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ALGEBRAIC METHODS

FOR

CONTROL SYSTEM ANALYSIS AND DESIGN

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Final Report on the National
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INTRODUCTION

The research on algebraic methods for analysis and design of control systems for the past two years, April, 1967 - April, 1969, can be conveniently divided into three major areas:

1. Linear System Design
2. Absolute Stability Analysis of Nonlinear Systems
3. Approximate Nonlinear Analysis

In this final report, the obtained results will be outlined for each area separately. Since most of the research was publishable, only a brief summary for each topic is given and the list of the references is included. For convenience, pertinent papers are added at the end of the report. The references that are not included in the report can be obtained on request from the investigators.

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CONTENT

1. LINEAR SYSTEM DESIGN

1.1. Multiparameter Analysis and Design.

The Parameter Method was basically a two parameter method, with some possible extensions to three parameter cases. Many problems are multiparameter problems for which analysis and design techniques are lacking. Investigations leading to multiparameter techniques have been initiated, starting with three dimensional studies as a first step. Worthwhile results have been obtained (1). Further extensions are concerned with geometric approach to the multiparameter design. A paper on basic geometric relationships (2) has been accepted for presentation at the Eighth International Symposium on Space Technology and Science in Tokyo. The geometric approach led to some algebraic techniques which appear to be useful for three, four and five parameter problems. A paper (3) presenting these results is under preparation.

1.2. Development of General Purpose Computer Programs.

The well known relationships between the coefficients of a polynomial and its roots can be written down in matrix form, and the coefficients are functions of the system parameters. This matrix relationship defines an N-dimensional parameter space, which appears to be worth studying in the future. Manipulation of the matrices permits reduction of the N-dimensional space to a two-parameter space. The resulting matrix equations are readily programmed using existing matrix subroutines, which are normally available in any computation center.*

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* A specific program has been developed for use with the NAVAL POSTGRADUATE SCHOOL IBM/360 System.

The program computes the parameter plane curves for cases in which the parameters are related linearly in the coefficients, and also for cases in which the parameters appear as a product. In addition, the program provides values of <u>all</u> roots of the polynomial at preselected points. Further, it can compute the sensitivities of the roots to parameter variations whenever desired.

The theoretical results have been presented in a short paper at Princeton University (4) and a full length paper is to be presented at the Eighth International Symposium on Space Technology and Science (5) in Tokyo and will be published in the proceedings of that Symposium.

A second general program** has been developed for the computation of Frequency Response as a function of parameters. The original development of the theory (6) has been extended considerably and future publications are expected as discussed in a later section.

The frequency response program computes certain characteristics of the frequency response as a function of one or two parameters. In all seven different kinds of functional relationships may be obtained (one of which is the well known Bode diagram). For example the program computes all parameter pairs that will guarantee a given magnitude ratio at a designated frequency; a special case of this would be a constant bandwidth curve, i.e., a locus of all points on the parameter plane which provide a magnitude ratio of -3db at any specified frequency. Details of the program are not given here but are available from the authors.

** This program also was developed for use with the NAVAL POSTGRADUATE SCHOOL IBM/360 System. (4)

1.3. Linear Time-Varying Systems.

It has been shown (7) that the parameter method along with the Krylov-Bogoliubov method can be used to predict transient responses of linear systems where parameters vary in time. Moreover, a time-varying parameter can be deliberately introduced into the system to improve its transient behavior. Computer simulation results are given to indicate the accuracy of the presented technique and its value as a design aid.

1.4. Squared-Error Minimization with Stability Constraints.

It is a well known fact that a minimization of the mean-squared error in linear closed-loop control systems may result in a poorly damped system response to deterministic inputs. To improve the results, it is suggested (8) to minimize the same performance index with a relative stability constraint so that all the characteristic roots have the relative damping coefficient greater or at least equal to a prescribed value. A solution of the constrained optimization was given in the parameter plane. Future work will be devoted to extension of this approach to multi-variable systems where it can lead to fundamental results in squared-error optimization of dynamic systems.

1.5. Frequency Response on the Parameter Plane.

The development of a program providing parameter plane studies of frequency response has provided the capability for investigating a number of interesting situations. A number of linear systems have been studied both in the analysis sense and with attention to design. It has also been determined that the curve families can be used to analyze and design nonlinear open loop systems and nonlinear closed loop systems. Papers presenting details are in preparation, but are not sufficiently advanced to permit inclusion with this report.

1.6. Analytic Design of Compensation.

It has been shown (9) that the basic Mitrović relationships can be combined with equations expressing performance specifications and the set of simultaneous equations thus obtained can be solved to determine a set of parameter values that provide the desired conditions. The original work was carried out with longhand solution of the simultaneous nonlinear algebraic equations on low order problems. A computer approach has been desired for higher order systems. Work is continuing with gradual improvement and it is hoped that a useful program will result within a year.

1.7. Specific Design Problems.

The Frequency Response Parameter Plane has been applied to a number of problems in the design of active and passive circuits. Most of these applications (10-13) have been too specialized for publication. One investigation (14) "Analysis and Design of Oscillators with Parameter Plane Methods" has been presented at the 1969 Princeton Conference on Information Sciences and Systems, and will be published in the Proceedings of that conference.

We have also applied the basic Parameter Plane to the study of Phase Locked Oscillators (15). A paper has been prepared and submitted to the IEEE Transactions on Communications Technology.

2. ABSOLUTE STABILITY ANALYSIS OF NONLINEAR SYSTEMS

2.1. Parameter Analysis.

In the absolute stability analysis of nonlinear systems (1), the nonlinear is not completely specified and it should only belong to a certain class of functions. On the other hand, the parameters of the linear part are specified numerically, which appears to be a quite unrealistic assumption. Therefore, a modification of the absolute stability definition was introduced (16) which relaxes the conditions on the linear part and allows system parameters to deviate from their nominal values. Both the analytical (16) and the graphical (17, 18) procedures were proposed to determine the absolute stability regions in the parameter space. In particular, a simple analytic test for absolute stability was derived which is based upon the Routh test (18-20). The test can be used to test positive realness (20).

2.2. Regions of Absolute Stability.

In posing the absolute stability problem, unrealistic assumptions are made regarding the structure of the differential equations in order to achieve the analytical simplicity of global stability. It is clear that physical systems are not globally stable, nor is there any practical reason to make them so: it is desirable only to make the region of asymptotic stability sufficiently large. Consequently, a modification of the absolute stability problem was proposed (21) to include cases with finite regions of asymptotic stability. In addition, exponential estimates are given, thus providing additional information about how fast the solutions approach the equilibrium position. Future work should be to extend the results to multiple nonlinearities.

2.3. Regions of Exponential Boundedness.

In the context of absolute stability, forced nonlinear systems were considered in which the nonlinear characteristic violates the sector condition in the neighborhood of the origin. It is shown (22) that the satisfaction of the Popov frequency condition and the boundedness of the forcing function and the nonlinearity (where the sector condition is violated) imply the exponential boundedness of the system motion. Quadratic Liapunov functions are used to obtain estimates of the region which system motions enter sooner than an exponential. Future research should be devoted to multiple nonlinearities.

2.4. Discrete-Time Systems.

By using the bilinear transformation, the absolute stability test of (18-20) was extended to discrete-time nonlinear systems (23). The research currently under way is devoted to conditions under which a discrete-time system is exponentially absolutely stable. It seems that the analytical test can be extended to verify these conditions.

3. APPROXIMATE NONLINEAR ANALYSIS

3.1. Nonlinear Transients.

Previous investigations (24, 25) have provided methods for calculating transients using the parameter plane and describing functions. It has been shown that reasonably accurate results obtain for systems with one or two nonlinearities when only one pair of complex roots need be considered. Extensions have been made to cases with real roots, and with combinations of complex roots and real roots. Under this grant these methods have been applied to cases in which two pairs of complex roots contribute significantly to the transients. No final conclusions are available, but the accuracy of the results does not seem commensurate with the labor required for computation.

3.2. Singular Lines and Self Adaptive Systems.

The theory of singular lines (1) has been developed in some detail, and methods have been developed for designing compensation that produces singular conditions in the system (26-28). Some of these results are being expanded for publication. It has been shown that the singular line condition, when it exists in a system, may be used for self-adaptive purposes. A paper presenting these results has been presented (29) at the IFAC Symposium on Sensitivity and Self Adaptivity in Dubrovnik, Yugoslavia, and was published in the proceedings of that meeting.

Recent developments (28) in the technique of designing systems to have singular lines have led to a need for continued studies in this area.

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CHARACTERISTICS OF THREE PARAMETER SPACE WITH APPLICATIONS TO SYSTEM SYNTHESIS*

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Abstract

Classical methods for analysis and design of systems normally restrict consideration to one (at most two) variable parameters, but most practical problems require consideration of three or more parameters. For the case of linear systems in which the coefficients of the characteristic equation are linear functions of three parameters (commonly encountered practically) a rectangular coordinate system may be used to define a three dimensional parameter space. Simple geometric correlations exist between specified roots and the sets of parameter values required to obtain them. A real root (one point on the s-plane) maps into a plane in parameter space. A pair of complex conjugate roots map into a straight line in parameter space.

Methods are developed which permit quantitative analysis and design of such systems using graphical techniques in two dimensions. These lead to an algebraic approach which permits design to obtain specified roots exactly, but with a capability for adjusting the locations of non-specified roots. Numerical illustrations are provided which demonstrate the procedure in detail.

1. Introduction

The design of dynamic systems requires that numerical values be chosen for many parameters. Classical design methods (frequency response and root locus) consider the effects of only one adjustable parameter, thus all other parameters are usually determined by trial and error methods which rely heavily on experience. Mitrovic's (Ref.2) introduction of a method for considering two coefficients of the characteristic polynomial provided means for rigorous study of some classes of two parameter problems, and extensions by Siljak et.al (Ref.4,5,6,7) greatly broadened the scope of coefficient plane - parameter plane methods, but still restricted them to two parameter problems.

The use of multi-parameter spaces for analysis and design is not easily justified because in general, we do not know how to process the data to obtain the kind of results needed for engineering work.

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This paper extends the two parameter plane to a three dimensional parameter space by showing the existence of certain types of geometric properties for some of the surfaces in three parameter space, and by applying these results to the analysis and design of broad classes of feedback control systems.

2. Mathematical Bases

Given a characteristic polynomial (linear)

$$F(s) = \sum_{k=0}^{n} a_k s^k = 0 \tag{1}$$

and expressing s in polar coordinates

$$s = \zeta\omega_n \mp j\, \omega_n\sqrt{1-\zeta^2} = \omega_n(\cos\theta + j\sin\theta) = \omega_n e^{j\theta} \tag{2}$$

where $\theta = \cos^{-1}(-\zeta)$, then

$$s^k = \omega_n{}^k\, e^{jk\theta} = \omega_n{}^k(\cos k\theta + j\sin k\theta) \tag{3}$$

By definition Chebyshev functions of the first and second kind are:

$$\begin{aligned} T_k(\zeta) &= \cos k\theta = \cos(k\cos^{-1}\zeta) \\ U_k(\zeta) &= \frac{\sin k\theta}{\sin\theta} = \frac{\sin(k\cos^{-1}\zeta)}{\sin(\cos^{-1}\zeta)} \end{aligned} \tag{4}$$

from which

$$s^k = \omega_n{}^k\left[(-1)^k T_k(\zeta) + j\sqrt{1-\zeta^2}(-1)^{k+1}\, U_k(\zeta)\right] \tag{5}$$

Substituting in Eq.(1) and requiring that reals and imaginaries become zero independently provides:

$$\begin{aligned} &\sum_{k=0}^{n} a_k\, \omega_n{}^k(-1)^k T_k(\zeta) = 0 \\ &\sum_{k=0}^{n} a_k\, \omega_n{}^k(-1)^{k+1}\, U_k(\zeta) = 0 \end{aligned} \tag{6}$$

$$\text{but } T_k(\zeta) = \zeta\left[U_k(\zeta)\right] - \left[U_{k-1}(\zeta)\right] \tag{7}$$

which permits reduction of Eq.(6) to

$$\begin{aligned} &\sum_{k=0}^{n} a_k\, \omega_n{}^k(-1)^k\, U_{k-1}(\zeta) = 0 \\ &\sum_{k=0}^{n} a_k\, \omega_n{}^k(-1)^k\, U_k(\zeta) = 0 \end{aligned} \tag{8}$$

Now the coefficients are defined to be linear functions* of three parameters

$$a_k = b_k\alpha + c_k\beta + d_k\gamma + f_k \tag{9}$$

Considering α, β, γ to be the independent variables, Eqs.(8) may be rewritten as

$$\begin{aligned} B_1\alpha + C_1\beta + D_1\gamma + F_1 &= 0 \\ B_2\alpha + C_2\beta + D_2\gamma + F_2 &= 0 \end{aligned} \tag{10}$$

where

$$\begin{aligned}
B_1 &= \sum_{k=0}^{n}(-1)^k b_k \omega_n^{\,k}\, U_{k-1}(\zeta) & B_2 &= \sum_{k=0}^{n}(-1)^k b_k \omega_n^{\,k}\, U_k(\zeta) \\
C_1 &= \sum_{k=0}^{n}(-1)^k c_k \omega_n^{\,k}\, U_{k-1}(\zeta) & C_2 &= \sum_{k=0}^{n}(-1)^k c_k \omega_n^{\,k}\, U_k(\zeta) \\
D_1 &= \sum_{k=0}^{n}(-1)^k d_k \omega_n^{\,k}\, U_{k-1}(\zeta) & D_2 &= \sum_{k=0}^{n}(-1)^k d_k \omega_n^{\,k}\, U_k(\zeta) \\
F_1 &= \sum_{k=0}^{n}(-1)^k f_k \omega_n^{\,k}\, U_{k-1}(\zeta) & F_2 &= \sum_{k=0}^{n}(-1)^k f_k \omega_n^{\,k}\, U_k(\zeta)
\end{aligned} \tag{11}$$

Equations(1)and(10)define the characteristics of the α, β, γ parameter space, and may be used to determine surfaces with known geometric properties.

3. Real Root Surfaces

For α, β, γ independent variables, Eq. (1) may be rewritten as

$$\alpha B(s) + \beta C(s) + \gamma D(s) + F(s) = 0 \tag{12}$$

The locus of all points in α, β, γ space for which (12) has a real root at $s = -\sigma_1$ is given by

$$\alpha B_R + \beta C_R + \gamma D_R + F_R = 0 \tag{13}$$

where B_R, C_R, D_R and F_R are the constant numbers obtained by substitution of $-\sigma_1$ for s in (12). But Eq. (13) is the equation of a plane in α, β, γ space; i.e., for a polynomial to have a specified real root, the parameters α, β, γ must be chosen such that the point thus defined lies on a particular plane in α, β, γ space.

* Nonlinear functions also occur in practice. These have been investigated for the two parameter case, but are beyond the scope of this paper.

The intercepts of the real root plane with the α, β, and γ axes are given by

$$\begin{aligned} &\alpha \text{ - axis intercept at } \alpha = -F/B \triangleq a \\ &\beta \text{ - axis intercept at } \beta = -F/C \triangleq c \\ &\gamma \text{ - axis intercept at } \gamma = -F/D \triangleq b \end{aligned} \tag{14}$$

Using these intercepts the plane can be defined graphically in a three dimensional drawing by its three traces in the coordinate planes as shown in Fig. 1. To obtain a useful geometric construction which provides two dimensional representation of lines in the root plane, note that generation of an auxiliary plane parallel to the α β coordinate plane and at coordinate $\gamma = \gamma_1$ defines a trace MN in this new plane such that the trace MN is parallel to the trace ac in the α β coordinate plane. Rotation of the axis into the $\alpha\beta$ plane such that γ and β are collinear but in opposite directions (i.e., $+ \gamma = -\beta$) provides a two dimensional representation of the root plane as in Fig.2.

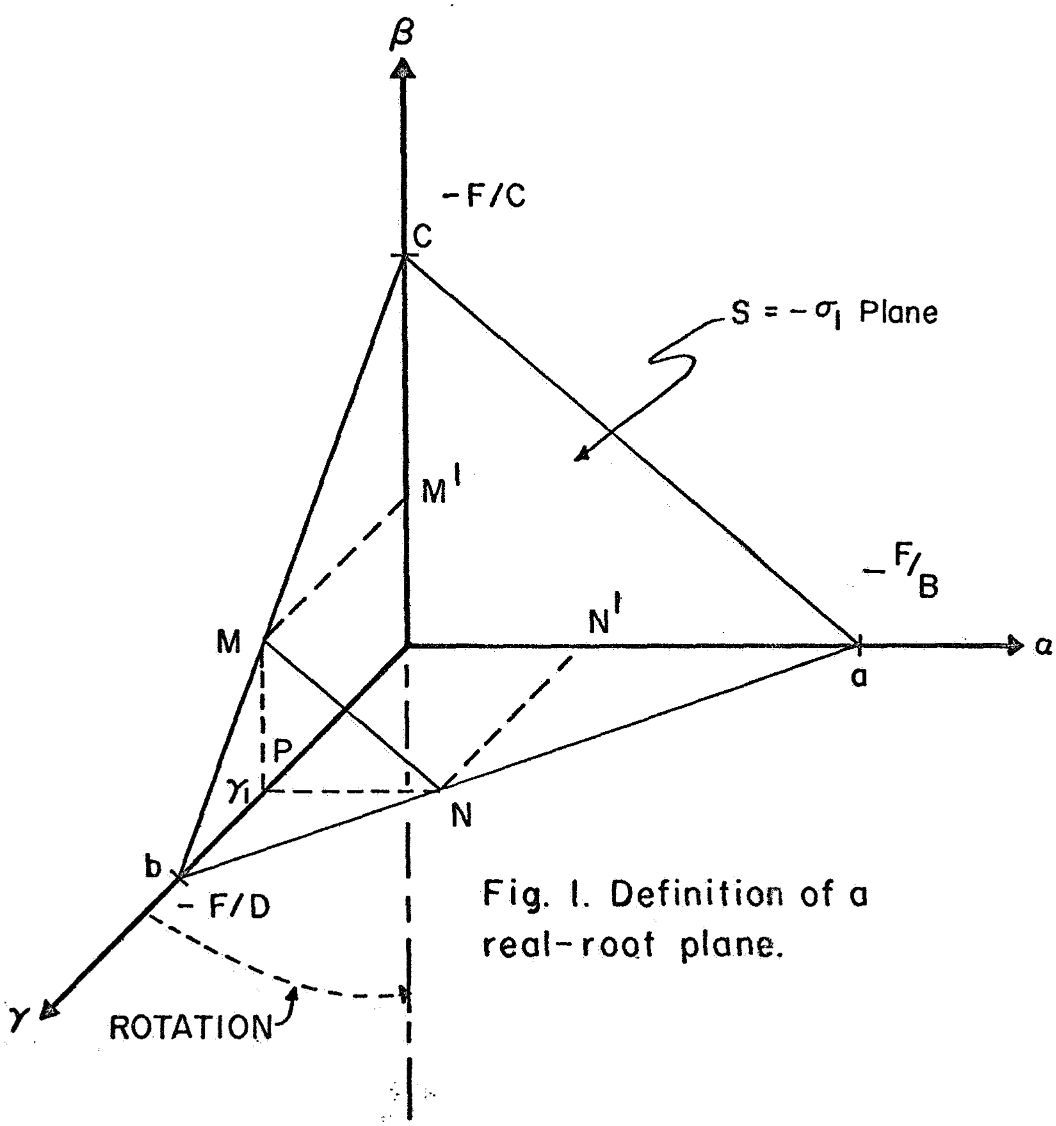

Fig. I. Definition of a real-root plane.

Note that on Fig. 1 the trace of the construction plane at $\gamma = \gamma_1$ intersects the root plane trace at point N in the $\alpha\gamma$ plane, and note particularly that the trace of the construction plane in the $\alpha\gamma$ plane is the line PN, which is parallel to the α - axis. Thus, on Fig. 2, locate the value of γ (say γ_1) at which a line in the root plane is desired and construct a line through this point but parallel to the α-axis, then the construction line intersects the ab trace at point N. A second construction line through N but parallel to the γ-axis locates which is the equation of the real root plane for s = -5. The graphical portrayal of the traces is shown on Fig. 3. Choice of parameter values is virtually unlimited, since the only constraint is a root at s = -5, and three parameters are adjustable. Assume that the range of adjustment for α and β is limited, and it is decided to try $\alpha = 0.5$, $\beta = 0.5$. A construction line is drawn through the desired point parallel to the $\alpha\beta$ trace, locating point c, this is projected parallel to the γ axis to intersect the $\alpha\gamma$ trace, thence projected to the γ axis to determine the required value of γ.

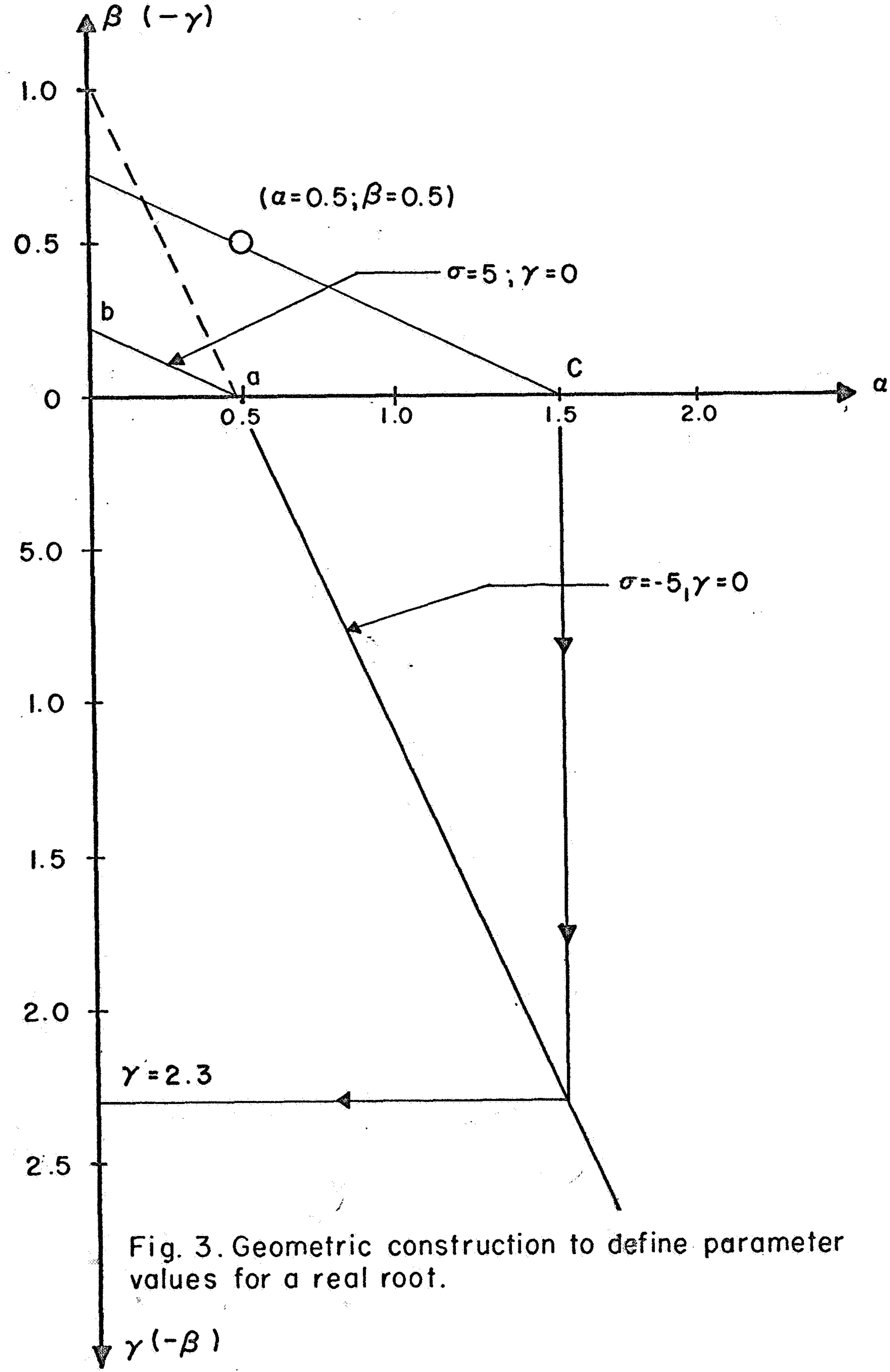

Fig. 3. Geometric construction to define parameter values for a real root.

point N' on the α-axis, and a line parallel to the trace ac, and passing through N' gives the line M' N' which is the projection of the trace MN (Fig.1) on the $\alpha\beta$ plane. The α-β coordinates of any point on M' N' may be read from the coordinate scales on Fig. 2, and if these values are used together with $\gamma = \gamma_1$, the characteristic equation must have a root at $s = -\sigma_1$. This construction permits projection on the $\alpha\beta$ plane of a real root line (for $s = -\sigma_1$) at any chosen value of γ. However, repeated use of the construction is not necessary since the relationships are linear and therefore proportional, after construction of only two lines as ac at $\gamma = 0$ and M' N' at $\gamma = \gamma_1$, all other lines representing this root are parallel to the constructed lines and with spacings proportional to the selected values of γ.

