THE APPLICATION OF SIGNAL DETECTION THEORY TO OPTICS

QUARTERLY PROGRESS REPORT

NASA GRANT 05 - 009 - 079

June 15, 1969

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SUMMARY

Lower bounds have been set on the mean-square errors in unbiased estimates of such parameters of an incoherently radiating object as its radiance, its position, and the frequency of its light. The bounds take into account the quantum nature of the light from both object and background.

A telescope has three primary functions: detection, resolution, and parameter estimation. Detection involves deciding whether objects of a specified class are present in the field of view. Resolution requires a decision whether two or more close objects are present or whether only one is present. Parameter estimation occurs when certain characteristics of the object, such as its position, its absolute radiance, or its size, are measured. Our work during the past quarter has concerned this third function of an optical system.

Estimates of object parameters are inevitably in error because of noise in detectors, graininess of film, background radiation, and the stochastic nature of the light from the object itself. Various means can be conceived of mitigating or avoiding some of these sources of error. Quieter detectors and finer emulsions might be sought. There remains, however, an irreducible minimum error that is due to the quantum nature of the electromagnetic field.

The radiant power P_0 of a star could be estimated, for instance, by somehow counting the total number n of photons that arrive from it during the observation interval. In the absence of background radiation, as when the star is isolated in a cold, black sky, the number n has a Poisson distribution with mean value N_s. Since N_s is proportional to the radiant power, it is an estimate of N_s that is needed, and the best estimate is simply the number n itself. This number n, though an unbiased estimate of N_s, will seldom equal N_s exactly. The mean-square error $\langle (n - N_s)^2 \rangle$, which is the variance of the Poisson distribution, is equal to N_s. The relative mean-square error in the estimate $\hat{P}_0$ of P_0 is therefore

$$\langle \hat{P}_0 - P_0 \rangle^2 / P_0^2 = N_s / N_s^2 = 1/N_s \ .$$

It can be shown that no scheme for obtaining an unbiased estimate of the radiant power P_0 can yield a smaller relative mean-square error than N_s^{-1} .

It is with problems of this nature that we have been dealing. We have treated estimates of the radiance of an object; of the frequency of its light, taken to be quasimonochromatic; and of its position. Lower bounds have been worked out for the mean-square errors in unbiased estimates of such parameters, as functions of the bandwidth of the object light, the duration of the observation interval, the coherence of the object light at the aperture of the instrument, and the mean numbers of photons received from object and background.

When the background vanishes, the errors in estimates are due only to the quantum fluctuations of the light from the object itself. The relative mean-square error in an estimate of radiance is then bounded below by N_s^{-1}, as just described, N_s being the average total number of photons received from the object. The mean-square error in an estimate of frequency is at least as great as $2W^2/N_s$, where W is the bandwidth of the object light. Both these bounds are independent of the degree of first-order coherence of the light.

The mean-square error of an estimate of the angular position of a point source is, under quantum-limited conditions, at least equal to $N_s^{-1}(ka)^{-2}$, where $k = 2\pi/\lambda$ is the propagation constant and a is the diameter of the aperture of the optical system. This bound is weakly dependent on the degree of coherence of the object light at the aperture when the object is larger than a point source.

When background light is also present, the minimum mean-square errors are greater. When the effective temperature of the background is very high, the formulas we have obtained become equivalent to those derived on the basis of classical statistics.

Details are given in the attached preprint.

Personnel note: The graduate assistant employed under this grant beginning with the Fall quarter, 1968, left school in mid-April because of the threat imposed by the draft.

Estimation of Object Parameters by a Quantum-Limited Optical System

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Abstract

By means of the quantum-mechanical form of the Cramér-Rao inequality, a lower bound is set to the mean-square error in an unbiased estimate of a parameter of an incoherently radiating object observed in the presence of thermal background light by an optical system admitting light through a finite aperture. Estimates of absolute radiance, frequency, and position of the object are specifically analyzed. The bounds reduce in the classical limit to those previously obtained, but are valid in the quantum limit as well. When the background vanishes, the bounds depend only weakly on the effective number of independent spatial and temporal degrees of freedom of the object light at the aperture.

An optical instrument such as a camera or telescope not only detects objects in its field of view, but also facilitates measuring certain of their parameters. Typical parameters are the absolute radiance of an object, its diameter, the coordinates of its center, and--if its light is quasimonochromatic--its frequency or wavelength. Estimates of parameters are always subject to random and systematic errors. If measurements are made on images recorded on photographic film, the granularity of the film and inaccuracies in specifying its H-D curves introduce error. When photosensitive surfaces are used, as in image intensifiers, emission fluctuations and dark currents create random noise. Many of these causes of error might in principle be eliminated, but one will always remain, the stochastic nature of the incident light itself.

The insurmountable error due to the stochastic properties of light can be assessed by regarding an optical instrument as processing the electromagnetic field at its aperture. We envision the class of all possible instruments that might analyze that field and produce an estimate of the parameter in question. The methods of statistics permit us to calculate a lower bound to the mean-square error in an unbiased estimate of a parameter in terms of the probability distributions of the incident light field. This has already been done for incoherent objects whose light is received in the presence of strong enough background light so that the net field can be treated by the methods of classical electromagnetism.[1,2] The bound was based on the Cramér-Rao inequality of conventional statistics.

Here we shall derive corresponding lower bounds that apply when the light is so weak that the quantum-mechanical properties of the field must be taken into account. The accuracy of parameter estimates is now limited by the quantum fluctuations arising from the photonic nature of the light from the object and the background.

The same conditions will be postulated as in our analysis of the detectability of incoherent objects by a quantum-limited optical system.[3] The light field at the aperture of the optical instrument is observed for a time T much longer than the reciprocal W^{-1} of the bandwidth of the object light. The diameter of the aperture is much greater than both a wavelength of the object light and the correlation length of the thermal background light. The bandwidth of the background light is much greater than that of the light from the object.

Section I will review the specification of the fields due to object and background. The notation is the same as in III.[3] In Section II the quantum-mechanical form of the Cramér-Rao inequality is used to derive a general formula for the lower bound on the mean-square error in an unbiased estimate of a parameter.[4,5] Section III treats parameter estimation when the object light is spatially incoherent at the aperture; Section IV takes the object light as having complete first-order coherence. Estimates of the absolute radiance of an object, its frequency, and its position are analyzed. Section V deals specifically with estimates of radiance and frequency of a uniform circular object whose light is received at a circular aperture.

I. The Aperture Field

The electromagnetic field, taken for simplicity as a scalar, is represented by the function $\psi(\underset{\sim}{r}', t)$ of coordinates $\underset{\sim}{r}' = (\underset{\sim}{r}, z)$ and time t. Here $\underset{\sim}{r}$ is a 2-vector of coordinates in a plane parallel to the aperture and normal to the z-axis, which points toward the object. The field is decomposed into its positive- and negative-frequency parts,

$$\psi(\underset{\sim}{r}', t) = \psi_+(\underset{\sim}{r}', t) + \psi_-(\underset{\sim}{r}', t) , \qquad (1.1)$$

which are hermitian-conjugate quantum-mechanical operators. The mutual coherence function of the field is defined by

$$\varphi(\underset{\sim}{r}_1', t_1 ; \underset{\sim}{r}_2', t_2 ; \theta) = \mathrm{Tr}\, \rho \psi_-(\underset{\sim}{r}_2', t_2)\, \psi_+(\underset{\sim}{r}_1', t_1) , \qquad (1.2)$$

where $\rho = \rho(\theta)$ is its quantum-mechanical density operator and "Tr" stands for the trace. The density operator and the mutual coherence function depend on the parameter θ to be estimated.

Onto the aperture falls light both from the background and from an incoherent object whose parameter θ is the estimandum. As the object and background radiate independently, the mutual coherence function in Eq. (1.2) is the sum of two corresponding terms,

$$\varphi(\underset{\sim}{r}_1', t_1 ; \underset{\sim}{r}_2', t_2 ; \theta) = \varphi_0(\underset{\sim}{r}_1', t_1 ; \underset{\sim}{r}_2', t_2) + \varphi_s(\underset{\sim}{r}_1', t_1 ; \underset{\sim}{r}_2', t_2 ; \theta) . \qquad (1.3)$$

(Henceforth the subscript 0 refers to the background, s to the object light, or "signal".) The object is assumed to lie in the plane $z = R$, far away in the z-direction; as a result, the mutual coherence function of its light has, at the aperture, the form

$$\varphi_s\,(\underset{\sim}{r}_1',\ t_1\ ;\ \underset{\sim}{r}_2',\ t_2\ ;\ \theta) =$$

$$\varphi_s'\,(\underset{\sim}{r}_1,\ t_1\ ;\ \underset{\sim}{r}_2,\ t_2\ ;\ \theta)\quad \exp\,[-i\Omega\,(t_1-t_2) - i\Omega\,(z_1-z_2)\ /\ c], \tag{1.4}$$

where Ω is the central angular frequency of its spectrum and c is the velocity of light.

Because the light from the object is a stationary stochastic process, φ_s' is a function of t_1 and t_2 only through $t_1 - t_2$, and it can be expressed in terms of its Fourier transform as

$$\varphi_s'\,(\underset{\sim}{r}_1,\ t_1\ ;\ \underset{\sim}{r}_2,\ t_2\ ;\ \theta) =$$

$$\int_{-\infty}^{\infty} \Phi_s(\underset{\sim}{r}_1,\ \underset{\sim}{r}_2\ ;\ \omega\ ;\ \theta)\quad \exp\,[-i\omega\,(t_1-t_2)]\ d\omega\ /\ 2\pi\ . \tag{1.5}$$

Under the assumption that the object light is cross-spectrally pure, the transform Φ_s can be decomposed into a spatial and a temporal part,[6]

$$\Phi_s\,(\underset{\sim}{r}_1,\ \underset{\sim}{r}_2\ ;\ \omega\ ;\ \theta) = \varphi_s\,(\underset{\sim}{r}_1,\ \underset{\sim}{r}_2\ ;\ \theta)\quad X\,(\omega\ ;\ \theta), \tag{1.6}$$

where the angular frequency ω is measured from Ω . The temporal spectral density $X\,(\omega\ ;\ \theta)$ is so normalized that

$$\int_{-\infty}^{\infty} X\,(\omega\ ;\ \theta)\quad d\omega\ /\ 2\pi = 1\ . \tag{1.7}$$

The bandwidth W of the object spectrum is defined by

$$W = \{\ \int_{-\infty}^{\infty} [\ X\,(\omega\ ;\ \theta)\ ]^2\quad d\omega\ /\ 2\pi\ \}^{-1} \tag{1.8}$$

--cf. I, Eq. (1.5). The total energy received from the object during the observation interval is

$$E_s = 2\Omega^2\ cT \int_A \varphi_s\,(\underset{\sim}{r},\ \underset{\sim}{r}\ ;\ \theta)\quad d^2\underset{\sim}{r}\quad , \tag{1.9}$$

where A stands for the aperture of the optical instrument.

