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M. SLEMROD AND E. F. INFANTE

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by

M. Slemrod

and

E. F. Infante

Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, Rhode Island

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1. Introduction

Elastic stability is usually discussed from strictly mechanical considerations. Recently, however, attempts have been made to analyze the influence of the usually neglected thermodynamic properties of elastic materials. More specifically, one may ask what effects the second law of thermodynamics has on the asymptotic stability of equilibrium of thermoelastic materials.

KOITER [1] has studied the general nonlinear thermoelastic problem, and for materials with internal friction he obtains asymptotic stability of the equilibrium solutions. ERICKSEN [2] has posed the question as to the asymptotic stability of the equilibrium solutions of elastic materials without imposing the assumption of internal friction. DAFERMOS [3] answered this question to some degree by obtaining a description of the states that the material approaches as $t \to \infty$.

This same question is studied here in a more general setting than was done by SLEMROD [4], but in the same spirit: it is shown that the results of [3] can be obtained as a simple application of <u>an invariance principle</u> for abstract dynamical systems [4,5].

2. Mathematical Preliminaries

The principal analytical tool to be used is a generalization due to HALE [5] for abstract dynamical systems of the well known <u>invariance principle</u> of LASALLE [6] for ordinary differential equations. The following brief presentation of this tool emphasizes notation and concepts to be used in studying the problem of thermoelastic stability.

Let $R^+ = [0,\infty)$ and let $\mathscr{B}$ be a Banach space with norm $\|\phi\|_{\mathscr{B}}$ for $\phi \in \mathscr{B}$. Then,

Definition 2.1. $\underline{u}$ is a dynamical system on a Banach space $\mathscr{B}$ if $\underline{u}$ is a function $u: R^+ \times \mathscr{B} \to \mathscr{B}$ such that $\underline{u}$ is continuous, $\underline{u}(0,\phi) = \phi$, $\underline{u}(t+\tau,\phi) = \underline{u}(t,\underline{u}(\tau,\phi))$ for all $t,\tau \geq 0$ and all ϕ in $\mathscr{B}$. The positive orbit $O^+(\phi)$ through ϕ in $\mathscr{B}$ is defined as $O^+(\phi) = \bigcup_{t\geq 0} \underline{u}(t,\phi)$. A point ψ in $\mathscr{B}$ is an equilibrium point if $O^+(\psi) = \psi$.

This set of definitions simply generalize familiar notions from the theory of differential equations to dynamical systems.

Definition 2.2. A set M in $\mathscr{B}$ is a positively invariant set of the dynamical system u if for each ϕ in M, $O^+(\phi) \subset M$.

Definition 2.3. A set M in $\mathscr{B}$ is an invariant set of the dynamical system $\underline{u}$ if for each ϕ in M there exists a function $\underline{U}(s,\phi)$, $\underline{U}(0,\phi) = \phi$ defined and in M for $s \in (-\infty,\infty)$ and such that $\underline{u}(t,\underline{U}(s,\phi)) = \underline{U}(t+s,\phi)$ for all $t \in R^+$.

Definition 2.2 is well known. The second definition is used to extend backward in time those solutions of the dynamical system which lie in an invariant set. It is clear that if a set M is invariant it is positively invariant but the converse is, in general, false.

Definition 2.4. If $\underline{u}$ is a dynamical system on $\mathscr{B}$ and V is a continuous scalar functional on $\mathscr{B}$, define the functional

$$\dot{V}(\phi) = \overline{\lim_{t \to 0}} \frac{1}{t} [V(u(t,\phi)) - V(\phi)].$$

Then

Definition 2.5. $V: \mathscr{B} \to R$ is said to be a Liapunov functional on a set G in $\mathscr{B}$ if V is continuous on $\overline{G}$, the closure of G, and if $\dot{V}(\phi) \leq 0$ for ϕ in G. Furthermore, denote by S the set $S = \{\phi$ in $\overline{G} | \dot{V}(\phi) = 0\}$ and let M be the largest invariant set in S for the dynamical system $\underline{u}$. With these definitions it is then possible to prove:

Theorem 2.1 (HALE [5]). Let $\underline{u}$ be a dynamical system on $\mathscr{B}$. If V is a Liapunov functional on G and a positive orbit $O^+(\phi)$ belongs to G and is in a compact set of $\mathscr{B}$ then $\underline{u}(t,\phi) \to M$ as $t \to \infty$.