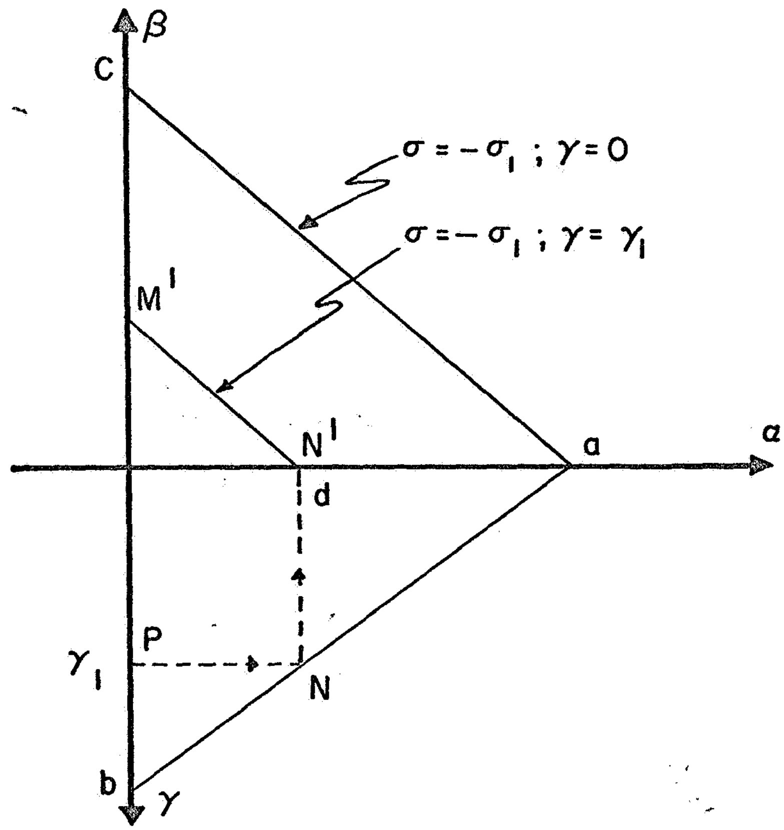

Fig. 2. Two Dimensional representation of the real root plant.

This construction method can be used to determine values of α, β, γ which will provide a desired real root for a control system. Consider a regulator with characteristic equation:

$$s^4+(2 \times 10^6K_{10}+ 1350)s^3+(3 \times 10^8K_{10}+ 4 \times 10^8K_{11}+3.85 \times 10^5)s^2+ \\ (10^{10}K_{10}+ 2 \times 10^{10}K_{11}+36 \times 10^6)s + 2 \times 10^{10}K_aK_p+2.1 \times 10^{10}= 0 \qquad (15)$$

Let $K_{10} \triangleq \alpha$, $K_{11} \triangleq \beta$, $K_aK_p \triangleq \gamma$ and rearrange to

$$(2 \times 10^6s^3+3 \times 10^8s^2+10^{10}s)\alpha +(4 \times 10^8s^2+ 2 \times 10^{10}s)\beta + \\ 2 \times 10^{10}\gamma+(s^4+1350s^3+3.85 \times 10^5s^2+36 \times 10^6s+2.1 \times 10^{10}) = 0 \qquad (16)$$

Next choose the desired real root value; let $s = -5$. Then (16) becomes

$$-4275\alpha - 9000\beta + 2000\gamma + 2.082.95 = 0 \qquad (17)$$

It it is desired to specify two real roots, then each root value is substituted into the characteristic equation resulting in an equation of a plane. The intersection of these two planes is a straight line, thus the only values of α, β, γ which provide both real roots are the coordinates of points on this line. The projection of the desired real root line onto the $\alpha\beta$ plane may be determined by graphical projection or by simultaneous equation solution. Proceeding graphically, find the point H at which the line pierces the $\alpha\beta$ plane by setting $\gamma = 0$ in both of the plane equations, and solve the resulting two equations in α and β for the coordinates of the pierce point. In like manner find the pierce point G in the $\alpha\gamma$ plane by setting $\beta = 0$ in the two equations and solving. Project this latter point onto the α-axis to obtain point F. The line FH is the desired projection of the line along which the real root planes intersect. The $\alpha\beta$ coordinates of any point on this line define two of the three required parameter values. The value of γ is determined by linear interpolation (or extrapolation) since a γ scale is readily established along the line, i.e., at point H it is known that $\gamma = 0$, and at point F the value of γ is the same as the γ coordinate of point G.

Alternately the $\alpha\beta$ pierce point is found as previously indicated by setting $\gamma = 0$ and solving the resulting equations for α and β. A second point on the desired intersection line is found by choosing another value for γ and again solving for α and β. A straight line on the $\alpha\beta$ plane which passes through these points is the desired projection.

If three real roots are specified each root defines an equation such as (13). Thus there are three equations in three parameters and an algebraic solution provides unique values of α, β, γ.

In general these methods guarantee that roots will be obtained at the specified locations. They do not guarantee that such roots will be dominant. Neither do they give any information as to the location of any unspecified roots. These factors must be considered when the methods are used for system synthesis.

4. Complex Roots

For a pair of complex roots at $s = -\zeta\omega_n \mp j\omega_n\sqrt{1-\zeta^2}$, application of equations (8) provide a pair of linear equations (10). Each of these equations defines a plane, and the intersection of these planes is a straight line. Therefore a pair of complex conjugate points in the s-plane map into a straight line in the α, β, γ parameter space. This straight line can be projected onto the $\alpha\beta$ plane, with γ scale for the line, by exactly the same procedure as used to project the intersection of two real root planes. Thus the values of parameters α, β, γ required to obtain a specified pair of complex roots can be read from a two dimensional plot.

If a third root (real root) is also specified, then it defines a third equation. Simultaneous solution of the three equations gives a unique result for α, β, γ.

5. Illustrative Design, Complex Roots

For the system of Fig. 4a the characteristic equation is

$$s^4 + 16.5s^3 + (\alpha+73)s^2 + (\beta+82.5)s + (\gamma+25) = 0 \tag{18}$$

where $\alpha = KK_a$, $\beta = KK_t$, $\gamma = KK_1$. For a desired root at $\zeta = 0.5$ and $\omega_n = 2.0$, the equations of the two planes become

$$\begin{aligned} 4\alpha - \gamma + 135 &= 0 \\ 4\alpha - 2\beta + 110 &= 0 \end{aligned} \tag{19}$$

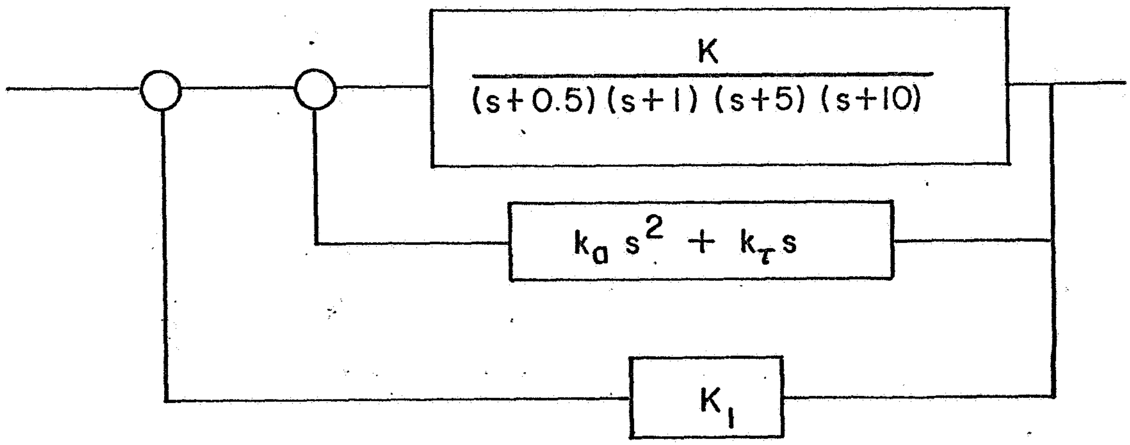

Fig. 4a. Block diagram of a three parameter system.

The equation $4\alpha-2\beta+110=0$ defines a plane parallel to the γ-axis, and is also the equation of its intersection with the α-β plane. Because this plane contains the desired line, the projection of this line on the $\alpha\beta$ plane is given by the same equation, and the γ scale on this line is defined by the equation of the other plane, i.e., $4\alpha-\gamma+135=0$. The desired line on the α-β plane is given on Fig.4b.

In terms of design if it is desired that all gains be positive, then a point must be chosen on that portion of the line which lies in the first quadrant on Fig.4b. However, it might be desirable to permit positive acceleration feedback (K_a negative), in which case the operating point could be chosen in the second quadrant. Any choice along this line defines values for α, β and γ which guarantee roots at $\zeta=-0.5$ and $\omega_n=2.0$. Values of the two unspecified roots are not indicated, and must be checked because their locations change with values of α, β and γ, and some choices of these parameters could provide unspecified roots with unsuitable values.

6. An Algebraic-Graphic Extension of the Design Procedure

A system with three independent parameters, α, β, γ, has three degrees of freedom. When specifications require that a certain pair of roots are to be obtained, two degrees of freedom must be used to guarantee this. The third degree of freedom may then be used to adjust the locations of all other roots. Because of this it is possible to define a functional relationship between all unspecified roots and any one of the parameters. The procedure is as follows:

a. Write the characteristic equation of the system and also the quadratic defined by the specified roots.

b. Divide the quadratic into the characteristic equation obtaining a quotient and two remainder terms.

c. In order for the specified roots to be roots of the characteristic equation the remainder terms must be identically zero for any s. Therefore the two remainder terms may each be set to zero, and together with the quotient they form three equations in three parameters.

d. Substitution of the remainder terms into the quotient reduce the quotient to a polynomial in s with <u>one</u> parameter. Then the quotient may be used to define a root locus which shows the locations of the unspecified roots as functions of one parameter.

e. The value of the third parameter is then chosen from inspection of the root locus.

Applying this procedure to the illustration of the preceding section, the characteristic equation is (18) and for roots at $\zeta = 0.5$

and $\omega_n = 2.0$ the quadratic is

$$s^2 + 2s + 4 = 0 \tag{20}$$

Dividing (20) into (18) the quotient is

$$s^2 + 14.5s + \alpha + 40 = 0 \tag{21}$$

and the remainder equations are

$$\begin{aligned} -4\alpha + \gamma - 135 &= 0 \\ -2\alpha + \beta - 55.5 &= 0 \end{aligned} \tag{22}$$

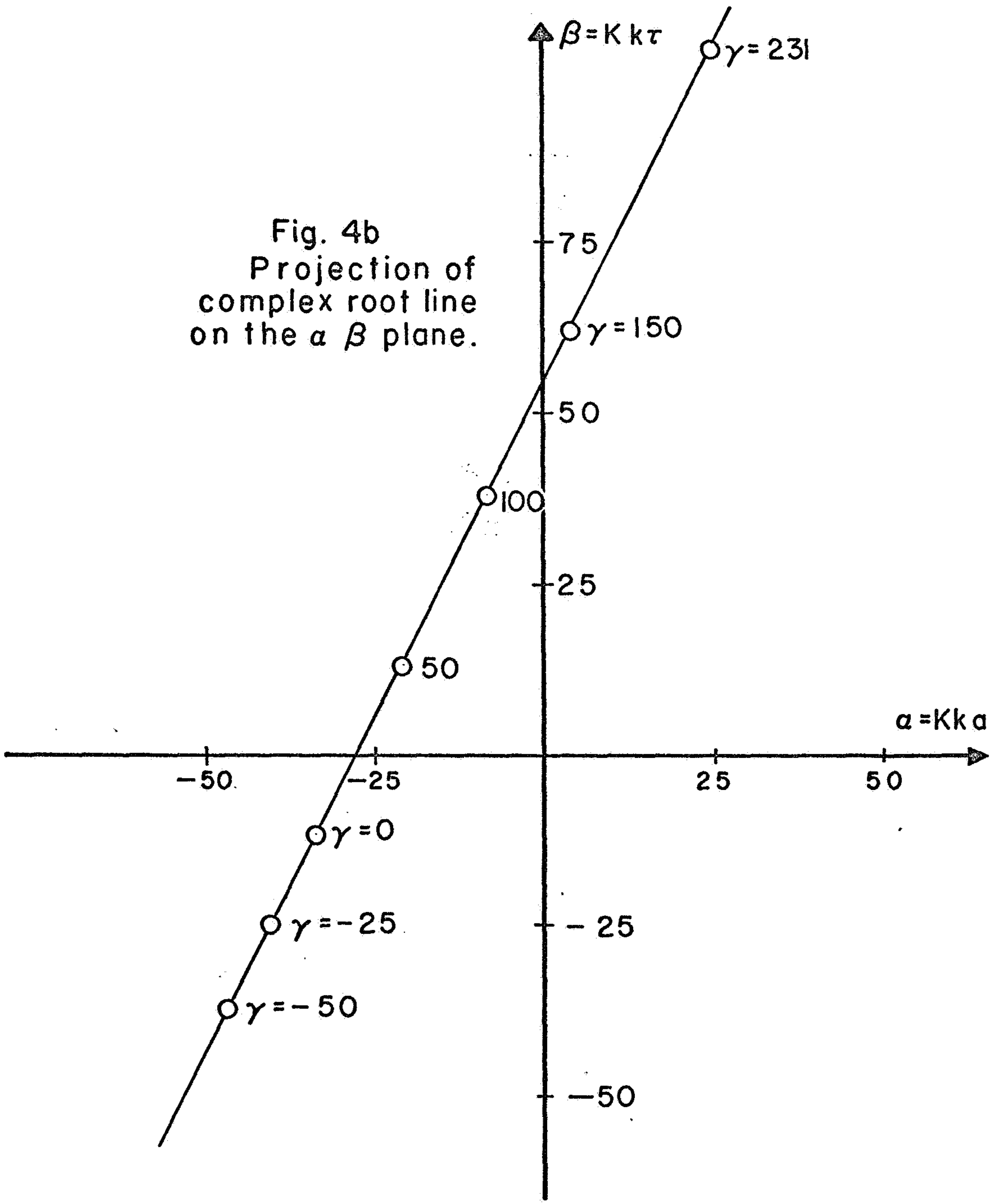

Fig. 4b
Projection of complex root line on the α β plane.

Since Eq. (21) happens to be a function of one variable only, no manipulation is needed. To get Eq. (21) in root locus form, rearrange to

$$\frac{\alpha}{s^2 + 14.5s + 40} = -1 \tag{23}$$

Factoring the denominator (23) becomes

$$\frac{\alpha}{(s + 10.995)(s + 3.705)} = -1 \tag{24}$$

The root locus for (24) is shown on Fig. 5, on which are also marked the specified roots. The unspecified roots may be forced to a non-dominant location such as $s = -7.35 \mp j\, 3.37$ by choosing $\alpha = 24$. Then substitution into Eq. (22) provides $\beta = 103.5$ and $\gamma = 231$.

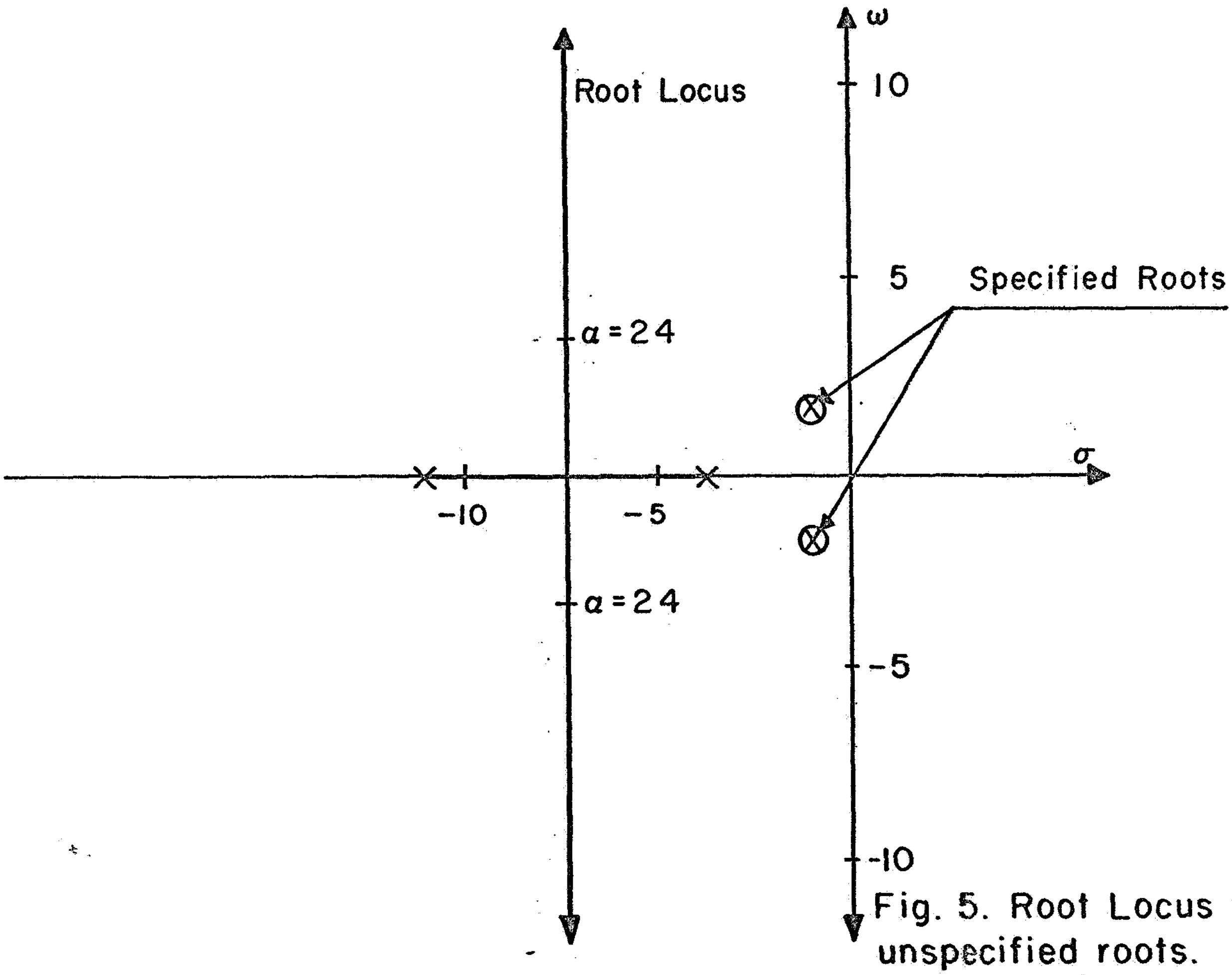

Fig. 5. Root Locus for unspecified roots.

7. Conclusions

Extension of parameter plane equations to define a three parameter space provides useful information for analysis and design. When the functional relationship between coefficient and parameters is linear, geometric interpretations are possible. A real root of the characteristic equation defines a plane in three space, and two real roots define a straight line which is the intersection of two planes. In like manner a pair of complex roots define a straight line in three parameter space.

Methods have been presented for obtaining projection of such lines on a plane and for using such projections in analysis and design. An algebraic procedure has been evolved from the graphical procedures which promises to be even more valuable in the design of multivariable systems.

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SYNTHESIS OF SYSTEMS WITH
DOMINANT COMPLEX ROOTS

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ABSTRACT: Systems with three (or more) free parameters are considered. Predetermined locations are assigned to one pair of complex roots, and their existence is guaranteed by assigning two degrees of freedom (two parameters) to constrain them to the desired location. The constraints are used to construct a reduced characteristic equation from which the assigned parameters have been eliminated. The polynomial thus obtained defines the locations of all roots except the specified complex pair, and (for the case of three parameters) a root locus study permits choice of a value of the third parameter such that the root pattern is compatible with a dominance requirement for the specified complex pair.

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INTRODUCTION

Classical methods for the design of feedback control systems normally strive to achieve dominance of a pair of complex roots. These roots may be designated indirectly (in terms of phase margin and gain margin) when using frequency response methods, or directly (in terms of ζ and ω_n or σ and ω) when using s-plane methods. When an error coefficient is specified solution of the design problem becomes difficult. If frequency response methods are used cascade compensators may be designed but trial and error is required and the answer is approximate, while the design of feedback compensation is still more difficult. Using Root Locus methods several techniques (2,3,4,5,6,10) have been developed which permit design of cascaded compensators to achieve a desired location of complex roots together with the specified error coefficient. These methods can give precise answers without excessive labor, but do not provide guidance to alternate solutions when the original solution is mathematically correct but practically undesirable.

In like manner the problem (of designing cascade compensation to achieve roots at designated locations while satisfying an error coefficient requirement) can be solved algebraically (2). Use of the Mitrovic's relationships provides two simultaneous equations which may be solved to determine the required locations of the pole and zero of the compensator.

All of these s-plane methods provide roots at the specified locations, and also provide the specified error coefficient, but they do not guarantee stability and they do not guarantee that the specified roots will be dominant. These requirements must be verified after the design has been carried out.

In general the methods developed thus far establish relationships between two parameters which must be satisfied in order to force two roots to desired locations.

Because there are only two degrees of freedom the solution is unique, and may be unacceptable if there has been an unfortunate choice of other parameters. In this paper a new concept is combined with well known techniques to guarantee the desired root locations and at the same time permit adjustments for stability and for dominance of the desired roots.

THEORY

The characteristic equation of a dynamic system may be developed from the system transfer functions (or other mathematical model) with coefficients expressed as functions of the system parameters. This characteristic equation may be expressed as

$$\sum_{k=0}^{n} a_k s^k = 0 \tag{1}$$

If m desired root locations are chosen ($m < n$), they define an mth order polynomial

$$\sum_{kl=0}^{m} a_{kl} s^{kl} = 0 \tag{2}$$

Formal division of (2) into (1) provides a quotient and a remainder

$$\left[\sum_{k=0}^{n} a_k s^k\right] \div \left[\sum_{kl=0}^{m} a_{kl} s^{kl}\right] =$$

$$a_{n-m} s^{n-m} + a_{n-m-1} s^{n-m-1} \ldots + a_o + \text{REMAINDER} \tag{3}$$

The system is guaranteed to have the m desired roots if the Remainder is identically zero.

There is nothing really new in relationships 1,2,3, but proper interpretation provides a new and useful technique for analysis and design. The new concept is to use the Remainder and the quotient to study the non-specified roots and place them in locations which guarantee stability of the system and dominance of some specified roots. Note that if the Remainder is zero, then the quotient is a polynomial the roots of which are the non-specified roots of the characteristic equation. The Remainder contains m terms, each of which must be zero for all values of s, and thus the Remainder defines m-equations in p-parameters. The quotient, when set to zero, provides one additional equation in p-parameters (or less than p).

If the system (i.e., the polynomial of Eq.(1)) contains exactly the same number of parameters as there are specifications, i.e., if $p \equiv m$, then the remainder defines m-equations in m-parameters which may be solved simultaneously for unique numerical values of these parameters. The coefficients of the quotient are thus defined numerically and no adjustments are possible. This situation is exactly the problem of Ref. 1-6, and the solution is also the same (i.e., the same values are obtained for the parameters). This method has some advantage in that it may be used for $m = p = 2,3,4$ etc., while the root locus methods are normally restricted to $m = p = 2$.

When $p > m$, then the m-equations from the remainder may be solved for m of the parameters, each of these solutions being in terms of the remaining p-m parameters. Substitution of these results into Eq.(3) provides a polynomial with coefficients expressed in terms of p-m parameters. Equation (3) becomes

$$D_\Delta s^\Delta + D_{\Delta-1} s^{\Delta-1} + \cdots D_o = 0 \tag{4}$$

where $\Delta \triangleq n-m$; $D = D(p-m)$

Equation 4 defines the <u>non-specified</u> roots of the system in terms of p-m parameters. By studying the locations of the roots of (4) as functions of the p-m parameters, a choice can be made for the values of these parameters. If there exists a range of parameter values such that all roots of (4) are in the left half of the s-plane, then a choice of parameter values within this range guarantees stability of the system. If, also, there is a sub-set within this range for which a pair of the specified roots are dominant, then the choice of the parameter values within this sub-set guarantees both stability and dominance. In either case the chosen p-m parameter values are then substituted into the solution of the remainder equations to determine the remaining m-parameter values. It is important to note that for <u>any</u> choice of the p-m parameters the specified roots are obtained exactly, in addition the non-specified roots are exactly those used in selecting the p-m parameter values.

For many practical cases study of the roots of Eq.(4) may be accomplished by well known methods. When p-m = 1, Eq. (4) provides a conventional root locus. When p-m = 2 algebraic manipulation removes all but two parameters from Eq.(4) which may then be studied by parameter plane methods. If p-m > 2 the particular methods used depend on the specific problem.

ILLUSTRATIONS

The manipulations involved in applying the method are very simple. They work surprisingly well on some rather difficult problems. The algebraic equations involved are non-linear, though, and thus the method does not always work easily. The best way to show this seems to be by specific illustrations:

CASE I. Type Zero System with three free parameters (Fig.1). Dominant complex roots desired with $\zeta = 0.5$ and $\omega_n = 2.0$. Evaluate p, k_a and k_t.

Solution: The system characteristic equation is

$$s^5+(p+50)s^4+(875+50p)s^3+(6250+875p+100k_a)s^2+(15000+6250p+100k_t)s + 15000p+100 = 0 \qquad (5)$$

The specified roots define the quadratic $s^2+2s+4 = 0$ (6)

Dividing (6) into (5) the quotient is

$$s^3+(48+p)s^2+(771+48p)s+4616+775p+100k_a \qquad (7)$$

with remainder:

$$(450p-200k_a+100k_t+2684)s+11900p-400k_a-18364 \qquad (8)$$

Each of these remainder coefficients must be zero, thus:

$$4508p - 200k_a + 100k_t + 2684 = 0 \qquad (9)$$

$$11900p - 400k_a - 18364 = 0 \qquad (10)$$

From (10) $\quad 100k_a = 2975p - 4591 \qquad (11)$

Substituting (11) in (7)

$$s^3 + (p+48)s^2 + (48p+771)s + 3750p + 25 = 0 \qquad (12)$$

Partitioning (12) $\quad \dfrac{p(s^2 + 48s + 3750)}{s^3 + 48s^2 + 771s + 25} = -1 \qquad (13)$

The root locus for (13) is shown on Fig.2. Note that this locus defines the values of the three unspecified roots as functions of p only. The specified complex roots have been superimposed on Fig.2 for convenience. p is chosen to insure dominance of the specified roots. For example, choosing a real root at $s = -30$ gives $p = 2.15$; substitution in (11) gives $k_a = 18.09$, and further substitution in (9) gives $k_t = -87.66$.

CASE II. Type I system with four free parameters (Fig.3). Dominant complex roots desired at $s = -1 \mp j3$. Error coefficient to be $K_v = 1.0$. Evaluate K_1, K_2, K_t, p.