The background light, on the other hand, is taken to have the properties of thermal light of absolute temperature $\mathcal{T}$. The average number $\mathcal{N}$ of photons per mode of the field is given by the Planck formula,

$$\mathcal{N} = [\exp (\hbar \Omega / K\mathcal{T}) - 1]^{-1} \quad , \tag{1.10}$$

where Ω is the frequency of the mode, K is Boltzmann's constant, and $\hbar$ is Planck's constant $h / 2\pi$.

II. The Quantum-Mechanical Cramér-Rao Inequality

In order to set a lower bound to the accuracy with which the parameter θ can be estimated, we suppose the optical instrument replaced by an ideal receiver, which consists of a large lossless cavity having the same aperture.[7] The cavity is initially closed and empty. During the observation interval $(0, T)$, the aperture is opened and the incident light allowed to enter the cavity. At time $t = T$ the aperture is closed, and at some time t thereafter, the best possible measurements are made of the field within the receiver. The field at that time is described by a density operator $\rho(\theta)$ depending on the estimandum θ.

Once again we specify the field in the receiver in terms of its normal modes $u_{\underset{\sim}{m}}(\underset{\sim}{r}) \exp(-i\omega_{\underset{\sim}{m}} t)$; the functions $u_{\underset{\sim}{m}}(\underset{\sim}{r})$ are solutions of the Helmholtz equation corresponding to angular frequency $\omega_{\underset{\sim}{m}}$ --see III, Eq. (3.11). The amplitudes of the modes are, as before, the operators $a_{\underset{\sim}{m}}$, so normalized that their commutation relations are[8]

$$[a_{\underset{\sim}{k}}, a_{\underset{\sim}{m}}^{+}] = a_{\underset{\sim}{k}} a_{\underset{\sim}{m}}^{+} - a_{\underset{\sim}{m}}^{+} a_{\underset{\sim}{k}} = \delta_{\underset{\sim}{k}\underset{\sim}{m}} ,$$
$$[a_{\underset{\sim}{k}}, a_{\underset{\sim}{m}}] = [a_{\underset{\sim}{k}}^{+}, a_{\underset{\sim}{m}}^{+}] = 0 . \qquad (2.1)$$

These mode amplitudes are linearly related to the field at the aperture during the interval $(0, T)$,

$$a_{\underset{\sim}{m}} = O_{\underset{\sim}{m}}(1)\, \psi_{+}(\underset{\sim}{r}_1', t_1) ,$$
$$a_{\underset{\sim}{m}}^{+} = O_{m}^{*}(1)\, \psi_{-}(\underset{\sim}{r}_1', t_1) , \qquad (2.2)$$

where $O_{\underset{\sim}{m}}(1)$ is the integro-differential operator defined in III, Eq. (3.21).

Both the background and the object consist of a myriad of independently and randomly radiating ions and electrons. The density operator of the field they generate, therefore, has in the P-representation a gaussian form specified by the correlation matrix[9]

$$\underset{\sim}{\varphi}(\theta) = \underset{\sim}{\varphi}_0 + \underset{\sim}{\varphi}_s(\theta)$$

of the amplitudes of the normal modes. Its elements are

$$\varphi_{\underset{\sim}{k}\underset{\sim}{m}}(\theta) = \mathrm{Tr}[\rho(\theta)\ a_{\underset{\sim}{m}}^{+}\ a_{\underset{\sim}{k}}] \quad , \tag{2.3}$$

which are bilinearly related, through the operators $O_{\underset{\sim}{k}}(1)$, $O_{\underset{\sim}{m}}{}^{*}(2)$, to the mutual coherence function $\varphi(\underset{\sim}{r}_1' , t_1 ; \underset{\sim}{r}_2' , t_2 ; \theta)$ of the field at the aperture during the interval $(0 , T)$.

A lower bound to the mean-square error in an unbiased estimate $\hat{\theta}$ of the parameter θ is set by the quantum-mechanical counterpart of the Cramér-Rao inequality,[4]

$$\underset{\sim}{E}(\hat{\theta} - \theta)^2 \geq [\mathrm{Tr}\ \rho L^2]^{-1} = [\mathrm{Tr}(L\ \partial\rho\ /\ \partial\theta)]^{-1}, \tag{2.4}$$

where $\underset{\sim}{E}$ stands for expectation and L is the symmetrized logarithmic derivative of $\rho(\theta)$ with respect to θ , defined by

$$2\ \partial\rho\ /\ \partial\theta = \rho L + L\rho \quad . \tag{2.5}$$

The derivatives are evaluated at the true value of the estimandum.

When the gaussian form of the density operator $\rho(\theta)$ is used, the inequality becomes[5]

$$\underset{\sim}{E}(\hat{\theta} - \theta)^2 \geq A_\theta{}^{-1} \quad , \tag{2.6}$$

$$A_\theta = \mathrm{Tr}[\underset{\sim}{\Lambda}_\theta(\partial\underset{\sim}{\varphi}_s\ /\ \partial\theta)] \tag{2.7}$$

where $\underset{\sim}{\Lambda}_\theta$ is the solution of the matrix equation

$$2\partial\underset{\sim}{\varphi}_s\ /\ \partial\theta = \underset{\sim}{\varphi}\underset{\sim}{\Lambda}_\theta(\underset{\sim}{I} + \underset{\sim}{\varphi}) + (\underset{\sim}{I} + \underset{\sim}{\varphi})\underset{\sim}{\Lambda}_\theta\ \underset{\sim}{\varphi} \quad , \tag{2.8}$$

$\underset{\sim}{I}$ being the identity matrix. It is convenient to put

$$A_\theta = \partial^2 H(\theta_1 , \theta_2) / \partial\theta_1 \partial\theta_2 |_{\theta_1 = \theta_2 = \theta} , \tag{2.9}$$

where

$$H(\theta_1 , \theta_2) = \mathrm{Tr}\, \underset{\sim}{L}(\theta_1)\, \underset{\sim}{\varphi}(\theta_2) \tag{2.10}$$

corresponds to the ambiguity function of signal estimation theory.[10] Here $\underset{\sim}{L}(\theta_1)$ is the solution of the matrix equation

$$2\, \underset{\sim}{\varphi}_s(\theta_1) = \underset{\sim}{\varphi}(\theta)\, \underset{\sim}{L}(\theta_1)\, [\underset{\sim}{I} + \underset{\sim}{\varphi}(\theta)] + [\underset{\sim}{I} + \underset{\sim}{\varphi}(\theta)]\, \underset{\sim}{L}(\theta_1)\, \underset{\sim}{\varphi}(\theta) , \tag{2.11}$$

and $\underset{\sim}{\Lambda}_\theta$ is $\partial \underset{\sim}{L}(\theta_1) / \partial\theta_1$ evaluated at $\theta_1 = \theta$.

It is now necessary to translate this prescription into a form involving the mutual coherence functions of the field at the aperture of the observing instrument. The analysis is much like that in III, Sections III and IV, and it is given in Appendix A. As in III, it is assumed that the bandwidth W of the object light is much smaller than that of the background light, $K\mathcal{T} / \hbar$. In addition, the diameter a of the aperture is much greater than both a wavelength $2\pi c / \Omega = 2\pi / k$ of the object light and the correlation length $\hbar c / K\mathcal{T}$ of the thermal field.[11] The condition $a \gg \hbar c / K\mathcal{T}$ is equivalent to $ka \gg \hbar\Omega / K\mathcal{T}$, and $ka \gg 1$. It is only at very low temperatures that $\hbar\Omega / K\mathcal{T}$ becomes comparable with ka , and Eq. (1.10) shows that the average number $\mathcal{N}(\Omega)$ of thermal photons per mode is thereupon exponentially small, so that whatever correlation the thermal background field may have will be inconsequential.

The result of our translation is most conveniently expressed in terms of the orthonormal set of eigenfunctions $\eta_k(\underset{\sim}{r})$ of the integral equation

$$\nu_k\, \eta_k(\underset{\sim}{r}_1) = (2\Omega c T / \hbar) \int_A \varphi_s(\underset{\sim}{r}_1 , \underset{\sim}{r}_2 ; \theta) \eta_k(\underset{\sim}{r}_2)\, d^2\underset{\sim}{r}_2 , \tag{2.12}$$

whose eigenvalues ν_k are pure numbers. The functions $\varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2 ; \theta_i)$, $i = 1,2$ are expanded in a double series of these eigenfunctions,

$$\varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2 ; \theta_i) = (\hbar / 2\Omega c T) \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \mu_{km}(\theta_i) \times \eta_k(\underset{\sim}{r}_1) \, \eta_m^*(\underset{\sim}{r}_2) , \tag{2.13}$$

and when $\theta_i = \theta$, as in III, Eq. (A25),

$$\mu_{km}(\theta) = \nu_k \, \delta_{km} . \tag{2.14}$$

The ambiguity function $H(\theta_1 , \theta_2)$ then becomes

$$H(\theta_1 , \theta_2) = T^{-1} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \mu_{km}(\theta_1) \; \mu_{mk}(\theta_2) \; X(\omega ; \theta_1) \, X(\omega ; \theta_2) \times \{[\mathcal{N} + \tfrac{1}{2} + \nu_k X(\omega ; \theta) T^{-1}] \, [\mathcal{N} + \tfrac{1}{2} + \nu_m X(\omega ; \theta) T^{-1}] - \tfrac{1}{4}\}^{-1} \, d\omega / 2\pi \tag{2.15}$$

as shown in Appendix A. In exceptional cases it may be possible to solve the integral equation (A11) and evaluate the ambiguity function by Eq. (A12).