It is self-evident that in applications to the problem of asymptotic stability of an equilibrium point ψ it is necessary to show that $M = \{\psi\}$. Moreover, it should be emphasized that the usefulness of this theorem in applications depends on the very relaxed assumptions imposed on the Liapunov functional V and its derivative $\dot{V}$. These conditions should be compared to the much stronger conditions imposed by standard asymptotic stability theorems (see, for example, PARKS [7]).

3. Constitutive Equations of Linear Thermoelasticity

A material point is identified by $\underline{x} = (x_1, x_2, x_3)$ in its state of equilibrium (no stresses, constant temperature $= \gamma_o$). The displacement field at some time t following an initial disturbance at time $t = 0$ is given by $\underline{u}(\underline{x},t)$ and the temperature deviation by $T(\underline{x},t)$; $\rho(\underline{x})$ denotes the density at $\underline{x}$ in the equilibrium state.

Let Ω be an open, bounded, connected set in E^3 which is properly regular [8]; let $\partial\Omega$ denote the boundary of Ω. The constitutive equations of thermoelasticity are taken then in the form

$$\rho\ddot{u}_i = (C_{ijk\ell}u_{k,\ell})_{,j} - (m_{ij}T)_{,j}, \qquad (3.1)$$

$$\rho C_D\dot{T} + m_{ij}\gamma_o\dot{u}_{i,j} = (K_{ij}T_{,j})_{,i}; \qquad (3.2)$$

where body forces and heat sources have been excluded. In these equations $C_{ijk\ell} = C_{jik\ell} = C_{k\ell ij}$, $m_{ij} = m_{ji}$, $K_{ij} = K_{ji}$ and C_D, ρ, $C_{ijk\ell}$, m_{ij} and K_{ij} are assumed to be smooth functions of $\underline{x}$.

Let now $t_o > 0$. By a classical solution of the mixed initial-boundary value problem in $\Omega \times (0,t_o)$ we mean a pair $(\underline{u},T)$ satisfying equations (3.1) and (3.2) together with the boundary conditions

$$\underline{u} = 0 \quad \text{on} \quad \partial\Omega \times (0,t_o) \quad \text{(closed boundary)}, \qquad (3.3)$$

$$T = 0 \quad \text{on} \quad \partial\Omega \times (0,t_o) \quad \text{(constant temperature)}; \qquad (3.4)$$

and with initial conditions

$$(\underline{u}(\underline{x},0),\dot{\underline{u}}(\underline{x},0), T(\underline{x},0)) = (\underline{u}_o(\underline{x}), \dot{\underline{u}}_o(\underline{x}), T_o(\underline{x})), \qquad (3.5)$$

where $\underline{u}_o(\underline{x})$, $\dot{\underline{u}}_o(\underline{x})$ and $T_o(\underline{x})$ are given functions on Ω.

4. The Thermoelastic Problem as a Dynamical System

In this section we show, by recalling some results of DAFERMOS [3], that the generalized solutions of the mixed initial boundary value problem described above can be viewed on an appropriate Banach space as a dynamical system (ZUBOV [9]). Once this is done, the application of Theorem 2.1 permits us to draw immediate conclusions on the asymptotic behavior of the solutions of our problem.

Consider the Sobolev spaces $W_2^{(k)}(\Omega)$ and $W_{20}^{(k)}(\Omega)$, $k = 1,2,\ldots$ (see, for instance SOBOLEV [10, 11], AGMON [12]). Assume that

$$\operatorname*{ess\,inf}_{\Omega} \rho(\underline{x}) > 0, \quad \operatorname*{ess\,inf}_{\Omega} C_D(\underline{x}) > 0, \tag{4.1}$$

$$K_{ij}\xi_i\xi_j \geq C_1\xi_i\xi_j, \quad C_1 > 0 \quad \text{constant}, \tag{4.2}$$

(a reformulation of the Clausius-Duhem inequality; LANDAU and LIFSHITZ [13]) and for all $v_i \in W_{20}^{(1)}(\Omega)$

$$\int_\Omega C_{ijk\ell} v_{i,j} v_{k,\ell} d\underline{x} \geq C_2 \int_\Omega v_{i,j} v_{i,j} d\underline{x}, \quad C_2 > 0 \quad \text{constant} \tag{4.3}$$

a general property of the tensor of the elastic modulii for infinitesimal elasticity (TRUESDELL and NOLL [14]).