Solution: The characteristic equation of the system is:

$$s^4+(3+p)s^3+(2+3p+K_1K_t)s^2+(2p+K_1+K_1K_t)s+K_1p+K_1K_2 + 0 \tag{14}$$

Note that the parameters appear in product combinations. The required roots define the polynomial

$$s^2 + 2s + 10 = 0 \tag{15}$$

Dividing (15) into (14) the quotient is:

$$s^2 + (1+p)s + (-10+p+K_1k_t) \tag{16}$$

and the remainder terms define

$$10 - 10p + K_1k_t(p-2) = 0 \tag{17}$$

$$100 - 10p - 10K_1k_t + K_1p + K_1K_2 = 0 \tag{18}$$

The K_v specification defines

$$K_1K_2 - 2p - 2K_1k_tp + K_1p = 0 \tag{19}$$

Solving (17), (18), (19) gives a unique result for p (this means that the four free parameters constitute only three independent parameters):

$$p = -1.5 \mp \sqrt{52.25} = 5.7287 \text{ or } -8.7287$$

Choosing the positive value provides:

$$p = 5.7287;\ K_1 = 10,\ k_t = 1.268,\ K_2 = 2.67$$

and the two remaining roots are $s = -3.36 \mp j1.7$. Note that in this case there were no "excess" free parameters which could be adjusted to obtain acceptable values for the unspecified roots; the solution obtained is unique, and the result may or may not be acceptable.

CASE III. Type 1 system with four free parameters (Fig.4). Evaluate K, z, p, k_t for roots specified at $-5 \mp j10$.

For this system the characteristic equation is

$$s^4 + (10+p)s^3 + (25+100k_t+10p)s^2 + (25p+100p\,k_t+100K)s + 100K_z = 0 \qquad (20)$$

The specified roots define the polynomial

$$s^2 + 10s + 125 \qquad (21)$$

Dividing (21) into (20) the quotient is

$$s^2 + ps + 100k_t - 100 \qquad (22)$$

and the remainder terms are

$$100p\,k_t - 100p + 100K - 100k_t + 1000 = 0 \tag{23}$$

$$100\,K_z - 12500k_t + 12500 = 0 \tag{24}$$

Since only two roots have been specified (23) and (24) define the relationships among the four parameters, but two such equations are not enough to permit elimination of one variable from (22). Thus it is easier to proceed by studying (22) on the parameter plane (7, 8, 9) of Fig.5. Note that any choice of p and k_t on Fig. 5 defines the locations of the third and fourth roots of (14) since the primary roots have already been constrained to $s = -5 \mp j10$. The remaining parameters, K and z, may be determined by substitution of p and k_t in the remainder constraints of (23) and (24).

In order to obtain acceptable values for the parameters with minimum trial and error, (23) and (24) may be plotted as shown on Fig.6. Inspection of Fig.6 shows that p and k_t cannot be chosen arbitrarily if K and z are to be positive numbers. It appears that the choice is restricted to $k_t > 1.0$; $p < 10$. An intelligent choice can be made by inspection, and a number of alternate possibilities can be explored rapidly.

Since the system has four free parameters and only two are required to constrain the dominant roots, additional constraints can be applied. For example, if it is desired to constrain the error coefficient the constraint equation is

$$K_v = \lim_{s \to 0} sG(s) = \frac{100K_z}{p(25+100k_t)} \tag{25}$$

from which

$$(25p+100pk_t)K_v = 100\ K_z \tag{25a}$$

but from (24) $\quad 100K_z = 12500(k_t-1) \qquad (26)$

Substituting (26) into (25a) and manipulating

$$k_t = -\frac{pK_v + 500}{4pK_v - 500} \tag{27}$$

Eq. (27) may be plotted as a single curve on normalized coordinates as in Fig.7a, or as a family of curves on parameter plane coordinates as in Fig.7b. Then, using the curves of Figs.5, 6, and 7, any choice of an operating point defines values for p, k_t, K, z, and K_v, while restricting the specified roots to $s = -5 \mp j10$.

CONCLUSIONS

A simple algebraic method has been presented which permits study and design of multiple parameter systems with pre specified locations for some of the characteristic roots. Determination of a suitable set of parameter values may be carried out in an entirely algebraic fashion, or with the assistance of graphical tools such as the root locus and the parameter plane.

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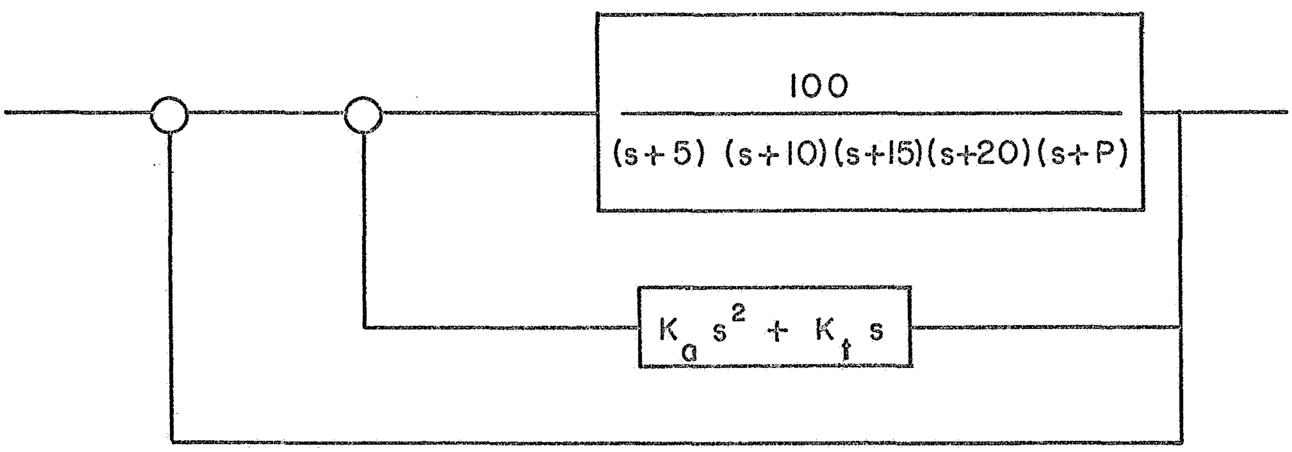

Fig. 1. Type Zero System with three parameters.

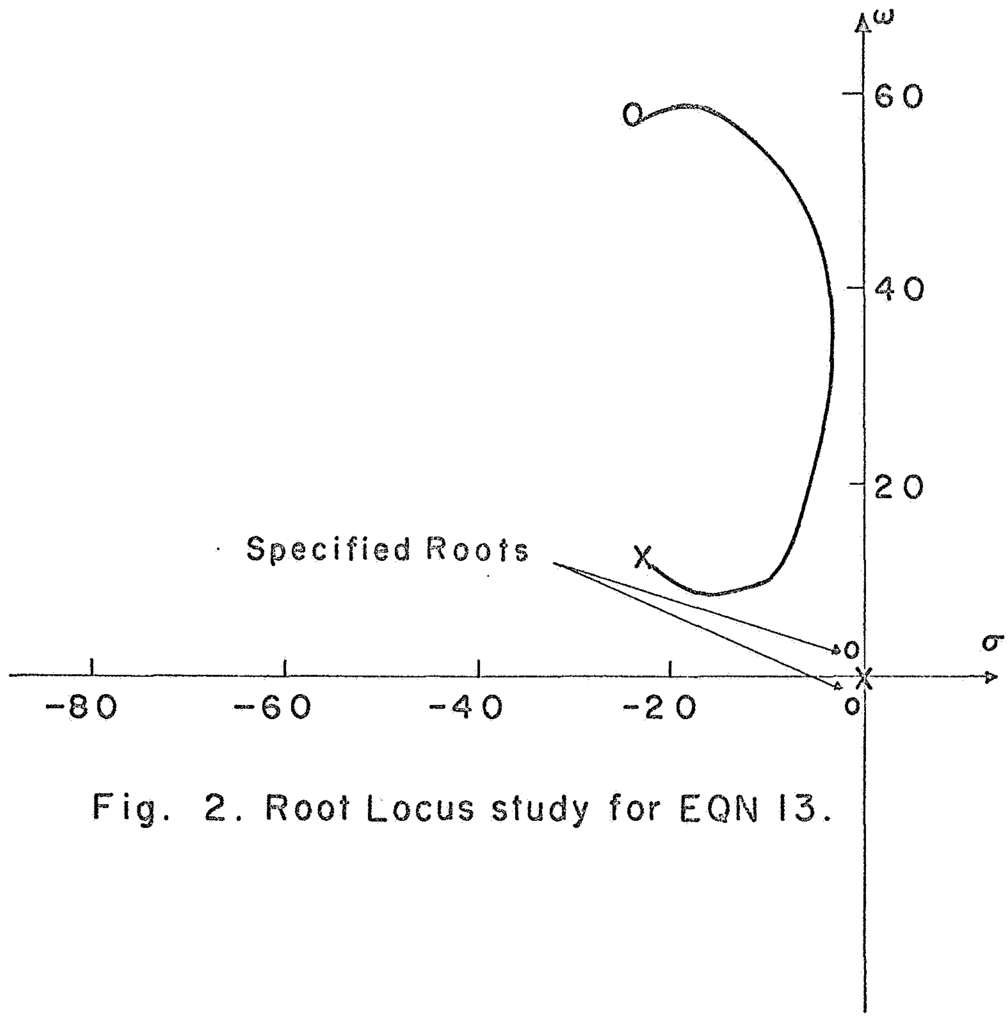

Fig. 2. Root Locus study for EQN 13.

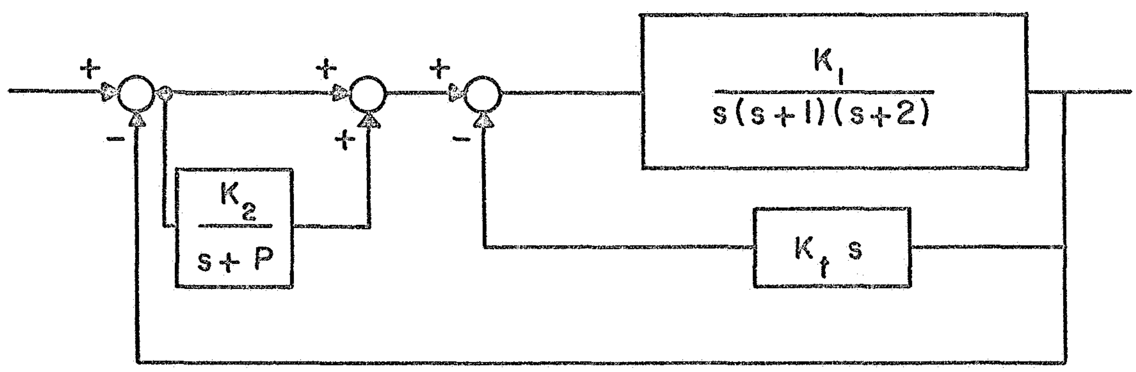

Fig. 3. Block Diagram for illustration, Case II.

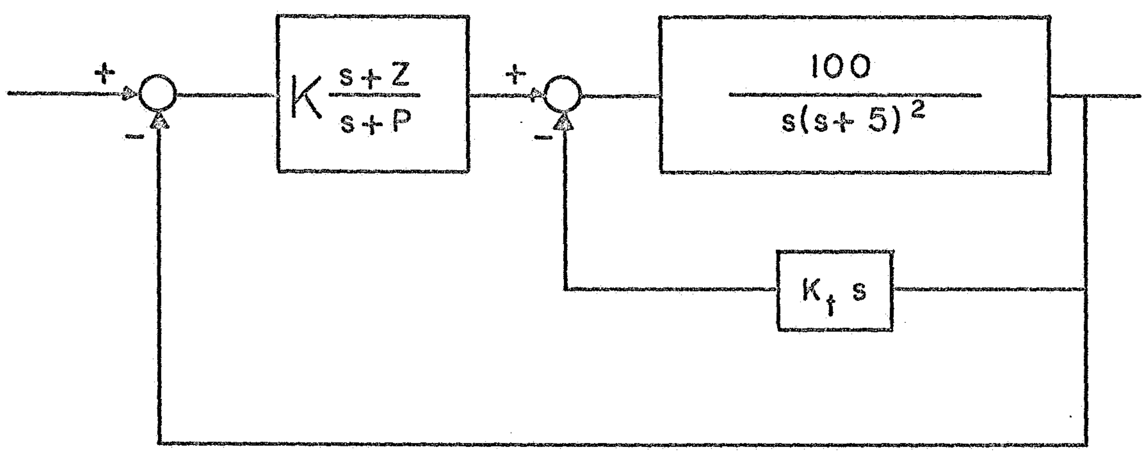

Fig. 4. Block Diagram for illustration, Case III.

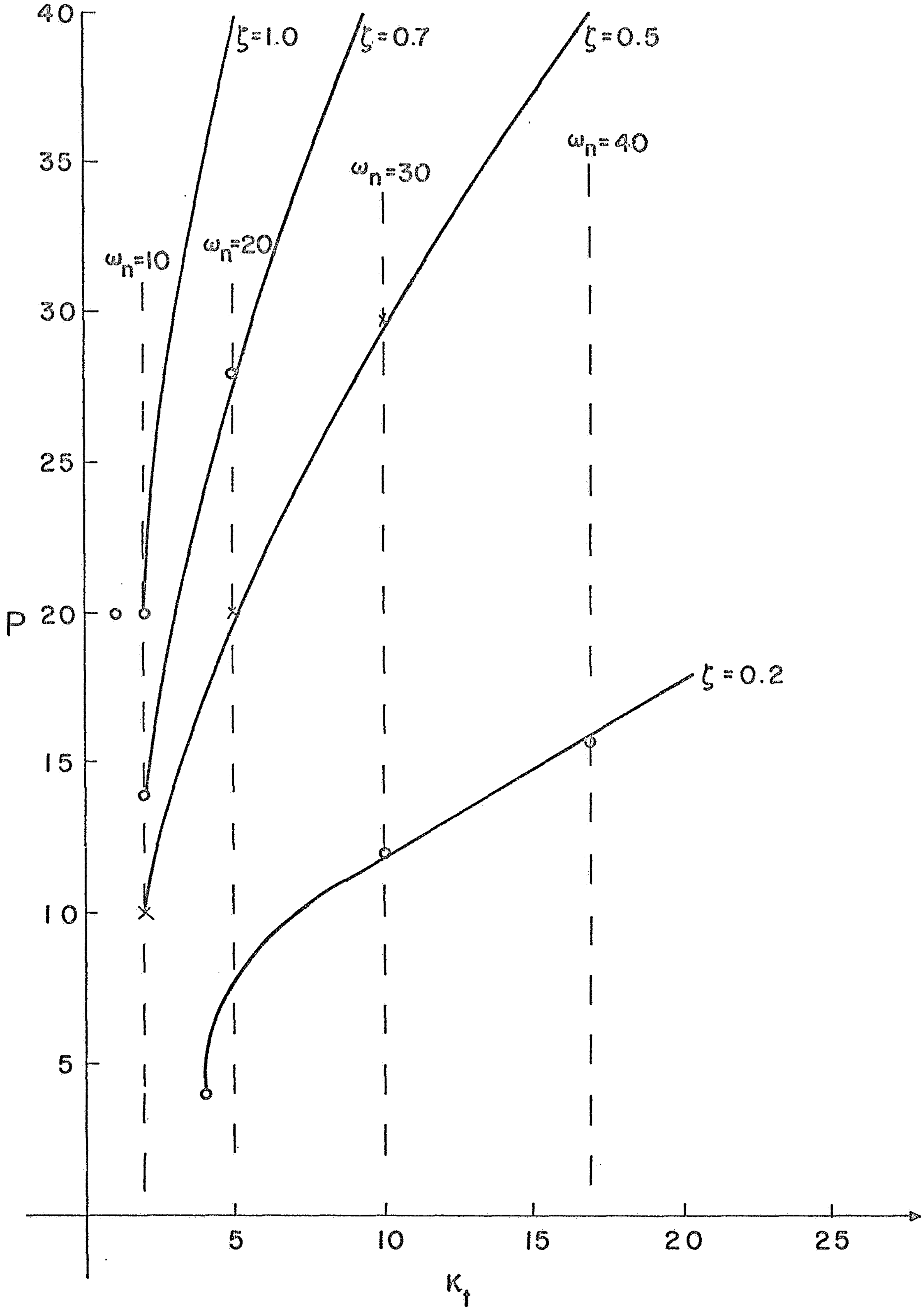

Fig. 5. Parameter Plane for $s^2 + Ps + 100K_t - 100 = 0$

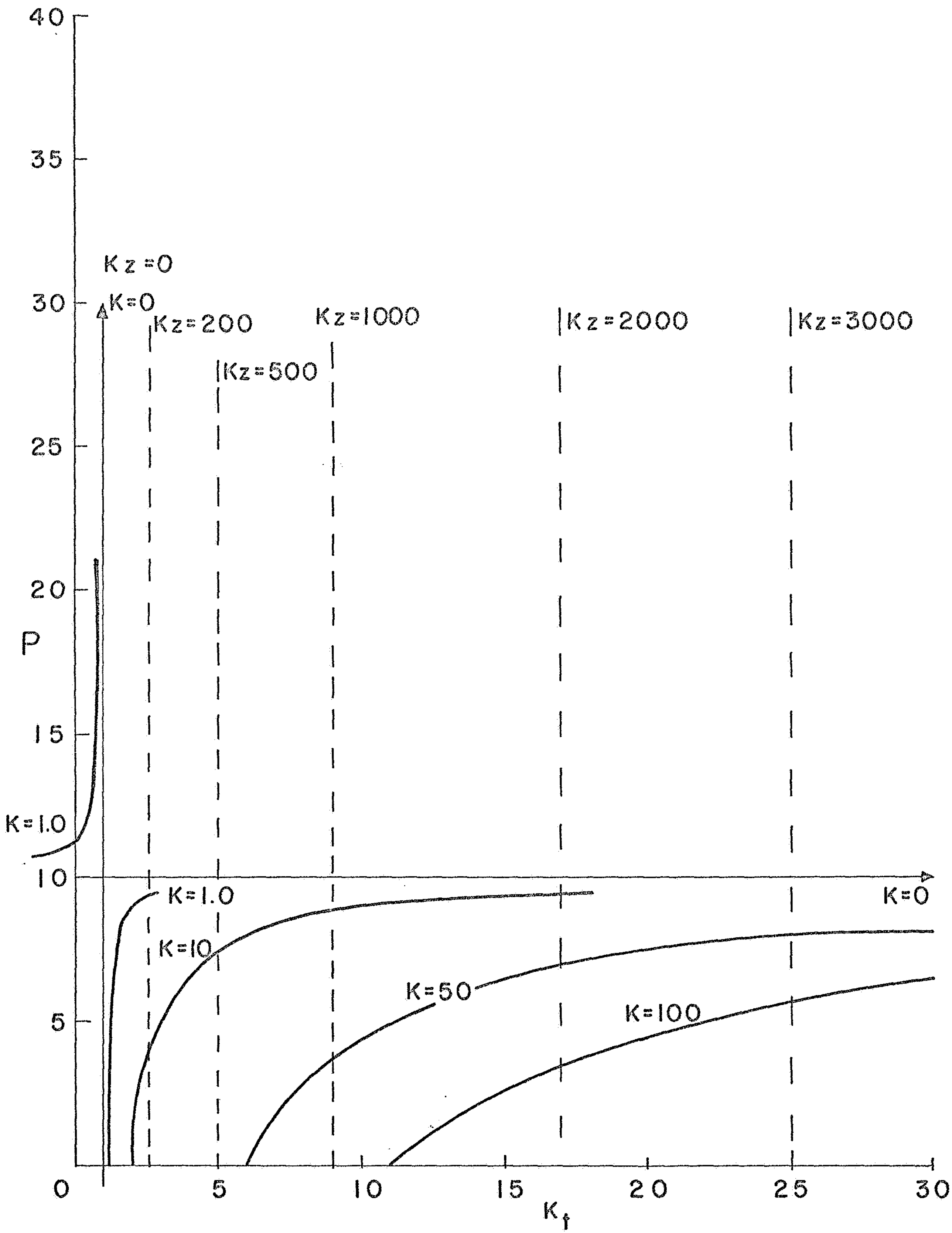

Fig. 6. Plot of Remainder Equations

$$100pk_t - 100_p + 100K - 1000k_t + 1000 = 0$$

$$100Kz - 12500k_t + 12500 = 0$$

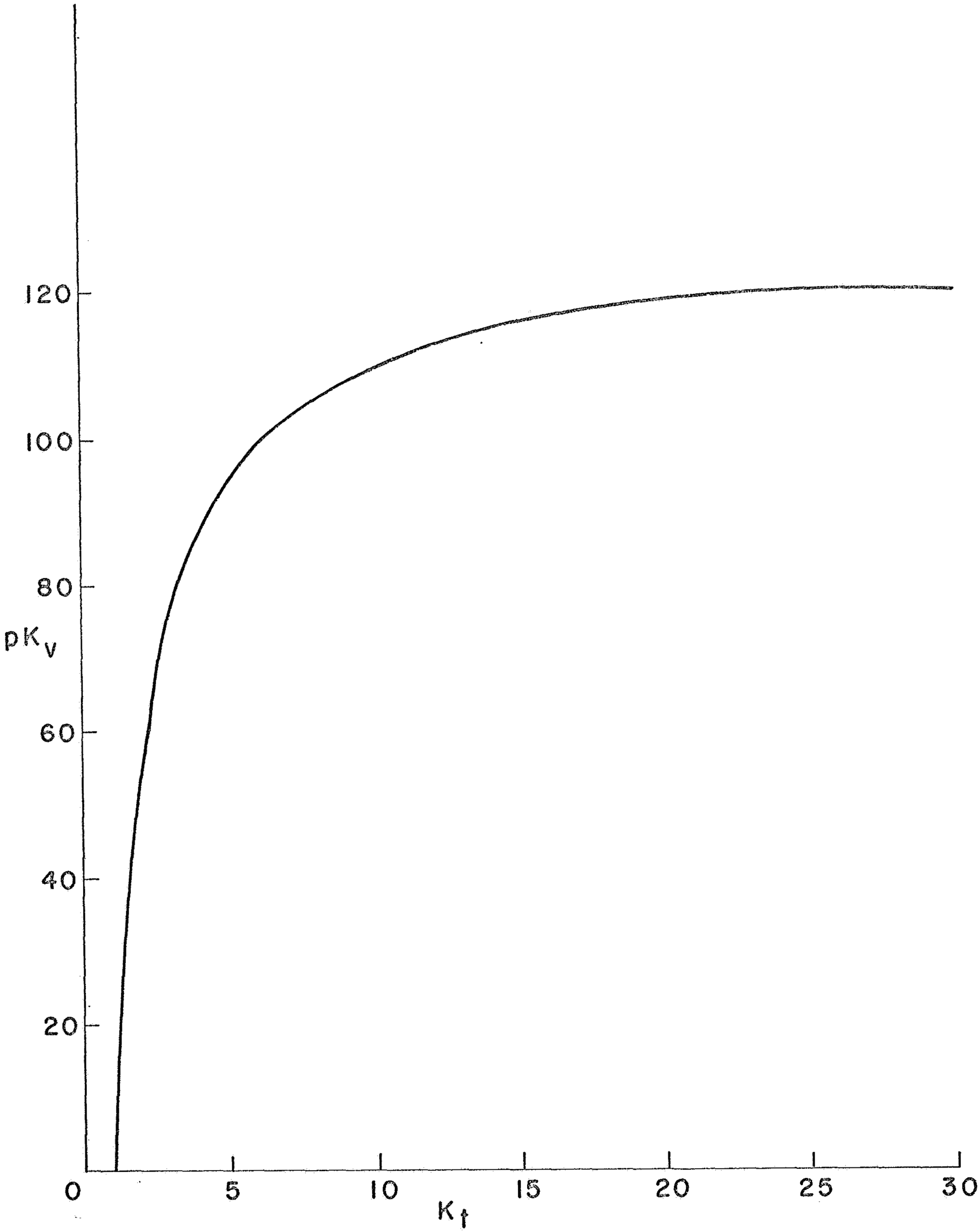

Fig. 7a. Plot of Constraint on K_v , normalized coordinates

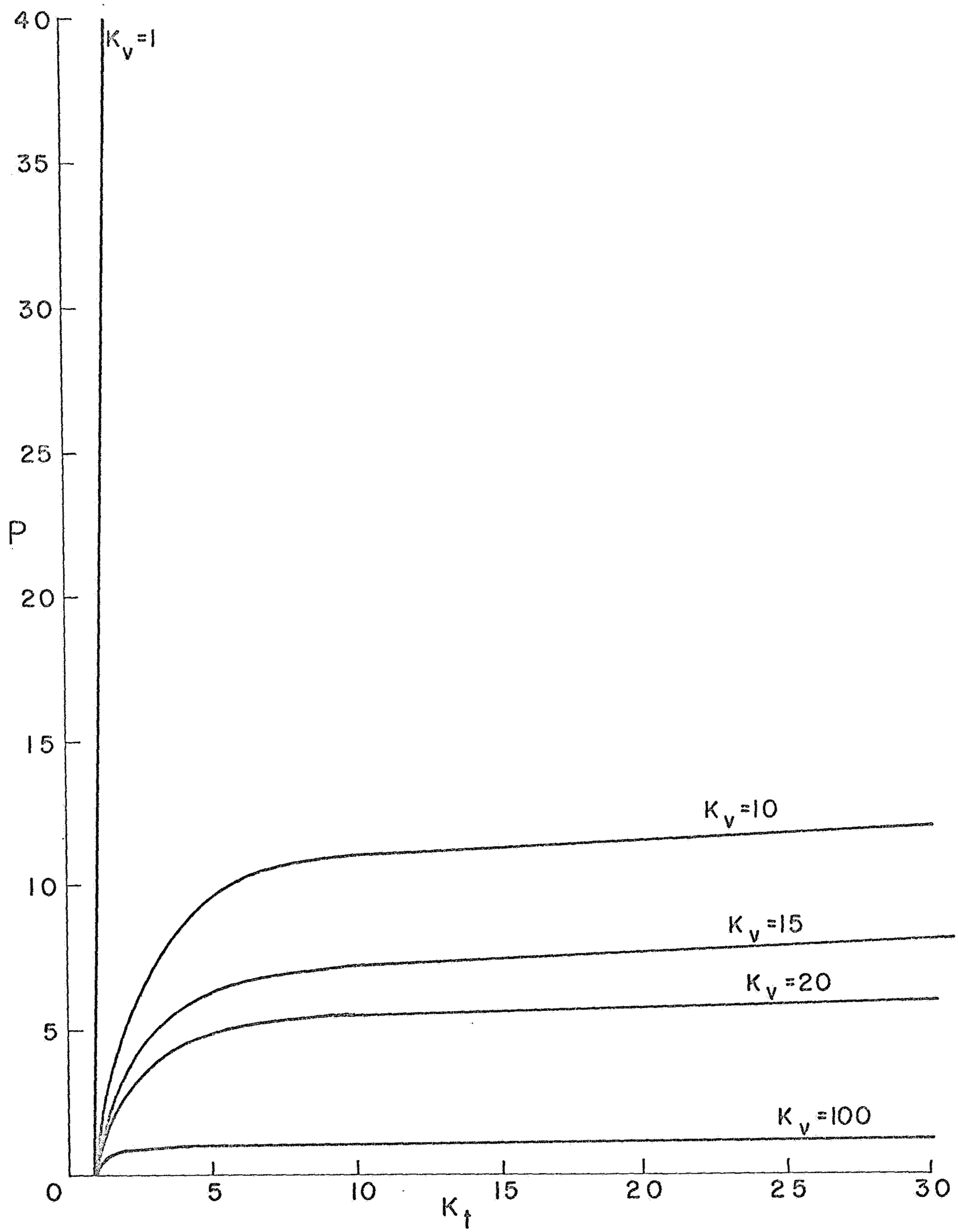

Fig. 7b. Plot of Constraint on K_v, parameter plane coordinates

LIST OF CAPTIONS

A MATRIX APPROACH TO
PARAMETER ANALYSIS OF DYNAMICAL SYSTEMS*

Jayant KARMARKAR**

George J. THALER***

Abstract

The response of a dynamic system to a command or a disturbance depends on the values of the parameters which comprise that system. Analysis of system performance in terms of the relationships between parameters and roots of the characteristic equation is therefore a very desirable procedure, and leads to an intelligent choice of parameters for a final design.