If there are several parameters $(\theta_1, \theta_2, \ldots , \theta_n) = \underset{\sim}{\theta}$ to be estimated at the same time, or if only one is to be estimated when the rest are unknown, generalized forms of the Cramér-Rao inequality[12,13] and its quantum counterpart[5] apply. The resulting bounds are best described in terms of the concentration ellipsoid, whose matrix can be derived from the ambiguity function $H(\underset{\sim}{\theta}_1 , \underset{\sim}{\theta}_2)$, defined as in Eq.(2.15), but with θ_1 and θ_2 replaced by the sets of parameters $\underset{\sim}{\theta}_1$ and $\underset{\sim}{\theta}_2$. The method of specifying the bounds is straightforward and has been described elsewhere.[5]

The Classical Limit

The sum of the eigenvalues of the integral equation (2.12) is[14]

$$\sum_{k=1}^{\infty} \nu_k = N_s = E_s / \hbar\Omega \quad , \tag{2.16}$$

where N_s is the average number of photons received from the object during $(0 , T)$. When the object light possesses complete first-order spatial coherence at the aperture, $\nu_1 = N_s$ and $\nu_k = 0, k > 1$. When, on the other hand, the object light has only slight spatial coherence, a rough approximation sets the M largest eigenvalues equal to $\bar{\nu} = N_s / M$ and the rest equal to zero, where[14]

$$M = \left[\int_A \varphi_s(\mathbf{r}, \mathbf{r} ; \theta) \, d^2\mathbf{r} \right]^2 \left[\iint_{AA} |\varphi_s(\mathbf{r}_1, \mathbf{r}_2 ; \theta)|^2 \, d^2 \mathbf{r}_1 \, d^2 \mathbf{r}_2 \right]^{-1} = \mathcal{F}^{-2} \tag{2.17}$$

with $\mathcal{F}$ the spatial factor for detection, defined by I, Eq. (3.8).

In the classical limit, $K\mathcal{T} \gg \hbar\Omega$ and $\mathcal{N} \gg 1$. For object light providing an effective signal-to-noise ratio of the order of 1--see III, Eq. (5.13)--, N_s is of the order of $\mathcal{N}(MWT)^{\frac{1}{2}}$. Since the spectral density $X(\omega ; \theta)$ is of the order of W^{-1} , the terms $\nu_k X(\omega ; \theta) T^{-1}$ in Eq. (2.15) will be of the order of $N_s / MWT \sim \mathcal{N}(MWT)^{-\frac{1}{2}} \ll \mathcal{N}$ when $MWT \gg 1$. For objects of moderate detectability, therefore, we can set the bracket in Eq. (2.15) equal to $\mathcal{N}^2$. The double summation can then be evaluated by using Eq. (2.13) and the orthonormality of the eigenfunctions $\eta_k(\mathbf{r})$ to obtain the ambiguity function

$$H(\theta_1, \theta_2) = (N_s / \mathcal{N})^2 \, T^{-1}$$

$$\times \iint\limits_{AA}\int_{-\infty}^{\infty} \Phi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2; \omega ; \theta_1) \;\; \Phi_s(\underset{\sim}{r}_2, \underset{\sim}{r}_1; \omega ; \theta_2) \;\; d^2 \underset{\sim}{r}_1 \, d^2 \underset{\sim}{r}_2 \;\; (d\omega / 2\pi)$$

$$\times \left[\int\limits_A \varphi_s(\underset{\sim}{r}, \underset{\sim}{r} ; \theta) \, d^2 \underset{\sim}{r} \right]^{-2} . \qquad (2.18)$$

When this is used in Eqs. (2.6) and (2.9), the same lower bound is obtained as in I, Section VI for the mean-square error in an unbiased estimate of a parameter θ of an object by a background-limited optical system.[15]

III. Estimation with Low Spatial Coherence

Turning now to the estimation of object parameters under a quantum limitation, we first consider object light with a low degree of spatial coherence at the aperture, and we work out a rough approximation to the ambiguity function $H(\theta_1, \theta_2)$. The number M of effectively independent spatial degrees of freedom is now large, and the first M eigenvalues ν_k can be set equal to $\bar{\nu} = N_s / M$ in Eq. (2.15). With $M \gg 1$, the double sum can again be evaluated as in Eq. (2.18), and we obtain

$$H(\theta_1, \theta_2) =$$
$$N_s^2 T^{-1} \iint_{AA} \varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2; \theta_1)\, \varphi_s(\underset{\sim}{r}_2, \underset{\sim}{r}_1; \theta_2)\, d^2 \underset{\sim}{r}_1\, d^2 \underset{\sim}{r}_2$$
$$\times \int_{-\infty}^{\infty} X(\omega; \theta_1)\, X(\omega; \theta_2)\, \{[\mathcal{N} + \tfrac{1}{2} + N_s X(\omega; \theta) M^{-1} T^{-1}]^2 - \tfrac{1}{4}\}^{-1} (d\omega / 2\pi)$$
$$\times \left[\int_A \varphi_s(\underset{\sim}{r}, \underset{\sim}{r}; \theta)\, d^2 \underset{\sim}{r}\right]^{-2} \qquad (3.1)$$

When quantum limitations are significant, $\mathcal{N} \ll 1$, but N_s and $\mathcal{N}MWT$ may be of the order of 1, with N_s / MWT of the order of $\mathcal{N}$ and very small.[14] Eq. (3.1) can then be simplified to read

$$H(\theta_1, \theta_2) =$$
$$N_s M \mathcal{D} W \iint_{AA} \varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2; \theta_1)\, \varphi_s(\underset{\sim}{r}_2, \underset{\sim}{r}_1; \theta_2)\, d^2 \underset{\sim}{r}_1\, d^2 \underset{\sim}{r}_2$$
$$\times \int_{-\infty}^{\infty} X(\omega; \theta_1)\, X(\omega; \theta_2)\, [1 + \mathcal{D} W X(\omega; \theta)]^{-1} (d\omega / 2\pi)$$
$$\times \left[\int_A \varphi_s(\underset{\sim}{r}, \underset{\sim}{r}; \theta)\, d^2 \underset{\sim}{r}\right]^{-2}, \qquad (3.2)$$

where

$$\mathcal{D} = N_s / \mathcal{N} MWT. \qquad (3.3)$$

For estimates of parameters θ, such as absolute radiance and position, that do not enter the spectral density $X(\omega ; \theta) = X(\omega)$, the lower bounds on the mean-square error derived from Eq. (3.2) have the same form as those obtained in I, Section VI, except that the ratio (N / E) must be replaced by $[N_s \, MWT \, f_1 \, (\mathcal{D})]^{-\frac{1}{2}}$, where the function

$$f_1 \, (\mathcal{D}) = \mathcal{D} W \int_{-\infty}^{\infty} [X(\omega)]^2 \, [1 + \mathcal{D} W X(\omega)]^{-1} \, d\omega \, / \, 2\pi \tag{3.4}$$

depends on the form of the spectral density of the object light. For $\mathcal{D} \ll 1$, $f_1 \, (\mathcal{D}) \doteq \mathcal{D}$, and (N / E) is replaced by $\mathcal{N}^{1/2} / N_s$. For $\mathcal{D} \gg 1$, $f_1 \, (\mathcal{D}) \doteq 1$. In particular, for an estimate of the absolute radiance B_0 of the object,

$$\underset{\sim}{E}(\hat{B}_0 - B_0)^2 \, / \, B_0^{\,2} \geq [N_s \, f_1 \, (\mathcal{D})]^{-1} \quad , \tag{3.5}$$

which becomes independent of the number MWT of spatial and temporal degrees of freedom when the background vanishes $(\mathcal{N} = 0)$.

When the object spectrum has a rectangular form,

$$X(\omega) = W^{-1} \; , \; |\omega| < \pi W \; ; \; X(\omega) = 0, \quad |\omega| > \pi W \; ,$$

$$f_1(\mathcal{D}) = \mathcal{D} / \, (\mathcal{D} + 1) \quad , \tag{3.6}$$

which is plotted as a dashed curve in Fig. 1. For a Lorentz spectrum,

$$X(\omega) = 2W(\omega^2 + W^2)^{-1} \; , \tag{3.7}$$

$$f_1(\mathcal{D}) = 2\mathcal{D}(1 + 2\mathcal{D})^{-\frac{1}{2}} \, [1 + (1 + 2\mathcal{D})^{\frac{1}{2}} \,]^{-1} \quad , \tag{3.8}$$

which is plotted as a solid curve in Fig. 1. The minimum mean-square error attainable is smaller when the spectrum is rectangular than when it is Lorentz.

Estimation of Frequency

In spectroscopy it may be necessary to measure the central frequency or wavelength of a weak spectral line. This corresponds to estimating the location ω_0 of the spectral density $X(\omega\ ;\ \omega_0) = X(\omega - \omega_0)$. According to Eqs. (2.6), (2.9), and (3.2), the relative mean-square error of an unbiased estimate $\hat{\omega}_0$ is bounded by

$$\underset{\sim}{E}(\hat{\omega}_0 - \omega_0)^2 / W^2 \geq [N_s f_2(\mathcal{D})]^{-1} \tag{3.9}$$

where

$$f_2(\mathcal{D}) = \mathcal{D} W^3 \int_{-\infty}^{\infty} [X'(\omega)]^2 [1 + \mathcal{D} W X(\omega)]^{-1} d\omega / 2\pi \quad , \tag{3.10}$$

the prime denoting differentiation with respect to ω .

For the Lorentz spectrum in Eq. (3.7) this is

$$f_2(\mathcal{D}) = [3 + (1 + 2\mathcal{D})^{\frac{1}{2}}] [1 + (1 + 2\mathcal{D})^{\frac{1}{2}}]^{-3} \quad , \tag{3.11}$$

which is plotted in Fig. 1. For $\mathcal{D} \ll 1$, $f_2(\mathcal{D}) \doteq \mathcal{D}/2$; for $\mathcal{D} \gg 1$, $f_2(\mathcal{D}) \doteq \frac{1}{2}$. The corresponding bound in the classical limit $\mathcal{N} \gg 1$ is

$$\underset{\sim}{E}(\hat{\omega}_0 - \omega_0)^2 / W^2 \geq 2(\mathcal{N}/N_s)^2 \, MWT \quad , \tag{3.12}$$

where $N_s / \mathcal{N} = E/N$, $M = \mathcal{F}^{-2}$ in the notation of I .

