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MATHEMATICAL PROGRAMMING FOR CONSTRAINED MINIMAL PROBLEMS
PART 1 - SEQUENTIAL GRADIENT - RESTORATION ALGORITHM

by

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Mathematical Programming for Constrained Minimal Problems

Part 1 - Sequential Gradient - Restoration Algorithm[1]

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Abstract. The problem of minimizing a function f(x) subject to a constraint $\varphi(x) = 0$ is considered. Here, f is a scalar, x is an n-vector, and φ is a p-vector. An iterative algorithm is presented, made up of the alternate succession of gradient phases and restoration phases. In the gradient phase, the first-order change of f is minimized subject to the linearized constraint and a quadratic constraint on the displacement Δx. In the restoration phase, the displacement modulus $|\delta x|$ is minimized subject to the linearized constraint. It is shown that, if α is the stepsize of the gradient phase, $\Delta x = O(\alpha)$ and $\delta x = O(\alpha^2)$. Therefore, for α sufficiently small, the restoration algorithm preserves the descent property of the gradient algorithm: the function f decreases between any two successive restoration phases.

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1. Introduction

In a previous paper (Ref. 1), a review of the ordinary gradient method for minimizing a function $f(x)$ of an unconstrained n-vector x was presented. The basic idea is to minimize the first-order change $\delta f(x)$ subject to a quadratic constraint on the displacement vector.

In this paper, the minimization of the function $f(x)$ subject to the constraint $\varphi(x) = 0$ is considered, where x is an n-vector and φ is a p-vector. Once more, the ordinary gradient method is employed. This involves the minimization of $\delta f(x)$ subject to $\delta\varphi(x) = 0$ and a quadratic constraint on the displacement vector. Since the constraint $\varphi(x) = 0$ is accounted for only to first order, the position vector $\tilde{x}$ at the end of the gradient phase is such that $\varphi(\tilde{x}) \neq 0$. This being the case, a restoration phase is needed prior to starting the next gradient phase. Specifically, one has to apply a small perturbation to $\tilde{x}$ to generate a new position vector $\tilde{\tilde{x}}$ such that $\varphi(\tilde{\tilde{x}}) = 0$. While there are infinite ways to perform this operation, the simplest way is that developed in Refs. 2-3: the constraint is restored to a preselected degree of accuracy subject to the least-square change of the position vector.

2. Statement of the Problem

We consider the problem of minimizing the function

$$f = f(x) \tag{1}$$

subject to the constraint

$$\varphi(x) = 0 \tag{2}$$

In the above equations, x is an n-vector and φ is a p-vector, respectively defined by

$$x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix}, \qquad \varphi(x) = \begin{bmatrix} \varphi^1(x) \\ \varphi^2(x) \\ \vdots \\ \varphi^p(x) \end{bmatrix} \tag{3}$$

3. Gradient Phase

Consider a displacement Δx leading from the nominal point x to the varied point $\tilde{x}$ such that

$$\tilde{x} = x + \Delta x \tag{4}$$

Assume that the nominal point x satisfies (2) exactly and that the varied point $\tilde{x}$ satisfies (2) to first-order. The first-order change of the function (1) is given by

$$\delta f(x) = g^T(f, x)\,\Delta x \tag{5}$$

where the symbol g(f, x) denotes the gradient of the scalar function f with respect to the vector x, that is,

$$g(f, x) = \begin{bmatrix} \partial f/\partial x^1 \\ \partial f/\partial x^2 \\ \vdots \\ \partial f/\partial x^n \end{bmatrix} \tag{6}$$

The symbol T denotes the transpose of a matrix. In turn, the first-order change of the constraint (2) is represented as

$$\delta\varphi(x) = G^T(\varphi, x)\,\Delta x = 0 \tag{7}$$

where $G(\varphi, x)$ denotes the gradient of the vectorial function φ with respect to the vector x. This is the matrix

$$G(\varphi, x) = [g(\varphi^1, x) \ , \ g(\varphi^2, x) \ , \ . \ . \ . \ , \ g(\varphi^p, x)] \tag{8}$$

whose jth column is the gradient $g(\varphi^j, x)$ of the scalar function φ^j with respect to the vector x. In extended form, (8) can be rewritten as the n x p matrix

$$G(\varphi, x) = \begin{bmatrix} \partial\varphi^1/\partial x^1 & \partial\varphi^2/\partial x^1 & \cdots & \partial\varphi^p/\partial x^1 \\ \partial\varphi^1/\partial x^2 & \partial\varphi^2/\partial x^2 & \cdots & \partial\varphi^p/\partial x^2 \\ \cdots & \cdots & \cdots & \cdots \\ \partial\varphi^1/\partial x^n & \partial\varphi^2/\partial x^n & \cdots & \partial\varphi^p/\partial x^n \end{bmatrix} \tag{9}$$

Next, consider the following quadratic constraint on the displacement:

$$K = \Delta x^T \Delta x \tag{10}$$

where K is a constant prescribed a priori. With this understanding, we formulate the following problem: Find the variation Δx which minimizes (5) subject to (7) and (10).