There has been considerable study of parameter analysis (Ref. 1,2,3,4,5,...) methods in recent years, the most effective of which have been the parameter plane (Ref. 1,2,3,...) methods which permit display of the parameter vs. root correlations with a two dimensional family of curves. Parameter plane techniques treat the problem as a two parameter problem. The resulting equations can be programmed and the curve families plotted, but special programs must be written. This paper approaches the problem from the viewpoint of an n-dimensional parameter space, (Ref.6) and formulates the problem in matrix form. The problem is then reduced to a two parameter problem in order to obtain the advantages of graphical presentation, but the matrix formulation permits direct solution using standard matrix subroutines so that the programming effort is minimal. Slight additional modifications permit simultaneous studies of sensitivity, stability and other features of interest.

1. Matrix Formulation of the Parameter Space

The dynamics of linear systems are defined by the roots of the characteristic equation. The characteristic polynomial may be defined as

$$\sum_{k=0}^{N} a_k s^k = 0 \tag{1}$$

where the coefficients, a_k, are functions of the parameters. Many functional relationships are possible, and each must be considered separately in a matrix formulation. In many physical systems the coefficients are linear functions of the parameters

$$a_k = b_k\alpha + c_k\beta + d_k\gamma + e_k\delta + \cdots + k_k \tag{2}$$

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The functional relationships between the coefficients and roots of a polynomial are well known and are the basis for the matrix formulation of the problem. When $a_N = 1.0$, the following relationships obtain

$$\begin{bmatrix} a_{N-1} \\ a_{N-2} \\ a_{N-3} \\ a_{N-4} \\ \vdots \\ a_0 \end{bmatrix} = \begin{bmatrix} b_{N-1} & c_{N-1} & d_{N-1} & e_{N-1} & \cdots & k_{N-1} \\ b_{N-2} & c_{N-2} & d_{N-2} & e_{N-2} & \cdots & k_{N-2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ b_0 & c_0 & d_0 & e_0 & & k_0 \end{bmatrix} * \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \vdots \\ 1. \end{bmatrix} = \begin{bmatrix} N_{R_1} \\ N_{R_2} \\ N_{R_3} \\ \vdots \\ N_{R_N} \end{bmatrix}$$

$$N_{R_1} = \sum_{i=1}^{N} r_i \qquad N_{R_2} = \sum_{\substack{i=1,j=1 \\ i \neq j}}^{N} r_i r_j \tag{3}$$

which may be rewritten

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} P \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \tag{3a}$$

In normal problems the A matrix and the R matrix are not known (i.e., numerical values are not known). The C matrix is completely known, and the P matrix, by definition, contains the variables to be studied. These relationships, then, define a N-dimensional parameter space. Specific solutions are not available, however, unless more information is available, i.e., unless specific constraints are placed on the roots.

To reduce the problem to a two dimensional space so that parameter plane curves can be obtained, two roots are specified as a complex conjugate pair, (which permits use of two parameters, say α and β). In addition, then, specific numerical values must be assigned for all additional parameters, γ, δ, etc. The matrices then become

$$\begin{bmatrix} b_{N-1} & c_{N-1} & d_{N-1} & \cdots & k_{N-1} \\ b_{N-2} & c_{N-2} & & & \vdots \\ \vdots & \vdots & & & \vdots \\ b_0 & c_0 & & & k_0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \vdots \\ 1 \end{bmatrix}$$

$$\cdot\cdot \begin{bmatrix} {}^{N-2}R_1 & 1 & 0 \\ {}^{N-2}R_2 & {}^{N-2}R_1 & 1 \\ {}^{N-2}R_3 & {}^{N-2}R_2 & {}^{N-2}R_1 \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 \\ 2\sigma \\ \sigma^2 + \omega^2 \end{bmatrix} \tag{4}$$

where all parameters except α and β are now known numerically, and the right hand side is obtained by removing the specific complex pair of roots* from the R matrix. The only unknown quantities are the two parameters α and β, and the root combinations ${}^{N-2}R_1$, ${}^{N-2}R_2 \ldots {}^{N-2}R_{N-2}$.

Each row of the matrix is rearranged, putting unknowns on the LHS and known numerical values on the RHS, yielding

$$\begin{bmatrix} b_{N-1} & c_{N-1} & -1 & 0 & \cdots & 0 \\ b_{N-2} & c_{N-2} & -2\sigma & 0 & & 0 \\ b_{N-3} & c_{N-3} & -(\sigma^2+\omega^2) & -1 & & 0 \\ \vdots & \vdots & \vdots & -2\sigma & & \vdots \\ b_2 & c_2 & 0 & \vdots & & -1 \\ b_1 & c_1 & 0 & \vdots & & -2 \\ b_0 & c_0 & 0 & \vdots & & -(\sigma^2+\omega^2) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ {}^{N-2}R_1 \\ {}^{N-2}R_2 \\ \vdots \\ \\ {}^{N-2}R_{N-2} \end{bmatrix} =$$

$$\begin{bmatrix} 2\sigma \\ \sigma^2+\omega^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} d_{N-1} & e_{N-1} & \cdots & k_{N-1} \\ d_{N-2} & e_{N-2} & & k_{N-2} \\ \vdots & \vdots & & \vdots \\ d_1 & \vdots & & k_1 \\ d_0 & \vdots & & k_0 \end{bmatrix} \begin{bmatrix} \gamma \\ \delta \\ \vdots \\ 1. \end{bmatrix} \qquad (5)$$

After rearrangement of the RHS of Eq.(5), the result is

$$\begin{bmatrix} b_{N-1} & c_{N-1} & -1 & 0 & \cdots \\ b_{N-2} & c_{N-2} & -2\sigma & -1 & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ b_0 & c_0 & & & \vdots \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \\ {}^{N-2}R_1 \\ \vdots \\ {}^{N-2}R_{N-2} \end{bmatrix} = \begin{bmatrix} 2\sigma-\bar{k}_{N-1} \\ (\sigma^2+\omega^2)-\bar{k}_{N-2} \\ \bar{k}_{N-3} \\ \vdots \\ k_0 \end{bmatrix} \qquad (6)$$

From this form solution for α and β is obtained using standard matrix subroutines.

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* Eq.(4) specifies complex roots (in rectangular coordinates) at $s = -\sigma \mp j\omega$. If polar coordinates are desired $s = -\zeta\omega_n \mp j\omega_n\sqrt{1-\zeta^2}$, and the corresponding matrix is $\begin{bmatrix} 1 \\ 2\zeta\omega_n \\ \omega_n^2 \end{bmatrix}$

2. Obtaining the Parameter Plane Curves

Parameter plane curves are simply mappings of s-plane contours in the α - β plane. Thus one chooses a constant σ-line (or constant ζ-line) substitutes the numerical value of σ into Eq.(6) and increments ω, solving (6) for α and β at each ω. This is repeated for as many lines as may be desired. Constant ω-lines are obtained in like manner.

3. Information about Other Roots

In addition to solutions for α and β, Eq.(6) also can provide solutions for ${}^{N-2}R_1$, ${}^{N-2}R_2$ ... ${}^{N-2}R_{N-2}$. These numbers are the coefficients of the reduced polynomial

$$s^{N-2} + {}^{N-2}R_1\, s^{N-3} + \cdots + {}^{N-2}R_{N-2} = 0 \tag{7}$$

The roots of (7) are precisely the remaining N-2 roots of Eq.(1). Having obtained the coefficients of (7) by solution of (6), (7) may be solved by any root-finding subroutine, thus evaluating all of the other roots of the polynomial for each chosen location of the specified roots. It is not convenient to plot these additional roots, but it is often convenient to have their values available.

Eq.(6) is formulated for a specified pair of complex roots. For some studies real roots are of importance and parameter plane curves for real roots are desirable. To determine real root lines, substitute $s = -r$ (r = a specified real root) into Eq.(1) obtaining

$$(-1)^N(r)^N + [(-1)^{N-1}r^{N-1}]a_{N-1} + [(-1)^{N-2}r^{N-2}]a_{N-2} + \cdots a_0 = 0 \tag{8}$$

Assuming that the a's are linear functions of only 2 parameters, α and β, (8) may be rearranged:

$$\begin{aligned}&[b_0-b_1r+b_2r^2- \cdots +(-1)^{N-1}b_{N-1}r^{N-1}]\alpha + \\ &\quad [c_0-c_1r+c_2r^2 \cdots +(-1)^{N-1}c_{N-1}r^{N-1}]\beta + \\ &\qquad k_0-k_1r+r_2r^2 \cdots +(-1)^{N-1}k_{N-1}r^{N-1}+(-1)^N r^N = 0\end{aligned} \tag{9}$$

which is the equation of a straight line. Eq.(9) requires its own simple program.

For specific values of σ and ω the determinant of the system matrix (LHS of Eq.(6)) may become zero. If this occurs the defining equations are no longer independent, and it is possible to have singular lines. The basic matrix solution of Eq.(6) cannot evaluate and plot these singular lines, but it can be made to print out a warning that a singular line may exist for a specific σ, ω pair.

4. Evaluation of Sensitivity

For many problems it is valuable to know the sensitivity of roots to variations in parameters. This information can be used to establish tolerance, to assist in self-adaptive schemes, etc. Several kinds of sensitivity have been defined. Root sensitivity is compatible with the matrix approach and is defined by

$$\begin{aligned} S_p^\sigma &\triangleq \frac{p}{\sigma}\frac{d\sigma}{dp} \\ S_p^\omega &\triangleq \frac{p}{\omega}\frac{d\omega}{dp}\end{aligned} \tag{10}$$

The parameters of interest here are α and β. Differentiate Eq.(6) with respect to α and note that:

$$\frac{\partial\beta}{\partial\alpha} = 0 \; ; \quad \frac{\partial\alpha}{\partial\alpha} = 1 \; ;$$

$$\frac{\partial(\sigma^2+\omega^2)}{\partial\alpha} = 2\sigma\,\frac{\partial\sigma}{\partial\alpha} + 2\omega\,\frac{\partial\omega}{\partial\alpha} \tag{11}$$

$$\frac{\partial b_i}{\partial\alpha} = \frac{\partial c_i}{\partial\alpha} = \frac{\partial\bar{k}_i}{\partial\alpha} = 0 \qquad i = 0,1,\ldots(N-1)$$

After rearranging the terms, the resulting matrix equation is

$$\begin{bmatrix} 2 & 0 & 1 & 0 & \cdots & 0 \\ 2(\sigma+{}^{N-2}R_1) & 2\omega & 2\sigma & 1 & & 0 \\ 2(\sigma^{N-2}R_1+{}^{N-2}R_2) & 2\omega^{N-2}R_1 & \sigma^2+\omega^2 & 2\sigma & & 0 \\ 2(\sigma^{N-2}R_2+{}^{N-2}R_3) & 2\omega^{N-2}R_2 & 0 & \sigma^2+\omega^2 & & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 2(\sigma^{N-2}R_{N-3}+{}^{N-2}R_{N-2}) & 2\omega^{N-2}R_{N-3} & 0 & 0 & & 2\sigma \\ 2\sigma^{N-2}R_{N-2} & 2\omega^{N-2}R_{N-2} & 0 & 0 & & \sigma^2+\omega^2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial\sigma}{\partial\alpha} \\ \frac{\partial\omega}{\partial\alpha} \\ \frac{\partial\,{}^{N-2}R_1}{\partial\alpha} \\ \vdots \\ \frac{\partial\,{}^{N-2}R_{N-2}}{\partial\alpha} \end{bmatrix} = \begin{bmatrix} b_{N-1} \\ b_{N-2} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix} \tag{12}$$

Eq.(12) may be solved for the sensitivities $\frac{\partial\sigma}{\partial\alpha}$ and $\frac{\partial\omega}{\partial\alpha}$ using the same procedure as for Eq.(6). Note that the first two columns of Eq.(12) may be obtained from the solution of Eq.(6). To obtain the sensitivities with respect to β, the column vector $[b_i]$ is replaced with the column vector $[c_i]$ and the variable notation is changed from α to β.

5. Matrix Formulation for Nonlinear Parameter Relationships

The functional relationship between coefficients and parameters is not always linear. This complicates the solution, of course, but a matrix approach is usually possible, if combined with a few additional simple steps. One such case arises when the coefficient-parameter relationship is of the form

$$a_k = b_k\alpha + c_k\beta + p_k\alpha\beta + k_k \tag{13}$$

Proceeding as in the linear case, the following matrix equation is obtained

$$\begin{bmatrix} p_{N-1} & c_{N-1} & -1 & \cdots & 0 & b_{N-1} & k_{N-1} \\ p_{N-2} & c_{N-2} & -2\sigma & & 0 & b_{N-2} & k_{N-2} \\ \vdots & \vdots & -(\sigma^2+\omega^2) & & 0 & \vdots & \vdots \\ \vdots & \vdots & 0 & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & -1 & \vdots & \vdots \\ p_1 & c_1 & 0 & & -2\sigma & b_1 & k_1 \\ p_0 & c_0 & 0 & \ddots & -(\sigma^2+\omega^2) & b_0 & k_0 \end{bmatrix} \begin{bmatrix} \alpha\beta \\ \beta \\ {}^{N-2}R_1 \\ {}^{N-2}R_2 \\ \vdots \\ {}^{N-2}R_{N-2} \\ \alpha \\ 1 \end{bmatrix} = 0 \qquad (14)$$

Eq. (14) is written for a specified σ, ω pair, and the matrix has (N+2) columns and N rows. Using a computer program for row operations provides a matrix equation of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & c_{11} & c_{12} \\ 0 & 1 & 0 & & & \vdots & \\ 0 & 0 & 1 & & & \vdots & \\ \vdots & & & \ddots & & \vdots & \\ \vdots & & & & \ddots & c_{N-1,N+1} & c_{N-1,N+2} \\ 0 & & & & 1 & c_{N,N+1} & c_{N,N+2} \end{bmatrix} \begin{bmatrix} \alpha\beta \\ \beta \\ {}^{N-2}R_1 \\ \vdots \\ \alpha \\ 1 \end{bmatrix} = 0 \qquad (15)$$

From the first two rows of (15), Eq. (16) provide a quadratic in α which can be solved for two values of α and the corresponding two values of β These solutions may then be used in Eq. (15) to evaluate ${}^{N-2}R_1 \ldots {}^{N-2}R_{N-2}$. Note that the only change in the sensitivity equation (corresponding to Eq. (12)) is an additional column vector on the RHS (see Eq. (17)).

$$\begin{aligned} \alpha\beta + c_{11}\alpha + c_{12} &= 0 \\ \beta + c_{21}\alpha + c_{22} &= 0 \end{aligned} \qquad (16)$$

$$= \begin{bmatrix} b_{N-1} \\ b_{N-2} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix} - \beta \begin{bmatrix} p_{N-1} \\ p_{N-2} \\ \vdots \\ p_1 \\ p_0 \end{bmatrix} \qquad (17)$$

Another case in which the parameters appear nonlinearly provides

$$a_k = b_k\alpha + c_k\beta + p_k\alpha\beta + q_k\alpha^2 + r_k\beta^2 + k_k \tag{18}$$

Matrix manipulations follow the same scheme as for the preceding case. In this case the parameter vector (see Eq.(14) and (15)) would be

$$\begin{bmatrix} \beta^2 \\ \alpha\beta \\ {}^{N-2}R_1 \\ \vdots \\ {}^{N-2}R_{N-2} \\ \beta \\ \alpha^2 \\ \alpha \end{bmatrix} \tag{19}$$

which contains N+4 terms. Continuing with the procedures used for Eq.(15) results in a biquadratic equation in α (comparable to Eq.(16)). The sensitivity equation now has two additional terms on the RHS (see Eq.(20)).

$$-\beta \begin{bmatrix} p_{N-1} \\ p_{N-2} \\ \vdots \\ p_1 \\ p_0 \end{bmatrix} \quad -2\alpha \begin{bmatrix} q_{N-1} \\ q_{N-2} \\ \vdots \\ q_1 \\ q_0 \end{bmatrix} \tag{20}$$

6. Illustration

A program has been developed which calculates and plots the α vs. β curves using the matrix techniques. Consider the system of Fig.1 for which

$$\frac{C}{R} = \frac{B(s+25)}{s^4+(20+A)s^3+(1700+20A)s^2+(1700+B)s+25B} \tag{21}$$

The denominator of Eq.(21) is the characteristic equation. Let $\alpha = A$ and $\beta = B$. Then the coefficients are linear in α and β, i.e.

$$a_k = b_k\alpha + c_k\beta + d_k \tag{22}$$

It is readily seen that

$b_o = 0,\quad c_o = 25,\quad d_o = 0$

$b_1 = 0,\quad c_1 = 1,\quad d_1 = 1700$

$b_2 = 20,\quad c_2 = 0,\quad d_2 = 1700$

$b_3 = 1,\quad c_3 = 0,\quad d_3 = 20$

Using the program provides the parameter plane curves of Fig.2.

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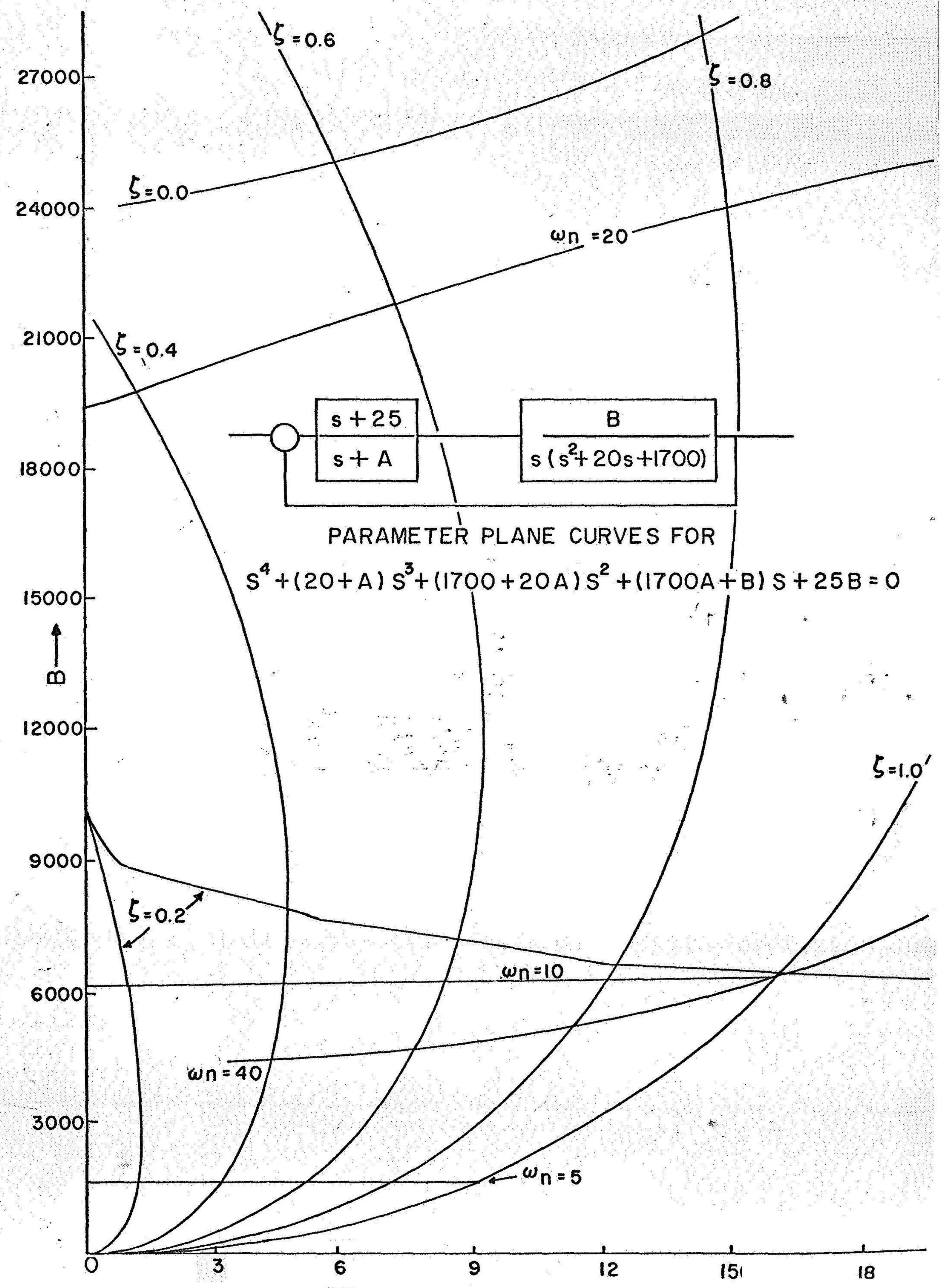

ζ=0.6
ζ=0.8
ζ=0.0
ωn =20
ζ=0.4
s + 25
s + A
B
s (s²+ 20s +1700)
PARAMETER PLANE CURVES FOR
S⁴+(20+A) S³+(1700+20A) S²+(1700A+B) S+25B=0
ζ=1.0
ζ=0.2
ωn=10
ωn=40
ωn=5
B
27000
24000
21000
18000
15000
12000
9000
6000
3000
0
3
6
9
12
15
18

from a real line into a closed curve[1] in the complex plane [3].

Stability analysis using this synchronous describing function is valid only for the specific synchronous inputs considered.

Conclusions

The circle of the circle criterion is determined by worst-case output distortion components and must be avoided. On the other hand, in describing function analysis, a closed region which must be avoided appears only in synchronous cases.

It seems that the conservative stability estimates of the circle criterion in the case of a general time-varying operator $N=N(x, t)$ are perhaps due to the possibility of synchronism connected with periodicities in N. Consideration of synchronous effects may lead to better stability conditions. It appears that intersection of the linear part $H(j\omega)$ with the stability circle may be permitted if synchronism cannot occur.

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[1] Incidentally, this closed curve is a circle in linear cases (e.g., when N is purely time varying), but takes a more complex form for nonlinear elements [3].

On a Class of Time-Varying Systems

Abstract—A class of time-varying systems whose matrices satisfy a certain matrix Riccati equation is discussed. For this class, the state transition matrix can be found in closed form.

Introduction

Consider the homogeneous linear time-varying system

$$\dot{X} = A(t)X(t) \qquad X(0) = X_0 \tag{1}$$

where X is an n vector and $A(t)$ is an $n\times n$ matrix that is nonsingular and whose elements are continuous and continuously differentiable functions of time. Closed-form solutions of (1) are discussed here for the class of $A(t)$ that satisfies the following matrix Riccati equation:

$$\dot{A}(t) + A^2(t) = B(t)A(t) \tag{2}$$

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where $B(t)$ belongs to certain classes of matrices to be discussed below. If (2) is satisfied, it can be shown that

$$\ddot{X} = B(t)\dot{X}. \tag{3}$$

For simplicity, the discussion that follows is limited to matrices $B(t)$ with real eigenvalues and one-dimensional invariant subspaces (distinct eigenvalues). The same approach can be extended to multiple eigenvalue cases and complex eigenvalues.

Closed-Form Solution

1) Matrix B is constant. If B is constant, $\dot{X}$ is an exponential function of B and

$$X(t) = [I + B^{-1}(e^{Bt} - I)A(0)]X(0). \tag{4}$$

As an extension of this case, suppose $B(t)$ is a function of time that satisfies the following Riccati equation:

$$\dot{B}(t) + B^2(t) = CB(t) \tag{5}$$

where C is a constant matrix; then the solution of (1) is given by

$$X(t) = [tA(0) + C^{-1}(C^{-1}e^{Ct} - tI)\dot{A}(0) + A^2(0) + I]X(0). \tag{6}$$

If C in itself is a function of time, but it satisfies a different equation of type (5), again solution of (1) exists in closed form. This procedure holds for a sequence of such Riccati equations of any length.

