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A STOCHASTIC MODEL OF AN INFORMATION CENTER

by

Jack Minker

UNIVERSITY OF MARYLAND
COMPUTER SCIENCE CENTER
COLLEGE PARK, MARYLAND

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Abstract

In a recent paper [1] the author investigated a stochastic model relevant to information handling centers best typified by computer utilities and document storage and retrieval centers. The growth characteristics of information centers were evaluated for retirement policies that govern when items are retired from a primary store to a less accessible store. The results obtained assumed that the primary store was of unbounded capacity. In this paper we remove this restriction and consider the case where the primary store has a finite capacity.

A set of integral equations is derived for the expected number of items in the primary store. The integral equations depend only upon the arrival distribution for documents, the request distribution, and the parameters associated with the retirement policy. No particular limiting assumptions have been made with respect to the form of the distributions.

The set of integral equations are solved for document arrivals that follow a Poisson distribution. The expected value of the size of the store approaches the result given in [1] as M, the size of the primary store, becomes unbounded.

1. INTRODUCTION

In this paper we shall be concerned with a mathematical model that describes a portion of the operation of an information center or a computer utility. Although many papers have been written concerning libraries and information centers, relatively few papers describe mathematical models. An extensive bibliography on papers discussing the use of libraries has been prepared by DeWeese [2]. The papers cited in that bibliography generally present a summary of data collected without reference to mathematical models for describing or predicting the use of books.

Jain [3] reviews some twelve mathematical models that predict the use of books. In addition, Jain develops a model of his own. The author [1] has developed a model of a library that makes use of the characteristics that describe book use to determine the expected size of a primary data store. This paper is an extension of the author's previous paper. The work differs from previous work in that others have developed models to describe how books are used, while we would employ the results of their work to determine the expected size of a primary store given a specific retirement policy for documents. The retirement policy for documents, described in section 2, considers both the age and the use history of a document.

In the information center under investigation in this paper, and in [1], two stores for documents are considered: a primary (active) store and a secondary (retirement) store. Current and frequently-used documents reside in the primary store, while less frequently used documents are placed in the secondary store. A retirement policy is specified that determines when an item in the primary store is to be retired to the secondary store, and when an item in the secondary store may be returned to the primary store.

Although no particular limiting assumptions were made upon the arrival and request distributions, it was assumed that there is only one class of documents. Hence, one arrival distribution and one request distribution are applicable for all documents. In [1] no limit has been placed on the size of the primary store. In this paper, we shall add a size limitation; specifically, the primary store may contain at most M items while the secondary store may become arbitrarily large.

In [1] the authors were able to develop an integral equation expressing the expected size of the primary store. In a similar manner, the expected number of items in the primary store under the condition that at most M items may reside there will be determined for arbitrary arrival and request distributions.

For a Poisson arrival distribution, an explicit expression is found for the expected member of items for an arbitrary request distribution. As $M \to \infty$, the expected size of the primary store approaches the result previously obtained.

2. STOCHASTIC MODEL DESCRIPTION OF THE INFORMATION HANDLING CENTER PROBLEM

Because of the bound, M, on the primary store size, some modifications must be made to the model described in [1]. For completeness, this paper contains all the assumptions and definitions as specified in [1]. The retirement and rebirth policy is defined as:

Definition 1: Retirement Policy and Rebirth Policy

a. An item in the primary store is retired if it arrived more than T years ago, or if it has been in the primary store at least X years ($X < T$) and has been used less than K-times in the past Y years ($X \geq Y$).

b. In the event that the primary store is filled to capacity, the oldest item in the store is retired whenever a new item arrives.

c. An item in the secondary (or retired) store is placed in the primary store if it has been requested at least K times in the previous Y years, provided that it did not arrive more than T years ago. If there are M items in the primary store, an item in the secondary store, eligible for the primary store, will replace the oldest item in the primary store if it is younger than that item.

d. If there are less than M items in the primary store, an eligible item from the secondary store will be shifted to the primary store.

The above definition modifies [1] to assure that there are at most M items in the primary store and that the youngest items are to be in the primary store. We can now define,

Definition 2: $P_M(w, t)$: Let $P_M(w, t)$ be the steady state probability that if an item was requested w years after it arrived, it is eligible to be in the primary store at time t.

Definition 2 differs from the definition of $P(w,t)$ given in [1] only in that it considers eligible items rather than definitely transferred items. However, since in [1] no limitation was made on the size of the primary store, then it follows that $P_M(w,t) \equiv P(w,t)$. We may, therefore, state the following lemma without proof, since it has been derived in [1].

Lemma 1: The steady state probability, $P(w,t)$, that an item requested w years after its arrival is eligible for the primary store $t-w$ years later is given by

$$(1)\quad P(w,t) = 1 \qquad \text{when } w < t < X$$

$$(2)\quad P(w,t) = 0 \qquad \text{when } w < t \qquad \text{and } t < T$$

$$(3)\quad P(w,t) = \int_{t-w-Y}^{t-w} R_1(r_1,w)dr_1 \int_0^{t-w-r_1} R_1(r_2,w+r_1)dr_2 \ldots \int_0^{t-w-\sum_{i=1}^{K-1} r_i} R_1(r_K,w+\sum_{i=1}^{K-1} r_i)dr_K$$

$$+ \int_0^{t-w-Y} R_1(r_0,w)\, P(w+r_0,t)dr_0 \quad \text{when } t \geq X,\ t-w \geq Y \text{ and } t \leq T$$

and where we define:

Definition 3: $R(r,u)$: $R(r,u)$ = probability that the time to the next request for an item is $\leq r$ if the item is requested u years after it arrived in the system ($u \geq 0$). $R_1(r,u)$ is defined as $R_1(r,u) = \frac{\partial}{\partial r}R(r,u)$.

Before proceeding with the derivation of the main results, we need the following definitions:

Definition 4: $E_M(z)$: Let $E_M(z)$ be the steady state expected number of items in the primary store z years after the last arrival of a new document for a finite primary store of size M.