3.1. Derivation of the Algorithm. Standard methods of the theory of maxima and minima show that the fundamental function of this problem is the scalar function

$$\Omega = g^T(f, x)\Delta x + \lambda^T[G^T(\varphi, x)\Delta x] + (1/2\alpha)\,\Delta x^T \Delta x \tag{11}$$

where $1/2\alpha$ is a scalar Lagrange multiplier and λ denotes the undetermined, constant vector Lagrange multiplier

$$\lambda = \begin{bmatrix} \lambda^1 \\ \lambda^2 \\ \vdots \\ \lambda^p \end{bmatrix} \tag{12}$$

If one introduces the augmented function

$$F(x) = f(x) + \lambda^T \varphi(x) \tag{13}$$

and observes that

$$g(F, x) = g(f, x) + G(\varphi, x)\lambda \tag{14}$$

and that

$$g^T(F, x) = g^T(f, x) + \lambda^T G^T(\varphi, x) \tag{15}$$

the fundamental function (11) becomes

$$\Omega = g^T(F, x)\,\Delta x + (1/2\,\alpha)\Delta x^T\,\Delta x \tag{16}$$

In Eqs. (14)-(16), $g(F, x)$ denotes the gradient of the augmented function F with respect to the vector x. The optimum change Δx satisfies the relation

$$g(\Omega, \Delta x) = 0 \tag{17}$$

where $g(\Omega, \Delta x)$ denotes the gradient of the fundamental function Ω with respect to the vector

Δx. The explicit form of (17) is the following:

$$\Delta x = -\alpha g(F, x) \tag{18}$$

and shows that the displacement vector Δx has the same direction as the gradient $g(F, x)$ of the augmented function F with respect to the vector x. Of course, $g(F, x)$ is known provided the multiplier λ is known (see following paragraph).

3.2. Lagrange Multiplier. If Eqs. (14) and (18) are combined, the displacement vector becomes

$$\Delta x = -\alpha[g(f, x) + G(\varphi, x)\lambda] \tag{19}$$

Next, we substitute (19) into the linearized constraint (7) and obtain the relation

$$G^T(\varphi, x)g(f, x) + [G^T(\varphi, x)G(\varphi, x)]\lambda = 0 \tag{20}$$

Since the vector $g(f, x)$ and the matrix $G(\varphi, x)$ are known at the nominal point x, Eq. (20) supplies the multiplier λ; this is precisely the value which guarantees satisfaction of the constraint (2) to first order at the end of the gradient phase.

3.3. Descent Property. The first variation of the augmented function F is given by

$$\delta F(x) = g^T(F, x)\Delta x \tag{21}$$

which, in the light of (18), can be written as

$$\delta F(x) = -\alpha g^T(F, x)g(F, x) \tag{22}$$

Since $g^T(F,x)g(F,x) > 0$, Eq. (22) shows that the first variation of F is negative for $\alpha > 0$. Therefore, if α is sufficiently small, the augmented function F decreases during the gradient phase.

Alternatively, the first variation of F can be written as

$$\delta F(x) = \delta f(x) + \lambda^T \delta\varphi(x) \qquad (23)$$

Because of (7), Eq. (23) reduces to the form

$$\delta F(x) = \delta f(x) \qquad (24)$$

which states that the functions F and f behave identically, to first order. Therefore, the descent property (22) also holds for the function f.

3.4. Stepsize. Since g(F,x) is known at the nominal point x, Eq. (18) shows that the correction Δx is proportional to α. This is why α is called the stepsize of the gradient method. Upon substituting (18) into (10), we see that

$$K = \alpha^2 g^T(F,x)\,g(F,x) \qquad (25)$$

Therefore, a one-to-one correspondence exists between the values of the constant K and the values of the stepsize α. This being the case, one can bypass prescribing K and reason directly on α, as in the considerations which follow.