2) Matrix $B(t)$ has constant eigenvectors. The matrix $B(t)$ in this case is decomposable as

$$B(t) = M\Gamma(t)M^{-1} \tag{7}$$

where M is a constant $n\times n$ matrix whose columns are the eigenvectors of $B(t)$ and $\Gamma(t)$ is a diagonal matrix with elements $\gamma_1(t), \cdots \gamma_n(t)$. One can show that

$$\begin{aligned} \dot{X}(t) &= M \exp\left[\int_0^t \Gamma(t)dt\right] M^{-1}A(0)X(0) \\ X(t) &= \int_0^t \dot{X}(t)dt + X(0) \end{aligned} \tag{8}$$

$$X(t) = \left[M\int_0^t \exp\left[\int_0^{t_1} \Gamma(t_2)dt_2\right] \cdot dt_1 M^{-1}A(0) + I\right] X(0). \tag{9}$$

The extension of the former case, namely, for a sequence of Riccati equations, in general applies here, except that the integration of the scalar exponentials appearing in (9) may not be very easy.

Note that condition (7) is equivalent to

$$B(t_1)B(t_2) = B(t_2)B(t_1)$$

for all t.

3) Matrix $B(t)$ has constant eigenvalues. The discussion here will be limited to matrices $B(t)$ of the form

$$B(t) = e^{Dt}\Gamma e^{-Dt} \tag{10}$$

where D is a constant matrix, and Γ is diagonal with constant elements. For this case.

$$\dot{X}(t) = e^{Dt}e^{(\Gamma-D)t}\dot{X}(0) \tag{11}$$

$$X(t) = [(e^{Dt}Fe^{(\Gamma-D)t} - F)A(0) + I]X(0) \tag{12}$$

where F is a constant matrix equal to

$$F = -\int_0^\infty e^{D\sigma}e^{(\Gamma-D)\sigma}d\sigma. \tag{13}$$

Again the extension to a sequence is valid, namely, solution of (1) can be found in closed form if $B(t)$ does not satisfy (10), but it satisfies a matrix Riccati equation such as (5) where C satisfies an equation of form (10) and so forth.

Conclusion

To two known classes of matrices $A(t)$ that render closed-form solutions for time-varying systems (matrices with constant eigenvectors or constant eigenvalues), a third class is added here—matrices that satisfy a certain Riccati equation.

The discussion concerning constant eigenvalue matrices was limited to a very special class (10). More research is necessary for extension of the present results to more general constant eigenvalue systems.

More research is desirable in analysis of structure of time-varying matrices with closed-form solutions in order to unify the present scattered results in this area.

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Squared-Error Minimization with Stability Constraints

Abstract—Minimization of the mean-squared error in linear closed-loop control systems may result in a poorly damped system response to deterministic inputs. To improve the results, it is suggested to minimize the same performance index with a relative stability constraint so that all the characteristic roots have the relative damping coefficient greater or at least equal to a prescribed value.

It is a well-known fact [1] that a minimization of the mean-squared error in linear closed-loop control systems may result in a poorly damped system response to deterministic inputs. To improve this situation, it is suggested that the same performance index be minimized with a relative stability constraint so that all the roots of the corresponding characteristic equation have the relative damping coefficient greater or at least equal to a prescribed value. The idea of constrained minimization is now illustrated in the parameter plane [2], [3].

Consider a control system, shown in the upper right corner of Fig. 1, with the specifications

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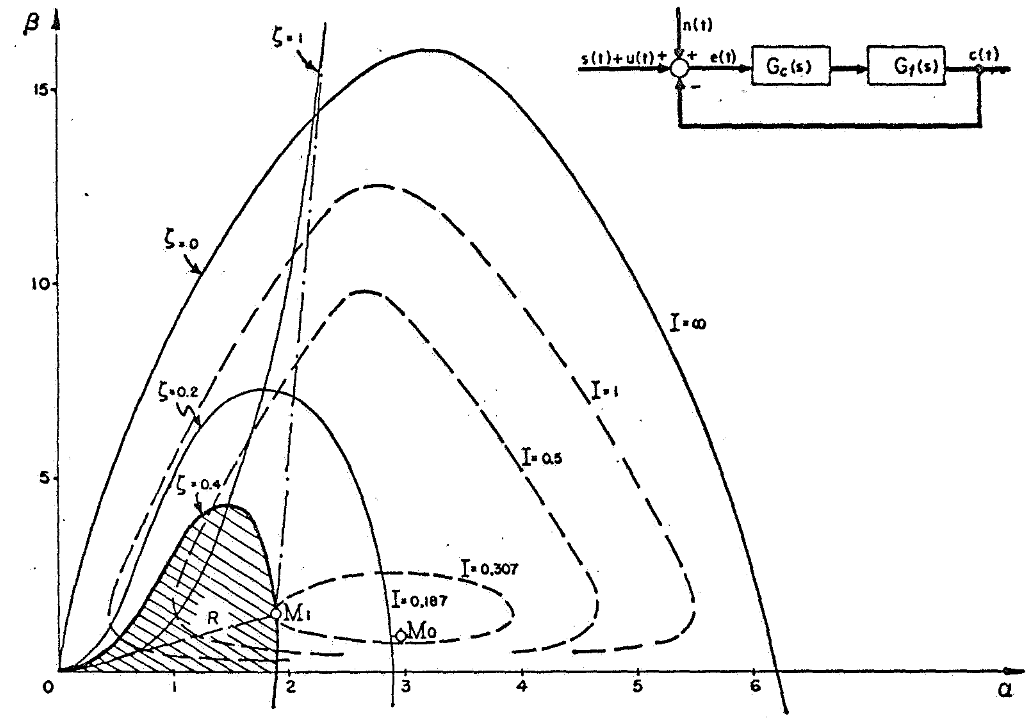

Fig. 1. Squared-error minimization in the parameter plane.

$$G_f(s) = \frac{1}{s(0.001s^2 + 0.025s + 0.25)}$$

$$G_c(s) = \frac{K(s+\lambda\delta)}{s+\delta}; \qquad \phi_S(s) = \frac{\gamma_S}{\pi s^2} \tag{1}$$

$u(t)$ = unit step function;

$$\phi_N(s) = \gamma_N/\pi.$$

It is required to determine the parameters K, λ, and δ of the integral compensator $G_c(s)$ so as to increase the system velocity constant and minimize the mean-squared error while maintaining the overshoot of the unit-step function response below 30 percent of its steady-state value. At first, the noise component is not considered.

$$I(\alpha, \beta) = \frac{0.025\beta^2 - 0.375\alpha\beta - 1.55\beta + 10^{-4}\alpha - 0.65\cdot 10^{-3}}{0.625\beta^2 + \alpha^2\beta - 6.25\alpha\beta}. \tag{4}$$

As known, the choice of the parameter $\delta(\ll 1)$ is not critical and the value 0.04 may be accepted. Then, if the numerical value of γ_S is 2π, the power density of error $\phi_E(s)$ which corresponds to the signal $s(t)$ given as

$$\phi_E(s) = \frac{0.002s^3 + 0.05s^2 + 0.5s + 0.02}{0.001s^4 + 0.025s^3 + 0.25s^2 + (K + 0.01)s + 0.04K\lambda}. \tag{2}$$

The denominator of $\phi_E(s)$ is the characteristic polynomial of the system under investigation. By substituting in the characteristic polynomial

$$K + 0.01 = \alpha$$
$$0.04K\lambda = \beta, \tag{3}$$

one can plot the ζ = constant curves on the parameter $\alpha\beta$ plane in the usual fashion [2], [3] as shown in Fig. 1. These curves determine in the $\alpha\beta$ plane the relative damping regions $R\{\zeta \geq \zeta_0\}$ which correspond to a given value of the relative damping coefficient ζ_0. Thus, for $\zeta_0 = 0.4$, the relative damping region $R\{\zeta \geq 0.4\}$ is determined by curve $\zeta = 0.4$ and is shown shaded on Fig. 1. For the values of α and β which lie inside $R\{\zeta \geq 0.4\}$, all the characterstic roots have $\zeta \geq 0.4$.

By using the substitution (3) and applying the well-known procedure [1], we can evaluate the mean-squared error I from (2) as function of α and β:

For different values of I, a family of the curves determined by (4) is plotted on Fig. 1. The minimum of I is found at the point M_0 which lies outside the damping region $R\{\zeta \geq 0.4\}$ and thus is unsatisfactory. The solution of the formulated control problem is at the point M_1 which corresponds to the constrained minimum of I on R. In addition, the diagram of Fig. 1 can be used to determine readily the characteristic roots related to M_1 [2], [3]. From the curve $\zeta = 0.4$ and tangents to curve $\zeta = 1$, the roots are

$$s_{1,2} = -4.1 \pm j9.39, \qquad s_3 = -0.864,$$
$$s_4 = -16.4. \tag{5}$$

The smaller real root and the zero of the corresponding closed-loop transfer function form a dipole whose effect may be neglected. The other real root is relatively large and its effect can be also neglected. The unit-step function response will be governed only by the pair of complex roots $s_{1,2}$ whose value of damping coefficient ($\zeta = 0.4$) ensures that the overshoot is less than 30 percent.

From the coordinates of the point $M_1(\alpha = 1.89;\ \beta = 1.52)$ and substitution (3), the values of the compensator parameters are $K = 1.88$ and $\lambda = 20.2$. The velocity constant of the compensated system is 38 times greater than that of the uncompensated system. The constrained minimum at M_1 is $I = 0.307$. The unconstrained minimum at M_0 is $I = 0.187$ and corresponds to $\zeta = 0.198$.

In a similar manner, the component of the mean-squared error related to the noise can also be expressed as a function of α and β which for $\gamma_N = 2\pi$ has the following form:

$$J(\alpha, \beta) = \frac{\alpha\beta - 25\alpha^2 - 6.25\beta}{\alpha^2 - 6.25\alpha + 0.625\beta}. \tag{6}$$

The same reasoning outlined above may now be applied to the noise case.

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ANALYSIS AND DESIGN OF OSCILLATORS WITH PARAMETER PLANE METHODS

G. V. Zorbas G. J. Thaler

INTRODUCTION

The ability of a circuit to oscillate depends on the values of its parameters, and the threshold of oscillation can be established for most circuits by linear analysis. The differential equations of the oscillator circuit are studied for small signal operation, which provides linear differential equations and thus a characteristic polynomial for the oscillator. The limit of stability for the characteristic polynomial is also the threshold of oscillation for the oscillator. Use of parameter plane methods permits study of the effects of parameter values on this stability limit and on the frequency of oscillation. These methods also give insight into the limits of adjustment of a given circuit, the effect of parameter values on the amplitude of oscillation, the effects of parameter variations on circuit performance (sensitivity), and the effects of parameter tolerance on operating conditions.

CHARACTERISTIC POLYNOMIAL OF A PHASE SHIFT OSCILLATOR

Fig. 1(a) shows a schematic diagram for a phase shift oscillator. Substituting the equivalent circuit for the transistor provides Fig. 1(b) and applying Thevenin's theorem provides Fig. 1(c). We choose to consider C_V and R_V as parameters (adjustable variables), all other elements have fixed numerical values. The loop equations are

$$I_1(R_1 + R + jX) - I_2R = E \tag{1}$$

$$-I_1R + I_2(2R + jX) - I_3R = 0 \tag{2}$$

$$-I_2R + I_3(R + R_V + jX_V) = 0 \tag{3}$$

By appropriate manipulation the characteristic polynomial is obtained and is:

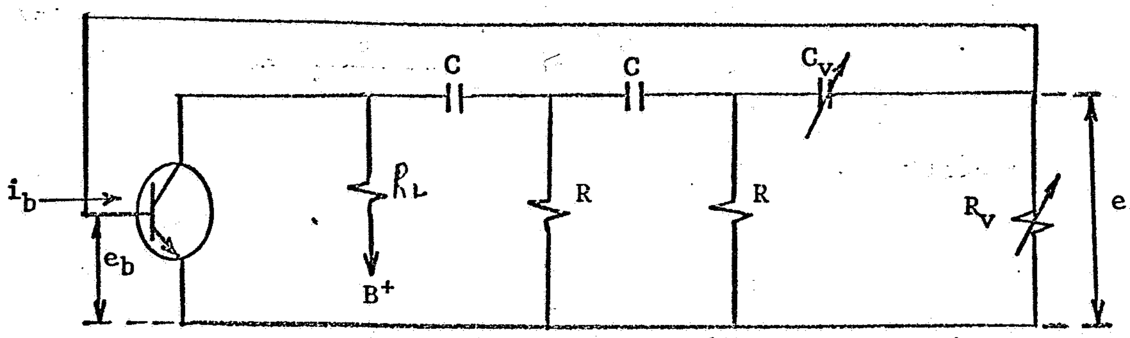
$$C^2\left[C_VR_V(AR^2 + R^2 + 2RR_1) + C_VR^2R_1\right] s^3 + \left[R_VC_V(3RC + R_1C) + (2R^2C + RR_1C) + (R^2C^2 + 2RR_1C^2)\right] s^2 + \left[R_VC_V + C_VR + 3RC + R_1C\right] s + 1 = 0 \tag{4}$$

ANALYSIS OF THE OSCILLATOR

To find the stability limit for this polynomial we must find all R_V, C_V pairs which provide a pair of conjugate imaginary roots, i.e., two of the three roots on the imaginary axis of the s-plane. Since this polynomial is only of third order, several techniques are available for finding the desired relationship. The curves in this paper were

2-12. Transistor phase shift oscillator.

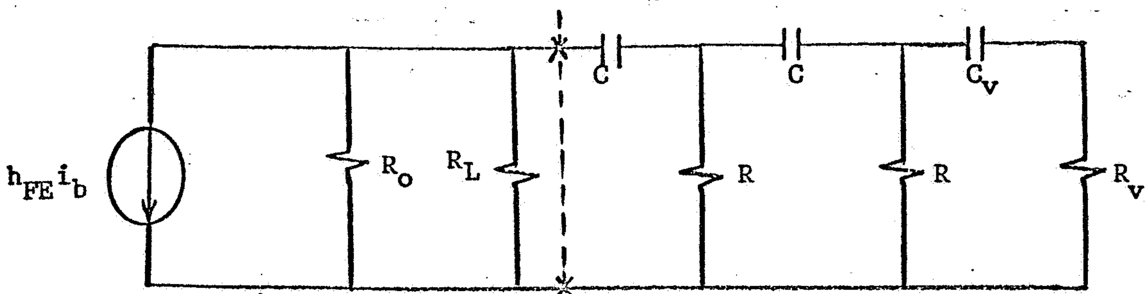
The same problem can be handled with transistors instead of tubes, with minor differences.



Transistor phase shift oscillator

Figure 2-43

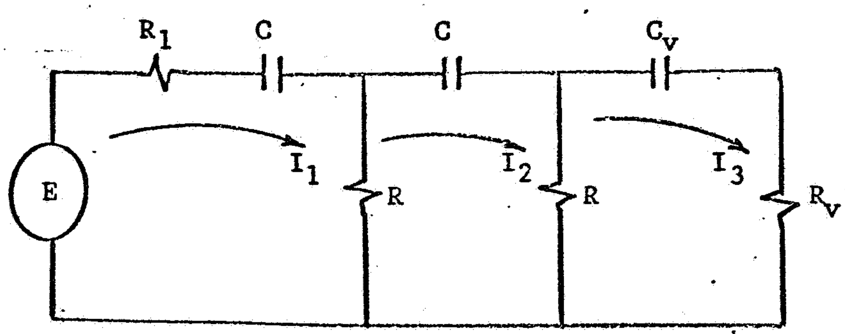
The equivalent circuit of Figure 2-43 is:



Transistor phase shift equivalent circuit

Figure 2-44

Using Thevenin equivalent to the left of points x-x we get:



Transistor phase shift equivalent circuit

Figure 2-45

where

$$R_1 = \frac{R_o R_L}{R_o + R_L}$$

R_o = output resistance

R_L = load resistance

obtained using the parameter plane method (1-5) which is summarized in Appendix I.

Let $C_v \triangleq \alpha$ and $R_v \triangleq \beta$, then the applicable equations are those in the appendix for the case where

$$a_k \triangleq b_k\alpha + c_k\beta + h_k\alpha\beta + d_k$$

Thus we can map the imaginary axis of the s-plane onto the α-β parameter plane through equations A.8. A computer program (6) is available for this and the results are shown on Fig. 2 for three different values of amplifier gain, A. Constant ω contours have been added.

Interpretation of these curves is as follows: each $\zeta = 0$ curve divides the parameter plane into two areas, for $\zeta < 0$ and $\zeta > 0$ (corresponding to the right half of the s-plane and the left half of the s-plane). Stability criteria defining these areas are in the literature (2, 3, 5) and are not repeated here. Choice of a point on the parameter plane (as an operating point) defines values for C_v and R_v as the coordinates of this point. If the chosen point lies <u>on</u> the $\zeta = 0$ curve, then the circuit is just at the stability limit, i.e., the circuit is just able to oscillate. If the chosen point lies in the $\zeta > 0$ area the circuit cannot oscillate, whereas if the chosen point is in the $\zeta < 0$ area the circuit oscillates with growing amplitude until the amplifier saturates and reduces its equivalent gain to an equilibrium value. For example, if the chosen operating point (on Fig. 2) is at $R_v = 2.7 \times 10^6$ and $C_v = 1.15 \times 10^{-9}$, then this point lies on the A = 30 curve

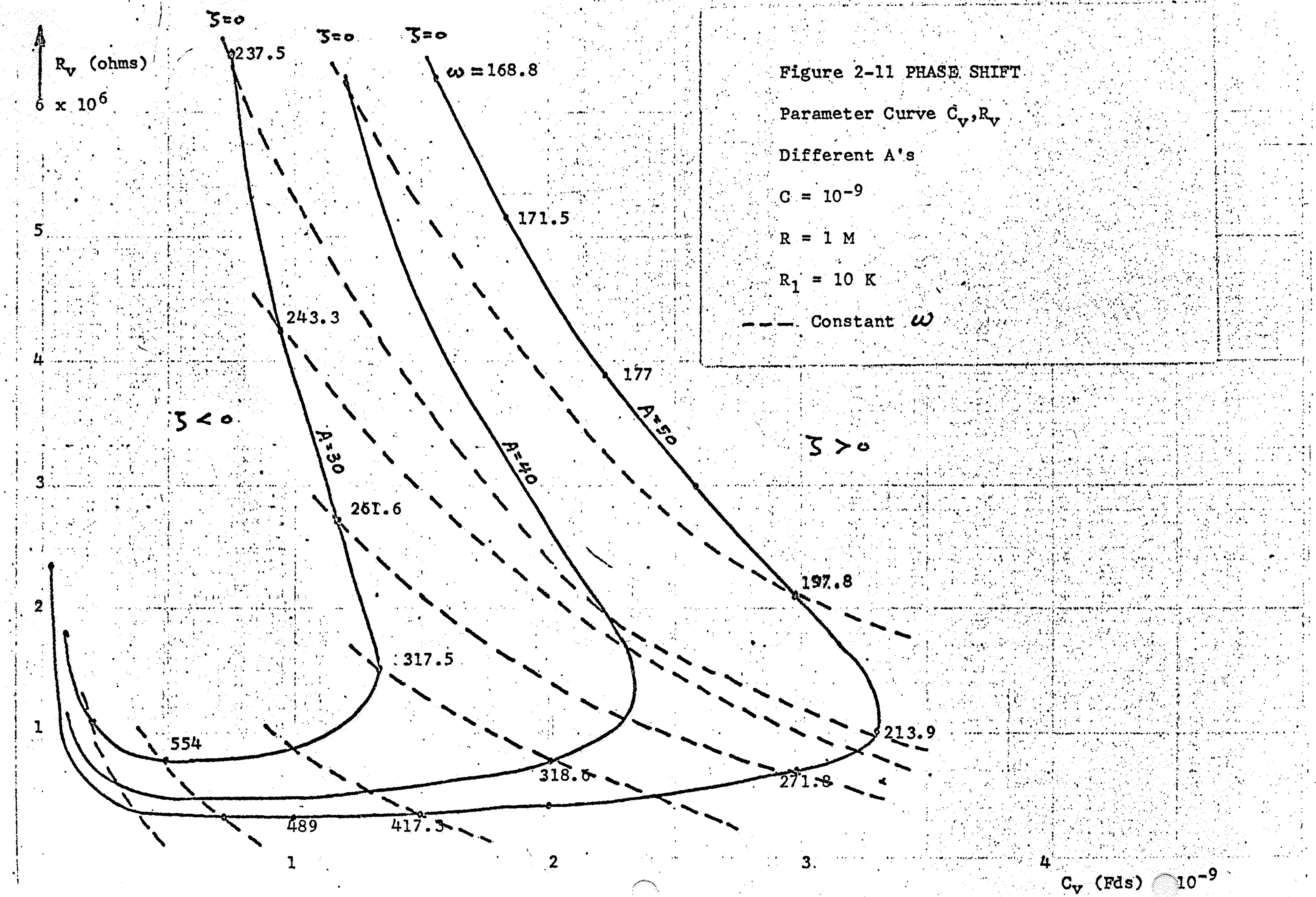


Figure 2-11 PHASE SHIFT

Parameter Curve C_v, R_v

Different A's

$C = 10^{-9}$

$R = 1$ M

$R_1 = 10$ K

– – – Constant ω

at $\omega = 261.6$. Assume, however, that the actual amplifier gain is $A = 40$; then the chosen point is in the $\zeta < 0$ area, the circuit will oscillate at $\omega = 261.6$, and the oscillations will increase in amplitude, saturating the amplifier and reducing the equivalent gain from $A = 40$ to $A = 30$.

The curves of Fig. 2 also indicate that for a considerable range of values for R_v and C_v the circuit will not oscillate. If $A = 40$, for example, the maximum useable value of C_v is about 2.65×15^9, and the minimum value for R_v is about 35×10^6.

Note that the curves of Fig. 2 actually involve <u>four</u> parameters - the coordinates R_v and C_v, the running parameter, ω, and the curve index, A. In using the parameter plane equations it is necessary that ω be retained as the sunning parameter, but any two of parameters R_v, C_v, A may be used as coordinates (α, β) with the third as index. Furthermore, the data can be crossplotted thus obtaining ω as one of the coordinates if this is advantageous.

SENSITIVITY CONSIDERATIONS (7, 8, 9)

In the design of an oscillator one is concerned with the sensitivity of the circuit to adjustment, and to tolerances. If, for example, C_v, R_v and A are close so that the oscillator operates with the amplifier saturated, then the frequency of oscillation is not sensitive to gain variations, but will be sensitive to variations in C_v and R_v.

The incremental change in ω due to an incremental change in either C_v or R_v can be determined by inspection of Fig. 2. Thus tolerance limits can be established for C_v and R_v in terms of permissible frequency deviation. Conversely, if desirable tolerances are known for C_v and R_v inspection of Fig. 2 permits choice of an operating point which provides the desired frequency of oscillation with minimum frequency sensitivity for the specified tolerances on C_v and R_v.

On the other hand, if it is desired to operate with minimal amplifier saturation then sensitivity considerations are much more important. By inspection of Fig. 2 operation at $\omega = 417$ provides a system which is very sensitive to changes in A and in R_v, in the sense that slight changes in either of these parameters may move the operating point across the $\zeta = 0$ line with the result that the system would not be able to oscillate. It may also be seen that operation at $\omega \cong 214$ provides an oscillator with minimum sensitivity to all three parameters. For required operation at $\omega = 417$ an obvious conclusion is that the designer should alter some of the other R or C values in the phase shift circuit to provide improved sensitivity conditions at $\omega = 417$.

In many cases the designer may be concerned with the sensitivity of the circuit to parameters which are not considered controllable or design parameters. For example the parameters of the active element such as R_1 may be subject to substantial variations. Sensitivity relationship may be established in the usual way, but it is more invormative and more useful to simply obtain another set of parameter plane curves with the quantity of interest as a new parameter; sensitivity with respect to the new parameter is then directly observable. Fig. 3 shows such a set of curves for parameters R_1 and C_v; constant ω lines have been added and the sensitivity of the oscillating frequency to variations in R_1 is readily observable.

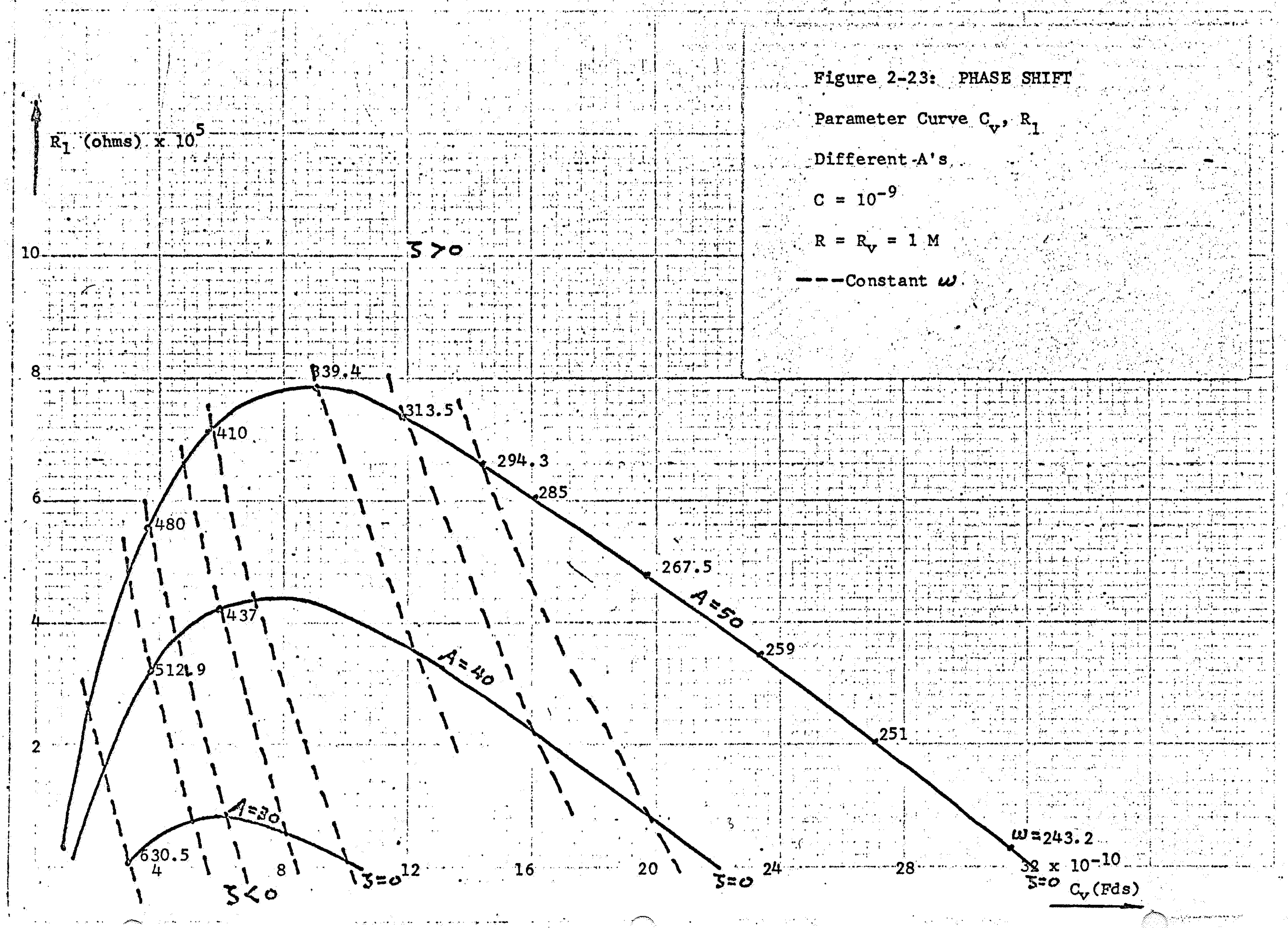


Figure 2-23: PHASE SHIFT

Parameter Curve C_v, R_1

Different A's

$C = 10^{-9}$

$R = R_v = 1$ M

---Constant ω

OTHER EXAMPLES

This technique can be applied to many oscillator circuits. Fig. 4a shows the well known Colpitts circuit for which the characteristic equation is

$$s^3 \frac{C_1C_2L}{y_o} + s^2 \left(C_1L + \frac{rC_1C_2}{y_o}\right) + s\left(C_1r + \frac{C_1 + C_2}{y_o}\right) + \left(A+1\right) = 0$$

where $y_o = \left(r_p + R_L\right)/r_pR_L$. Fig. 4b presents a parameter plane plot showing stability limits with C_1 and L as parameters; Fig. 4c presents a similar plot with C_2 and L as parameters.