Definition 5: E_M: Let E_M be the steady state expected number of items in the primary store for a finite primary store of size M.

Definition 6: $S_{M,i}(z|R)$: Let $S_{M,i}(z|R)$ be the steady-state probability that there are exactly i items in the primary store z years after the arrival of a new item, for a primary store of size M, and given that z is in region R. The region R may take on one of three values:

$z < X$; $X \le z \le T$, which we shall condense to $X \le z$; and finally $z > T$.

In [1], where M was really unbounded, it was possible to derive the expected size of the primary store directly by first finding $E(z)$. However, with a finite M, it is not possible to find an expression for $E_M(z)$ directly. Once the probabilities $S_{M,i}(z|R)$, $i = 1,\ldots,M$ are found, it is a straightforward matter to obtain $E_M(z)$ and E_M, as noted in the following section.

3. STOCHASTIC INTEGRAL EQUATIONS

In this section we shall derive integral equations to determine the probabilities $S_{M,i}(z|R)$ that there are exactly i items in the primary store of size M,z years after the last arrival of a new item given that z is in region R. It will be convenient to use a shorthand notation for probabilistic statements. The shorthand notation can be interpreted directly and is easy to manipulate.

Definition 7: ξ-Notation. We shall define an infinite sequence, $\xi = (\xi_1, \xi_2,\ldots, \xi_i,\ldots)$,to represent probabilities of various different states, starting from the current state ξ_1, at a particular time and ranging backwards in time. The parameters ξ_i may take on the following values.

(a) ξ_i ='1': denotes the probability that the entry under consideration is in the primary store. If there are less than M items in the primary store, then the probability of this state is given by P(w, t).

(b) ξ_i ='$\bar{1}$': denotes the probability that the item under consideration is not in the primary store. If there are less than M items in the store, the state probability is given by [1-P(w, t)].

(c) $\xi_i = \emptyset$: denotes the probability one, i.e., a state in which the i^{th} item and all earlier items no longer play a role in system operation, and hence, any state is allowed.

(d) $\underline{\xi_i = A_R}$: denotes the probability that the $i^{\underline{th}}$ document arrives in the region R.

(e) $\underline{\xi_i = \bar{A}_R}$: denotes the probability that the $i^{\underline{th}}$ document does not arrive in the region R.

(f) $\underline{\xi_i = A_R.'1'}$: denotes the probability that the $i^{\underline{th}}$ item arrived in the region R and the document is still in the primary store.

(g) $\underline{\xi_i = A_R.'\bar{1}'}$: denotes the probability that the $i^{\underline{th}}$ item arrived in the region R and the document is no longer in the primary store.

Definition 8: Convolution Type Operator *. The convolution type operator, defined by the operator (*) is to be interpreted as:

$$A(t) * F(t) = \int_{B_{1,2}} dA(t)\, F\,(t + \tau)$$

integrated over some region $B_{1,2}$. The function $A(t)$ may be replaced either by the function $A_{R_2}(t)$ or the function $A_{R_1}(t)$ and is to be interpreted as the function $A(t)$ without the subscript in the region R_2 or R_1, respectively.

With the above definitions, we can now state and prove the following theorem:

Theorem 1: The probabilities $S_{M,i}(z|R)$ that there are exactly i items in the primary store of size M, z years after the arrival of a new document, given that z is in region R is expressed by the following recursion relations, where R_1 is the region $z < X$ and R_2 the region $X \leq z \leq T$.

1.1 The region R_1: $z < X$

(1) $S_{1,1}(z|R_1) = 1$

$$(2.1)\quad S_{2,1}\,(z|R_1) = \int_{T-z}^{\infty} dA(t) + \int_{X-z}^{T-z} dA(t)\,[1-P(0,\, t+z)]\, F_{2,1}\,(t+z)$$

where

$$F_{2,1}\,(t+z) = \int_{T-t-z}^{\infty} dA(t) + \int_{0}^{T-t-z} dA(\xi)\,[1-P(0,\, t+z+\xi)]\, F_{2,1}\,(t+z+\xi)$$

$$(2.2)\quad S_{2,2}\,(z|R_1) = \int_{X-z}^{T-z} dA(t)\, S_{1,1}\,(t+z|R_2) + \int_{0}^{X-z} dA(t)$$

and (3)

$$\begin{cases} S_{M,1}\,(z|R_1) = S_{2,1}(z|R_1) & \text{for } M = 2,3\ldots \\ S_{M,2}\,(z|R_1) = S_{3,2}(z|R_1) & \\ \vdots & \\ S_{M,M-2}\,(z|R_1) = S_{M-1,M-2}\,(z|R_1) & \end{cases}$$

$$(4)\quad S_{M,M-1}\,(z|R_1) = \int_{X-z}^{T-z} dA(t)\, S_{M-1,M-2}\,(t+z|R_2) + \int_{0}^{X-z} dA(t)\, S_{M-1,M-2}(t+z|R_1)$$

$$(5)\quad S_{M,M}(z|R_1) = \int_{X-z}^{T-z} dA(t)\, S_{M-1,M-1}(t+z|R_2) + \int_{0}^{X-z} dA(t)\, S_{M-1,M-1}(t+z|R_1)$$

For the region R_2 we have the following recursion relations:

1.2 The region R_2: $X \le z \le T$.

$$(6)\quad S_{1,1}(z|R_2) = P(o,z) + [1-P(o,z)] \int_{0}^{T-z} dA(t)\, S_{1,1}(t+z|R_2)$$

$$(7.1)\quad S_{2,1}(z|R_2) = H_{2,1}(z) + [1-P(o,z)] \int_{0}^{T-z} dA(t)\, S_{2,1}(t+z|R_2)$$

where

$$H_{2,1}(z) = P(o,z) \int_{T-z}^{\infty} dA(t) + P(o,z) \int_{0}^{T-z} dA(t)[1-P(o,t+z)]\, \frac{H_{2,1}(t+z)}{P(o,t+z)}$$

$$(7.2)\quad S_{2,2}(z|R_2) = P(o,z) \int_{0}^{T-z} dA(t)\, S_{1,1}(t+z|R_2) + [1-P(o,z)] \int_{0}^{T-z} dA(t)\, S_{2,2}(t+z|R_2)$$

and (8)

$$\begin{cases} S_{M,1}(z|R_2) = S_{2,1}(z|R_2) & M = 2,3,\ldots \\ S_{M,2}(z|R_2) = S_{3,2}(z|R_2) & \\ \vdots & \\ S_{M,M-2}(z|R_2) = S_{M-1,M-2}(z|R_2) & \end{cases}$$

(9) $$S_{M,M-1}(z|R_2) = \int_{X-z}^{T-z} dA(t)\, S_{M-1,M-2}(t+z|R_2) + \int_0^{X-z} dA(t)\, S_{M-1,M-2}(t+z|R_2)$$

(10) $$S_{M,M}(z|R_2) = P(o,z) \int_0^{T-z} dA(t) S_{M-1,M-1}(t+z|R_2) + [1-P(o,z)] \int_0^{T-z} dA(t) S_{M,M}(t+z|R_2)$$

1.3 The region R_3: $z > T$

(11) $S_{M,i}(z|R_3) = 0$ for all M and all i.

Proof:

We shall prove the theorem in several stages. We first consider the case where M = 1, and derive equations (1) and (6). Second, we take the case M = 2, and derive equations 2.1, 2.2, 7.1 and 7.2. Arbitrary values of M are then considered and the remaining equations are derived.

a. M = 1.

a.1. The region $z < X$. The derivation of $S_{1,1}(z|z<X)$ is obvious. Since at time 0 an arrival occurred, the probability that there is one entry in the primary store is one since $z < X$ and the item must stay in the store at least X years, unless a subsequent arrival comes along to replace it.

a.2. The region $X \leq z \leq T$.

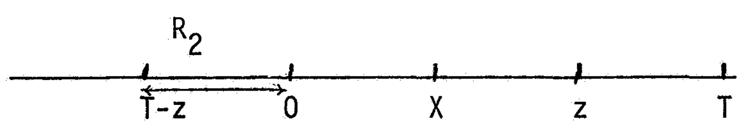


Figure 1

Since M = 1, we have to consider the various cases in which we could get an entry in the primary store z years after the last arrival. The following equations expressed in the ξ-notation account for the various cases:

$$S_{1,1}(z|X\_<z\leq T) = \begin{cases} ('1', \emptyset, \emptyset, \ldots) \\ + ('\bar{1}', A_{R_2}'1', \emptyset, \emptyset, \ldots) \\ + ('\bar{1}', A_{R_2}'\bar{1}', A_{R_2}'1', \emptyset, \emptyset, \ldots) \\ + \quad \vdots \end{cases}$$

The first term represents the probability that the last item is in the primary store and we don't care about any other state since, regardless of what occurs, there can be only one item in the primary store. The second term represents the probability that the item that arrived last is not in the store, the item that arrived previously came in the region R_2 and is in the primary store, and we don't care about subsequent items. The third term represents the probability that the last item to arrive is not in the primary store, the one previous to that one arrived in the region R_2, and is not in the primary store, and the one previous to that item arrived in the region R_2 after that item and is in the primary store, and we don't care about subsequent arrivals. Subsequent cases are clear. From the above discussion and the equation describing $S_{1,1}(z|X\leq z\leq T)$, it is readily seen that we may write

$$S_{1,1}(z|X\leq z\leq T)='1'+'\bar{1}'(A^*'1')+'\bar{1}'(A^*'\bar{1}'(A^*'1'))+'\bar{1}'(A^*'\bar{1}'(A^*'\bar{1}'(A^*'1')))+\ldots,$$

where the operator * is given in Definition 8.

It may be seen, readily, that the above series can be factored. One may merely return to the probabilistic interpretation of the series to develop the factorization. Thus,

$$S_{1,1}(z|X\leq z\leq T) = '1'+'\bar{1}'[A^*\{'1'+'\bar{1}'(A^*'1')+'\bar{1}'(A^*'\bar{1}'(A^*'1'))\}]$$

$$S_{1,1}(z|X\leq z\leq T) = '1'+'\bar{1}'[A^*\{S_{1,1}(z|X\leq z\leq 1)\}].$$

Translating this expression into probabilistic terms we have,

$'1' \equiv P(o,z)$

$'\bar{1}' \equiv [1-P(o,z)]$

Then,

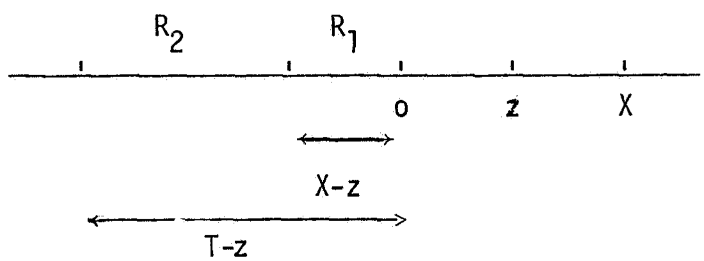
$$(6) \quad S_{1,1}(z|R_2) = P(o,z) + [1-P(o,z)] \int_0^{T-z} dA(t)\, S_{1,1}(t+z|R_2)$$

Thus, we have proved equation (6).

b. <u>M = 2</u>

b.1. The region $z < X$, and $S_{2,1}(z|z<X)$.

We first derive $S_{2,1}(z|z<X)$. Having developed the significance of each term in the previous case (derived in a.2), we shall merely employ the notation developed to derive our result.