The next step is to assign a value to the stepsize α. If Eqs. (4) and (18) are combined, the position vector at the end of the gradient phase becomes

$$\tilde{x} = x - \alpha g(F,x) \qquad (26)$$

For each point x, Eq. (26) defines a one-parameter family of points $\tilde{x}$ for which the functions f and F take the form

$$f(\tilde{x}) = f[x - \alpha g(F,x)] = f(\alpha)$$
$$F(\tilde{x}) = F[x - \alpha g(F,x)] = F(\alpha) \tag{27}$$

These equations are particular cases of a more general equation of the form

$$\Psi(\tilde{x}) = \Psi[x - \alpha g(F,x)] = \Psi(\alpha) \tag{28}$$

where $\Psi = f$ if one reasons in terms of the function to be minimized and $\Psi = F$ if one reasons in terms of the augmented function.

For the sake of discussion, we assume that the function Ψ has a relative minimum with respect to α. Under these conditions, the greatest decrease of the function Ψ occurs if the parameter α satisfies the following necessary condition:

$$\dot{\Psi}(\alpha) = 0 \tag{29}$$

where the dot denotes the derivative with respect to α. On account of (28), the following relation holds:

$$\dot{\Psi}(\alpha) = - g^T(\Psi, \tilde{x}) g(F,x) \tag{30}$$

Therefore, Eq. (29) becomes

$$g^T(\Psi, \tilde{x}) g(F,x) = 0 \tag{31}$$

and shows that, at the point where Ψ is a minimum, the gradient $g(\Psi, \tilde{x})$ is orthogonal to the gradient $g(F, x)$. In the light of (18), Eq. (31) can also be rewritten as

$$g^T(\Psi, \tilde{x})\,\Delta x = 0 \tag{32}$$

and shows that the gradient $g(\Psi, \tilde{x})$ is orthogonal to the correction Δx. This is due to the fact that Δx and $g(F, x)$ are parallel.

In conclusion, the optimum value of α must be determined by solving Eq. (29), in general, by approximate methods. In order to prevent α from becoming too large (that is, in order to limit the constraint violation), the solution of (29) is to be subordinated to either of the following inequalities:

$$\alpha \leq \epsilon_1 \qquad \text{or} \qquad P(\tilde{x}) \leq \epsilon_2 \tag{33}$$

where

$$P(x) = \varphi^T(x)\varphi(x) \tag{34}$$

denotes the <u>performance index</u> and where ϵ_1 and ϵ_2 are small quantities prescribed <u>a priori</u>. Incidentally, Ineqs. (33) are of fundamental importance in those cases where $\Psi(\alpha)$ is monotonically decreasing, that is, those cases where Eq. (29) has no real solution.

4. Search Technique

In this section, we present methods to solve the equation

$$\dot{\Psi}(\alpha) = 0 \quad , \quad \ddot{\Psi}(\alpha) > 0 \tag{35}$$

where Ψ can be either f or F. Since the methods in question involve the consideration of the first derivative $\dot{\Psi}(\alpha)$ and perhaps the second derivative $\ddot{\Psi}(\alpha)$, we summarize these derivatives below.

The derivative $\dot{\Psi}(\alpha)$ is given by Eq. (30) as

$$\dot{\Psi}(\alpha) = -g^T(\Psi,\tilde{x})g(F,x) \tag{36}$$

and the derivative $\ddot{\Psi}(\alpha)$ is given by

$$\ddot{\Psi}(\alpha) = g^T(F,x)H(\Psi,\tilde{x})g(F,x) \tag{37}$$

where $H(\Psi,x)$ denotes the n x n matrix of second derivatives

$$H(\Psi,x) = \begin{bmatrix} \partial^2\Psi/\partial x^1\partial x^1 & \partial^2\Psi/\partial x^1\partial x^2 & \cdots & \partial^2\Psi/\partial x^1\partial x^n \\ \partial^2\Psi/\partial x^2\partial x^1 & \partial^2\Psi/\partial x^2\partial x^2 & \cdots & \partial^2\Psi/\partial x^2\partial x^n \\ \cdots & \cdots & \cdots & \cdots \\ \partial^2\Psi/\partial x^n\partial x^1 & \partial^2\Psi/\partial x^n\partial x^2 & \cdots & \partial^2\Psi/\partial x^n\partial x^n \end{bmatrix} \tag{38}$$

If the second-derivative matrix is not explicitly available, one can use the difference scheme

$$\ddot{\Psi}(\alpha) = \frac{1}{2\eta}\left\{g^T[\Psi,\tilde{x} + \eta g(F,x)] - g^T[\Psi,\tilde{x} - \eta g(F,x)]\right\} g(F,x) \tag{39}$$

In practice, one may choose

$$\eta = \epsilon_3 / |g(F, x)| \tag{40}$$

where ϵ_3 is a small number.

