is sufficient to formulate the problem for the quarter plane $x > 0$ and $y > 0$ subjected to the mixed boundary conditions on the edge y = 0:

$$u_x(x,0) = 0, \quad x>a$$
$$S_y(x,0) = - S_o P(x), \quad x<a \tag{17}$$

and

$$T_{xy}(x,0) = Z_y(x,0) = 0 \text{ for all } x \tag{18}$$

Since the domain under consideration is unbounded, eqs. (17) and (18) need to be supplemented by the regularity condition that the displacements and stresses are finite as $(x,y)\to\infty$.

The method of integral transforms will now be used to solve eqs. (13). Omitting the details, the displacements in the transformed domain are, where s is the transform variable,

$$(1-\nu)Du_x^s = 2(1-\nu)\alpha^4\beta m s^3 A(s) \exp(-msy)$$
$$-\alpha^2 s A(s) \operatorname{Im}[(1+i\beta\kappa)(q/p)\exp(-psy)]$$
$$(1-\nu)Du_y^c = 2(1-\nu)\alpha^4\beta s^3 A(s) \exp(-msy) \tag{19}$$
$$-\alpha^2 s A(s) \operatorname{Im}[(1+i\beta\kappa)q \exp(-psy)]$$
$$(1-\nu)Du_z^c = \alpha^2 A(s) \operatorname{Im}[(1-i\beta\kappa)(q/p) \exp(-psy)]$$

which are consistent with the regularity requirement at infinity and the boundary conditions of eqs. (18). The transforms of the quantities describing the variation of the in-plane stresses as a function of x and y are

$$S_x^c = 2(1-\nu)\alpha^4\beta m s^4 A(s) \exp(-msy)$$

$$-A(s)\ \text{Im}\{[-\frac{\nu\beta}{\beta - i\sqrt{1-\nu^2}} + (1+i\beta\kappa)\alpha^2 s^2](q/p)\exp(-psy)\}$$

$$S_y^c = -2(1-\nu)\alpha^4\beta m s^4 A(s)\exp(-msy)$$

$$+\frac{1}{\sqrt{1-\nu^2}} A(s)\ \text{Im}\ [(q^2/p)\exp(-psy)] \qquad (20)$$

$$T_{xy}^s = -(1-\nu)\alpha^2\beta s^2(1+2\alpha^2 s^2)\ A(s)\exp(-msy)$$

$$+ A(s)\ \text{Im}\ [(1+i\beta\kappa)q\exp(-psy)]$$

and the transform expressions of Z_x, Z_y, Z_z may be written as

$$Z_x^s = (1-\nu)\alpha^2\beta m s^3\ A(s)\exp(-msy)$$

$$-sA(s)\ \text{Im}[(q/p)\exp(-psy)]$$

$$Z_y^c = (1-\nu)\alpha^2\beta s^3\ A(s)\exp(-msy) \qquad (21)$$

$$-sA(s)\ \text{Im}\ [q\exp(-psy)]$$

$$Z_z^c = \frac{\sqrt{1-\nu^2}}{\alpha^2(1-\nu^2+\beta^2)} A(s)\text{Im}[(\sqrt{1-\nu^2}-i\beta)(q/p)\exp(-psy)]$$

The auxiliary quantities m, p and q appearing in eqs. (19) to (21) are defined by

$$m^2 = 1+(\alpha s)^{-2}, \quad q = \sqrt{1-\nu^2}\ [1+(1+i\beta\kappa)\alpha^2 s^2]$$

$$p = |p|\ \exp(-i\upsilon/2)$$

where

$$|p| = \{(1-\nu^2+\beta^2)^{-1}[(1-\nu^2)m^4+\beta^2]\ \}^{1/4}$$

and

$$\upsilon = \tan^{-1}[\frac{\beta}{\beta^2+(\beta^2+1)\alpha^2 s^2}]\ ,\ 0\leq\upsilon<\pi/2$$

The foregoing expressions are derived on the basis that β is real and hence the inequality

$$I_1 > (2/3)\, I_2^2$$

must hold. This restriction in no way affects the fundamental character of the solution. The superscripts s and c in eqs. (19) to (21) are used to identify the sine and cosine tranform of a given function. The appropriate inverse transforms of the displacements and stresses follow immediately from the inversion theorem as

$$u_x(x,y) = \frac{2}{\pi}\int_0^\infty u_x^s\,(s,y)\,\sin(sx)ds$$

$$u_y(x,y) = \frac{2}{\pi}\int_0^\infty u_y^c\,(s,y)\,\cos(sx)ds \tag{22}$$

etc. which constitute the complete solution to the problem once the unknown A(s) is found.