<u>Figure 2</u>

$$S_{2,1}(z|z<X) = \begin{cases} ('1', \bar{A}_{R_{1+2}}, \emptyset, \ldots) \\ + ('1', A_{R_2} '\bar{1}', \bar{A}_{R_2}, \emptyset, \ldots) \\ + ('1', A_{R_2} '\bar{1}', A_{R_2} '\bar{1}', \bar{A}_{R_2}, \emptyset, \ldots) \\ \vdots \end{cases}$$

The above are the only cases in which there can be one item in the primary store, given a maximum primary store of two items and $z < X$. The above equations can be rewritten, using the operator * as,

$$S_{2,1}(z|R_1) = \bar{A}_{R_1+R_2} + A_{R_2} * ('\bar{1}'\bar{A}_{R_2}) + A_{R_2} * ('\bar{1}'(A_{R_2} * ('\bar{1}'\bar{A}_{R_2})) + \ldots$$

$$S_{2,1}(z|R_1) = \bar{A}_{R_1+R_2} + A_{R_2} * '1' \{\bar{A}_{R_2} + A_{R_2} * ('\bar{1}'\bar{A}_{R_2}) + A_{R_2} * ('\bar{1}'(A_{R_2} * ('\bar{1}'\bar{A}_{R_2}))) + \ldots\}$$

We note that the term in the brackets of the last equation may be written as,

$$F_{2,1}(z) = \bar{A}_{R_2} + A_{R_2} * ('\bar{1}'\ F_{2,1}(z)) \ .$$

Transforming the last two equations to probability distribution and integral forms, we have,

$$(2.1)\ \begin{cases} S_{2,1}(z|R_1) = \int_{T-z}^{\infty} dA(t) + \int_{X-z}^{T-z} dA(t)\ [1-P(o,t+z)]\ F_{2,1}(t+z) \\ F_{2,1}(t+z) = \int_{T-t-z}^{\infty} dA(\xi) + \int_{o}^{T-t-z} dA(\xi)\ [1-P(o,t+z+\xi)]\ F_{2,1}(t+z+\xi) \end{cases}$$

as was to be shown.

b.2. The region $z<X$ and $S_{2,2}(z|z<X)$.

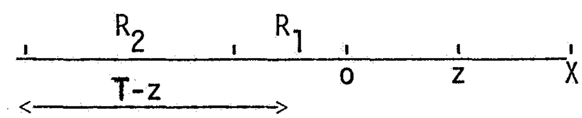


Figure 3

We must distinguish two regions, and hence two cases. In the first case, following the last arrival, there is an arrival in region R_1 and, in the second case at least one item that arrived in region R_2, but not in region R_1 is in the primary store. These are represented by

Case 1: $(A_{R_1}, \emptyset, \emptyset, \ldots \qquad)$

Case 2: $(A_{R_2}'1', \emptyset, \emptyset, \ldots \qquad)$

$+ (A_{R_2}'\bar{1}', A_{R_2}'1', \emptyset, \ldots \qquad)$

$+ (A_{R_2}'\bar{1}', A_{R_2}'\bar{1}', A_{R_2}'1', \emptyset, \ldots)$

$\vdots$

We note, however, that Case 2 is related directly to $S_{1,1}(z|X\le z\le T)$.
We may then write,

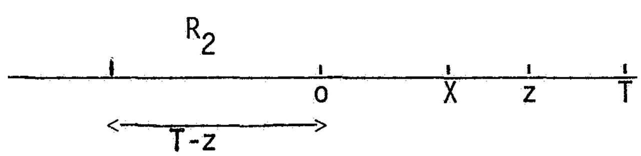
$$S_{2,2}(z|R_1) = A_{R_1} + A_{R_2} * S_{1,1}(z|R_2).$$

Therefore,

$$(2.2)\quad S_{2,2}(z|R_1) = \int_{0}^{X-z} dA(t) + \int_{X-z}^{T-z} dA(t)\ S_{1,1}(t+z|R_2)$$

which is the desired result.

b.3. <u>The region $X\le z\le T$ and $S_{2,1}(z|X\le z\le T)$.</u>



<u>Figure 4</u>

Using our notation, we may write down the equation directly. Hence,

$$\begin{aligned}
S_{2,1}(z|R_2) = {} & ('1', \bar{A}_{R_2}, \emptyset, \ldots) \\
& + ('1', A_{R_2}\ 'T', \bar{A}_{R_2}, \emptyset, \ldots) \\
& \quad \vdots \\
& + ('T', A_{R_2}\ '1', \bar{A}_{R_2}, \emptyset, \ldots) \\
& + ('T', A_{R_2}\ '1', A_{R_2}\ 'T', \bar{A}_{R_2}, \emptyset, \ldots) \\
& \quad \vdots \\
& + ('T', A_{R_2}\ 'T', A_{R_2}\ '1', \bar{A}_{R_2}, \emptyset, \ldots) \\
& + ('T', A_{R_2}\ 'T', A_{R_2}\ '1', A_{R_2}\ 'T', \bar{A}_{R_2}, \emptyset, \ldots) \\
& \quad \vdots
\end{aligned}$$

Regrouping terms (Def. 8), and using the convolution operator, we have,

$$S_{2,1}(z|R_2) = H_{2,1}(z) + 'T'\ [A*\{S_{2,1}(z|R_2)\}],$$

where

$$H_{2,1}(z) = \text{'1'}\bar{A}_{R_2} + \text{'1'} \left[A^*_{R_2}\text{'}\bar{1}\text{'}\left\{\frac{H_{2,1}(z)}{\text{'1'}}\right\}\right].$$

In probabilistic and integral terms we have

$$(7.1)\begin{cases} S_{2,1}(z|R_2) = H_{2,1}(z|R_2) + [1-P(o,z)] \int_0^{T-z} dA(t)\ S_{2,1}(t+z|R_2) \\ H_{2,1}(z) = P(o,z) \int_{T-z}^{\infty} dA(t) + P(o,z) \int_0^{T-z} dA(t)\ [1-P(o,t+z)] \dfrac{H_{2,1}(t+z)}{P(o,t+z)} \end{cases}$$

as was to be shown.