4.1. <u>Cubic Interpolation</u>. Let the values of the function $\Psi(\alpha)$ and its derivative $\dot{\Psi}(\alpha)$ be computed for two different values of α, namely α_1 and α_2, with $\alpha_2 > \alpha_1 \geq 0$. If α_1 and α_2 are such that

$$\dot{\Psi}(\alpha_1) < 0 \ , \quad \dot{\Psi}(\alpha_2) > 0 \tag{41}$$

then the minimum of the function $\Psi(\alpha)$ occurs for some value α in the range

$$\alpha_1 < \alpha < \alpha_2 \tag{42}$$

In this range, we represent the function $\Psi(\alpha)$ with the cubic

$$\Psi(\alpha) = A + B(\alpha - \alpha_1) + C(\alpha - \alpha_1)^2 + D(\alpha - \alpha_1)^3 \tag{43}$$

whose first and second derivatives are given by

$$\dot{\Psi}(\alpha) = B + 2C(\alpha - \alpha_1) + 3D(\alpha - \alpha_1)^2 \ , \quad \ddot{\Psi}(\alpha) = 2C + 6D(\alpha - \alpha_1) \tag{44}$$

The scalar coefficients A, B, C, D are determined by requiring (43) to match the ordinate and the slope of the curve $\Psi(\alpha)$ at α_1 and α_2. Therefore, one has to solve the linear equations

$$\begin{aligned}
\Psi(\alpha_1) &= A \\
\Psi(\alpha_2) &= A + B(\alpha_2 - \alpha_1) + C(\alpha_2 - \alpha_1)^2 + D(\alpha_2 - \alpha_1)^3 \\
\dot{\Psi}(\alpha_1) &= B \\
\dot{\Psi}(\alpha_2) &= B + 2C(\alpha_2 - \alpha_1) + 3D(\alpha_2 - \alpha_1)^2
\end{aligned} \tag{45}$$

Once the coefficients of the cubic (43) are known, the optimum value of α is determined by the conditions (35). Therefore, in the light of (44), one arrives at the solution

$$\alpha = \alpha_1 + (1/3D)[-C + \sqrt{(C^2 - 3BD)}] \tag{46}$$

At this point, one recomputes the function $\Psi(\alpha)$ and the derivative $\dot{\Psi}(\alpha)$. Then, the process is iterated until Eq. (35-1) is satisfied to a desired degree of accuracy, that is, until

$$|\dot{\Psi}(\alpha)| \leq \theta \tag{47}$$

In practice, one may choose

$$\theta = \epsilon_4 |\dot{\Psi}(0)| \tag{48}$$

where ϵ_4 is a small number.

4.2. Quasilinearization. An alternate technique for computing the optimum stepsize, that of quasilinearization with built-in safeguards to ensure that the function decreases at every step of the iterative search, is now presented. Let

$$\delta\alpha = \alpha - \alpha_o \tag{49}$$

denote the correction to α starting from an arbitrary nominal value α_o. If quasilinearization is applied to Eq. (35-1), one obtains the linear algebraic equation

$$\ddot{\Psi}(\alpha_o)\delta\alpha + \dot{\Psi}(\alpha_o) = 0 \tag{50}$$

Next, we imbed Eq. (50) in the more general equation

$$\ddot{\Psi}(\alpha_o)\delta\alpha + \mu\dot{\Psi}(\alpha_o) = 0 \tag{51}$$

where μ denotes a scaling factor and ρ a direction factor such that

$$0 \leq \mu \leq 1 \quad , \quad \rho = \pm 1 \tag{52}$$

Equation (51) admits the solution

$$\delta\alpha = -\mu\rho\dot{\Psi}(\alpha_o)/\ddot{\Psi}(\alpha_o) \tag{53}$$

The direction factor ρ is determined in such a way that the first variation

$$\delta\Psi(\alpha_o) = \dot{\Psi}(\alpha_o)\delta\alpha \tag{54}$$

is negative. From (53)-(54), we obtain

$$\delta\Psi(\alpha_o) = -\mu\rho\dot{\Psi}^2(\alpha_o)/\ddot{\Psi}(\alpha_o) \tag{55}$$