There remains the satisfaction of eqs. (17) which render the dual integral equations

$$\int_0^\infty sA(s)\,\cos(sx)ds = 0\,, \quad x>a$$

$$\int_0^\infty s^2 f(s)\,A(s)\,\cos(sx)ds = -\frac{\pi S_0}{\alpha^2\beta}\,p(x),\ x<a \tag{23}$$

for the determination of A(s) as f(s) is a known function given by

$$f(s) = \frac{2\sqrt{1-\nu^2}}{\alpha^2\beta s^2|p|}\{-2\alpha^4\beta\kappa m s^4|p|$$

$$+[(1+\alpha^2 s^2)^2+(\beta\kappa)^2(\alpha s)^4]\sin\frac{\upsilon}{2}+[2(1-\nu)(1+\alpha^2 s^2)\alpha^2\beta s^2]\cos\frac{\upsilon}{2}\} \tag{24}$$

A procedure for solving dual integral equations of the type shown in eqs. (23) has already been thoroughly discussed in [12] and thus only the essential steps will be outlined merely to preserve the continuity of the analysis.

With the aim of reducing the problem to a standard integral equation, a new function

$$w(x) = -\frac{1}{\pi}\,\alpha^2\beta\int_0^\infty sA(s)\,\cos(sx)ds \tag{25}$$

is introduced. Upon application of the Fourier inversion theorem and the first of eqs. (21), it is not difficult to arrive at the result

$$sA(s) = -\frac{2}{\alpha^2\beta}\int_0^a w(x)\cos(sx)dx \tag{26}$$

The structure of the function $w(x)$ is determined by the singularities inherent in the physical problem. For the present case, the representation

$$w(x) = \int_x^a \frac{\psi(t)\,t\,dt}{\sqrt{t^2-x^2}}, \quad x<a \tag{27}$$

where $w(x) = 0$ for $x>a$ is admitted. The function ψ is to be defined on the interval $[0,a]$ and is permitted to depend on the parameters α, β, ν, etc. Eliminating $w(x)$ between eqs. (26) and (27) yields

$$sA(s) = -\frac{\pi}{\alpha^2\beta}\int_0^a \psi(t)\,J_0(st)\,t\,dt \tag{28}$$

where J_0 is the zero-order Bessel function of the first kind. Inserting eq. (28) into the second of eqs. (23) gives

$$\int_0^\infty sf(s)\cos(sx)\,ds \int_0^a \psi(t)\,J_0(st)\,t\,dt = S_0P(x),\quad x<a \tag{29}$$

Further, by letting

$$f(s) = \frac{(1-\nu^2)(\beta^2+1)}{1-\nu^2+\beta^2}[1+g(s)]$$

such that $g(s) = 0(s^{-2})$ as $s\to\infty$, eq. (29) can be put into a standard Fredholm integral equation of the second kind:

$$\Phi(\xi) + \int_0^1 F(\xi,\eta)\,\Phi(\eta)\,d\eta = \frac{2}{\pi}\sqrt{\xi}\int_0^\xi \frac{P(a\eta)d\eta}{\sqrt{\xi^2-\eta^2}},\quad 0<\xi<1 \tag{30}$$

in which the kernel

$$F(\xi,\eta) = \sqrt{\xi\eta}\int_0^\infty s\,g(\tfrac{s}{a})\,J_0(\xi s)\,J_0(\eta s)ds,\quad 0<\xi\leq 1;\ 0<\eta\leq 1 \tag{31}$$

is symmetric with respect to the dimensionless variables ξ and η, which are normalized against the half crack length a. The unknown Φ in eq. (30) is related to ψ by

$$\Phi(\xi) = \frac{(1-\nu^2)(1+\beta^2)}{1-\nu^2+\beta^2}\,\frac{1}{S_0}\,\sqrt{\xi}\,\psi(a\xi) \tag{32}$$