In like manner Fig. 5a shows the tuned plate oscillator circuit, for which the characteristic equation is

$$s^2 \frac{CL}{y_o} + s\left(AM + L + \frac{Cr}{y_o}\right) + \left(r + \frac{1}{y_o}\right) = 0$$

where $y_o = (r_p + R_L)/r_pR_L$. Fig. 5b shows the parameter plane relationships.

A very interesting case is the Hartley circuit, shown in Fig. 6a, for which the characteristic equation is

$$s^3 \ (ACL_1L_2 - ACM^2 + CL_1L_2 - CM^2) +$$

$$s^2\left(ACL_1r_2 + ACL_2r_1 + CL_1r_2 + CL_2R_1 + \frac{CL_1 + CL_2 + 2CM}{y_o}\right) +$$

$$s\left(ACr_1r_2 - AM + Cr_1r_2 + L_2 + \frac{Cr_1 + Cr_2}{y_o}\right) + r_2 + \frac{1}{y_o} = 0$$

where $y_o = (r_p + R_L)/r_pR_L$. Fig. 6b shows typical parameter plane curves for parameters C and L_1. For parameters C and L_2, however, the $\zeta = 0$ curve has two branches as shown on Fig. 7. Between these branches is an area marked "Dead Zone". This is a misnomer, of course, because the curves are actually continuous across this zone. It may be observed from the figure that the "Dead Zone" occurs between the low frequency terminations of the two branches of each curve. This is simply an indication that there is a minimum frequency below which the circuit will not oscillate. The frequency at the curve termination is just slightly above this minimum frequency, and the computer program (which increments ω and computes values for C and L_2) chooses too low a value for ω and computes complex values for C and L_2.

VERIFICATION OF THEORY

All circuits discussed were simulated using the CDC 1604 computer or a DONNER ANALOG COMPUTER. The theoretical predictions were verified in all cases, including studies of the "dead zone" phenomenon for the Hartley circuit.

4. Colpitt's Oscillator

4-1. Derivation of characteristic equation.

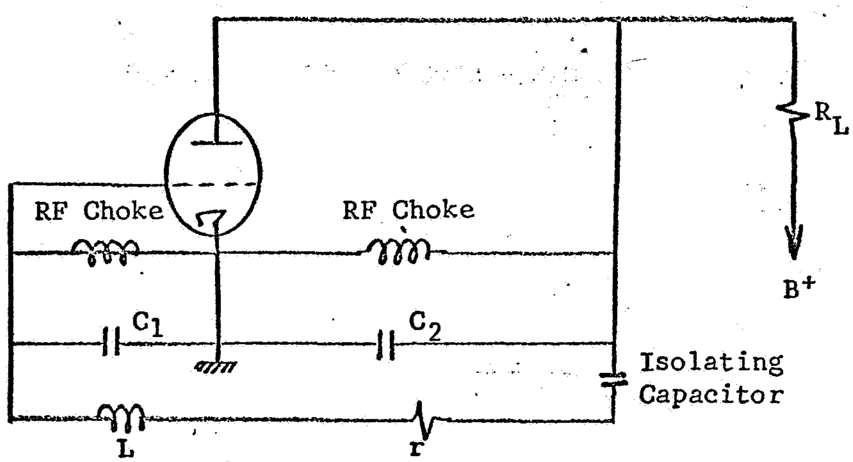


Figure 4-1. Colpitt's oscillator

Characteristic equation:

$$y_f Z_f + y_o Z_i + 1 = 0$$

$$y_f = g_m$$

$$Z_f = -\frac{1}{\omega^2 C_1 C_2 Z} = \frac{1}{S^2 C_1 C_2 Z}$$

$$Z_i = -\frac{1}{Z}\left[\frac{1}{\omega^2 C_1 C_2} + j\left(\frac{r}{\omega C_2}\right) - \frac{L}{C_2}\right]$$

$$= \frac{1}{Z}\left[\frac{1}{S^2 C_1 C_2} + \frac{r}{SC_2} + \frac{L}{C_2}\right] = \frac{1}{Z} \quad \frac{1 + SC_1 r + S^2 C_1 L}{S^2 C_1 C_2}$$

$$Z = r + SL + \frac{1}{SC_1} + \frac{1}{SC_2} + \frac{S^2 r C_1 C_2 + S^3 C_1 C_2 L + SC_2 + SC_1}{S^2 C_1 C_2}$$

$$= \frac{S^2 C_1 C_2 L + S\, r C_1 C_2 + C_1 + C_2}{SC_1 C_2}$$

Plug into the characteristic equation:

$$g_m \frac{1}{S^2 C_1 C_2 Z} + y_o\left[\frac{1}{Z} \frac{(1 + SC_1 r + S^2 C_1 L)}{S^2 C_1 C_2}\right] + 1 = 0$$

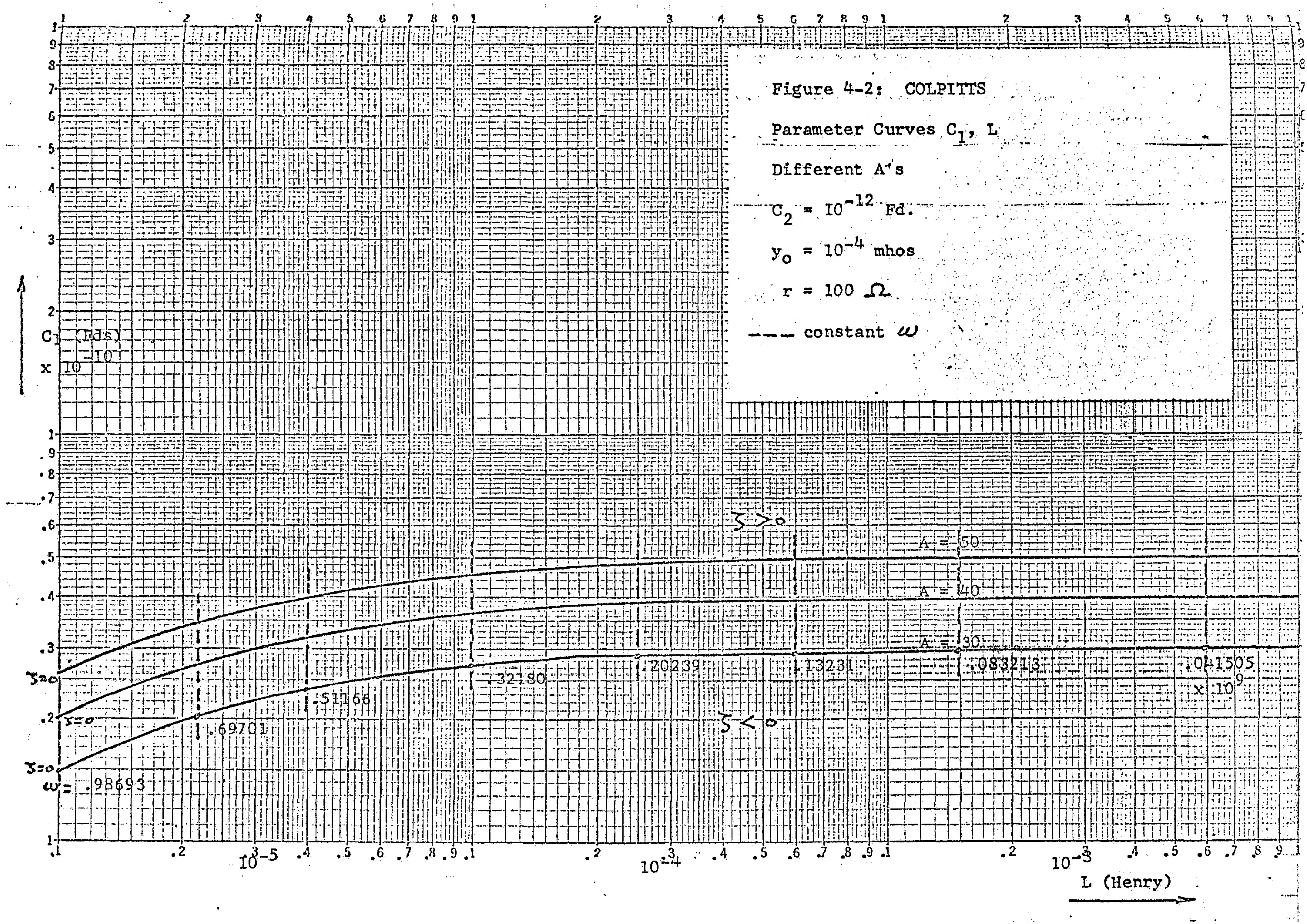


Figure 4-2: COLPITTS
Parameter Curves C_1, L
Different A's
$C_2 = 10^{-12}$ Fd.
$y_o = 10^{-4}$ mhos
$r = 100\ \Omega$
--- constant ω
C_1 (Fds) x 10^{-10}
L (Henry)
10^{-5}
10^{-4}
10^{-3}
A = 50
A = 40
A = 30
$\zeta > 0$
$\zeta < 0$
$\zeta = 0$
$\zeta = 0$
$\zeta = 0$
ω = .98693
.69701
.51166
.32180
.20239
.13231
.083213
.041505
x 10^9

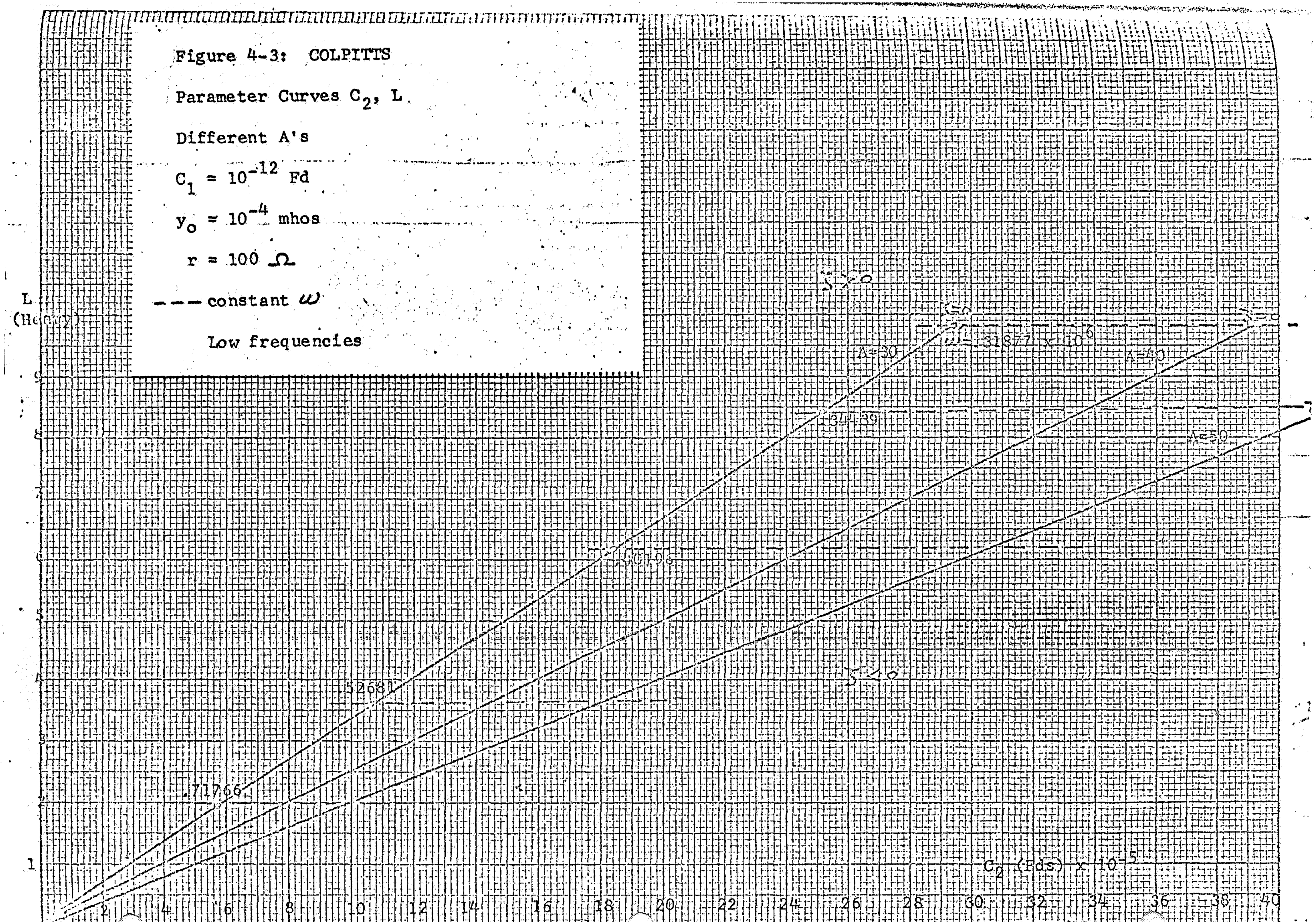


Figure 4-3: COLPITTS

Parameter Curves C_2, L

Different A's

$C_1 = 10^{-12}$ Fd

$y_o = 10^{-4}$ mhos

$r = 100\ \Omega$

--- constant ω

Low frequencies

7. Tube Tuned Plate Oscillator

7-1. Derivation of the characteristic equation.

It has been found [2] that:

$$Z_i = \frac{1}{\omega^2 C^2 Z} - j\frac{1}{\omega C} = -\frac{1}{S^2C^2Z} + \frac{1}{SC} \quad \text{letting } S = j\omega$$

$$Z_f = \frac{M}{CZ}$$

$$Z = r + SL + \frac{1}{SC} = \frac{SCr + S^2LC + 1}{SC}$$

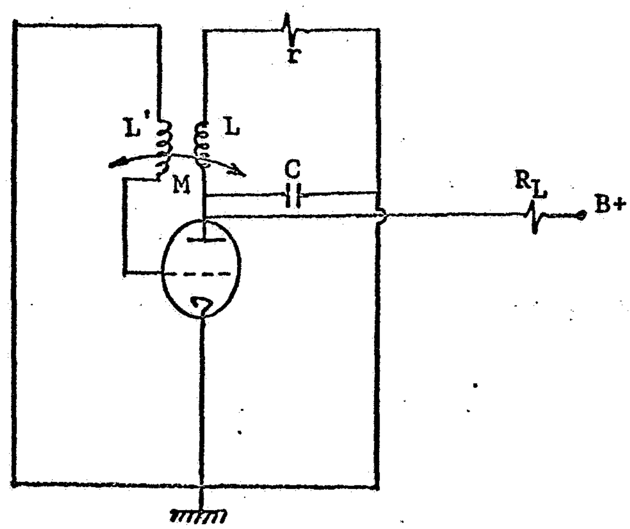


Figure 7-1. Tuned plate oscillator

and equivalently:

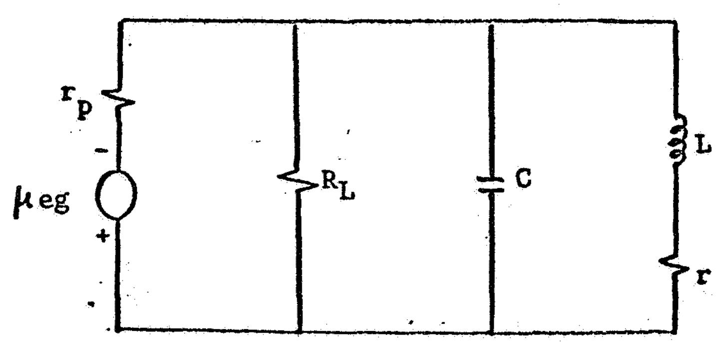


Figure 7-2. Tuned plate equivalent circuit

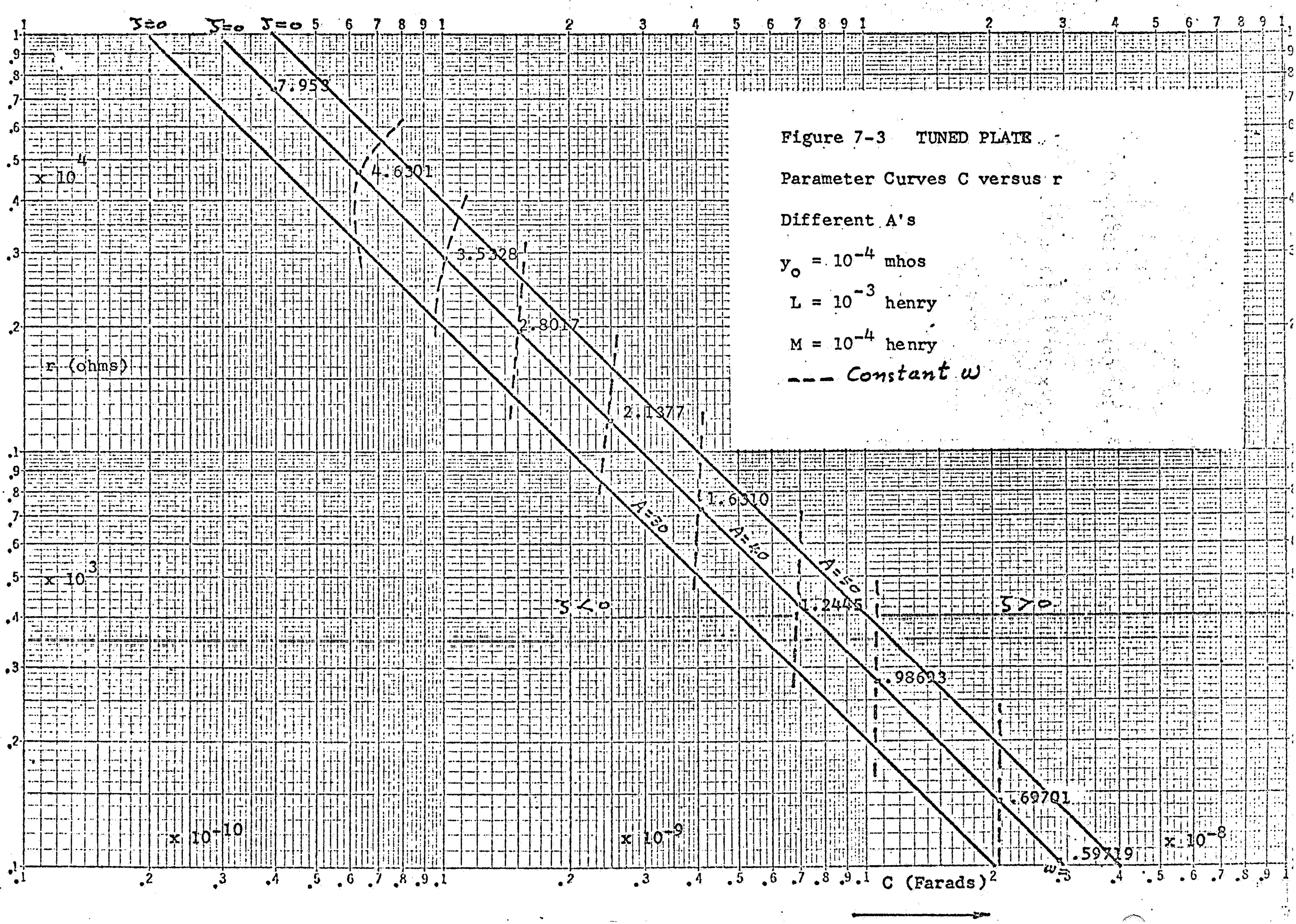


Figure 7-3 TUNED PLATE
Parameter Curves C versus r
Different A's
$y_o = 10^{-4}$ mhos
$L = 10^{-3}$ henry
$M = 10^{-4}$ henry
--- Constant ω
r (ohms)
C (Farads)
x 10^4
x 10^3
x 10^-10
x 10^-9
x 10^-8
ζ=0
ζ=0
ζ=0
ζ<0
ζ>0
A=30
A=40
A=50
7.953
4.6301
3.5328
2.8017
2.1377
1.6310
1.2445
.98603
.69701
.59719
ω=3

5. Hartley Oscillator.

5-1. Derivation of characteristic equation.

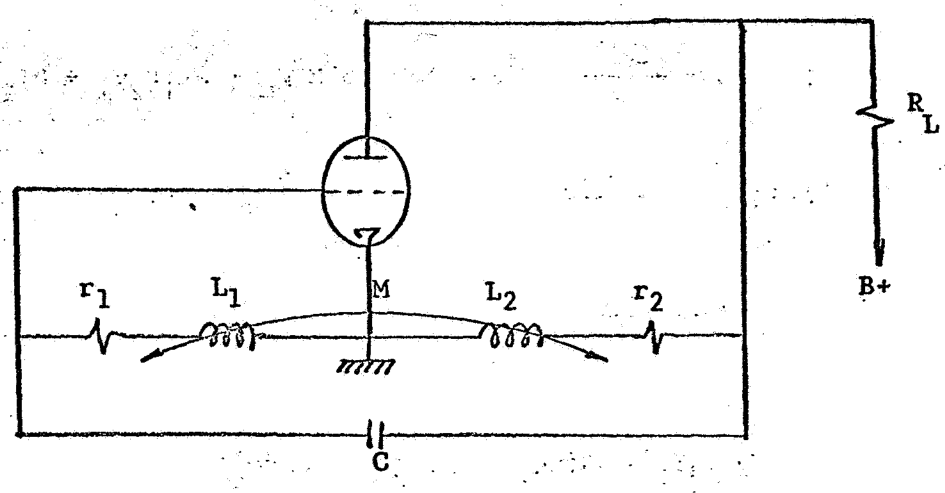


Figure 5-1. Hartley Oscillator

Characteristic equation:

$$y_f Z_f + y_o Z_i + 1 = 0 \qquad y_f = g_m$$

$$y_o = \frac{1}{r_p // R_L} = \frac{r_p + R_L}{r_p R_L}$$

$$Z_f = \frac{1}{Z}\left[r_1 r_2 - \frac{M}{C} - \omega^2 L_1 L_2 + \omega^2 M^2\right] + \frac{j}{Z}(L_1 r_2 + L_2 r_1)$$

$$= \frac{1}{Z}\left[r_1 r_2 - \frac{M}{C} + S^2 L_1 L_2 - S^2 M^2 + S L_1 r_2 + S L_2 r_1\right]$$

$$Z_i = \frac{1}{Z}\left[r_1 r_2 + \frac{L_2}{C} + S^2 L_1 L_2 - S^2 M^2 + S L_1 r_2 + S L_2 r_1 + \frac{r_2}{SC}\right]$$

$$Z = r_1 + r_2 + S L_1 + S L_2 + 2SM + \frac{1}{SC}$$

Plug into the characteristic equation:

$$g_m\left[\frac{1}{Z}(r_1 r_2 - \frac{M}{C} + S^2 L_1 L_2 - S^2 M^2 + S L_1 r_2 + S L_2 r_1)\right]$$