Therefore, $\delta\Psi(\alpha_o)$ is negative if the direction factor ρ is chosen as follows:

$$\rho = \text{sign}\,\ddot{\Psi}(\alpha_o) \tag{56}$$

Because of this choice, the correction (53) becomes

$$\delta\alpha = -\mu\dot{\Psi}(\alpha_o)/|\ddot{\Psi}(\alpha_o)| \tag{57}$$

To perform the search, a nominal value must be given to α_o. Then, one sets $\mu = 1$, computes $\delta\alpha$ from Eq. (57) and α from Eq. (49). If $\Psi(\alpha) < \Psi(\alpha_o)$, the scaling factor $\mu = 1$ is acceptable. If $\Psi(\alpha) > \Psi(\alpha_o)$, the previous value of μ must be replaced by some smaller value in the range $0 \leq \mu \leq 1$ until the condition $\Psi(\alpha) < \Psi(\alpha_o)$ is met; this can be obtained through bisection, that is, by successively dividing the value of μ by 2. At this point, the search step is completed. The value obtained for α becomes the nominal value α_o for the next search step, and the procedure is repeated until a desired degree of accuracy is obtained, that is, until Ineq. (47) is satisfied. In the absence of better information, the first step in the search procedure can be made with $\alpha_o = 0$.

5. Restoration Phase

At the end of the gradient phase, the point $\tilde{x}$ is known. If the constraint is linear, the relation $\varphi(\tilde{x}) = 0$ holds. On the other hand, if the constraint is nonlinear, $\varphi(\tilde{x}) \neq 0$, which means that some degree of dissatisfaction exists. Therefore, a restoration phase is needed prior to starting the next gradient phase. Specifically, one has to apply a small variation δx to $\tilde{x}$ to generate a new position vector

$$\tilde{\tilde{x}} = \tilde{x} + \delta x \tag{58}$$

such that $\varphi(\tilde{\tilde{x}}) = 0$. While there are infinite ways to perform the restoration, the simplest is that developed in Refs. 2-3: the constraint is restored to a preselected degree of accuracy subject to the least-square change of the position vector.

If quasilinearization is employed, Eq. (2) is approximated by

$$\varphi(\tilde{x}) + G^T(\varphi, \tilde{x})\delta x = 0 \tag{59}$$

In order to prevent the variations δx from becoming too large, we imbed Eq. (59) into the one-parameter family

$$k\varphi(\tilde{x}) + G^T(\varphi, \tilde{x})\delta x = 0 \tag{60}$$

where

$$0 \leq k \leq 1 \tag{61}$$

denotes a scaling factor.

In the light of the previous discussion, we seek the minimum of the function

$$J = \frac{1}{2}\delta x^T \delta x \tag{62}$$

subject to the linearized constraint (60). Standard methods of the theory of maxima and minima show that the fundamental function of this problem is given by

$$\omega = \frac{1}{2}\delta x^T \delta x + \sigma^T [k\varphi(\tilde{x}) + G^T(\varphi, \tilde{x})\delta x] \tag{63}$$

where σ, a p-vector, denotes the undetermined, constant Lagrange multiplier

$$\sigma = \begin{bmatrix} \sigma^1 \\ \sigma^2 \\ \vdots \\ \sigma^p \end{bmatrix} \tag{64}$$

The optimum change δx satisfies the relation

$$g(\omega, \delta x) = 0 \tag{65}$$

where $g(\omega, \delta x)$ denotes the gradient of the scalar function ω with respect to the vector δx. The explicit form of (65) is the following:

$$\delta x = -G(\varphi, \tilde{x})\sigma \tag{66}$$

The Lagrange multiplier is obtained by combining (60) and (66) to eliminate δx. This yields the relation

$$k\varphi(\tilde{x}) - [G^T(\varphi, \tilde{x})G(\varphi, \tilde{x})]\sigma = 0 \tag{67}$$

For any given k in the range (61), Eq. (67) supplies the Lagrange multiplier vector σ. Once σ is known, the correction δx is given by Eq. (66), and the corrected position vector $\tilde{\tilde{x}}$ is given by Eq. (58). Of course, the restoration phase must be performed iteratively until a desired degree of accuracy is obtained, that is, until the inequality

$$P(\tilde{\tilde{x}}) \leq \epsilon_5 \tag{68}$$

is satisfied, where ϵ_5 is a small number.