In connection with the numerical solution of eq. (30), it is desirable to further set

$$g(s) = \frac{C}{(\alpha s)^2+n^2} + \frac{h(s)}{\alpha s}$$

where

$$C = \frac{1}{2}\,\frac{(1-\nu^2)+(2-\nu+3\nu^2)\beta^2+(1+\nu)^{-1}\beta^4}{(1+\beta^2)(1-\nu^2+\beta^2)}$$

$$n^2 = \frac{1}{4} \frac{(1-\nu^2)^2+(4-6\nu+15\nu^2)(1-\nu^2)\beta^2+(5-8\nu-5\nu^2)\beta^4+2(1+\nu)^{-1}\beta^6}{(1-\nu^2+\beta^2)[(1-\nu^2)+(2-\nu+3\nu^2)\beta^2+(1+\nu)^{-1}\beta^4]}$$

such that eq. (31) becomes

$$F(\xi,\eta) = (a/\alpha)\sqrt{\xi\eta}\,[(Ca/\alpha)\, I_0(\tfrac{a}{\alpha} n\xi)\, K_0(\tfrac{a}{\alpha} n\eta) + \int_0^\infty h(\tfrac{s}{a})\, J_0(\xi s)\, J_0(\eta s)\, ds], \quad 0<\xi\leq\eta \tag{33}$$

with I_0 and K_0 being the zero-order modified Bessel functions of the first and second kind. The advantage of this alternative expression of F as compared to eq. (31) is apparent from

$$h(s) = O(s^{-5}) \text{ as } s\to\infty$$

Finally, returning to the solution of the original set of dual integral equations, A(s) takes the form

$$A(s) = -\frac{\pi(1-\nu^2+\beta^2)\, S_0 a}{(1-\nu^2)\alpha^2\beta(1+\beta^2)\, s^2}\{\Phi(1)\, J_1(sa) - \int_0^1 \frac{d}{d\xi}[\frac{\Phi(\xi)}{\sqrt{\xi}}]\, J_1(sa\xi)\, \xi\, d\xi\} \tag{34}$$

where J_1 is the first-order Bessel function of the first kind. Eq. (34) may be put into eqs. (19) to (21) for obtaining the nine unknowns u_x, u_y, etc. from which the displacements and stresses at every point in the plate may be calculated once Φ is evaluated from the Fredholm integral equation and the function f(2z/h) is determined.

Stress Variation Across Plate Thickness

Before entering the discussion of numerical results, it is necessary to find the stress variation in the z-direction or the function $f(\zeta)$. A possible way of determining $f(\zeta)$ is to require that the local stress field interior to the plate be in a state of plane strain as dictated by the exact analysis in [5,6]. Thus, by putting the appropriate stress components from eqs. (36) into the condition

$$\sigma_z = \nu(\sigma_x + \sigma_y) \tag{38}$$

there results the harmonic equation

$$f''(\zeta) + \frac{h^2}{4\alpha^2(1+\beta^2)} f(\zeta) = 0, \quad |\zeta|<1-\varepsilon \tag{39}$$

whose solution is

$$f(\zeta) = B \cos\left[\frac{h\zeta}{2\alpha\sqrt{1+\beta^2}}\right], \quad |\zeta|<1-\varepsilon \tag{40}$$

since $f(\zeta)$ must be even in ζ for the plane extension problem.