The lower bounds on the mean-square errors in the estimates of both absolute radiance B_0 and frequency ω_0 become independent of the number MWT of spatial and temporal degrees of freedom when the background vanishes ($\mathcal{N} = 0$). As there may be some question about the limit processes involved, an independent derivation is presented in Appendix B, where it is shown that these results are valid when $MWT \gg N_s$.

Estimation of Position

For the estimates of the coordinates $(\varepsilon_x , \varepsilon_y)$ of the center of a uniformly radiating object of radius b, the mean-square error is subject in the classical limit to the lower bound

$$\underset{\sim}{E}(\hat{\varepsilon}_x - \varepsilon_x)^2/\delta^2 \geq 2(\mathcal{N}/N_s)^2 \; TW[\mathcal{V}(\alpha)]^{-1} \quad ,$$
$$\alpha = b/\delta \; , \; \delta = R/ka \quad , \tag{3.13}$$

where $\mathcal{V}(\alpha)$ is given in I, Eq. (6.5), and a is the radius of the circular aperture. For $\alpha \gg 1$ the integral there can be approximated to yield

$$\mathcal{V}(\alpha) \cong 64/3\pi^2\alpha^3 \quad . \tag{3.14}$$

By I, Eq. (3.10), on the other hand, the number M of independent spatial degrees of freedom is, for $\alpha \gg 1$, given by[16]

$$\mathcal{F}^2 = M^{-1} \cong 4/\alpha^2 \quad . \tag{3.15}$$

Hence the classical bound is

$$\underset{\sim}{E}(\hat{\varepsilon}_x - \varepsilon_x)^2/\delta^2 \geq \tfrac{3}{4}\pi^2(\mathcal{N}/N_s)^2 \; TWM^{3/2}. \tag{3.16}$$

In the quantum limit, therefore, for $M \gg 1$,

$$\underset{\sim}{E}(\hat{\varepsilon}_x - \varepsilon_x)^2/\delta^2 \geq \tfrac{3}{4}\pi^2[N_s f_1(\mathcal{D})]^{-1} \; M^{1/2} \; , \tag{3.17}$$

which in the limit $\mathcal{N} \to 0, \mathcal{D} \doteq 1$ depends weakly on the number M of independent spatial degrees of freedom, and not at all on WT.

IV. Estimation with First-Order Coherence

When the object is a point source far from the aperture, its light upon reaching the aperture will possess first-order spatial coherence. Let a source of total radiant power B_0 be located at the point $(\underset{\sim}{\varepsilon}, R)$. At the aperture its light will have a mutual coherence function whose spatial part is

$$\varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2; \underset{\sim}{\varepsilon}) = (\hbar N_s/2\Omega cTA)\, \mathcal{E}(\underset{\sim}{r}_1, \underset{\sim}{\varepsilon})\, \mathcal{E}^*(\underset{\sim}{r}_2, \underset{\sim}{\varepsilon}) \quad , \tag{4.1}$$

with

$$\mathcal{E}(\underset{\sim}{r}, \underset{\sim}{\varepsilon}) = \exp(ik|\underset{\sim}{r} - \underset{\sim}{\varepsilon}|^2/2R) \quad , \quad k = \Omega/c \quad . \tag{4.2}$$

Here, by Eq. (1.9),

$$N_s = B_0 AT/4\pi R^2 \hbar\Omega \tag{4.3}$$

is the average total number of photons received during the interval $(0, T)$. In order to derive Eq. (4.1), we put $B(\underset{\sim}{u}) = B_0\delta(\underset{\sim}{u} - \underset{\sim}{\varepsilon})$ into I, Eq. (1.8) and use the point-spread function in I, Eq. (1.9), dividing by $2\Omega^2 c$ to convert to the normalization in this paper.

When the object light has complete first-order spatial coherence, the first eigenvalue of Eq. (2.12) is $\nu_1 = N_s$, and its associated eigenfunction is

$$\eta_1(\underset{\sim}{r}) = A^{-1/2}\, \mathcal{E}(\underset{\sim}{r}, \underset{\sim}{0}) \quad . \tag{4.4}$$

The remaining eigenvalues vanish, and as their eigenfunctions we can take an arbitrary set of functions orthonormal among themselves and to $\eta_1(\underset{\sim}{r})$ over the aperture A.

For estimates of the radiant power B_0 and the frequency ω_0 of a point source located on the z-axis $(\underset{\sim}{\varepsilon} = \underset{\sim}{0})$, Eq. (2.14) applies, and the series for the ambiguity function in Eq. (2.15) contains a single term,

$$H(\theta_1 , \theta_2) = T^{-1} \nu_1(\theta_1)\nu_1(\theta_2)$$

$$\times \int_{-\infty}^{\infty} X(\omega ; \theta_1) X(\omega ; \theta_2) \{[\mathcal{N} + \tfrac{1}{2} + \nu_1 X(\omega ; \theta)T^{-1}]^2 - \tfrac{1}{4}\}^{-1} d\omega/2\pi \quad . \tag{4.5}$$

For an estimate of radiant power B_0 , $\nu_1(\theta_i) = (B_i/B_0)\,\nu_1$, $i = 1,2$, and after substitution into Eq. (2.9) and the differentiations with respect to $\theta_1 = B_1$, $\theta_2 = B_2$, we put $B_1 = B_2 = B_0$. We find that an unbiased estimate $\hat{B}_0$ of radiance has a relative mean-square error bounded below by

$$\underset{\sim}{E}(\hat{B}_0 - B_0)^2/B_0^{\,2} \geq [N_s f_1(\mathcal{D})]^{-1} \quad , \tag{4.6}$$

where $f_1(\mathcal{D})$ is given in Eq. (3.4), but $\mathcal{D}$ is now

$$\mathcal{D} = N_s/\mathcal{N}WT \quad . \tag{4.7}$$

Before integrating, we have again passed to the quantum limit and set $\mathcal{N} \ll 1$, $N_s/WT \ll 1$ in Eq. (4.5). The graphs in Fig. 1 apply with $M = 1$.

For an estimate of frequency, $\nu_1(\theta_1) = \nu_1$ and $X(\omega ; \theta_i) = X(\omega - \omega_i)$, $i = 1,2$, with ω_1 and ω_2 set equal to ω_0 after the differentiations in Eq. (2.9). We find in the quantum limit,

$$\underset{\sim}{E}(\hat{\omega}_0 - \omega_0)^2/W^2 \geq [N_s f_2(\mathcal{D})]^{-1} \, , \tag{4.8}$$

where $f_2(\mathcal{D})$ is given by Eq. (3.10) and graphed in Fig. 1. Thus the approximate lower bounds on the mean-square errors in estimates of absolute radiance and frequency obtained in Section III for $M \gg 1$ become exact for point objects if M is set equal to 1.

Estimation of Position

The coordinates $(\varepsilon_x, \varepsilon_y) = \underset{\sim}{\varepsilon}$ of the point source in the plane $z = R$ can be estimated independently when the aperture is circular and the object is located near the optical axis. It is now necessary to evaluate the coefficients

$$\mu_{km}(\underset{\sim}{\varepsilon}_i) = (2\Omega cT/\hbar) \iint\limits_{AA} \eta_k^*(\underset{\sim}{r}_1)\, \varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2\, ;\, \underset{\sim}{\varepsilon}_i)\, \eta_m(\underset{\sim}{r}_2)\, d^2\underset{\sim}{r}_1 d^2\underset{\sim}{r}_2 \;, \qquad i = 1, 2 \;, \tag{4.9}$$

of the expansion in Eq. (2.13). After the ambiguity function is substituted into Eq. (2.9), we shall differentiate these coefficients with respect to ε_{ix} or ε_{iy} $(i = 1, 2)$, and set $\underset{\sim}{\varepsilon}_1 = \underset{\sim}{\varepsilon}_2 = \underset{\sim}{0}$.

We find from Eq. (4.1),

$$\mu_{km}(\underset{\sim}{\varepsilon}_i) = N_s\, e_k(\underset{\sim}{\varepsilon}_i)\, e_m^*(\underset{\sim}{\varepsilon}_i) \;, \tag{4.10}$$

$$e_k(\underset{\sim}{\varepsilon}_i) = A^{-1/2} \int\limits_A \eta_k^*(\underset{\sim}{r})\, \mathcal{E}(\underset{\sim}{r}, \underset{\sim}{\varepsilon}_i)\, d^2\underset{\sim}{r} \;. \tag{4.11}$$

In particular, from Eq. (4.4),

$$e_1(\underset{\sim}{\varepsilon}_i) = A^{-1} \int\limits_A \mathcal{E}^*(\underset{\sim}{r}, \underset{\sim}{0})\, \mathcal{E}(\underset{\sim}{r}, \underset{\sim}{\varepsilon}_i)\, d^2\underset{\sim}{r} = \mathcal{G}_A^*(\underset{\sim}{\varepsilon}_i) \tag{4.12}$$

in terms of the Fourier transform

$$\mathcal{G}_A(\underset{\sim}{v}) = A^{-1} \int\limits_A \exp(ik\underset{\sim}{v}\cdot\underset{\sim}{r}/R)\, d^2\underset{\sim}{r} \tag{4.13}$$

of the indicator function of the aperture--see I, Eq. (A4). For a circular aperture of radius A,

$$\mathcal{G}_A(\underset{\sim}{\varepsilon}) = 2(\delta/|\underset{\sim}{\varepsilon}|)\, J_1(|\underset{\sim}{\varepsilon}|/\delta) \doteq 1 - (\varepsilon_x^{\,2} + \varepsilon_y^{\,2})/8\delta^2 + O(|\varepsilon|^4/\delta^4) \;, \tag{4.14}$$

where $\delta = R/ka$ and $J_1(x)$ is the Bessel function of order 1.

The term $\mu_{11}(\underset{\sim}{\varepsilon}_1)\mu_{11}(\underset{\sim}{\varepsilon}_2) = N_s^2|e_1(\underset{\sim}{\varepsilon}_1)|^2\ |e_1(\underset{\sim}{\varepsilon}_2)|^2$ in the sum in Eq. (2.15) will thus yield 0 after substitution into Eq. (2.9), differentiation, and the setting of $\underset{\sim}{\varepsilon}_1$ and $\underset{\sim}{\varepsilon}_2$ equal to $\underset{\sim}{0}$.