b.4. <u>The region $X\leq z\leq T$ and $S_{2,2}(z|X\leq z\leq T)$</u>

The cases for finding exactly two items in the primary store given a capacity of two items in the primary store and $X\leq z\leq T$ are enumerated as:

$$\begin{array}{l} (\text{'1'}, A\text{'1'}, \emptyset, \ldots \qquad\qquad) \\ +(\text{'1'}, A\text{'}\bar{1}\text{'}, A\text{'1'}. \emptyset, \ldots \qquad) \\ +(\text{'1'}, A\text{'}\bar{1}\text{'}, A\text{'}\bar{1}\text{'}, A\text{'1'}, \emptyset, \ldots) \\ \qquad\qquad \vdots \\ +(\text{'}\bar{1}\text{'}, A\text{'1'}, A\text{'1'}, \emptyset, \ldots \qquad) \\ +(\text{'}\bar{1}\text{'}, A\text{'1'}, A\text{'}\bar{1}\text{'}, A\text{'1'}, \emptyset, \ldots) \\ \qquad\qquad \vdots \\ +(\text{'}\bar{1}\text{'}, A\text{'}\bar{1}\text{'}, A\text{'1'}, A\text{'1'}, \emptyset, \ldots \qquad) \\ +(\text{'}\bar{1}\text{'}, A\text{'}\bar{1}\text{'}, A\text{'1'}, A\text{'}\bar{1}\text{'}, A\text{'1'}, \emptyset, \ldots) \\ \qquad\qquad \vdots \end{array}$$

It may be seen readily that these cases reduce to:

$$S_{2,2}(z|R_2) = \text{'1'}\ [A^*\{S_{1,1}(z|R_2)\}] + \text{'}\bar{1}\text{'}\ [A^*\{S_{2,2}(z|R_2)\}].$$

Hence,

$$(7.2) S_{2,2}(z|R_2)=P(o,z) \int_0^{T-z} dA(t)\ S_{1,1}(t+z|R_2) + [1-P(o,z)] \int_0^{T-z} dA(t)\ S_{2,2}(t+z|R_2).$$

c. <u>General value of M > 2.</u>

c.1. <u>The region $z < X$, and $S_{M,i}(z|z<X)$ for $i = 1,2,\ldots M-2$.</u>

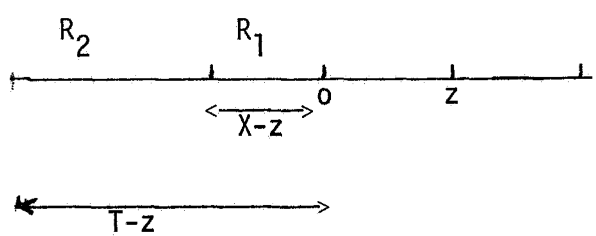
When $i = 1, 2,\ldots, M-2$, it is clear that the value of $S_{M,i}(z|z<X)$ is dependent upon i and not M. Thus, for example, $S_{3,1}(z|z<X)$ is identical to $S_{2,1}(z|z<X)$ since the value of $M = 3$ had no bearing upon the probability that there is one item in the store. The limiting factor was i, rather than M. The equations (3) therefore are valid for $i = 1, 2,\ldots, M - 2$, and we must then derive $S_{M,M-1}(z|z<X)$ and $S_{M,M}(z|z<X)$.

c.2. <u>The region $z < X$, and $S_{M,M-1}(z|z<X)$</u>

Using the convolution operation (Def. 8), $S_{M,M-1}(z|z<X)$ is given by

$$S_{M,M-1}(z|R_1) = A_{R_2} * S_{M,M-2}(z|R_2) + A_{R_1} * A_{R_2} * S_{M,M-3}(z|R_2)$$
$$+\ldots+ \overbrace{A_{R_1} * A_{R_1} * \ldots * A_{R_1}}^{M-3} * A_{R_2} * S_{M,1}(z|R_2)$$
$$+ \overbrace{A_{R_1} * \ldots * A_{R_1}}^{M-2} * S_{M,1}(z|R_1)$$

That the above is correct may be seen by considering the following:



<u>Figure 5</u>

In the first term we note that A_{R_2} represents the probability that there is an arrival in the region R_2 preceding the last arrival, and the term $S_{M,M-2}(z|R_2)$ denotes the conditional probability that there are exactly

M-2 items in the primary store from that region if z is in R_2. The second term represents the probability that there is an arrival in the region R_1 followed by all others in region R_2. The next to last term follows the same pattern. The last term represents the probability that the M-2 arrivals following the last arrival are in the region R_1. There are no other cases that need to be expressed.

We may rewrite the above equation as

$$S_{M,M-1}(z|R_1)=A_{R_2}*S_{M,M-2}(z|R_2)+A_{R_1}*\{A_{R_2}*S_{M,M-3}(z|R_2)+\ldots+A_{R_1}*\ \cdots\ *A_{R_1}*A_{R_2}*S_{M,1}(z|R_2)\}$$
$$+\ A_{R_1}*\ldots*A_{R_1}*S_{M,1}(z|R_1).$$

We note, however, that the term in brackets is simply,

$$S_{M-1,M-2}(z|R_1)-\overbrace{A_{R_1}*\ldots*A_{R_1}}^{M-3}*S_{M-1,M-2}(z|R_1).$$

Then, we may rewrite our equation as,

$$S_{M,M-1}(z|R_1) = A_{R_2}*S_{M,M-2}(z|R_2 +A_{R_1}*S_{M-1,M-2}(z|R_1).$$

As will be noted in Section c.4, $S_{M,M-2}(z|R_2) = S_{M-1,M-2}(z|R_2)$. Therefore,

$$(4)\quad S_{M,M-1}(z|R_1) = \int_{X-z}^{T-z} dA(t)S_{M-1,M-2}(t+z|R_2) + \int_{0}^{X-z} dA(t)S_{M-1,M-2}(t+z|R_1)$$

is proved as soon as we show $S_{M,M-2}(z|R_2) = S_{M-1,M-2}(z|R_2)$.