In the above expression, B is a constant and the quantity $h/2\alpha\sqrt{1+\beta^2}$ in the argument of the cosine function can be related to $f(\zeta)$ as

$$\frac{h}{2\alpha\sqrt{1+\beta^2}} = \left\{ \frac{\int_{-1}^{1} [f'(\zeta)]^2 d\zeta}{\int_{-1}^{1} [f(\zeta)]^2 d\zeta} \right\}^{1/2} \tag{41}$$

Combining eqs. (40) and (41), it is seen that $f(\zeta)$ cannot be determined completely and is valid only for $|\zeta|<1-\varepsilon$, where $\varepsilon h/2$ is used to denote the thickness of the boundary layer close to the plate surface within which the plane strain condition is not satisfied. This layer is introduced with the intent of emphasizing that the stress solution on the surface layer of the plate is not determined by the present theory. The function $f(\zeta)$ for $(1-\varepsilon)<|\zeta|<1$, however, can be constructed arbitrarily with the requirement that $f(\zeta)$ and its first and second derivatives are continuous at $|\zeta| = 1-\varepsilon$ and that the free-surface conditions $f(\pm 1) = f'(\pm 1) = 0$ are satisfied.

Although the determination of $f(\zeta)$ involves a considerable amount of algebra, it can be calculated numerically without difficulty. A typical set of curves showing the variations of the in-plane stresses σ_x, σ_y, τ_{xy}; the transverse shear stresses τ_{xz}, τ_{yz}; and the transverse normal stress σ_z through the plate thickness is illustrated in Fig.3. The function $f(\zeta)$depicted in Fig. 3 gives nearly constant values of the in-plane stresses along the z-axis, deviating from the constant only in a layer of thickness $\varepsilon h/2$ measured from the plate surface. Within this layer, σ_x, σ_y and τ_{xy} rise rapidly and change from compression on the plate surface to tension in the interior region of the plate. The transverse normal stress is compressive owing to contraction in the thickness direction caused by the stretching load. Curves of different shapes may be drawn for various values of β ranging from 0 to ∞. Once β (which is inversely related to ε) corresponding to certain z-distribution of the stresses is found, the Fredholm integral equation in eq. (30) may be solved for Φ. This, in turn, determines the variations of the intensity of the three-dimensional stress field with the dimensionless parameter α/a which is related to the plate thickness.

Discussion of Numerical Results

The requisite numerical results are obtained by solving the integral equation (30) for Φ on the computer. These computations are carried out for a Poisson's ratio $\nu = 0.30$ and various β-values; the corresponding results are summarized graphically.

In Fig. 4, the normalized stress-intensity factor k_1 in eq. (37) is plotted as a function of α/a for five typical values of β in the interval from zero to infinity. Each value of β refers to a particular set of curves for the stress distribution across the plate thickness as in Fig. 3. In general, as α/a departs from zero, all the curves increase in magnitude very sharply at the beginning and then level off steadily as α/a continues to grow. Notice that for small values of α/a or thin plates, a significant change in the k_1-factor may be observed for a slight variation of the thickness $\varepsilon h/2$ of the boundary layer through the parameter β.

It is interesting to note that the curve for $\beta = \infty$, which corresponds to a vanishingly small boundary layer thickness $\varepsilon \to 0$, differs from all the others in that it has a non-zero limit at $\alpha/a = 0$. This limit can be verified analytically for when $\beta \to 0$, $|p| \to 1, \upsilon \to 0$, and hence

$$g(s) \to 4\kappa(1+\alpha^2 s^2)-1$$

From the above result with $\alpha=0$, the kernel $F(\xi,\eta)$ in eq. (31) simplifies to

$$F(\xi,\eta) = (4\kappa-1) \quad \delta(\xi-\eta)$$

where δ is the Dirac delta generalized function. It follows immediately that

$$4\kappa \quad \Phi(\xi) = \sqrt{\xi}$$

or

$$\Phi(1) \to \frac{1}{4\kappa} \qquad \text{for} \qquad \alpha = 0;\ \beta = \infty$$

It should be emphasized that although the present results are considered to be a refinement over those of generalized plane stress they remain an approximation to the three-dimensional problems. In addition, the theory neglects the effect of plastic flow near the crack front. Hence, experimental verification of the theory can only be carried out for brittle materials.

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Figure Captions

Figure 1 - Geometry of a Flat Plate.