Since $\nu_k = 0,\ k > 1$, we can write Eq. (2.15) as applied to the estimation of ε_x or ε_y in the form

$$H(\underset{\sim}{\varepsilon}_1, \underset{\sim}{\varepsilon}_2) = [\mathcal{N}(\mathcal{N}+1)]^{-1}\ T^{-1}$$

$$\times \int_{-\infty}^{\infty} [X(\omega)]^2 \ \{ \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \mu_{km}(\underset{\sim}{\varepsilon}_1)\ \mu_{mk}(\underset{\sim}{\varepsilon}_2)$$

$$- (\mathcal{N}+\tfrac{1}{2})\ N_s X(\omega)\ T^{-1}\{(\mathcal{N}+\tfrac{1}{2})\ [\mathcal{N}+\tfrac{1}{2}+N_s X(\omega)\ T^{-1}] - \tfrac{1}{4}\}^{-1}$$

$$\times \sum_{k=1}^{\infty} [\mu_{k1}(\underset{\sim}{\varepsilon}_1)\ \mu_{1k}(\underset{\sim}{\varepsilon}_2) + \mu_{1k}(\underset{\sim}{\varepsilon}_1)\ \mu_{k1}(\underset{\sim}{\varepsilon}_2)]\}\ d\omega/2\pi$$

$$+ \text{term} \propto \mu_{11}(\underset{\sim}{\varepsilon}_1)\ \mu_{11}(\underset{\sim}{\varepsilon}_2) \quad , \tag{4.15}$$

the last term contributing 0 to the bound. As in Eq. (2.18),

$$\sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \mu_{km}(\underset{\sim}{\varepsilon}_1)\ \mu_{mk}(\underset{\sim}{\varepsilon}_2) =$$

$$(2\Omega cT/\hbar)^2 \iint_{AA} \varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2\ ;\ \underset{\sim}{\varepsilon}_1)\ \varphi_s(\underset{\sim}{r}_2, \underset{\sim}{r}_1\ ;\ \underset{\sim}{\varepsilon}_2)\ d^2\underset{\sim}{r}_1 d^2\underset{\sim}{r}_2 =$$

$$N_s^{\ 2}|\mathcal{J}_A(\underset{\sim}{\varepsilon}_2 - \underset{\sim}{\varepsilon}_1)|^2 \doteq N_s^{\ 2}(1 - |\underset{\sim}{\varepsilon}_2 - \underset{\sim}{\varepsilon}_1|/4\delta^2)\ . \tag{4.16}$$

In addition, we obtain by using the orthonormality of the eigenfunctions $\eta_k(\underset{\sim}{r})$

$$\sum_{k=1}^{\infty} \mu_{k1}(\underset{\sim}{\varepsilon}_1)\ \mu_{1k}(\underset{\sim}{\varepsilon}_2) =$$

$$(2\Omega cT\hbar)^2 \iiint_{AAA} \eta_1{}^*(\underset{\sim}{r}_1)\ \varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2\ ;\ \underset{\sim}{\varepsilon}_2)\ \varphi_s(\underset{\sim}{r}_2, \underset{\sim}{r}_3\ ;\ \underset{\sim}{\varepsilon}_1)\ \eta_1(\underset{\sim}{r}_3)$$

$$\times\ d^2\underset{\sim}{r}_1 d^2\underset{\sim}{r}_2 d^2\underset{\sim}{r}_3 =$$

$$(N_s^{\,2}/A^3) \iiint\limits_{AAA} \mathcal{E}^*(\underset{\sim}{r}_1,\ \underset{\sim}{0})\ \ \mathcal{E}(\underset{\sim}{r}_1,\ \underset{\sim}{\varepsilon}_2)\ \ \mathcal{E}^*(\underset{\sim}{r}_2,\ \underset{\sim}{\varepsilon}_2)\ \ \mathcal{E}(\underset{\sim}{r}_2,\ \underset{\sim}{\varepsilon}_1)\ \ \mathcal{E}^*(\underset{\sim}{r}_3,\ \underset{\sim}{\varepsilon}_1)$$

$$\times\ \mathcal{E}(\underset{\sim}{r}_3,\ \underset{\sim}{0})\ d^2\underset{\sim}{r}_1 d^2\underset{\sim}{r}_2 d^2\underset{\sim}{r}_3 =$$

$$N_s^{\,2}\ \mathcal{G}_A(-\underset{\sim}{\varepsilon}_2)\ \ \mathcal{G}_A(\underset{\sim}{\varepsilon}_2 - \underset{\sim}{\varepsilon}_1)\ \ \mathcal{G}_A(\underset{\sim}{\varepsilon}_1) \doteq$$

$$N_s^{\,2}(1 - |\underset{\sim}{\varepsilon}_2|^2/8\delta^2 - |\underset{\sim}{\varepsilon}_1|^2/8\delta^2 - |\underset{\sim}{\varepsilon}_2 - \underset{\sim}{\varepsilon}_1|^2/8\delta^2)\ . \tag{4.17}$$

When these are put into Eq. (4.15) and the result into Eq. (2.9), we find

$$\underset{\sim}{E}(\hat{\varepsilon}_x - \varepsilon_x)^2/\delta^2 = \underset{\sim}{E}(\hat{\varepsilon}_y - \varepsilon_y)^2/\delta^2 \geq$$

$$2T\ N_s^{-2}\{\int_{-\infty}^{\infty} [X(\omega)]^2\ [\mathcal{N}(\mathcal{N}+1) + (\mathcal{N}+\tfrac{1}{2})\ N_s X(\omega)\ T^{-1}]^{-1}\ d\omega/2\pi\}^{-1}. \tag{4.18}$$

In the classical limit $\mathcal{N} \gg 1$, this becomes

$$\underset{\sim}{E}(\hat{\varepsilon}_x - \varepsilon_x)^2/\delta^2 \geq 2(\mathcal{N}/N_s)^2\ WT\ \ , \tag{4.19}$$

as in Eq. (3.13), in which $\mathcal{V}(\alpha) = 1$ because the object light is completely coherent in first order.

In the quantum limit, on the other hand, $\mathcal{N} \ll 1$, $N_s/WT \ll 1$, the bound becomes

$$\underset{\sim}{E}(\hat{\varepsilon}_x - \varepsilon_x)^2/\delta^2 \geq N_s^{-1}[f_3(\mathcal{D})]^{-1} \tag{4.20}$$

where

$$f_3(\mathcal{D}) = f_1(\mathcal{D}/2). \tag{4.21}$$

The function $f_3(\mathcal{D})$ is plotted versus $\mathcal{D}$ in Fig. 1 for the Lorentz spectrum in Eq. (3.7).

The position of a point source might be estimated by focusing its light on a photosensitive surface and processing the numbers of photoelectrons emitted from each of many elements of the surface. The joint probability distribution of these numbers would be maximized with respect to the parameters ε_x and ε_y to provide maximum-likelihood estimates. The mean-square error of such estimates has been calculated[17]; in our present notation it is

$$\underset{\sim}{E}(\hat{\varepsilon}_x - \varepsilon_x)^2/\delta^2 = \underset{\sim}{E}(\hat{\varepsilon}_y - \varepsilon_y)^2/\delta^2$$
$$\cong 18\pi^2(9\pi^2 - 64)^{-1}\ (\eta N_s \mathcal{D})^{-1} = 7.16/\eta N_s \mathcal{D}, \tag{4.22}$$

where η is the quantum efficiency of the surface, $0 < \eta < 1$. It was assumed that the object light is much weaker than that from the background, but that this weakness is compensated by a long observation time T. In addition, the spectrum was taken as rectangular of width W. In the corresponding circumstances, our bound in Eq. (4.19) becomes

$$\underset{\sim}{E}(\hat{\varepsilon}_x - \varepsilon_x)^2/\delta^2 \geq 2/N_s \mathcal{D}, \tag{4.23}$$

which lies below the mean-square error given by Eq. (4.22). General Cramér-Rao bounds on the mean-square error in estimating the position of a stellar image on a photosensitive surface have been worked out by Farrell, with numerical results for images with a gaussian pattern of illuminance.[18] The difference between those essays and the present one lies in the nature of the primary data. There the data are numbers of photoelectrons emitted from the image plane; here they are the values of the electromagnetic field at the aperture of the optical instrument, whatever it may be.

V. Estimation for Circular Objects

A quasimonochromatic circular object of uniform radiance B_0 and radius b is centered at $(\underset{\sim}{0}, R)$. Its light falls on a circular aperture of radius a. We seek lower bounds on the mean-square errors in unbiased estimates of its radiance B_0 and its frequency ω_0. The spatial part $\varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2)$ of the mutual coherence function of its light at the aperture is, by (I), Eqs. (1.8) and (1.9),

$$\varphi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2) = (B_0/8\pi\Omega^2 cR^2) \exp\left[\frac{ik}{2R}(\underset{\sim}{r}_1{}^2 - \underset{\sim}{r}_2{}^2)\right]$$

$$\times \int_O \exp\left[\frac{ik}{R}\, \underset{\sim}{u}\cdot(\underset{\sim}{r}_2 - \underset{\sim}{r}_1)\right] d^2\underset{\sim}{u} \quad , \qquad (5.1)$$

where O denotes the circular object.

The integral equation (2.12) can now be identified with the two-dimensional one studied by Slepian.[19] The eigenfunctions of Slepian's Eq. (12) are

$$\psi_k(\underset{\sim}{x}) = \eta_k(a\underset{\sim}{x}) \exp(-ika^2\underset{\sim}{x}^2/2R) \quad , \qquad (5.2)$$

and the eigenvalues are

$$\lambda_k = (\alpha^2/4)\ \nu_k/N_s , \qquad (5.3)$$

where $\alpha = kab/R$ is equivalent to Slepian's parameter c. The kernel of Slepian's integral equation (12) is

$$K_c(\underset{\sim}{x}) = (\alpha/2\pi)^2 \int_C \exp(i\alpha\underset{\sim}{z}\cdot\underset{\sim}{x})\ d^2\underset{\sim}{z} \quad , \qquad (5.4)$$

where C denotes the unit circle.