c.3. <u>The region $z < X$ and $S_{M,M}(z|z<X)$.</u>

As in Section c.2, above, we may write,

$$S_{M,M}(z|R_1) = A_{R_2}*S_{M-1,M-1}(z|R_2)+A_{R_1}*A_{R_2}*S_{M-2,M-2}(z|R_2)$$
$$+\ldots+\overbrace{A_{R_1}*A_{R_1}*\ldots*A_{R_1}}^{M-2}*A_{R_2}*S_{1,1}(z|R_2)+\overbrace{A_{R_1}*\ldots*A_{R_1}}^{M-1}.$$

This equation may then be written as,

$$S_{M,M}(z|R_1) = A_{R_2}*S_{M-1,M-1}(z|R_2)+A_{R_1}*S_{M-1,M-1}(z|R_1).$$

Hence,

$$(5)\quad S_{M,M}(z|R_1)=\int_{X-z}^{T-z} dA(t)\ S_{M-1,M-1}(t+z|R_2)+\int_{0}^{X-z} dA(t)\ S_{M-1,M-1}(t+z|R_1).$$

c.4. The region $X\leq z\leq T$ and $S_{M,i}(z|R_2)$, $i = 1,\ldots, M-2$.

The discussion of section c.1, above, applies, where instead of region R_1 we replace it with region R_2. Hence, equations (8) apply, and equation 4 is now proved.

c.5. The region $X\leq z\leq T$ and $S_{M,M-1}(z|R_2)$.

For this region we may write the equation expressing $S_{M,M-1}(z|R_2)$ directly as,

$$S_{M,M-1}(z|R_2)='1'\ [A_{R_2}*S_{M-1,M-2}(z|R_2)]+'\bar{1}'\ [A_{R_2}*'1'\ A_{R_2}*S_{M-1,M-2}(z|R_2)]$$
$$+'\bar{1}'\ A_{R_2}\ '\bar{1}'A_{R_2}*\{S_{M-1,M-2}(z|R_2)\}+\ldots+'\bar{1}'\ A_{R_2}*'\bar{1}'A_{R_2}*\ldots*A_{R_2}*A_{R_2}*S_{M-1,M-2}(z|R_2).$$

Therefore,

$$S_{M,M-1}(z|R_2)='1'\ A_{R_2}*\{S_{M-1,M-2}(z|R_2)\}+'\bar{1}'\ A_{R_2}*\{S_{M,M-1}(z|R_2)\}.$$

Hence,

$$(9)\ S_{M,M-1}(z|R_2)=P(o,z)\int_{0}^{T-z} dA(t)S_{M-1,M-2}(t+z|R_2)+[1-P(o,z)]\int_{0}^{T-z} dA(t)S_{M,M-1}(t+z|R_2).$$

Therefore, (9) has been shown.

c.6. The region $X\leq z\leq T$ and $S_{M,M}(z|R_2)$.

$S_{M,M}(z|R_2)$ is given by

$$S_{M,M}(z|R_2) = '1' * A_{R_2}\{S_{M-1,M-1}(z|R_2)\}+'\bar{1}' * A_{R_2}'1' *A_{R_2}\{S_{M-1,M-1}(z|R_2)\}$$
$$+\ldots + '\bar{1}' * A_{R_2}'\bar{1}' *\ldots* A_{R_2}'\bar{1}' * A_{R_2}'1' *A_{R_2}\{S_{M-1,M-1}(z|R_2)\}.$$

Therefore,

$$S_{M,M}(z|R_2) = '1' *A_{R_2}\{S_{M-1,M-1}(z|R_2)\} + '1' *A_{R_2}\{S_{M,M}(z|R_2)\}.$$

Hence,

$$(10)S_{M,M}(z|R_2)=P\ (o,z)\int_{0}^{T-z} dA(t)S_{M-1,M-1}(t+z|R_2)+[1-P\ (o,z)]\int_{0}^{T-z} dA(t)S_{M,M}(t+z|R_2).$$

We have, therefore, proved Theorem 1.

The probabilities $S_{M,i}(z|R)$ are only of interest to us to help obtain the expected size of the primary store. The following theorem will now be shown to be valid.

Theorem 2: The expected size of the primary store for a given bounded primary store of size M is given by the recursion relationship:

$$(11)\quad E_M = E_{M-1} + (M-1)\,[S_{M+1,M}(o|R_1) - S_{M,M}(o|R_1)] + M\,S_{M+1,M+1}(o|R_1).$$

Equation (11) is valid for an arbitrary arrival distribution.

Proof:

Since we have, by Theorem 1, the probabilities $S_{M,i}(z|R)$ for $i = 1,\ldots,M$ and $R = R_1, R_2$, the expected number in the store z years after the last arrival for a particular region R is, by definition, the sum of the probability that there are exactly i items in the store times the number of items i. Thus, by definition,

$$(12)\quad E_M(z|R) = \sum_{i=1}^{M} i\, S_{M,i}(z|R).$$

Again, by definition, we have

$$E_M = \int_{R_1} E_M(z|R_1)dA(z) + \int_{R_2} E_M(z|R_2)dA(z).$$

Hence,

$$(13)\quad E_M = \sum_{i=1}^{M} i\,\{\int_0^X S_{M,i}(z|R_1)dA(z) + \int_X^T S_{M,i}(z|R_2)dA(z)\}\,.$$