5.1. Descent Property. The first variation of the performance index is given by

$$\delta P(\tilde{x}) = 2\varphi^T(\tilde{x})G^T(\varphi, \tilde{x})\,\delta x \tag{69}$$

and, because of Eq. (60), reduces to

$$\delta P(\tilde{x}) = -2k\varphi^T(\tilde{x})\varphi(\tilde{x}) \tag{70}$$

which, in the light of (34), becomes

$$\delta P(\tilde{x}) = -2kP(\tilde{x}) \tag{71}$$

Since $P(\tilde{x}) > 0$, Eq. (71) shows that the first variation of the performance index is negative for $k > 0$. Therefore, if k is sufficiently small, the decrease of the performance index is guaranteed.

6. Sequential Gradient-Restoration Algorithm

The algorithm presented in this report consists of the alternate succession of gradient phases and restoration phases. A summary of the algorithm and its properties is given below.

6.1. Gradient Phase: (a) Select a nominal point x such that $\varphi(x) = 0$; (b) at this nominal point, compute the gradient $g(f,x)$ with Eq. (6) and the matrix $G(\varphi,x)$ with Eq. (9); (c) solve Eq. (20) in order to obtain the multiplier λ; (d) compute the gradient $g(F,x)$ with Eq. (14); (e) the displacement vector Δx is supplied by Eq. (18) and is known except for the stepsize α, to be determined by the search technique.

6.2. Search Technique: (a) Establish a search criterion $\Psi = f$ or $\Psi = F$; (b) consider the one-parameter family of points $\tilde{x}$ defined by (26); (c) for these points, consider the function $\Psi(\alpha)$ and search for the minimum of this function using, for instance, cubic interpolation or quasilinearization; (d) if the minimum exists, stop the search when Ineq. (47) is satisfied; the resulting value of α is subordinated to the further satisfaction of either of Ineqs. (33); (e) if the minimum does not exist, use the value of α for which either of (33) becomes an equality; (f) once α is known, the gradient correction Δx can be computed with Eq. (18); (g) the position vector $\tilde{x}$ at the end of the gradient phase is determined with Eq. (4).

6.3. Restoration Phase: (a) At the point $\tilde{x}$, compute the vector $\varphi(\tilde{x})$ with Eq. (3-2) and the matrix $G(\varphi,\tilde{x})$ with Eq. (9); (b) assuming $k = 1$, determine the multiplier σ with Eq. (67); (c) once σ is known, the correction δx is supplied by Eq. (66); (d) the new position vector $\tilde{\tilde{x}}$ is given by Eq. (58); (e) if $P(\tilde{\tilde{x}}) < P(\tilde{x})$, the scaling factor $k = 1$ is acceptable; if $P(\tilde{\tilde{x}}) > P(\tilde{x})$, the previous value of k must be replaced by some smaller value in the range

(61), until the condition $P(\tilde{\tilde{x}}) < P(\tilde{x})$ is met; this can be achieved through successive bisections of k; (f) return to step (a) and repeat the restoration algorithm using $\tilde{\tilde{x}}$ as the starting point $\tilde{x}$ for the subsequent iteration; (g) terminate the restoration algorithm when the stopping condition (68) is satisfied; (h) once the restoration algorithm is completed, verify the inequality

$$f(\tilde{\tilde{x}}) < f(x) \tag{72}$$

If Ineq. (72) is satisfied, start the next gradient phase. If Ineq. (72) is violated, return to the previous gradient phase and reduce the stepsize α until, after restoration, Ineq. (72) is satisfied.

6.4. Stopping Conditions. The combined gradient-restoration algorithm is terminated when

$$Q(\tilde{\tilde{x}}) \le \epsilon_6 \tag{73}$$

where

$$Q(x) = g^T(F,x)g(F,x) + \varphi^T(x)\varphi(x) \tag{74}$$

and where ϵ_6 denotes a small number.

6.5. Order of Magnitude Analysis. The position vector $\tilde{\tilde{x}}$ at the end of the restoration phase and the position vector x at the beginning of the gradient phase are related by

$$\tilde{\tilde{x}} = x + \Delta x + \delta x \tag{75}$$

where Δx is the gradient correction and δx is the restoration correction. From Eq. (18), we see that the gradient correction has the order

$$\Delta x = O(\alpha) \tag{76}$$

Next, we observe that

$$\varphi(\tilde{x}) = \varphi(x) + G^T(\varphi, x)\Delta x + h(\varphi, x, \Delta x) \tag{77}$$

where $h(\varphi, x, \Delta x)$ denotes the p-vector

$$h(\varphi, x, \Delta x) = \begin{bmatrix} \Delta x^T H(\varphi^1, x)\Delta x \\ \Delta x^T H(\varphi^2, x)\Delta x \\ \vdots \\ \Delta x^T H(\varphi^p, x)\Delta x \end{bmatrix} \tag{78}$$