Define now the spaces $H_o(\Omega) \approx W_{20}^{(1)}(\Omega) \times L_2(\Omega) \times L_2(\Omega)$ with norm $|(v_i, w_i, R)|_o^2 = \int_\Omega [\rho w_i w_i + C_{ijk\ell} v_{i,j} v_{k,\ell} + \frac{\rho C_D}{\gamma_o} R^2] d\underline{x}$ and $H(\Omega) = W_{20}^{(1)}(\Omega) \times W_{20}^{(1)}(\Omega) \times W_{20}^{(1)}(\Omega)$. Define the map $P\colon H_o(\Omega) \xrightarrow{\text{onto}} H_1(\Omega)$ sending $(v_i, w_i, R) \in H_o(\Omega)$ onto $(u_i, v_i, T) \in H_1(\Omega) \subset H(\Omega)$ where $(u_i, T) \in W_{20}^{(1)}(\Omega) \times W_{20}^{(1)}(\Omega)$ is defined by the solution of the system

$$\int_\Omega C_{ijk\ell} u_{k,\ell} \theta_{i,j} d\underline{x} = -\int_\Omega [\rho w_i \theta_i + m_{ij} T \theta_{i,j}] d\underline{x}$$

$$\int_\Omega K_{ij} T_{,j} D_{,i} d\underline{x} = \int_\Omega [\rho C_D R + m_{ij} \gamma_o v_{i,j}] D \, d\underline{x}$$

for every $D, \theta_i \in W_{20}^{(1)}(\Omega)$. The mapping P is linear, well defined on $H_o(\Omega)$ and one to one. Hence, defining $P_m = \overbrace{P \circ P \circ \cdots \circ P}^{m}$ let $H_m(\Omega)$ denote the range of the map P_m. It is clear that P_m^{-1} exists and maps $H_m(\Omega)$ onto $H_o(\Omega)$. Let $\psi \in H_m(\Omega)$ and define $|\psi|_m = |P_m^{-1}\psi|_o$. Then:

Lemma 4.1 (DAFERMOS [3]). H_m is a Banach space with norm $|\cdot|_m$. $H_o(\Omega) \supset H(\Omega) \supset \ldots \supset H_m(\Omega)$ algebraically and topologically. Furthermore, $H_m(\Omega)$ is dense in $H_\ell(\Omega)$ for $m > \ell$ and the imbedding $I: H_m(\Omega) \to H_\ell(\Omega)$ is compact.

Let us now define appropriately a generalized solution of our problem:

Definition 4.1. $(u_i, \dot{u}_i, T)$ will be called a generalized solution of (3.1) - (3.5) on $\Omega \times (0,t_o)$ if for all smooth test functions (v_i, R) with compact support on Ω and vanishing on $\Omega \times 0$.

$$\begin{aligned}\int_o^{t_o}\int_\Omega \{ &(t-t_o)[\rho\dot{u}_i\dot{v}_i - C_{ijk\ell}u_{k,\ell}\dot{v}_{i,j} + m_{ij}T\dot{v}_{i,j} + \\ &+ \frac{\rho C_D}{\gamma_o} T\dot{R} + m_{ij}u_{i,j}\dot{R}] + \rho\dot{u}_i\dot{v}_i + \rho\frac{C_D}{\gamma_o} TR + \\ &+ m_{ij}u_{i,j}R - \frac{1}{\gamma_o}\int_o^t (K_{ij}R_{,i})_{,j}T\, dt\} d\underline{x}dt = \\ &= -t_o\int_\Omega[\rho\dot{u}_{o_i}\dot{v}_i|_{t=0} + \frac{\rho C_D}{\gamma_o}T_oR|_{t=0} + m_{ij}u_{o_{i,j}}R|_{t=0}]d\underline{x}\end{aligned} \tag{4.4}$$

With this definition it follows that:

Theorem 4.1 (DAFERMOS [3]). Under assumptions (4.1) - (4.3) the triple $(u_i, \dot{u}_i, T)$ describes a dynamical system on $H_m(\Omega)$, $m = 0,1,2,\ldots$, where $(u_i, \dot{u}_i, T)$ is the generalized solution to the equations of linear thermoelasticity satisfying equation (4.4). Furthermore, for t in $(0,t_o)$

$$\begin{aligned}|(u_i, \dot{u}_i, T)(t)|_m^2 &+ \frac{1}{\gamma_o}\int_o^t\int_\Omega K_{ij}T^{(m)}_{,i}T^{(m)}_{,j}\,d\underline{x}d\tau = \\ &= |(u_{i_o}, \dot{u}_{i_o}, T_o)|_m^2\end{aligned} \tag{4.5}$$

where $T^{(m)}(\underline{x},t)$ denotes the generalized mth derivative in time of $T(\underline{x},t)$.

5. Stability Analysis

The problem of thermoelastic stability has now been put in a setting appropriate for the application of Theorem 2.1, which allows us to obtain stability results in a simple and direct manner.

For this purpose, fix $m \geq 1$. Then, by Theorem 4.1 and (4.5) it follows that for any initial data in $H_m(\Omega)$ the trajectory $(u_i, \dot{u}_i, T)(t)$ will lie in a bounded set of $H_m(\Omega)$ for any $t > 0$. Hence, by Lemma 4.1 the trajectory remains in a compact set G of $H_\ell(\Omega)$, $\ell < m$. But then all the hypothesis of Theorem 2.1 are met with $\mathcal{B} = H_\ell(\Omega)$. For simplicity let $\ell = 0$ and $V = |(u_i, \dot{u}_i, T)|_o^2$. From (4.2) and (4.5) it immediately follows that $\dot{V} = -\frac{1}{\gamma_o}\int_\Omega K_{ij}T_{,i}T_{,j}d\underline{x} \leq -c_3\|T\|^2_{L_2}$, $c_3 > 0$; therefore the set S is the set $S = \{(u_i, \dot{u}_i, T) \in H_o(\Omega) \mid \|T\|_{L_2} = 0\}$. Let now M^+ be the largest positively invariant set in S. By the definition of generalized solution (4.4) it follows that $M^+ = \{(u_i, \dot{u}_i, T) \in H_o(\Omega) \mid \|T(\cdot,t)\|_{L_2} = 0 \text{ for } t \geq 0\}$. Choosing now $v_i \equiv 0$ in (4.4) it follows that for $(u_i, \dot{u}_i, T) \in M^+$ it is necessary that

$$\int_o^{t_o}\int_\Omega \frac{d}{dt}[(t-t_o)R]m_{ij}u_{i,j}d\underline{x}dt = -t_o\int_\Omega m_{ij}u_{o_{i,j}}R\, d\underline{x}$$

for every test function R. Choosing this function as $R(\underline{x},t) = \dfrac{\omega(t)\eta(\underline{x})}{t-t_o}$ where $\eta(\underline{x})$ is an arbitrary test function and $\omega(t)$ is the C^∞ "bump" function of Serrin [15], it follows that

$$\int_\Omega m_{ij}(\underline{x})u_{i,j}(\underline{x},t)d\underline{x} = \int_\Omega m_{ij}(x)u_{oi,j}(x)d\underline{x}, \quad t \geq 0$$

for $(u_i, \dot{u}_i, T)$ in M^+. Hence, Theorem 2.1 applied to this context yields the desired result:

<u>Theorem 5.1</u>: For any initial data $(u_{o_i}, \dot{u}_{o_i}, T)$ in H_m, $m \geq 1$, and under assumptions (4.1) - (4.3) $(u_i, \dot{u}_i, T)(t)$ approaches asymptotically the set $\{(v_i, \dot{v}_i, R) \in W_{20}^{(1)}(\Omega) \times L_2(\Omega) \times L_2(\Omega) \mid \int_\Omega m_{ij}(\underline{x})v_{i,j}(\underline{x},t)dx = \int m_{ij}(\underline{x})u_{o_{i,j}}(\underline{x})d\underline{x}$, $t \geq 0$, $R = 0\}$ in the norm of the space $W_{20}^{(1)} \times L_2(\Omega) \times L_2(\Omega)$.

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