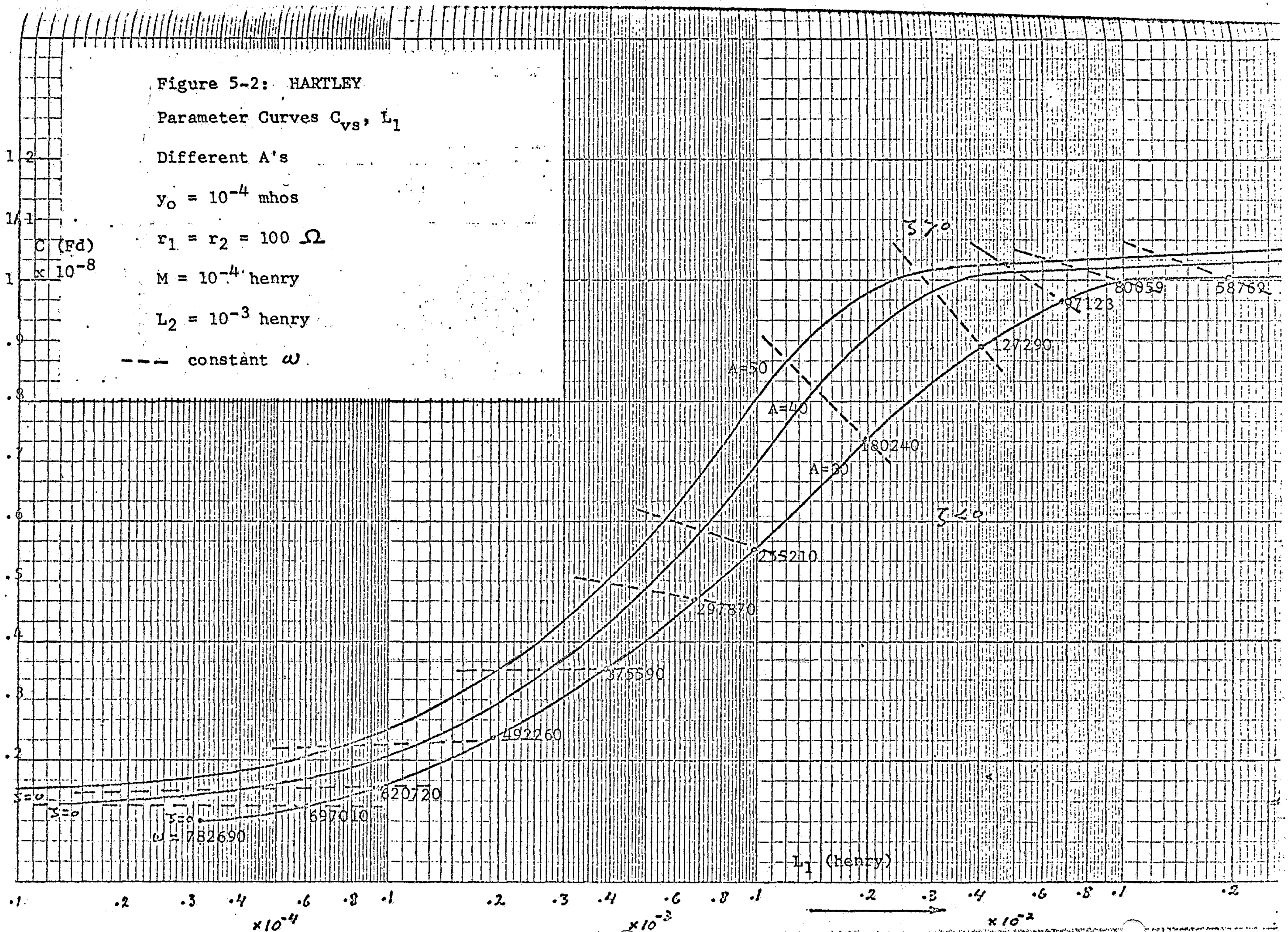


Figure 5-2: HARTLEY
Parameter Curves C_{vs}, L_1
Different A's
$y_o = 10^{-4}$ mhos
$r_1 = r_2 = 100\ \Omega$
$M = 10^{-4}$ henry
$L_2 = 10^{-3}$ henry
--- constant ω
C (Fd)
x 10^{-8}
A=50
A=40
A=30
$\zeta > 0$
$\zeta < 0$
$\zeta = 0$
$\omega = 782690$
697010
620720
492260
375590
297870
235210
180240
127290
97123
80059
58762
L_1 (henry)
$\times 10^{-4}$
$\times 10^{-3}$
$\times 10^{-2}$

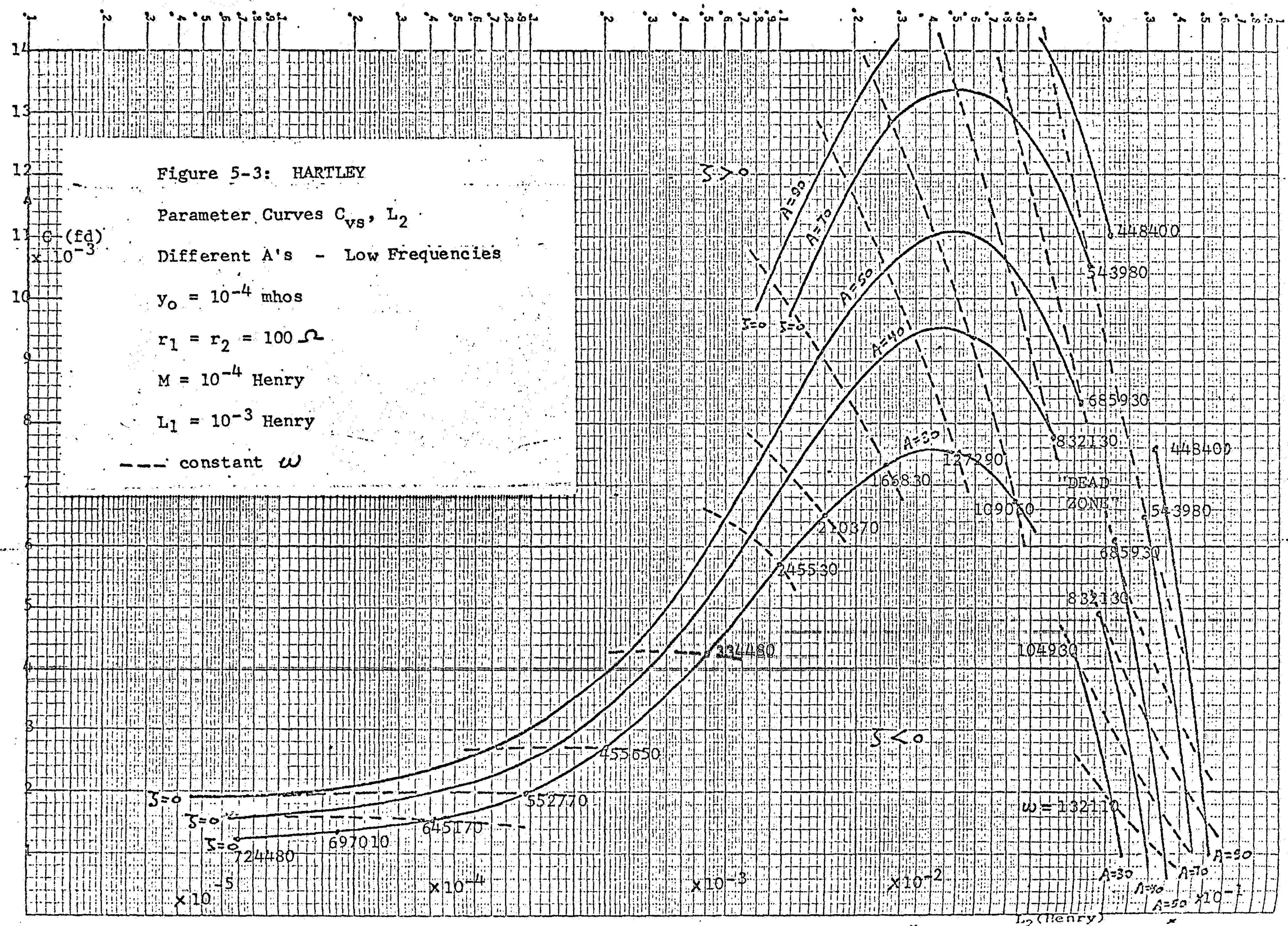


Figure 5-3: HARTLEY

Parameter Curves C_{vs}, L_2

Different A's - Low Frequencies

$y_o = 10^{-4}$ mhos

$r_1 = r_2 = 100\ \Omega$

$M = 10^{-4}$ Henry

$L_1 = 10^{-3}$ Henry

--- constant ω

CONCLUSIONS:

The choice of parameter values for design or oscillators is expedited by the use of the parameter plane. Ranges of values of the parameters suitable for operation at a given frequency are easily obtained. The sensitivity of the oscillating frequency to parameter variations is available from the plots. As a result optimization of the design is simplified.

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APPENDIX I

Derivation of Parameter Plane Relationships:

Consider the polynomial

$$\sum_{k=0}^{n} a_k s^k = 0 \tag{A.1}$$

where $a_k \triangleq b_k\alpha + c_k\beta + d_k$ (A.1a)

b_k, c_k, d_k are numbers,

α, β are parameters

$s \triangleq \omega_n(\cos\theta + j\sin\theta) = \omega_n e^{jk\theta}$

where $\theta = \cos^{-1}(-\zeta)$

Then $s^k = \omega_n{}^k e^{jk\theta} = \omega_n{}^k(\cos k\theta + j\sin k\theta)$

$$= \omega_n{}^k\left[(-1)^k T_k(\zeta) + j\sqrt{1-\zeta^2}(-1)^{k+1}\bar{U}_k(\zeta)\right] \tag{A.2}$$

where $T_k(\zeta)$ and $\bar{U}_k(\zeta)$ are Chebysher functions and are

$$T_k(\zeta) \triangleq \cos k\theta = \cos(k\cos^{-1}\zeta)$$

$$\bar{U}_k(\zeta) \triangleq \frac{\sin k\theta}{\sin\theta} = \frac{\sin(k\cos^{-1}\zeta)}{\sin(\cos^{-1}\zeta)}$$

Inserting A.2 in A.1 and requiring that the reals and imaginaries go to zero independently provides:

$$\sum_{k=0}^{n} a_k\omega_n{}^k(-1)^k T_k(\zeta) = 0$$

$$\sum_{k=0}^{n} a_k\omega_n{}^k(-1)^{k+1}\bar{U}_k(\zeta) = 0 \tag{A.3}$$

but $T_k(\zeta) = \zeta\,[\bar{U}_k(\zeta)] - \bar{U}_{k-1}(\zeta)$ (A.4)

Substituting A.4 into A.3 provides

$$\sum_{k=0}^{n}(-1)^k a_k \omega_n{}^k\, \bar{U}_{k-1}(\zeta) = 0$$

$$\sum_{k=0}^{n}(-1)^k a_k \omega_n{}^k\, \bar{U}_k(\zeta) = 0 \tag{A.5}$$

Inserting A.1a and collecting terms

$$\begin{aligned} \alpha B_1 + \beta C_1 + D_1 &= 0 \\ \alpha B_2 + \beta C_2 + D_2 &= 0 \end{aligned} \tag{A.6}$$

where

$$B_1 = \sum_{k=0}^{n}(-1)^k\, b_k \omega_n{}^k\, \bar{U}_{k-1}(\zeta); \quad B_2 = \sum_{k=0}^{n}(-1)^k\, b_k \omega_n{}^k\, \bar{U}_k(\zeta)$$

$$C_1 = \sum_{k=0}^{n}(-1)^k\, c_k \omega_n{}^k\, \bar{U}_{k-1}(\zeta) \quad C_2 = \sum_{k=0}^{n}(-1)^k\, c_k \omega_n{}^k\, \bar{U}_k(\zeta)$$

$$D_1 = \sum_{k=0}^{n}(-1)^k\, d_k \omega_n{}^k\, \bar{U}_{k-1}(\zeta) \quad D_2 = \sum_{k=0}^{n}(-1)^k\, d_k \omega_n{}^k\, \bar{U}_k(\zeta)$$

Equation A.6 may be solved for α and β (using Cramer's rule)

$$\alpha = \frac{C_1D_2 - C_2D_1}{B_1C_2 - B_2C_1} \qquad \beta = \frac{B_2D_1 - D_2B_1}{B_1C_2 - B_2C_1} \tag{A.7}$$

For any chosen point on the s-plane (not on the real axis) ζ and ω_n are defined, $B_1B_2, C_1, C_2, D_1, D_2$ are the numbers, so α and β are determined numerically. Thus any line on the s-plane may be mapped onto the α-β parameter plane through Eq. A.7.

Equations A.7 obtain when the coefficients, a_k, are linear in α and β. Comparable results can be obtained when the a_k are not linear in α and β. A case of interest here is the case when

$$a_k = b_k\alpha + c_k\beta + h_k\alpha\beta + d_k \tag{A.1b}$$

for this case a solution is

$$\begin{aligned} \alpha_{1,2} &= \frac{-e \pm \sqrt{e^2-4ac}}{2a} \\ \beta_{1,2} &= \frac{\beta_1\alpha_{1,2}+D_1}{H_1\alpha_{1,2}+C_1} = -\frac{B_2\alpha_{1,2}+D_2}{H_2\alpha_{1,2}+C_2} \end{aligned} \tag{A.8}$$

where $B_1, C_1, D_1, B_2, C_2, D_2$ areas previously defined and

$$H_1 = \sum_{k=0}^{n}(-1)^k\, h_k\omega_n{}^k\, \bar{U}_{k-1}(\zeta) \qquad H_2 = \sum_{k=0}^{n}(-1)^k\, h_k\omega_n{}^k\, \bar{U}_k(\zeta)$$

$$a = B_2H_1 - B_1H_2 \qquad c = C_1D_2 - C_2D_1$$

$$b = C_2H_1 - C_1H_2 \qquad d = B_1D_2 - B_2D_1$$

$$e = C_2B_1 - B_2C_1 + H_1D_2 - H_2D_1 \quad f = C_2B_1 - B_2C_1 + H_1D_2 - H_2D_1$$

DESIGN OF PHASE LOCKED LOOPS ON THE PARAMETER PLANE*

J. S. Karmarkar**   G. J. Thaler***

ABSTRACT: Phase locked loops are considered from a "nonlinear systems" viewpoint, utilizing parameter plane and describing function techniques. Pertinent specification curves together with the region of stability are displayed in the plane of the parameters. Procedures are presented for predicting transient response in the absence of noise, for estimating statistical parameters in the presence of noise, for predicting limit cycles, and for studying root sensitivity to parameter variation. The design problem is, in the main, reduced to one of inspection of the pertinent parameter plane curves.

Second and third order systems are analyzed to illustrate the technique, although the procedure is directly applicable to systems of higher order.

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INTRODUCTION: Phase locked loops, of the type widely used in communication systems, may be represented by the block diagram of Fig.1a. For this representation the input signal and output signal are

$$\begin{aligned} \text{INPUT SIGNAL} &= \sqrt{2}\,A\sin\theta(t) \\ \text{OUTPUT SIGNAL} &= \sqrt{2}\,K_1\cos\theta'(t) \end{aligned} \tag{1}$$

The multiplication of signals in Fig.1a can be replaced by a summation and a nonlinear element as shown in Fig.1b, and the signals subtracted at the summer are the phases of the input and output waves. In Fig.1b:

$$\begin{aligned} \theta_1(t) &= \theta(t) - \omega_o t \\ \theta_2(t) &= \theta'(t) - \omega_o t \\ \phi(t) &= \theta_1(t) - \theta_2(t) \\ \omega_o &= \text{quiescent oscillator frequency.} \\ \eta'(t) &= \text{noise, if any is to be considered.} \end{aligned} \tag{2}$$

Note that $\theta(t)$ and $\theta'(t)$ are defined in Eq.(1).

If a describing function can be derived for the nonlinear block of Fig.1b, and if transfer functions are known for the linear parts of the loop, then analysis and design on the parameter plane may be accomplished.

DERIVATION OF DESCRIBING FUNCTION

The nonlinear block operates on its input signal and produces an output which is A times the sin of the input signal.

The input signal is $\phi(t) = \phi_m \sin \omega t$ (3)

Then the output is

$$\begin{aligned} E(t) &= A\sin[\phi_m \sin \omega t] \\ &= A*2\sum_{n=0}^{\infty} J_{2n+1}(\phi_m)\sin(2n+1)\omega t \\ &= 2A[J_1(\phi_m)\sin\omega t + J_3(\phi_m)\sin 3\omega t + \ldots] \end{aligned} \tag{4}$$

where the J's are Bessel functions.

Truncating* the series the ratio of output over input is the describing function

$$N = \frac{2A\ J_1(\phi_m)\sin\omega t}{\phi_m \sin\omega t} = \frac{2A\ J_1(\phi_m)}{\phi_m} \tag{5}$$

Table 1 gives the values of ϕ_m, $J_1(\phi_m)$ and N for A = 1.

* We note that $2J_1(\phi_m) = .881$; $2J_3(\phi_m) = .06$; $2J_5(\phi_m) = .0005$ so we may reasonably neglect J_3 and J_5.

TABLE I (8)

DESCRIBING FUNCTION OF THE NONLINEAR BLOCK

ϕ_m	$J_1(\phi_m)$	N
0	0	1.0
.1	.05	1.0
.2	.0995	.995
.3	.148	.987
.4	.196	.98
.5	.242	.968
.6	.287	.957
.7	.329	.940
.8	.369	.922
.9	.406	.902
1.0	.440	.88
1.1	.471	.855
1.2	.498	.832
1.3	.522	.804
1.4	.542	.775
1.5	.558	.745
1.6	.570	.714
1.7	.577	.68
1.8	.581	.636
1.9	.581	.612
2.0	.576	.576

FORMULATION OF THE PARAMETER PLANE CURVES (3)

The parameter plane method is basically a linear systems method, but can be used in conjunction with describing functions to analyze and design many nonlinear systems. The method portrays the s-plane location of the roots of the characteristic equation as controlled by two parameters α and β. If transfer functions are used the characteristic equation of a phase locked loop is readily obtained and may be expressed as

$$\sum_{k=0}^{N} a_k s^k = 0 \tag{6}$$

where $a_k = b_k\alpha + c_k\beta + d_k$

b_k, c_k, d_k are constants

α, β are parameters

Using the root-coefficient relationships

$$\begin{bmatrix} a_{N-1} \\ a_{N-2} \\ a_{N-3} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} b_{N-1} & c_{N-1} & d_{N-1} \\ b_{N-2} & c_{N-2} & d_{N-2} \\ \vdots & & \vdots \\ b_0 & c_0 & d_0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ 1 \end{bmatrix} = \begin{bmatrix} N_{R_1} \\ N_{R_2} \\ \vdots \\ N_{R_N} \end{bmatrix} \tag{7}$$

where

$$N_{R_1} = \sum_{i=1}^{N} r_i \;; \quad N_{R_2} = \sum_{\substack{i=1,j=1 \\ i\neq j}}^{N} r_i r_j \quad \text{etc.}$$

To specify the relationship between a pair of complex roots and the parameters α and β, define the coordinates of the complex pair by (σ, ω) and rearrange the right hand side of Eq.(7):

$$\begin{bmatrix} b_{N-1} & c_{N-1} & d_{N-1} \\ b_{N-2} & c_{N-2} & d_{N-2} \\ \vdots & \vdots & \vdots \\ b_0 & c_0 & d_0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{N-2}R_1 & 1 & 0 \\ {}^{N-2}R_2 & {}^{N-2}R_1 & 1 \\ {}^{N-2}R_3 & {}^{N-2}R_2 & {}^{N-2}R_1 \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 \\ 2\sigma \\ \sigma^2 + \omega^2 \end{bmatrix} \tag{8}$$

which manipulates to

$$\begin{bmatrix} b_{N-1} & c_{N-1} & -1 & 0 \ldots & 0 \\ b_{N-2} & c_{N-2} & -2\sigma & -1 \ldots & 0 \\ \vdots & \vdots & -(\sigma^2+\omega) & -2\sigma & \vdots \\ \vdots & \vdots & \vdots & \vdots & -1 \\ \vdots & \vdots & \vdots & \vdots & -2\sigma \\ b_0 & c_0 & 0 & 0 & -(\sigma^2+\omega^2) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ {}^{N-2}R_1 \\ \vdots \\ {}^{N-2}R_{N-2} \end{bmatrix} = \begin{bmatrix} 2\sigma - d_{N-1} \\ (\sigma^2+\omega^2) - d_{N-2} \\ -d_{N-3} \\ \vdots \\ -d_0 \end{bmatrix} \tag{9}$$

For chosen values of σ and ω Eq.(9) may be processed by standard computer subroutines to evaluate α and β (and also ${}^{N-2}R_1$, etc. if desired). Thus constant σ-lines may be chosen on the s-plane and mapped onto the $\alpha - \beta$ parameter plane, as may constant ω lines. A basic use of this mapping is that one may locate a desired point (σ_1, ω_1) on the $\alpha - \beta$ plane, and the coordinates of the point specify the values of α and β which must be used if the characteristic equation is to have roots with the desired σ_1 and ω_1.

Using the transformations

$$\sigma = \zeta\omega_n$$
$$\omega = \omega_n\sqrt{1-\zeta^2}$$

We can also obtain constant ζ and ω_n lines if needed, and matrix equations for evaluating root sensitivities are also available.

TABLE II PERFORMANCE RELATIONSHIPS

Received Signal Phase $\phi(t)$	Order of Loop	F(s)	Closed Loop Transfer Function H(s)	Steady State Error	Noise Bandwidth of Loop -- BL	Mean Squared Error
$\omega t+\theta_o$	1	1	$\frac{N'}{s+N'}$	$\frac{\omega-\omega_o}{N'}$	$\frac{N'}{4}$	$\frac{1}{2N'}$
$\omega t+\theta_o$	2	$1+\frac{a}{s}$	$\frac{N'(s+a)}{s^2+N's+N'a}$	0	$\frac{N'+a}{4}$	$\frac{1}{2N'a}$
$\omega t+\theta_o$	2	$\frac{s+a}{s+\epsilon}$	$\frac{N'(s+a)}{s^2+(N'+\epsilon)s+N'a}$	$\frac{\epsilon}{a}(\frac{\omega-\omega_o}{N'})$	$\frac{N'(N'+a)}{4(N'+\epsilon)}$	$\frac{N'a+\epsilon^2}{2(N'+\epsilon)N'a}$
$\frac{1}{2}Rt^2+\omega t+\theta_o$	2	$1+\frac{a}{s}$	$\frac{N'(s+a)}{s^2+N's+N'a}$	$\frac{R}{aN'}$	$\frac{N'+a}{4}$	$\frac{1}{2N'a}$
$\frac{1}{2}Rt^2+\omega t+\theta_o$	3	$1+\frac{a}{s}+\frac{b}{s^2}$	$\frac{N'(s^2+as+b)}{s^3+N's^2+aN's+bN'}$	0	$\frac{N'(aN'+a^2-b^2)}{4(aN'-b)}$	$\frac{a}{2(aN'-b)}$

where

$$BL = \frac{1}{2\pi}\int_0^\infty |H(i\omega)|^2 \, d\omega$$

$$MSE = \int_{-\infty}^{\infty} |1-H(i\omega)|^2 \, S_{\theta_1}(\omega) \frac{d\omega}{2\pi}$$

We assume $S_{\theta_1}(\omega)$ [input spectral density] to correspond to a step change in the time domain

$$\text{ie} \quad S_{\theta_1}(\omega) = \frac{1}{i\omega}$$

using numerical integration techniques MSE curves for any input spectral density may be obtained.

PHASE LOCKED LOOP ANALYSIS

Fig. 2 gives a modified form of the block diagram of Fig. 1b. The characteristic polynomial is readily obtained when F(s) is specified. For example let $F(s) = (s+a)/(s+\alpha)$ and let $N' = \beta$. Then the characteristic equation of the loop is

$$s^2 + (\alpha+\beta)s + a\beta = 0$$

The real root line (3) for $s = 0$ is $\beta = 0$ and the parameter plane curves are calculated using Eq.(9) and appear on Fig.3. By inspection a pair of complex roots can be chosen (to satisfy loop criteria for stability and damping) from which α and β are determined.

Selection of the operating point on Fig. 3 requires consideration of all of the performance specifications. Many of these can be described by algebraic functions of α and β, and thus can be plotted on Fig. 2 to act as guides in the design. These relationships are not derived here because they are well known (Ref.1 and 7) but the results are tabulated in Table II using the notation of Fig. 2. Curves for each of these performance relationships have been added to Fig. 3. It is emphasized that these relations are derived for the linear model.

In addition to these factors the nonlinear nature of the loop must be considered. For the system of Fig.3 a numerical value for "a" must be chosen before analysis can proceed. For this paper $a = 5.0$ was chosen. (If results are not acceptable analysis can be repeated with another value of "a".) Since the ordinate of the plot on Fig. 3 is $\beta = N'$, the value of β varies with the signal into the nonlinear element, but the value of α does not change; therefore the motion of the roots due to the nonlinearity can be described on the parameter plane by the line N! A scale may be added to this line using Eq.(5), and this scale may be adjusted by altering AK.

Choosing $\alpha = 2.5$, the nonlinear operating line has been drawn on Fig. 3A and Fig. 3B for A = 1 and K = 1. From Table I, $N' = 1$ when $\phi = 1.5$. The other lines may be similarly scaled for K = 2 (see Fig. 3A) and K = 10 (see Fig. 3B). For any selected α, A and K analysis consists of reading the root values and other performance characteristics from the parameter plane. A basic design method is to try various combinations of α, A and K until an acceptable set of performance characteristics is obtained.

PERFORMANCE OPTIMIZATION

When noise is present system performance deteriorates and it is desired to choose an operating point which in some way optimizes the performance. In steady state the phase error variance due to noise is given by

$$\sigma^2_{\phi N} = \frac{N_o B_L}{A^2} \quad \text{(for a linear model)}$$

Since the phase error due to noise is independent of that due to modulation, thus

$$\sigma^2_{\phi} = \sigma^2_{\phi N} + MSE$$

but the phase error due to noise is proportional to the loop bandwidth

$$\sigma^2_{\phi N} \propto B_L$$

It is convenient to choose a performance criterion, Q, such that

$$Q = B_L + \lambda(MSE)$$

where λ is a weighting factor. Then the performance is said to be optimized when Q is a minimum.

Examination of Fig. 3A shows that for K = 2, MSE is less than 0.3, while for K = 1, MSE is less than 0.5. On the other hand, for K = 2, $B_L \triangleq 0.75$ while for K = 1, B_L is less than 0.5. Thus for different values of the weighting factor, λ, different operating points are required to minimize the performance criterion. Furthermore, the transient characteristics of the system may be estimated; for K = 2 the roots range from $\zeta = 0.73$ to $\zeta = 0.71$, while for K = 1 they range from $\zeta = 0.85$ to $\zeta = 0.79$. (It is of interest to note the $\alpha = 2.5$ was chosen for this example because the system is underdamped for lower values of α and overdamped for larger values of α). Stability need not be considered because the loop is only of second order.

It is not intended that the performance criterion, Q, be used to determine an absolute optimum operating point but rather it is intended to serve as a guideline for choice of an operating point; eg., for $\lambda = 3$ it may be shown that K = 2 is slightly better than K = 1.