Since the nominal point x satisfies (2) exactly and the variation Δx satisfies (2) to first order, the first two terms on the right-hand side of (77) vanish. Therefore, in the light of (76) and (78), we conclude that

$$\varphi(\tilde{x}) = O(\alpha^2) \tag{79}$$

Next, we turn our attention to Eq. (67). If

$$k = O(1) \tag{80}$$

then, a Taylor expansion of $G(\varphi, \tilde{x})$ shows that σ has the same order of magnitude as $\varphi(\tilde{x})$, that is,

$$\sigma = O(\alpha^2) \tag{81}$$

Therefore, from (66) and the Taylor expansion of $G(\varphi, \tilde{x})$, we see that δx has the same order of magnitude as σ, that is,

$$\delta x = O(\alpha^2) \tag{82}$$

In conclusion, if the gradient correction is of the order α, the restoration correction is of the order α^2. This guarantees that, for sufficiently small α,

$$|\delta x| \ll |\Delta x| \tag{83}$$

6.6. Descent Property. Finally, we consider the points x and $\tilde{\tilde{x}}$, both satisfying the constraint (2). To first order, the difference of the values of the function f at these points is given by

$$f(\tilde{\tilde{x}}) - f(x) = g^T(f, x)[\Delta x + \delta x] \tag{84}$$

For α sufficiently small, Ineq. (83) applies and Eq. (84) can be approximated by

$$f(\tilde{\tilde{x}}) - f(x) \cong g^T(f, x)\Delta x = -\alpha g^T(F, x) g(F, x) \tag{85}$$

Therefore, for α sufficiently small, the restoration algorithm preserves the descent property of the gradient algorithm: the function f decreases between any two successive restoration phases.

7. Examples

In order to illustrate the theory, two numerical examples are now supplied. For simplicity, all the symbols employed in this section denote scalar quantities. We refer to the problem of minimizing the function

$$f=f(x,y,z) \tag{86}$$

subject to the scalar constraint

$$\varphi(x,y,z) = 0 \tag{87}$$

The gradient algorithm is represented by

$$\tilde{x} = x + \Delta x \quad , \quad \tilde{y} = y + \Delta y \quad , \quad \tilde{z} = z + \Delta z \tag{88}$$

where

$$\begin{aligned} \Delta x &= -\alpha(f_x + \lambda\varphi_x) \\ \Delta y &= -\alpha(f_y + \lambda\varphi_y) \\ \Delta z &= -\alpha(f_z + \lambda\varphi_z) \end{aligned} \tag{89}$$

and where the multiplier λ is defined as

$$\lambda = -(f_x\varphi_x + f_y\varphi_y + f_z\varphi_z)/(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) \tag{90}$$

The stepsize α is determined in such a way that either the function f or the function F is minimized along the line represented by Eqs. (88)-(89). This is done by employing quasi-linearization, as explained in Section 4.2. The search is terminated when

$$|\dot{f}(\alpha)| \leq 10^{-3} |\dot{f}(0)| \tag{91}$$

or when

$$|\dot{F}(\alpha)| \leq 10^{-3} |\dot{F}(0)| \tag{92}$$

with the understanding that

$$\alpha \leq 1 \quad \text{or} \quad P(\tilde{x}, \tilde{y}, \tilde{z}) \leq 1 \tag{93}$$

The restoration algorithm is represented by

$$\tilde{\tilde{x}} = \tilde{x} + \delta x \quad , \quad \tilde{\tilde{y}} = \tilde{y} + \delta y \quad , \quad \tilde{\tilde{z}} = \tilde{z} + \delta z \tag{94}$$

where

$$\delta x = - \sigma\varphi_x \quad , \quad \delta y = - \sigma\varphi_y \quad , \quad \delta z = - \sigma\varphi_z \tag{95}$$

and where the multiplier σ is given by

$$\sigma = k\varphi/(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) \tag{96}$$

The restoration algorithm is terminated when

$$P(\tilde{\tilde{x}}, \tilde{\tilde{y}}, \tilde{\tilde{z}}) \leq 10^{-12} \tag{97}$$

It should be understood that, for every combined gradient-restoration phase,

$$f(\tilde{\tilde{x}}, \tilde{\tilde{y}}, \tilde{\tilde{z}}) < f(x, y, z) \tag{98}$$

If Ineq. (98) is satisfied, the next gradient phase is started; if Ineq. (98) is violated, the stepsize α is bisected successively until (98) is met.