Now, (13) can be rewritten by first summing from i=1 to M-2 and noting that due to the relationships (3) and (8), we can replace M by M-1 in $S_{M,i}(z|R)$. Then, adding and subtracting an appropriate term to account for $S_{M-1,M-1}(z|R)$, we obtain

$$(14)\quad E_M = E_{M-1} + (M-1)\int_0^X dA(t)\,\{S_{M,M-1}(t|R_1) - S_{M-1,M-1}(t|R_1)\}$$

$$+ (M-1)\int_X^T dA(t)\,\{S_{M,M-1}(t|R_2) - S_{M-1,M-1}(t|R_2)\}$$

$$+ M\int_0^X dA(t)\; S_{M,M}(t|R_1) + M\int_X^T dA(t)\; S_{M,M}(t|R_2).$$

Now, setting M = M+1 in equation (4), and subtracting equation (4) and setting z = o, we obtain

$$(15)\quad \int_0^X dA(t)\,\{S_{M,M-1}(t|R_1)-S_{M-1,M-1}(t|R_1)\}=[S_{M+1,M}(o|R_1)-S_{M,M}(o|R_1)]$$

$$+\int_X^T dA(t)\,\{S_{M-1,M-1}(t|R_1)-S_{M,M-1}(t|R_1)\}.$$

Furthermore, from equation (5) we obtain, by setting M = M+1 and z = o,

$$(16)\quad S_{M+1,M+1}(o|R_1) = \int_0^X dA(t)\; S_{M,M}(t|R_1) + \int_X^T dA(t)\; S_{M,M}(t|R_2).$$

Substituting equations (15) and (16) into (14), we obtain equation (11) which proves the theorem. We note that no limiting assumptions were made concerning the form of the distribution A(t).

4. THE EXPECTED SIZE OF THE PRIMARY STORE FOR THE CASE OF POISSON ARRIVALS OF NEW DOCUMENTS

Using the results of the previous section, the expected size of the primary store is calculated for a Poisson distribution of arrivals. As shown in (1), if the primary store is permitted to be unbounded in size, then

$$(17)\quad \rho = E_\infty = \alpha X + \alpha\int_X^T P(o,u)du,$$

where

$$(18)\quad A(t) = 1 - e^{-\alpha t}\ .$$

The expression for E_M will be shown to be related closely to E_∞.

We first define,

$$(19) \qquad H_{M,i}(z|R_1) = \int_0^{X-z} dA(t)\ S_{M,i}(t+z|R_1)$$

$$H_{M,i}(z|R_2) = \int_0^{T-z} dA(t)\ S_{M,i}(t+z|R_2).$$

The following theorem then applies.

Theorem 3: Let A(t) be a Poisson distribution of arrivals of new documents to a primary store, where A(t) is given by (18). Then, we have[1)]

$$(20) \quad \begin{cases} S_{1,1}(z|R_1) = 1 \\ H_{1,1}(z|R_1) = 1-e^{-\alpha X}\ e^{\alpha z} \\ S_{1,1}(z|R_2) = 1 - [1-P\ (o,z)]\ \exp(-\alpha \int_z^T P(o,v)dv) \\ H_{1,1}(z|R_2) = 1 - \exp(-\alpha \int_z^T P(o,v)dv) \end{cases}$$

$$(21) \quad \begin{cases} S_{2,1}(z|R_1) = e^{-\rho}\ e^{\alpha z} \\ H_{2,1}(z|R_1) = \alpha[X-z]\ e^{-\rho}\ e^{\alpha z} \\ H_{2,2}(z|R_1) = 1 - e^{-\alpha X}\ e^{\alpha z} - \alpha[X-z]e^{-\rho}\ e^{\alpha z} \\ S_{2,2}(z|R_1) = 1 - e^{-\rho}\ e^{\alpha z} \\ S_{2,1}(z|R_2) = \{P(o,z) + \alpha \int_z^T P(o,u)du - \alpha P(o,z) \int_z^T P(o,u)du\}\ \exp(-\alpha \int_z^T P(0,v)dv) \\ H_{2,1}(z|R_2) = \alpha \int_z^T P(o,u)du\ \exp(-\alpha \int_z^T P(0,v)dv) \\ S_{2,2}(z|R_2) = 1 - \{1 + \alpha[1-P\ (o,z)] \int_z^T P(o,u)du\}\ \exp(-\alpha \int_z^T P(o,u)du) \\ H_{2,2}(z|R_2) = 1 - \{1 + \alpha \int_z^T P(o,u)du\}\ \exp(-\alpha \int_z^T P(o,u)du) \end{cases}$$

and,

1) For convenience, the notation "exp" and "e" will be used interchangeably to denote the exponential function.

$$S_{M,M-1}(z|R_1) = \frac{\rho^{M-2} e^{-\rho} e^{\alpha z}}{(M-2)!} + \sum_{j=1}^{M-2} (-1)^j \frac{(\alpha z)^j}{j\,!} e^{-\rho} e^{\alpha z}$$

$$H_{M,M-1}(z|R_1) = \frac{\rho^{M-1}}{(M-1)!} e^{-\rho} e^{\alpha z} + \sum_{j=1}^{M-1} \frac{(-1)^j (\alpha z)^j}{j\,!} e^{-\rho} e^{\alpha z} - \frac{e^{-\rho} e^{\alpha z} \left[\alpha \int_X^T P(o,u)du\right]^{M-1}}{(M-1)!}$$

$$S_{M,M}(z|R_1) = 1 + \sum_{j=o}^{M-2} (-1)^{j+1} \frac{(\alpha z)^j}{j\,!} \sum_{i=o}^{M-2-j} \frac{\rho^i}{i!} e^{-\rho} e^{\alpha z}$$

$$H_{M,M}(z|R_1) = 1-e^{-\alpha X} e^{-\alpha z} + \sum_{j=0}^{M-1} (-1)^{j+1} \frac{(\alpha z)^j}{j!} \sum_{i=o}^{M-1-j} \frac{\rho^i}{i!} e^{-\rho} e^{\alpha z}$$

$$+ \sum_{i=o}^{M-1} \frac{\left[\int_X^T P(o,u)du\right]^i}{i\,!} e^{-\rho} e^{\alpha z}$$