In the quantum limit $\mathcal{N} \ll 1$, the mean-square errors in unbiased estimates $\hat{B}_0$ and $\hat{\omega}_0$ of radiance and frequency are bounded below by

$$\underset{\sim}{E}(\hat{B}_0 - B_0)^2/B_0^{\;2} \geq [N_s \mathcal{V}_1(\alpha; \mathcal{D})]^{-1} \quad , \tag{5.5}$$

$$\underset{\sim}{E}(\hat{\omega}_0 - \omega_0)^2/W^2 \geq [N_s \mathcal{V}_2(\alpha; \mathcal{D})]^{-1} \quad , \tag{5.6}$$

with

$$\mathcal{D} = N_s/\mathcal{N}WT \quad , \tag{5.7}$$

where by Eq. (2.15), (3.8) and (3.10),

$$\mathcal{V}_i(\alpha; \mathcal{D}) = \sum_{k=1}^{\infty} (\nu_k/N_s)\; f_i(\nu_k \mathcal{D}/N_s), \qquad i = 1, 2, \tag{5.8}$$

the functions f_1 and f_2 being given by Eqs. (3.4) and (3.10). When a single eigenvalue is significant, we get the bounds in Eqs. (4.6) and (4.8); when $M \gg 1$, those in Eqs. (3.5) and (3.9) appear.

The twelve largest eigenvalues λ_k have been tabulated by Slepian[19] as functions of α.[20] For values of α missing from the tables we calculated the eigenvalues by Lagrange's interpolation formula applied to $\ell n(-\ell n\; \lambda_k)$. The eigenvalues of higher order are small, and for them the approximations

$$f_1(\nu_k \mathcal{D}/N_s) \cong \nu_k \mathcal{D}/N_s \quad , \tag{5.9}$$

$$f_2(\nu_k \mathcal{D}/N_s) \cong \nu_k \mathcal{D}/2\, N_s \tag{5.10}$$

can be made. Indicating by a prime the summation over these remaining eigenvalues, we write their contribution to $\mathcal{V}_1(\alpha; \mathcal{D})$ as

$$\Delta \mathcal{V}_1(\alpha; \mathcal{D}) \cong \sum{}' (\nu_k/N_s)^2 \mathcal{D} \quad ; \tag{5.11}$$

a similar formula holds for $\Delta \mathcal{V}_2(\alpha; \mathcal{D})$.

We observe that the squared spatial factor $\mathcal{F}^2$ is given by Eq. (2.17) and (III), Eq. (5.10) as

$$[\mathcal{F}(\alpha)]^2 = \sum_{k=1}^{\infty} (\nu_k/N_s)^2 \quad ,$$

and this we have calculated from (I), Eq. (3.10).[16] By summing the squares of the eigenvalues given in Slepian's tables, multiplying by $(4/\alpha^2)^2$, and subtracting from $[\mathcal{F}(\alpha)]^2$, we can evaluate $\sum' (\nu_k/N_s)^2$ in Eq. (5.11) and thus supply approximately the missing terms in Eq. (5.8).

By this procedure the functions $\mathcal{V}_i(\alpha; \mathcal{D})$, $i = 1, 2$, have been calculated for three representative values of $\mathcal{D} = N_s/\mathcal{N}WT$; they are plotted versus $\alpha = kab/R$. A Lorentz spectrum was postulated. The curves show that the larger $\mathcal{D}$ (the smaller $\mathcal{N}$), the less sensitive the bounds are to the number M of spatial degrees of freedom of the object light at the aperture.

Appendix A

Derivation of Bounds

The translation of Eqs. (2.10) and (2.11) into forms involving the field at the aperture is effected by means of the function

$$L(\underset{\sim}{r}_1', t_1 ; \underset{\sim}{r}_2', t_2 ; \theta_1) = L_1(\underset{\sim}{r}_1', t_1 ; \underset{\sim}{r}_2', t_2) =$$

$$(\hbar/2) \sum_{\underset{\sim}{k}} \sum_{\underset{\sim}{m}} (\omega_{\underset{\sim}{k}} \omega_{\underset{\sim}{m}})^{-\frac{1}{2}} L_{\underset{\sim}{k}\underset{\sim}{m}}(\theta_1) u_{\underset{\sim}{k}}(\underset{\sim}{r}_1') u_{\underset{\sim}{m}}(\underset{\sim}{r}_2')$$

$$\times \exp(-i\omega_{\underset{\sim}{k}} t_1 + i\omega_{\underset{\sim}{m}} t_2) \tag{A1}$$

--cf. III, Eq. (4.3)--and the associated operators

$$L_1(1, 2) = \sum_{\underset{\sim}{k}} \sum_{\underset{\sim}{m}} O_{\underset{\sim}{k}}^*(1) L_{\underset{\sim}{k}\underset{\sim}{m}}(\theta_1) O_{\underset{\sim}{m}}(2) \tag{A2}$$

and $\mathcal{L}_1(1, 2)$, respectively defined like $Q(1, 2)$ and $\mathfrak{Q}(1, 2)$ of III, Eqs. (4.2) and (4.5).

We observe the similar structures of Eq. (2.11) and III, Eq. (2.7) and conclude by a derivation of the same type as in III, §IV that the function $L_1(\underset{\sim}{r}_1', t_1 ; \underset{\sim}{r}_2', t_2)$ is the solution of an integro-differential equation of the same form as III, Eq. (4.7),

$$2\varphi_s(\underset{\sim}{r}_1', t_1 ; \underset{\sim}{r}_2', t_2) = \mathcal{L}_1(1, 3)\, \varphi(\underset{\sim}{r}_3', t_3 ; \underset{\sim}{r}_2', t_2 ; \theta)$$
$$+ \mathcal{L}_1^*(3, 2)\, \varphi(\underset{\sim}{r}_1', t_1 ; \underset{\sim}{r}_3', t_3 ; \theta) +$$
$$2L_1(3, 4)\, \varphi(\underset{\sim}{r}_1', t_1 ; \underset{\sim}{r}_3', t_3 ; \theta)\, \varphi(\underset{\sim}{r}_4', t_4 ; \underset{\sim}{r}_2', t_2 ; \theta). \tag{A3}$$

Furthermore, we can use Eq. (A2) and III, Eq. (4.10) to write Eq. (2.10) as

$$H(\theta_1, \theta_2) = \sum_{\underset{\sim}{k}} \sum_{\underset{\sim}{m}} L_{\underset{\sim}{k}\underset{\sim}{m}}(\theta_1)\, \varphi_{\underset{\sim}{m}\underset{\sim}{k}}(\theta_2) =$$

$$\sum_{\underset{\sim}{k}} \sum_{\underset{\sim}{m}} L_{\underset{\sim}{k}\underset{\sim}{m}}(\theta_1)\, O_{\underset{\sim}{m}}(2)\, O_{\underset{\sim}{k}}^*(1)\, \varphi(\underset{\sim}{r}_2', t_2 ; \underset{\sim}{r}_1', t_1 ; \theta_2) =$$

$$L_1(1, 2)\ \varphi(\underset{\sim}{r}_2', t_2 ; \underset{\sim}{r}_1', t_1 ; \theta_2) =$$

$$(c^4/\hbar^2) \int_0^T\int_0^T dt_1 dt_2 \iint_{AA} d^2\underset{\sim}{r}_1 d^2\underset{\sim}{r}_2$$

$$\times [L_1(\underset{\sim}{r}_1', t_1 ; \underset{\sim}{r}_2', t_2)\ (\underset{\sim}{n}.\nabla_1)\ (\underset{\sim}{n}.\nabla_2)$$
$$- \underset{\sim}{n}.\nabla_1\ L_1(\underset{\sim}{r}_1', t_1 ; \underset{\sim}{r}_2', t_2)\ (\underset{\sim}{n}.\nabla_2)$$
$$- \underset{\sim}{n}.\nabla_2\ L_1(\underset{\sim}{r}_1', t_1 ; \underset{\sim}{r}_2', t_2)\ (\underset{\sim}{n}.\nabla_2)$$
$$+ (\underset{\sim}{n}.\nabla_1)\ (\underset{\sim}{n}.\nabla_2)\ L_1(\underset{\sim}{r}_1', t_1 ; \underset{\sim}{r}_2', t_2)]\ \varphi(\underset{\sim}{r}_2', t_2 ; \underset{\sim}{r}_1', t_1 ; \theta_2)\Big|_{z_1=z_2=0} , \qquad \text{(A4)}$$

where $\underset{\sim}{n}$ is a unit vector normal to the aperture.

Because of Eq. (1.4), we can argue as in III, Section IV that the function L_1 has the form

$$L_1(\underset{\sim}{r}_1', t_1 ; \underset{\sim}{r}_2', t_2) = L_1'(\underset{\sim}{r}_1, t_1 ; \underset{\sim}{r}_2, t_2)$$
$$\times \exp[-i\Omega(t_1 - t_2) - i\Omega(z_1 - z_2)/c] \qquad \text{(A5)}$$

and that we can put for the mutual coherence function of the thermal background light

$$\varphi_0(\underset{\sim}{r}_1', t_1 ; \underset{\sim}{r}_2', t_2) =$$
$$\varphi_0'(\underset{\sim}{r}_1, t_1 ; \underset{\sim}{r}_2, t_2) \exp[-i\Omega(t_1 - t_2) - i\Omega(z_1 - z_2)/c] , \qquad \text{(A6)}$$
$$\varphi_0'(\underset{\sim}{r}_1, t_1 ; \underset{\sim}{r}_2, t_2) =$$
$$(\hbar/2\Omega c)\ \mathcal{N}(\Omega)\ \delta_2(\underset{\sim}{r}_1 - \underset{\sim}{r}_2)\ \delta(t_1 - t_2) , \qquad \text{(A7)}$$

where $\mathcal{N} = \mathcal{N}(\Omega)$ is given by Eq. (1.10). At this point the assumptions are again being made that the diameter of the aperture is much greater than the correlation length $(\hbar c/K\mathcal{T})$ of the thermal light, and that the bandwidth W of the object light is much less than the effective bandwidth $K\mathcal{T}/\hbar$ of the thermal light.[11]