Figure 2 - Polar Coordinates Around a Rectangular Crack In an Infinite Plate.

Figure 3 - Stress Variations Across the Plate Thickness.

Figure 4 - Dimensionless Stress-Intensity Factor versus Normalized Plate Thickness.

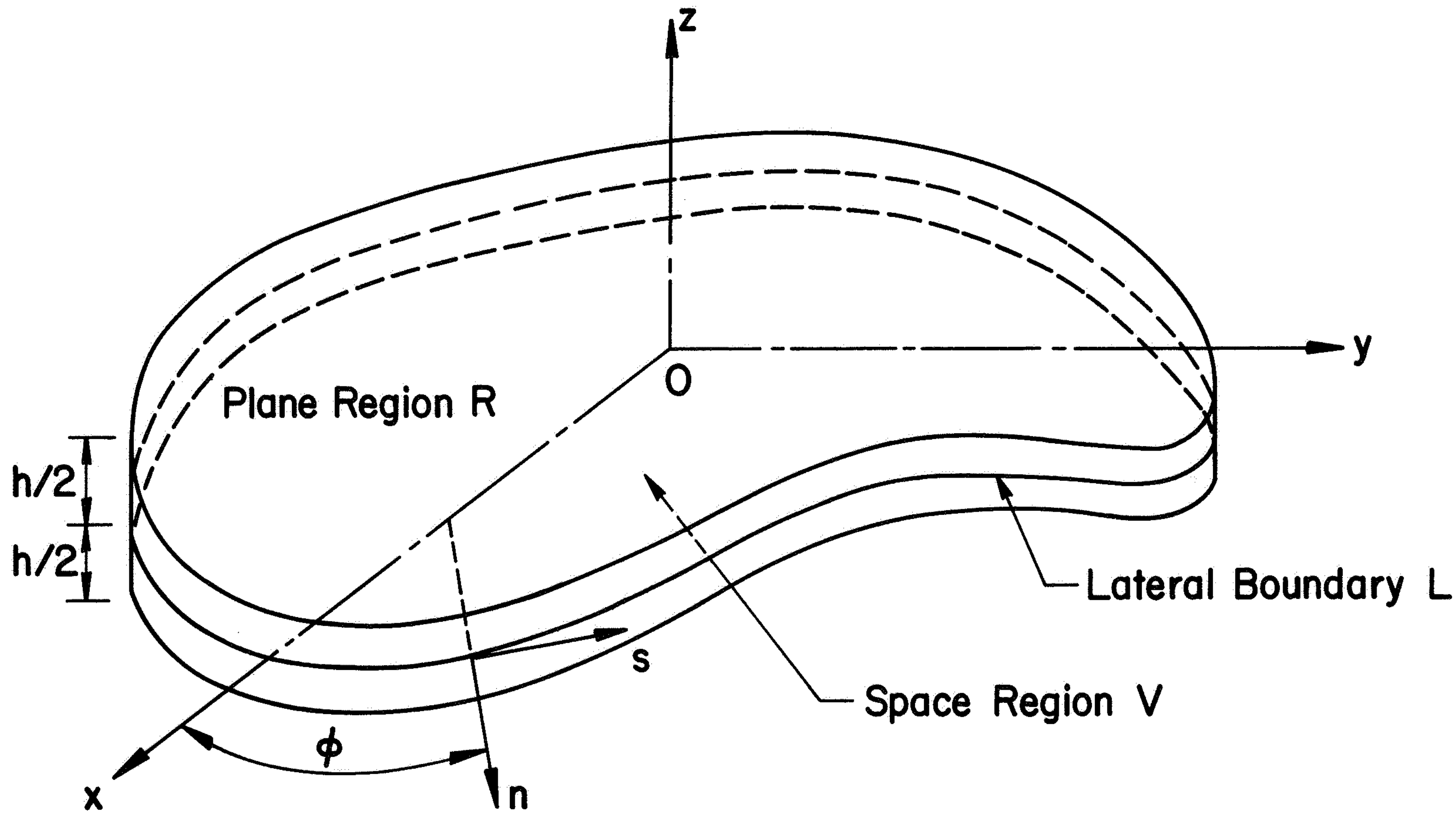

Figure 1 - Geometry of a Flat Plate