(22)
$$S_{M,M-1}(z|R_2) = \frac{\left[\alpha \int_z^T P(o,u)du\right]^{M-1}}{(M-1)!} \exp(-\alpha \int_z^T P(o,u)du)$$

$$+ P(o,z)\frac{\left[\alpha \int_z^T P(o,u)du\right]^{M-2}}{(M-2)!} - P(o,z) \frac{\left[\alpha \int_z^T P(o,u)du\right]^{M-1}}{(M-1)!} \exp(-\alpha \int_z^T P(o,u)du)$$

$$H_{M,M-1}(z|R_2) = \frac{\left[\alpha \int_z^T P(o,u)du\right]^{M-1}}{(M-1)!} \exp(-\alpha \int_z^T P(o,v)dv)$$

$$S_{M,M}(z|R_2) = 1 - \sum_{j=o}^{M-1} \frac{\left[\alpha \int_z^T P(o,u)du\right]^j}{j\,!} \exp(-\alpha \int_z^T P(o,v)dv)$$

$$+ \frac{P(o,u)\left[\alpha \int_z^T P(o,u)du\right]^{M-1}}{(M-1)!} \exp\,(-\alpha \int_z^T P(o,v)dv)$$

$$H_{M,M}(z|R_2) = 1 - \sum_{j=o}^{M-1} \frac{\left[\alpha \int_z^T P(o,u)du\right]^j}{j\,!} \exp(-\alpha \int_z^T P(o,v)dv)$$

Proof:

That the equations (20),(21), and (22) are correct may be seen by substituting these equations into equations (1) - (10). Because of the length of the proof, we shall provide, therefore, only a sketch of how the formulae were derived.

For M=1 and M=2, the integral equations expressing $S_{M,i}(z|R)$ were solved by transforming the equations into first order differential equations. The pattern for the general formulae then became evident. The general case was then solved by induction on M. More particularly, we may write, from equation (10), and (19),

$$S_{M,M}(z|R_2) = P(o,z)\ H_{M-1,M-1}(z|R_2) + [1-P(o,z)] \int_0^{T-z} dA(t)\ S_{M,M}(t+z|R_2).$$

Since A(t) is given by (18), we may differentiate both sides of the equation and, using both the equation and its derivative, we can transform this equation into the form

$$\frac{dS_{M,M}(z|R_2)}{dz} + \left\{ \frac{\frac{dP(o,z)}{dz}}{1-P(o,z)} - \alpha P(o,z) \right\} S_{M,M}(z|R_2) =$$

$$P(o,z)\ \frac{dH_{M-1,M-1}}{dz}(z|R_2) + H_{M-1,M-1}(z|R_2)\left\{ \frac{\frac{dP(o,z)}{dz}}{[1-P(o,z)]} - \alpha P(o,z) \right\}$$

This equation is a first order linear differential equation whose solution can be determined most readily to be:

$$S_{M,M}(z|R_2) = H_{M-1,M-1}(z|R_2) - [1-P(o,z)]\ \exp(\alpha \int_T^z P(o,u)du)$$

$$\int_T^z \exp(-\alpha \int_T^v P(o,u)du)\ [\frac{dH_{M-1,M-1}(v|R_2)}{dv}]dv$$

Since $S_{M,M}(z|R_2)$ is dependent upon $H_{M-1,M-1}(z|R_2)$, by induction we may obtain the formula for $S_{M,M}(z|R_2)$.

Finally, we have:

Theorem 4: Let $A(t)$ be a Poisson distribution of arrivals given by (18). Then, the expected number of documents in the primary store given a maximum size of the primary store of M documents, is given by:

$$(23) \quad E_M = E_{M-1} + 1 - e^{-\rho} \sum_{j=o}^{M-1} \frac{\rho^j}{j!}$$

or

$$(24) \quad E_M = M - e^{-\rho} \sum_{j=o}^{M-1} \frac{(M-j)}{j!} \rho^j$$

or

$$(25) \quad E_M = \rho + \sum_{j=M}^{\infty} (-1)^j \frac{\rho^j}{j!} \sum_{i=0}^{M-1} (-1)^i (M-i) \frac{j!}{i!\,(j-i)!}$$

where $\rho = \alpha X + \alpha \int_X^T P(o,u)du = E_\infty$ and $M \geq 1$.

We define $E_o = 0$.

Proof:

Equation (23) can be determined readily. From Theorem 2 we have that E_M is given by equation (11). From equation (22) we know explicit formulae for $S_{M,M-1}(z|R_1)$ and $S_{M,M}(z|R_1)$. If we set $z = o$ and substitute these expressions into (11), we get

$$E_M = E_{M-1} + (M-1) \, [S_{M+1,M}(o|R_1) - S_{M,M}(o|R_1)] + M \, S_{M+1,M+1}(o|R_1)$$

$$= E_{M-1} + (M-1) \left[\frac{\rho^{M-1} e^{-\rho}}{(M-1)!} - (1 - \sum_{i=o}^{M-2} \frac{\rho^i}{i!} e^{-\rho}) \right] + M \left[1 - \sum_{i=o}^{M-1} \frac{\rho^i}{i!} e^{-\rho} \right],$$

and hence, performing the algebra, we obtain (23).

Equations (24) and (25) are then obtained most readily by algebraic manipulation.

It can be seen from (25), that E_M may be written as

$$E_M = E_\infty + \sum_{j=M}^{\infty} B_j \rho^j \ , \quad \text{where the } B_j \text{ are derived from (25).}$$

As $M \to \infty$, we obtain the result found in [1]. For M sufficiently large, we may approximate E_M by E_∞ and thereby save considerable computation time.

5. SUMMARY

The work started in [1] concerning the determination of the expected size of the primary store of a two level storage system has been extended to the case of a bounded primary store of size M. Given a probability distribution describing how requests are made on documents stored in the system, equation (23) gives a simple recursion formula for determining the expected size of the store. Lemma 1, derived in [1], provides an integral equation for determining P(w,t), required in equation (23). Unfortunately, for some realistic distribution of R(r,u), it will be difficult to solve the integral equation expressed in Lemma 1. It would appear that the equations would have to be solved numerically by digital computer.

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