Eqs. (A3) and (A4) can then be simplified to read

$$\varphi_s'(\underset{\sim}{r}_1, t_1 ; \underset{\sim}{r}_2, t_2 ; \theta_1) = (\Omega c/\hbar) \int_0^T dt_3 \int_A d^2\underset{\sim}{r}_3$$

$$\times [L_1'(\underset{\sim}{r}_1, t_1 ; \underset{\sim}{r}_3, t_3) \varphi'(\underset{\sim}{r}_3, t_3 ; \underset{\sim}{r}_2, t_2 ; \theta) + \varphi'(\underset{\sim}{r}_1, t_1 ; \underset{\sim}{r}_3, t_3 ; \theta) L_1'(\underset{\sim}{r}_3, t_3 ; \underset{\sim}{r}_2, t_2)] +$$

$$(2\Omega c/\hbar)^2 \int_0^T\int_0^T dt_3 dt_4 \iint_{AA} d^2\underset{\sim}{r}_3 d^2\underset{\sim}{r}_4$$

$$\times \varphi'(\underset{\sim}{r}_1, t_1 ; \underset{\sim}{r}_3, t_3 ; \theta) L_1'(\underset{\sim}{r}_3, t_3 ; \underset{\sim}{r}_4, t_4) \varphi'(\underset{\sim}{r}_4, t_4 ; \underset{\sim}{r}_2, t_2 ; \theta) , \tag{A8}$$

$$H(\theta_1, \theta_2) = (2\Omega c/\hbar)^2 \int_0^T\int_0^T dt_1 dt_2 \iint_{AA} d^2\underset{\sim}{r}_1 d^2\underset{\sim}{r}_2$$

$$\times L_1'(\underset{\sim}{r}_1, t_1 ; \underset{\sim}{r}_2, t_2) \varphi_s'(\underset{\sim}{r}_2, t_2 ; \underset{\sim}{r}_1, t_1 ; \theta_2) , \tag{A9}$$

where

$$\varphi'(\underset{\sim}{r}_1, t_1 ; \underset{\sim}{r}_2, t_2 ; \theta) = \varphi_0'(\underset{\sim}{r}_1, t_1 ; \underset{\sim}{r}_2, t_2) + \varphi_s'(\underset{\sim}{r}_1, t_1 ; \underset{\sim}{r}_2, t_2 ; \theta) . \tag{A10}$$

The temporal stationarity of the light fields and the great length of the observation interval (0, T) as compared with the reciprocal bandwidths of object and background light permit us to replace the temporal part of Eq. (A8) by Fourier transforms. As a result, $L_1(\underset{\sim}{r}_1, t_1 ; \underset{\sim}{r}_2, t_2)$ will to good approximation be a function only of $t_1 - t_2$, and it will have a Fourier transform $L_1(\underset{\sim}{r}_1, \underset{\sim}{r}_2 ; \omega)$ defined as in Eq. (1.5). Eqs. (A8) and (A9) can now be rewritten as

$$\Phi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2 ; \omega ; \theta) =$$

$$(\Omega c/\hbar) \int_A d^2r_3 \; [L_1(\underset{\sim}{r}_1, \underset{\sim}{r}_3 ; \omega) \Phi(\underset{\sim}{r}_3, \underset{\sim}{r}_2 ; \omega; \theta) + \Phi(\underset{\sim}{r}_1, \underset{\sim}{r}_3 ; \omega; \theta) L_1(\underset{\sim}{r}_3, \underset{\sim}{r}_2 ; \omega)]$$

$$+ (2\Omega c/\hbar)^2 \iint_{AA} d^2\underset{\sim}{r}_3 d^2\underset{\sim}{r}_4 \; \Phi(\underset{\sim}{r}_1, \underset{\sim}{r}_3 \; ; \omega; \theta) \; L_1(\underset{\sim}{r}_3, \underset{\sim}{r}_4 \; ; \omega) \; \Phi(\underset{\sim}{r}_4, \underset{\sim}{r}_2 \; ; \omega; \theta) \quad , \tag{A11}$$

and

$$H(\theta_1, \theta_2) = (2\Omega c/\hbar)^2 \; T \int_{-\infty}^{\infty} (d\omega/2\pi) \iint_{AA} d^2\underset{\sim}{r}_1 d^2\underset{\sim}{r}_2$$

$$\times \; L_1(\underset{\sim}{r}_1, \underset{\sim}{r}_2 \; ; \omega) \; \Phi_s(\underset{\sim}{r}_2, \underset{\sim}{r}_1 \; ; \omega; \theta_2) \quad , \tag{A12}$$

where by Eqs. (A6), (A7),

$$\Phi(\underset{\sim}{r}_1, \underset{\sim}{r}_2 \; ; \omega; \theta) = \Phi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2 \; ; \omega; \theta) + \Phi_0(\underset{\sim}{r}_1, \underset{\sim}{r}_2 \; ; \omega) \quad ,$$

$$\Phi_0(\underset{\sim}{r}_1, \underset{\sim}{r}_2 \; ; \omega) = (\hbar/2\Omega c) \; \mathcal{N}(\Omega) \; \delta_2(\underset{\sim}{r}_1 - \underset{\sim}{r}_2) \; . \tag{A13}$$

We now expand all functions in terms of the orthonormal eigenfunctions $\eta_k(\underset{\sim}{r})$ of the two-dimensional integral equation (2.12). In particular, we write

$$L_1(\underset{\sim}{r}_1, \underset{\sim}{r}_2 \; ; \omega) = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \lambda_{km}(\omega \; ; \theta_1) \; \eta_k(\underset{\sim}{r}_1) \; \eta_m{}^*(\underset{\sim}{r}_2) \quad , \tag{A14}$$

$$\Phi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2 \; ; \omega; \theta_i) =$$
$$(\hbar/2\Omega cT) \; X(\omega \; ; \theta_i)$$
$$\times \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \mu_{km} (\theta_i) \; \eta_k(\underset{\sim}{r}_1) \; \eta_m{}^*(\underset{\sim}{r}_2) \quad , \quad i = 1, 2, \tag{A15}$$

and by Eq. (A13) and the closure property of the eigenfunctions $\eta_k(\underset{\sim}{r})$,

$$\Phi_0(\underset{\sim}{r}_1, \underset{\sim}{r}_2 \; ; \omega) = (\hbar/2\Omega c) \; \mathcal{N}(\Omega) \sum_{k=1}^{\infty} \eta_k(\underset{\sim}{r}_1) \; \eta_m{}^*(\underset{\sim}{r}_2). \tag{A16}$$

When these are substituted into Eq. (A11) and the orthonormality of the eigenfunctions is used, the equation

$$(\hbar/2\Omega cT) \; \mu_{km}(\theta_1) \; X(\omega; \theta_1) =$$
$$\lambda_{km}(\omega; \theta_1) \; \{\mathcal{N} + \tfrac{1}{2}(\nu_k + \nu_m) \; X \; (\omega; \theta) \; T^{-1}$$
$$+ [\mathcal{N} + \nu_k \; X(\omega; \theta) \; T^{-1}] \; [\mathcal{N} + \nu_m \; X(\omega; \theta) \; T^{-1}]\} \tag{A17}$$

is obtained. From it we get the coefficients $\lambda_{km}(\omega;\theta_1)$, which are substituted along with Eqs. (A14) and (A15) into Eq. (A12) to obtain Eq. (2.15).

Appendix B

The Bound in the Absence of Background

In the absence of any background light, the integral equation, Eq. (A11), when written for an estimate of absolute radiance B_0, is

$$(B_1/B_0)\,\Phi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2\,;\,\omega;\,B_0) =$$
$$(\Omega c/\hbar)\int_A d^2\underset{\sim}{r}_3[L_1(\underset{\sim}{r}_1, \underset{\sim}{r}_3\,;\,\omega)\,\Phi_s(\underset{\sim}{r}_3, \underset{\sim}{r}_2\,;\,\omega;\,B_0) +$$
$$\Phi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_3\,;\,\omega;\,B_0)\,L_1(\underset{\sim}{r}_3, \underset{\sim}{r}_2\,;\,\omega)] +$$
$$(2\Omega c/\hbar)^2\iint_{AA} d^2\underset{\sim}{r}_3 d^2\underset{\sim}{r}_4\,\Phi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_3\,;\,\omega;\,B_0)\,L_1(\underset{\sim}{r}_3, \underset{\sim}{r}_4\,;\,\omega)\,\Phi_s(\underset{\sim}{r}_4, \underset{\sim}{r}_2\,;\,B_0)\;; \tag{B1}$$

we have put $\theta_1 = B_1$, $\theta_2 = B_2$, $\theta = B_0$. This equation can be solved by iteration; its solution takes the form of a series,

$$L_1(\underset{\sim}{r}_1, \underset{\sim}{r}_2\,;\,\omega) = (B_1/B_0)\,\{(\hbar/2\Omega c)\,\delta_2(\underset{\sim}{r}_1 - \underset{\sim}{r}_2)$$
$$- \Phi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2\,;\,\omega;\,B_0) +$$
$$(2\Omega c/\hbar)\int_A \Phi_s(\underset{\sim}{r}_1, \underset{\sim}{r}\,;\,\omega;\,B_0)\,\Phi_s(\underset{\sim}{r}, \underset{\sim}{r}_2\,;\,\omega;\,B_0)\,d^2\underset{\sim}{r} - \ldots\} \tag{B2}$$

as can be verified by substituting it into Eq. (B1). The ambiguity function is now, by Eq. (A12) with $\Phi_s(\underset{\sim}{r}_2, \underset{\sim}{r}_1\,;\,\omega;\,B_2) = (B_2/B_0)$ $\times\,\Phi_s(\underset{\sim}{r}_2, \underset{\sim}{r}_1\,;\,\omega;\,B_0)$,

$$H(B_1, B_2) = B_0^{-2} B_1 B_2 (2\Omega cT/\hbar)$$
$$\times\,\{\int_A \Phi_s(\underset{\sim}{r}, \underset{\sim}{r}\,;\,\omega;\,B_0)\,d^2\underset{\sim}{r}$$
$$- (2\Omega c/\hbar)\iint_{AA} |\Phi_s(\underset{\sim}{r}_1, \underset{\sim}{r}_2\,;\,\omega;\,B_0)|^2\,d^2\underset{\sim}{r}_1 d^2\underset{\sim}{r}_2 + \ldots\}. \tag{B3}$$

Using Eq. (1.6) with $\theta = B_0$, the definition of the number M of spatial degrees of freedom, Eq. (2.17), and the definition of the bandwidth W, Eq. (1.8), we finally obtain

$$H(B_1, B_2) = B_0^{-2} B_1 B_2 N_s [1 - (N_s/MWT) + \ldots]. \tag{B4}$$

When this is substituted into Eq. (2.9) and the result into Eq. (2.6), we find the lower bound

$$\underset{\sim}{E}(\hat{B}_0 - B_0)^2/B_0^2 \geq N_s^{-1}[1 + (N_s/MWT) + \ldots]. \tag{B5}$$

When $MWT \gg N_s$, this reduces to the result in Eq. (3.5) with $f_1(\mathcal{D}) = 1$.