Example 7.1. We consider the problem of minimizing the function

$$f = x^2 + y^2 + z^2 \tag{99}$$

subject to the constraint

$$x + y^2 - 1 = 0 \tag{100}$$

This function admits the relative minimum

$$f = 3/4 \tag{101}$$

at the points defined by

$$x = 1/2 \ , \quad y = \pm 1/\sqrt{2} \ , \quad z = 0 \tag{102}$$

The nominal point chosen for the descent process is the point of coordinates

$$x = -3 \ , \quad y = 2 \ , \quad z = 1 \tag{103}$$

consistent with (100). Since the function is quadratic and the constraint is quadratic, the search performed by quasilinearization yields the optimum value of α in one step. Since the minimum (101) is known a priori, the stopping condition (73) is replaced by

$$|f - 3/4| \leq 10^{-6} \tag{104}$$

Computations were performed with a Burroughs B-5500 computer in double-precision arithmetic. The results are summarized in Tables 1 and 2, where N denotes the number of iterations (each iteration includes a gradient phase and a restoration phase) and N_R the number of restoration cycles per iteration. As can be seen, the number of restoration cycles decreases as the algorithm progresses toward termination.

Example 7.2. We consider the problem of minimizing the function

$$f = (x - y)^2 + (y - z)^4 \tag{105}$$

subject to the constraint

$$x(1 + y^2) + z^4 - 3 = 0 \tag{106}$$

This function admits the relative minimum

$$f = 0 \tag{107}$$

at the point defined by

$$x = y = z = 1 \tag{108}$$

The nominal point chosen for the descent process is the point of coordinates

$$x = -13/5 , \quad y = 2 , \quad z = 2 \tag{109}$$

consistent with (106). Since the minimum (107) is known *a priori*, the stopping condition (73) is replaced by

$$f \leq 10^{-6} \tag{110}$$

Computations were performed with a Burroughs B-5500 computer in double-precision arithmetic. The results are summarized in Tables 3 and 4, where N denotes the number of iterations (each iteration includes a gradient phase and a restoration phase) and N_R the number of restoration cycles per iteration. As can be seen, the number of restoration cycles decreases as the algorithm progresses toward termination.

Table 1 ($\Psi = f$)

N	N_R	x	y	z	f
0		-3.0000	2.0000	1.0000	14.0000000
1	3	0.1769	0.9072	0.0000	0.8543922
2	2	0.4191	0.7621	0.0000	0.7565319
3	1	0.4752	0.7244	0.0000	0.7506136
4	1	0.4919	0.7127	0.0000	0.7500646
5	1	0.4973	0.7089	0.0000	0.7500070
6	1	0.4991	0.7077	0.0000	0.7500008

Table 2 ($\Psi = F$)

N	N_R	x	y	z	f
0		-3.0000	2.0000	1.0000	14.0000000
1	3	0.2701	0.8543	-0.0328	0.8039208
2	2	0.4840	0.7182	0.0092	0.7503381
3	1	0.4977	0.7087	-0.0025	0.7500116
4	1	0.4994	0.7075	0.0003	0.7500004

Table 3 ($\Psi = f$)

N	N_R	x	y	z	f
0		-2.6000	2.0000	2.0000	0.21×10^{2}
1	5	-0.3517	0.0226	1.3530	0.32×10^{1}
3	2	0.4890	0.5104	1.2425	0.28×10^{0}
4	2	0.5317	0.7694	1.2113	0.94×10^{-1}
7	1	0.8222	0.8203	1.1289	0.90×10^{-2}
13	1	0.9060	0.9053	1.0781	0.89×10^{-3}
27	1	0.9474	0.9472	1.0471	0.99×10^{-4}
68	1	0.9708	0.9713	1.0271	0.99×10^{-5}
194	0	0.9838	0.9839	1.0154	0.99×10^{-6}

Table 4 (Ψ = F)

N	N_R	x	y	z	f
0		-2.6000	2.0000	2.0000	0.21×10^{2}
1	5	-0.3368	0.0130	1.3515	0.33×10^{1}
2	2	-0.2234	0.6155	1.3486	0.99×10^{0}
4	2	0.6173	0.7954	1.1880	0.55×10^{-1}
7	1	0.8334	0.8352	1.1220	0.67×10^{-2}
12	1	0.8995	0.9148	1.0774	0.93×10^{-3}
25	0	0.9483	0.9484	1.0463	0.91×10^{-4}
59	0	0.9712	0.9712	1.0270	0.97×10^{-5}
161	0	0.9839	0.9839	1.0155	0.99×10^{-6}

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