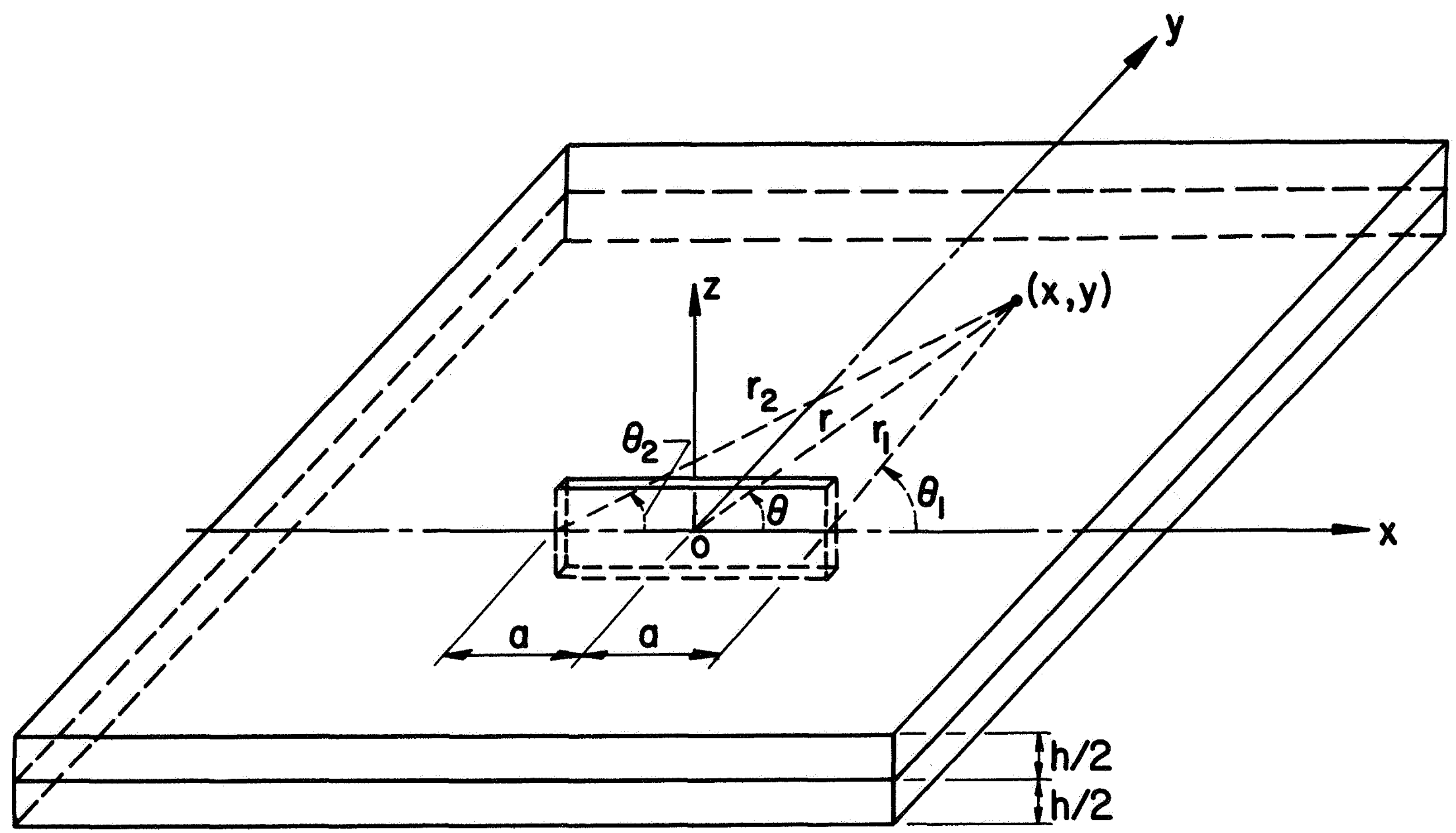

Figure 2 - Polar Coordinates Around a Rectangular Crack In An Infinite Plate

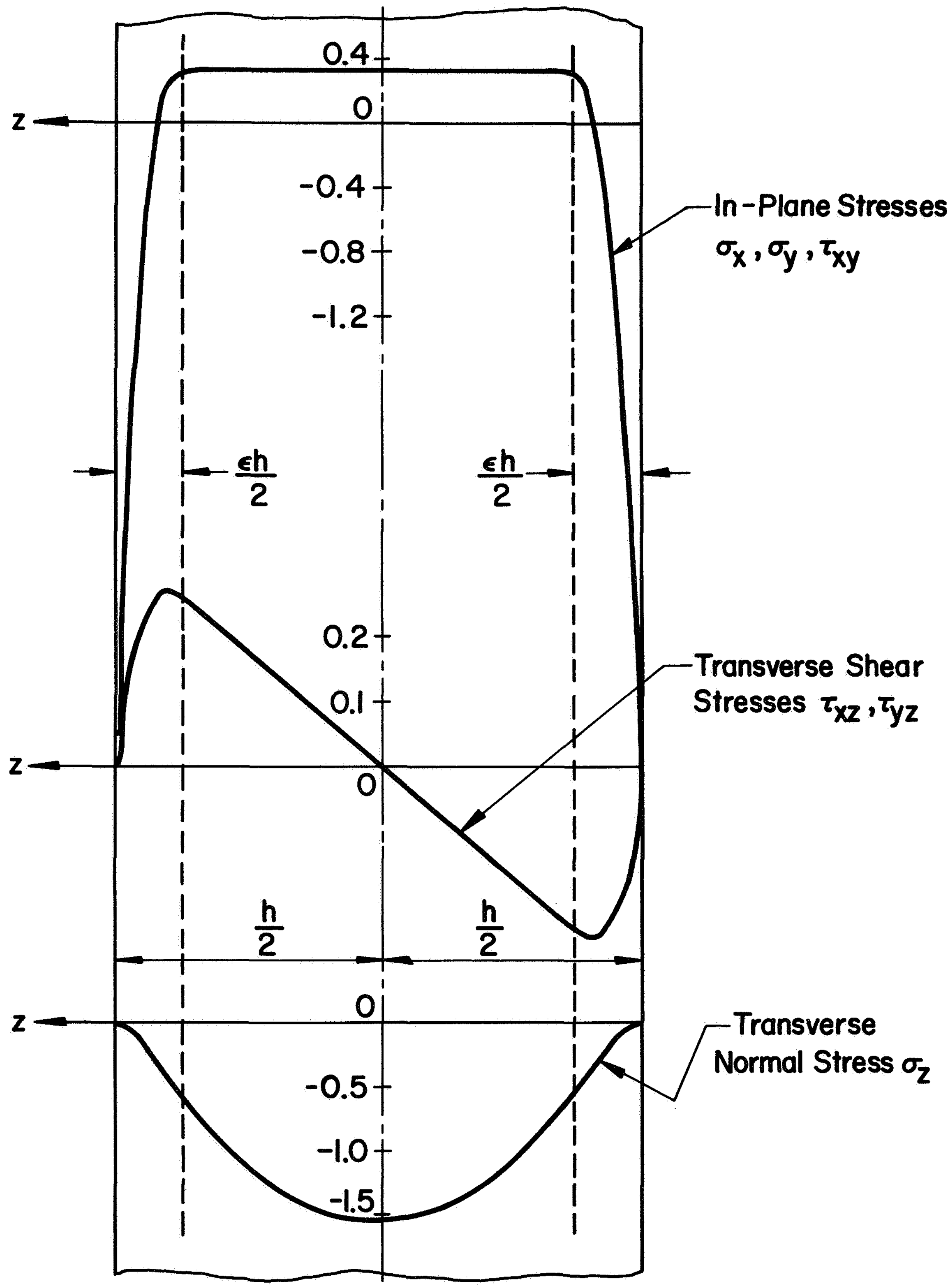