A similar derivation can be carried through for an estimate of frequency ω_0, but not for an estimate of position.

Footnotes

* This research has been carried out under NASA grant NGR 05-009-079.

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14. See III, Section V.

15. In comparing with I, Eq. (6.2), recall that the functions φ_s in I are equal to $2\Omega^2 c$ times those used here and that the spatio-temporal spectral density N in I is equal to $K\mathcal{T}$ for thermal light of absolute temperature $\mathcal{T}$.

16. In I, Eq. (3.10), $\mathcal{F}$ should be $\mathcal{F}^2$.

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20. The first four of those tabulated have multiplicity 1; the rest have multiplicity 2 and must be taken twice in all summations.

Figure Captions

Fig. 1. Functions $f_i(\mathscr{D})$, $i = 1, 2, 3$, appearing in lower bounds on mean-square errors in unbiased estimates of absolute radiance, frequency, and position, versus $\mathscr{D} = N_s/\mathscr{N}MWT$. Dashed curve: rectangular spectrum; solid curves: Lorentz spectrum.

Fig. 2. Bounding functions $\mathscr{V}_i(\alpha;\mathscr{D})$ for mean-square errors in estimates of radiance $(i = 1)$ and frequency $(i = 2)$ of a circular object of uniform radiance, versus $\alpha = kab/R$. $\mathscr{D} = N_s/\mathscr{N}WT = 0.1, 1, 10.$

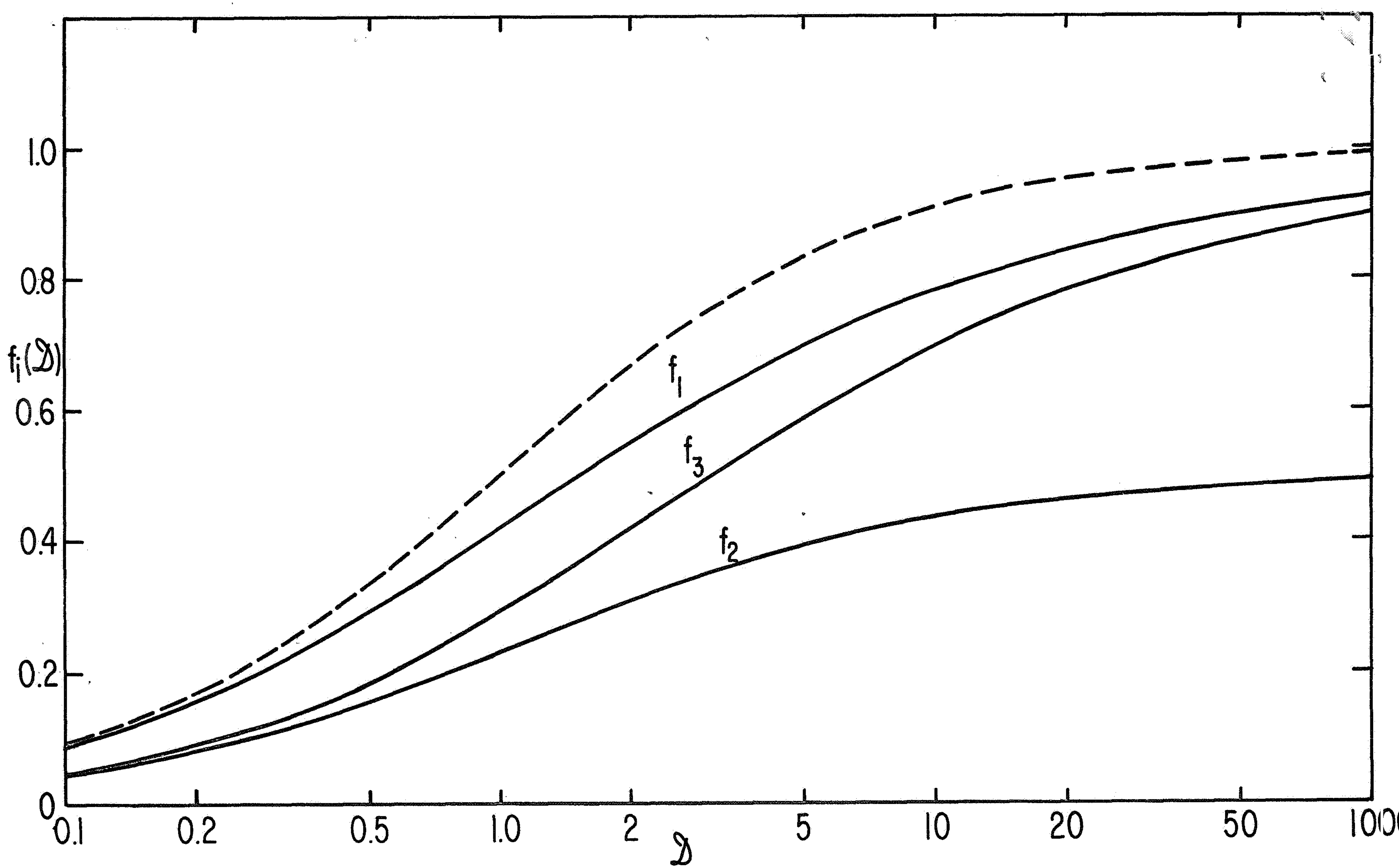

Fig. 1. Functions $f_i(\mathcal{D})$, $i = 1, 2, 3$, appearing in lower bounds on mean-square errors in unbiased estimates of absolute radiance, frequency, and position, versus $\mathcal{D} = N_s/\mathcal{N}MWT$. Dashed curve: rectangular spectrum; solid curves: Lorentz spectrum.

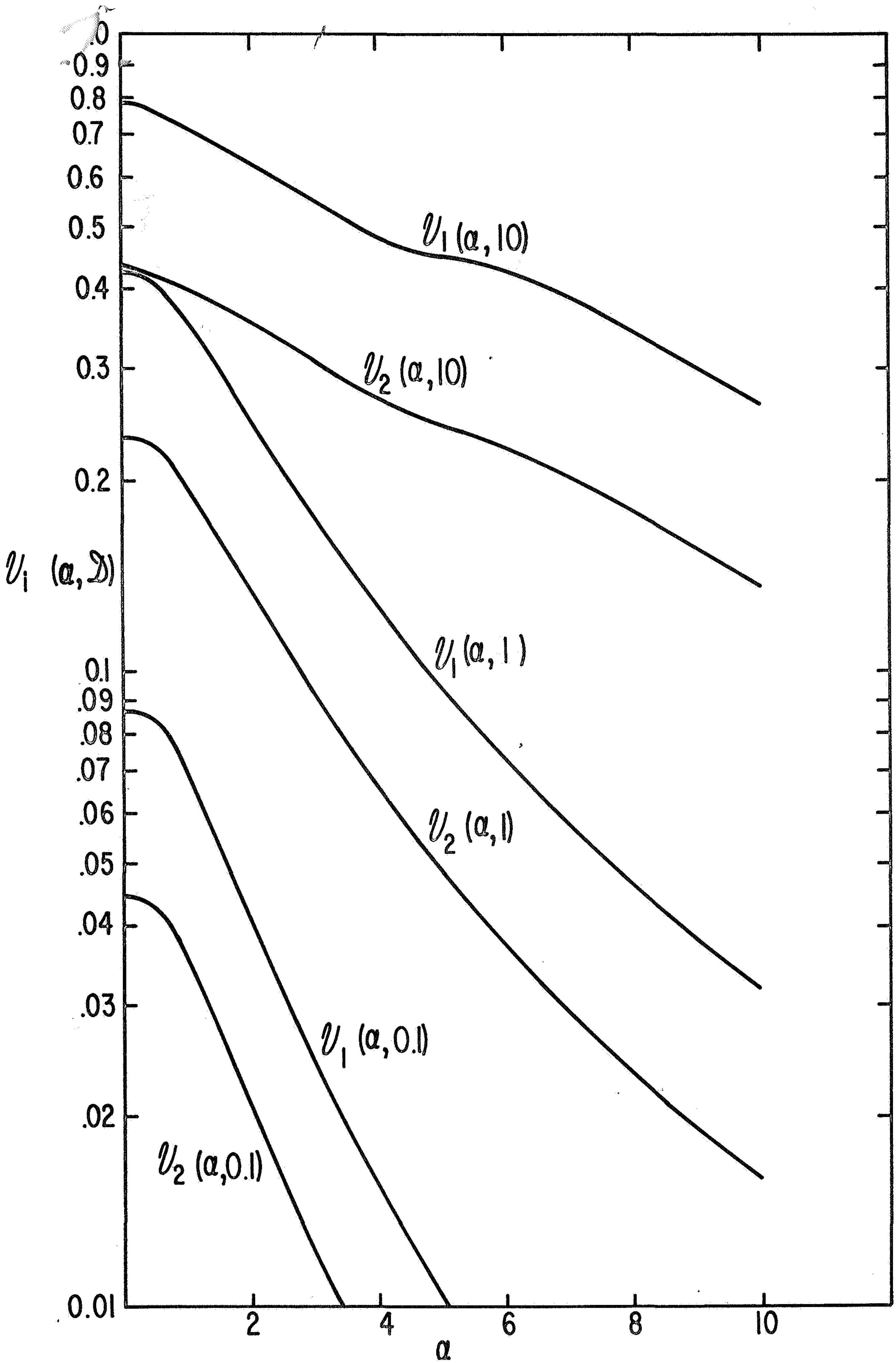

Fig. 2. Bounding functions $\mathcal{V}_i(\alpha;\mathcal{D})$ for mean-square errors in estimates of radiance (i = 1) and frequency (i = 2) of a circular object of uniform radiance, versus α = kab/R . $\mathcal{D} = N_s/\mathcal{N}WT$ = 0.1, 1, 10.