Figure 3 - Stress Variations Across the Plate Thickness

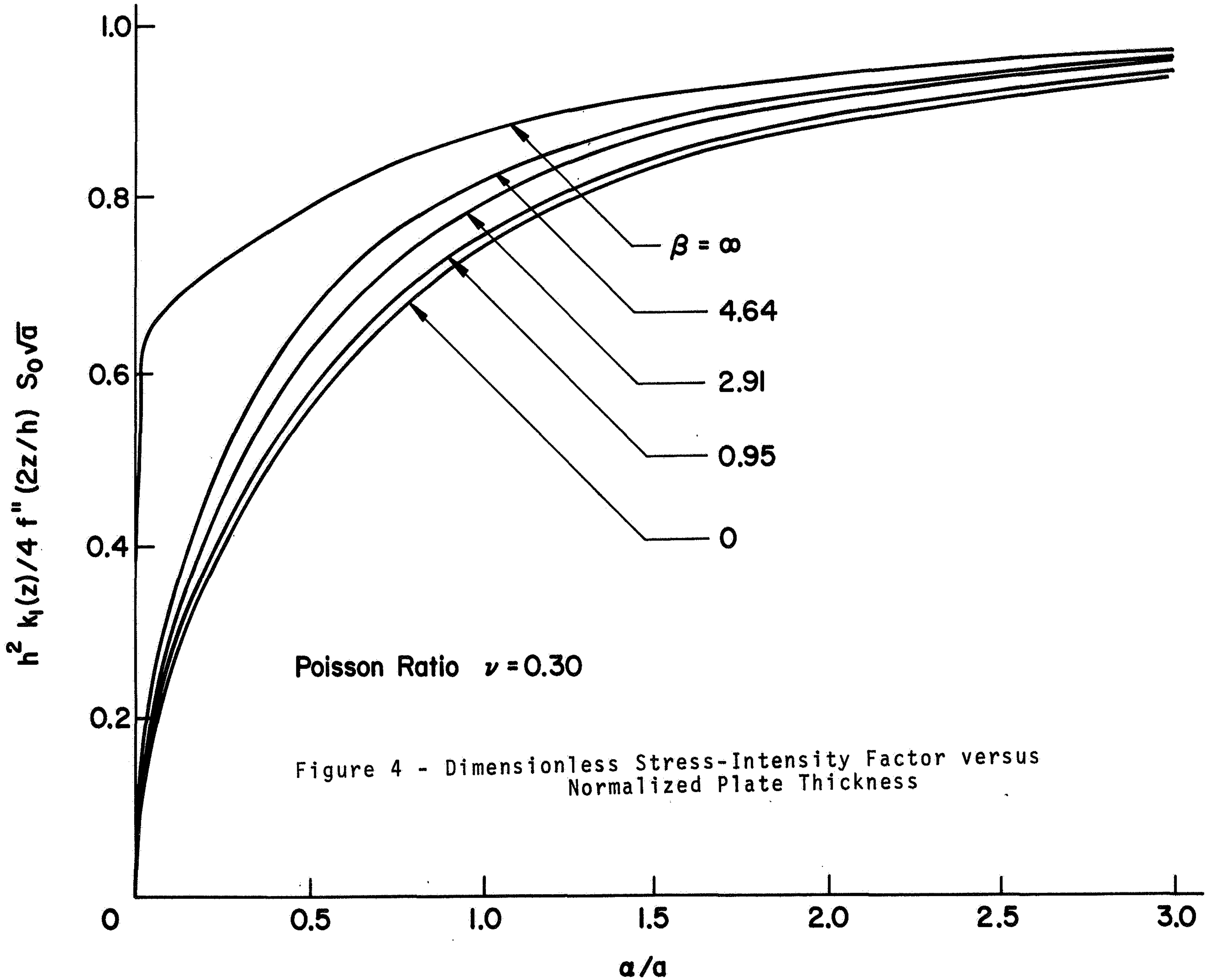

Figure 4 - Dimensionless Stress-Intensity Factor versus Normalized Plate Thickness