For cases where transient response is important, it should be noted that parameter plane methods can be used to calculate the step response of the nonlinear system with good accuracy. Details are given in Ref. 4.

A THIRD ORDER SYSTEM

Consider a third order loop for which

$$F(s)\ 1 + \frac{a}{s} + \frac{b}{s^2}$$

Define $\beta \triangleq N' \triangleq N\,A\,K$

$$\alpha = bN' = b\beta$$

$$a = 1.0$$

Then the characteristic equation is

$$s^3 + \beta s^2 + \beta s + \alpha = 0$$

The real root line (3) for $s = 0$ is $\alpha = 0$. Fig. 4 shows the parameter plane for this system. The performance curves for constant MSE and constant B_L are shown separately on Fig. 5. The optimization procedure is carried out as in the case of the second order loop, so details are not presented here. For the third order loop, however, it is possible that instability may occur, so operating conditions must be chosen to avoid this possibility.

Consider the conditions if the operating point is chosen at $\alpha = \beta = 1.25$, so that $b = 1.0$. If the loop were linear the parameter plane predicts that it should be stable. When the nonlinear operating line is constructed it is seen that this line crosses the stability boundary ($\sigma = 0$ curve), so that for sufficiently large initial condition the system is unstable and will not lock. The value of ϕ for which the limit of stability is achieved is predicted to be $\phi = 1.31$, which should provide an unstable limit cycle (see Ref. 5). Simulation on the digital computer provides the results of Fig. 6a,b,c, which verify the existence of the unstable limit cycle, but define the required value of ϕ as $\phi = 1.545$. This discrepancy between the predicted and actual values of ϕ is due to the inaccuracies and approximations inherent in the describing function method. Techniques for improving the accuracy are available (Ref.5) if needed.

PERFORMANCE SPECIFICATIONS

Besides the specifications of MSE and B_L used in the preceding examples, numerous other criteria, some empirical and some exact, are available in the literature. These include relationships for acquisition, tracking limits, tracking error, etc. For example constant error coefficient curves may be readily superimposed on the parameter plane plot (Ref.7), and these permit definition of a hold-in range (empirical, see Ref.2, p.168):

$$\Delta w_H = \pm K_V \text{ rad/sec.}$$

CONCLUSION:

Methods for the analysis and design of phase locked loops using the parameter plane have been presented. This technique permits the designer to consider several design constraints simultaneously. Suitable system parameters are picked by inspection of the pertinent curves so that "optimum" performance is obtained.

It is of interest to note that quasi-static estimates of statistical parameters are also placed in evidence. In addition stability analysis is readily performed when needed, as for third and higher order systems where phase plane techniques are not applicable.

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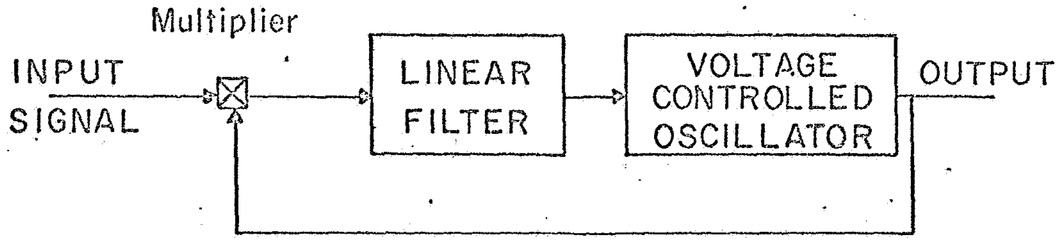
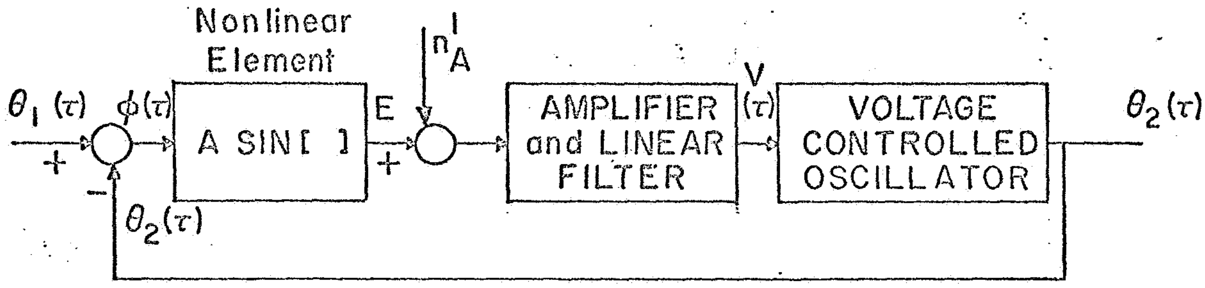
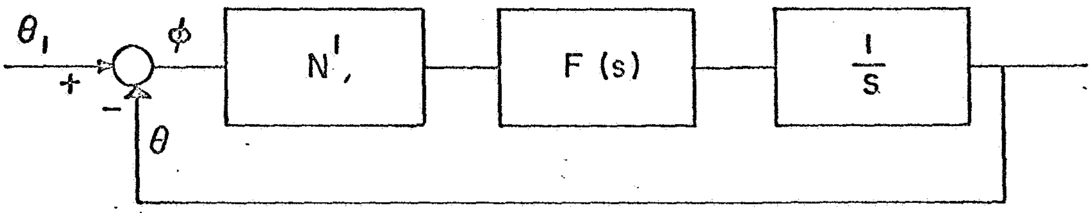


a) Basic Representation



b) A Derived Diagram

Fig. 1. Block diagram of a phase locked loop.



N' = Describing function of Nonlinearity

F (s) = Transfer function of Linear-Filter

Fig. 2. Modified block diagram of Phase Locked Loop.

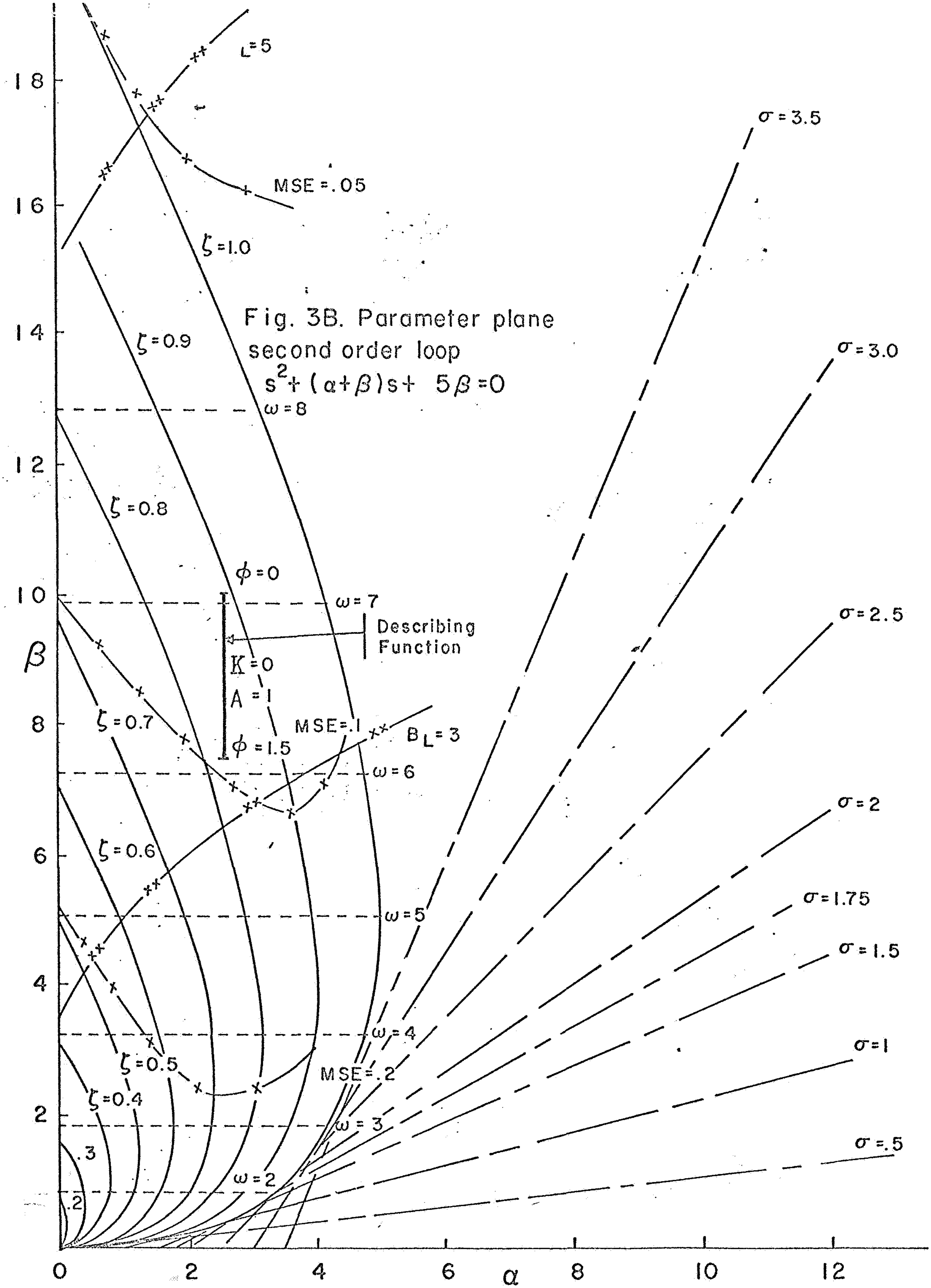


Fig. 3B. Parameter plane second order loop
$s^2 + (\alpha+\beta)s + 5\beta = 0$
β
α
$B_L = 5$
MSE = .05
ζ = 1.0
ζ = 0.9
ζ = 0.8
ζ = 0.7
ζ = 0.6
ζ = 0.5
ζ = 0.4
.3
.2
ω = 8
ω = 7
ω = 6
ω = 5
ω = 4
ω = 3
ω = 2
ϕ = 0
Describing Function
K = 0
A = 1
ϕ = 1.5
MSE = .1
$B_L = 3$
MSE = .2
σ = 3.5
σ = 3.0
σ = 2.5
σ = 2
σ = 1.75
σ = 1.5
σ = 1
σ = .5
0
2
4
6
8
10
12
14
16
18

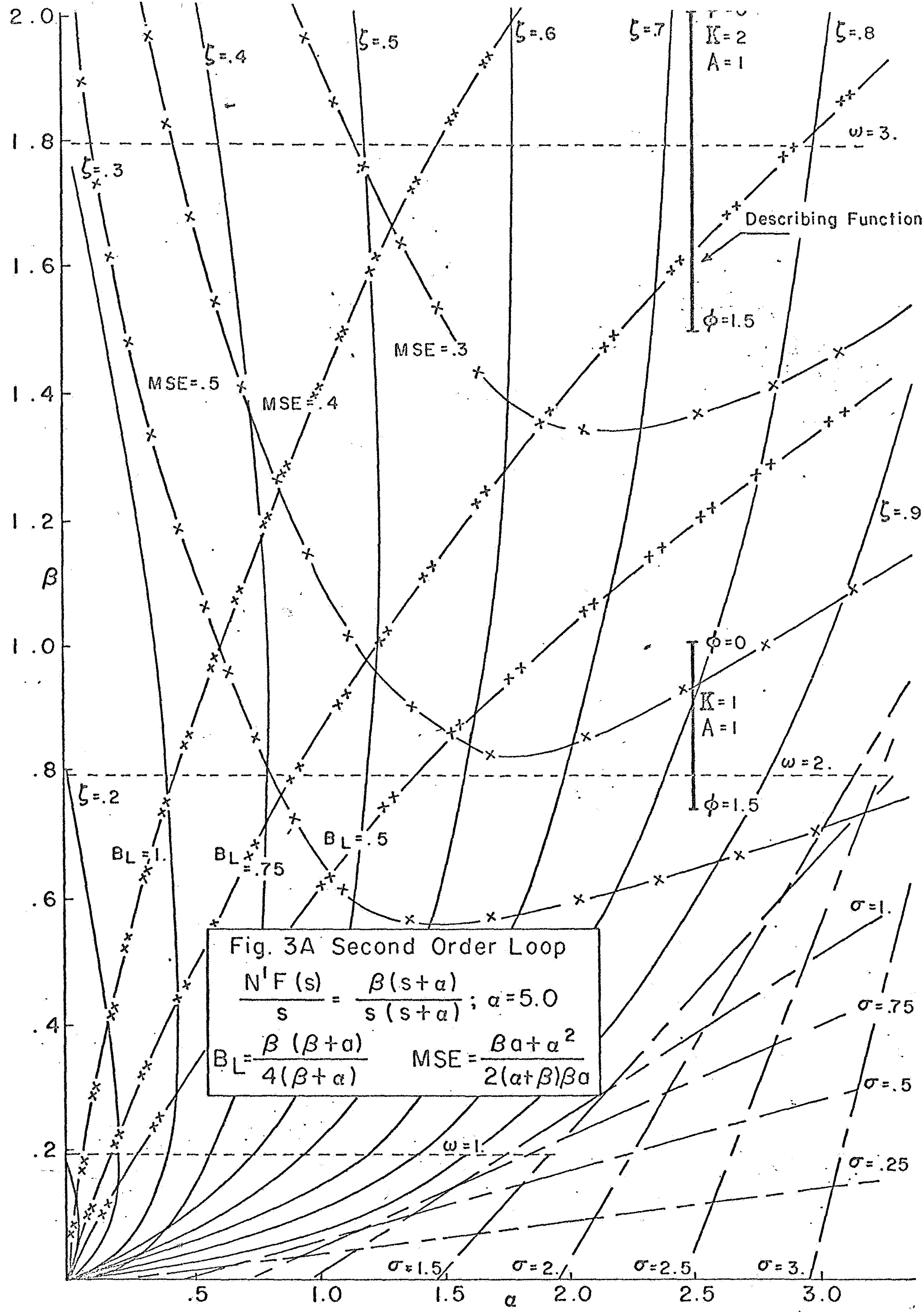


Fig. 3A Second Order Loop

$$\frac{N'F(s)}{s} = \frac{\beta(s+\alpha)}{s(s+\alpha)};\ \alpha = 5.0$$

$$B_L = \frac{\beta(\beta+\alpha)}{4(\beta+\alpha)} \qquad MSE = \frac{\beta\alpha+\alpha^2}{2(\alpha+\beta)\beta\alpha}$$

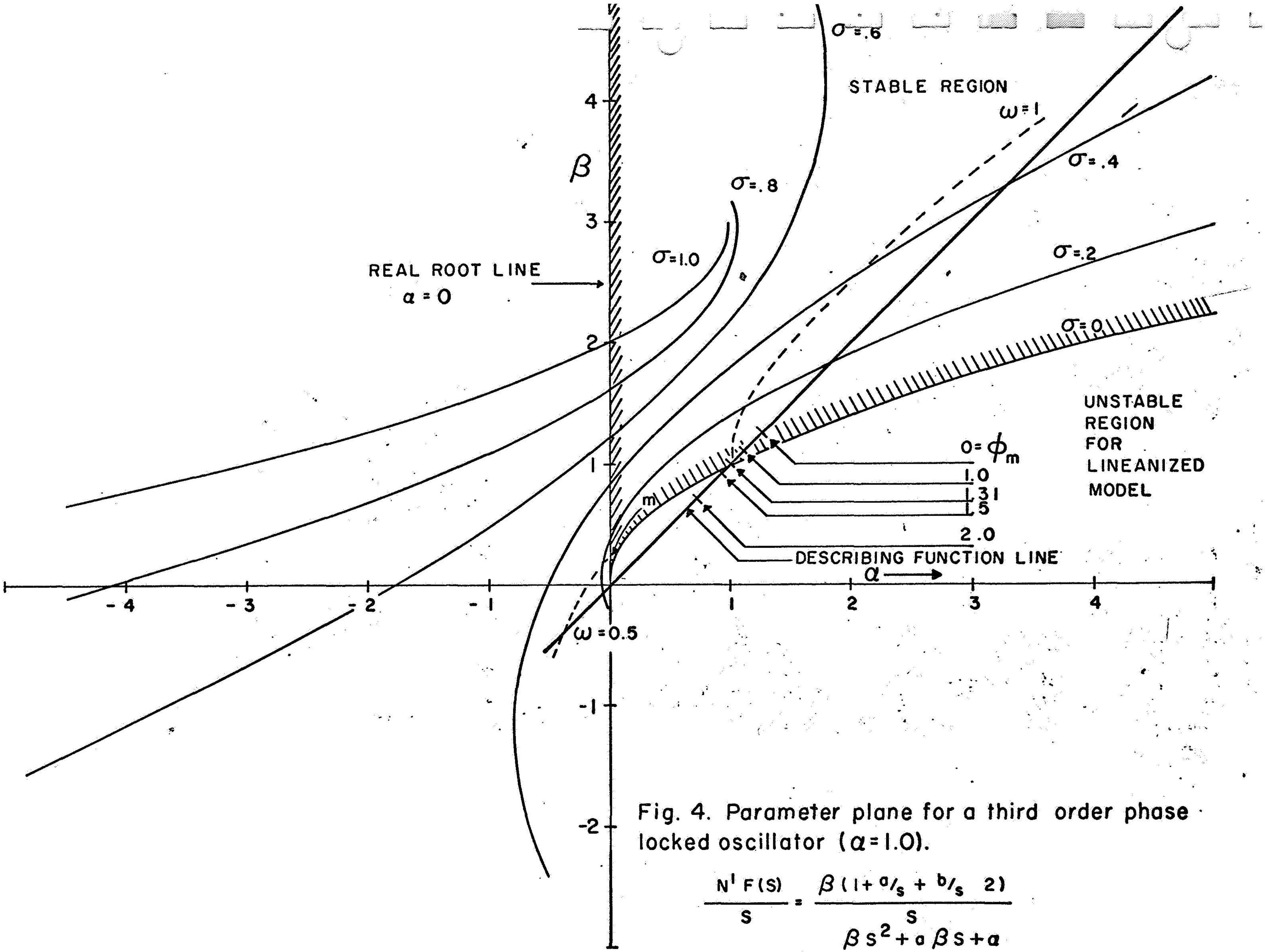


Fig. 4. Parameter plane for a third order phase locked oscillator ($\alpha = 1.0$).

$$\frac{N^{1} F(S)}{S} = \frac{\beta\,(1 + a/_{s} + b/_{s}\ 2)}{S} = \frac{\beta s^{2} + a \beta s + a}{s^{3}}$$

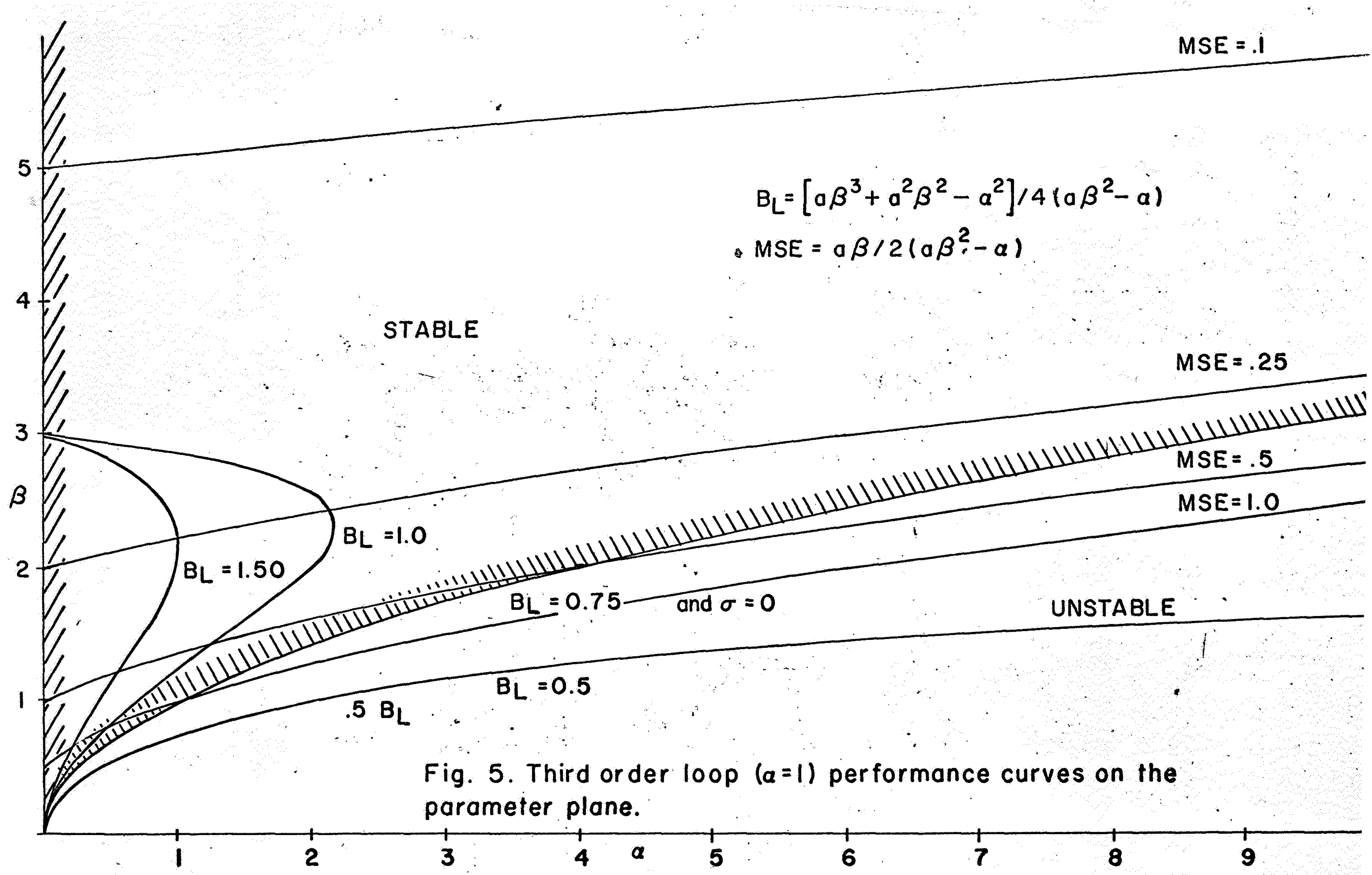


Fig. 5. Third order loop (α=1) performance curves on the parameter plane.

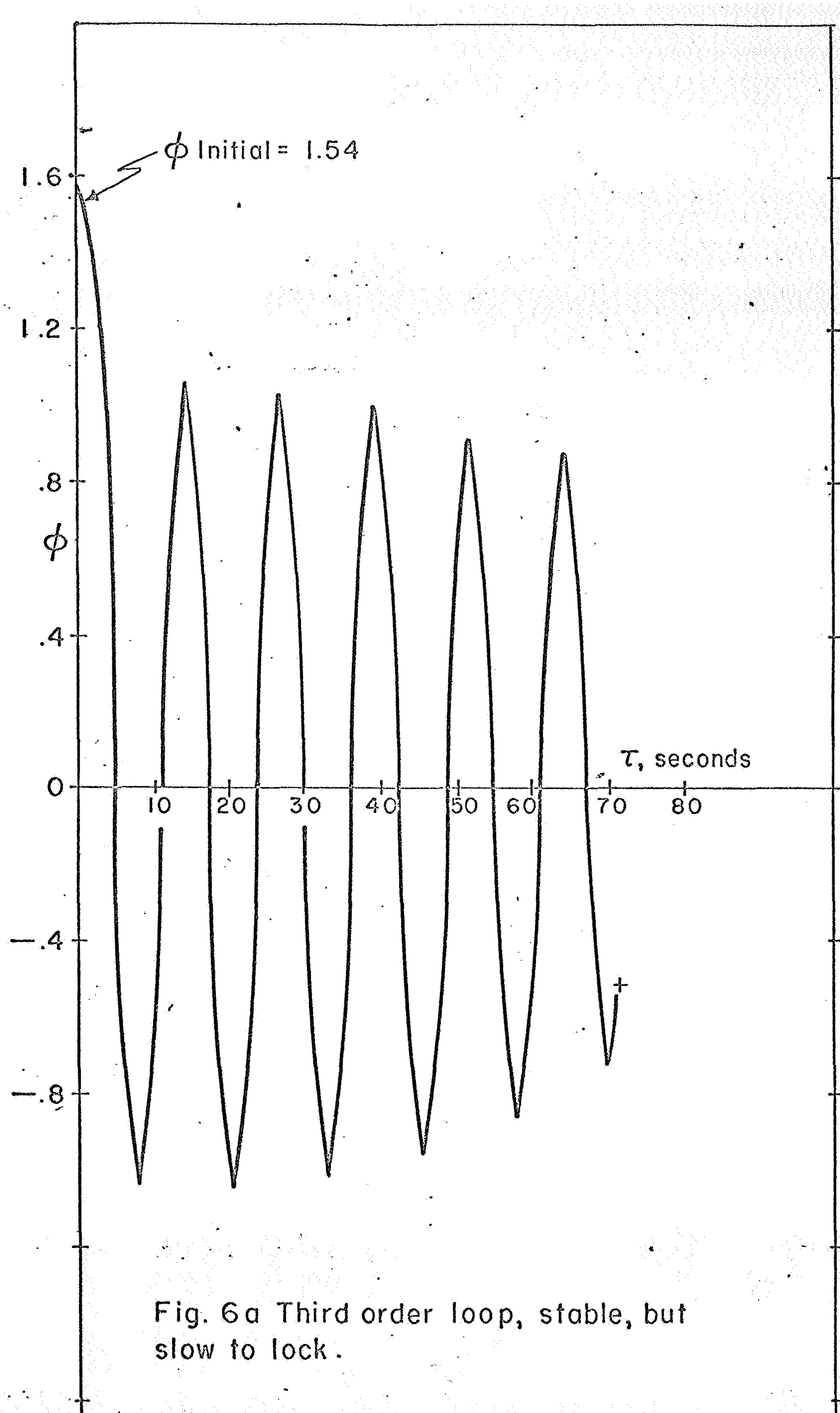


Fig. 6a Third order loop, stable, but slow to lock.

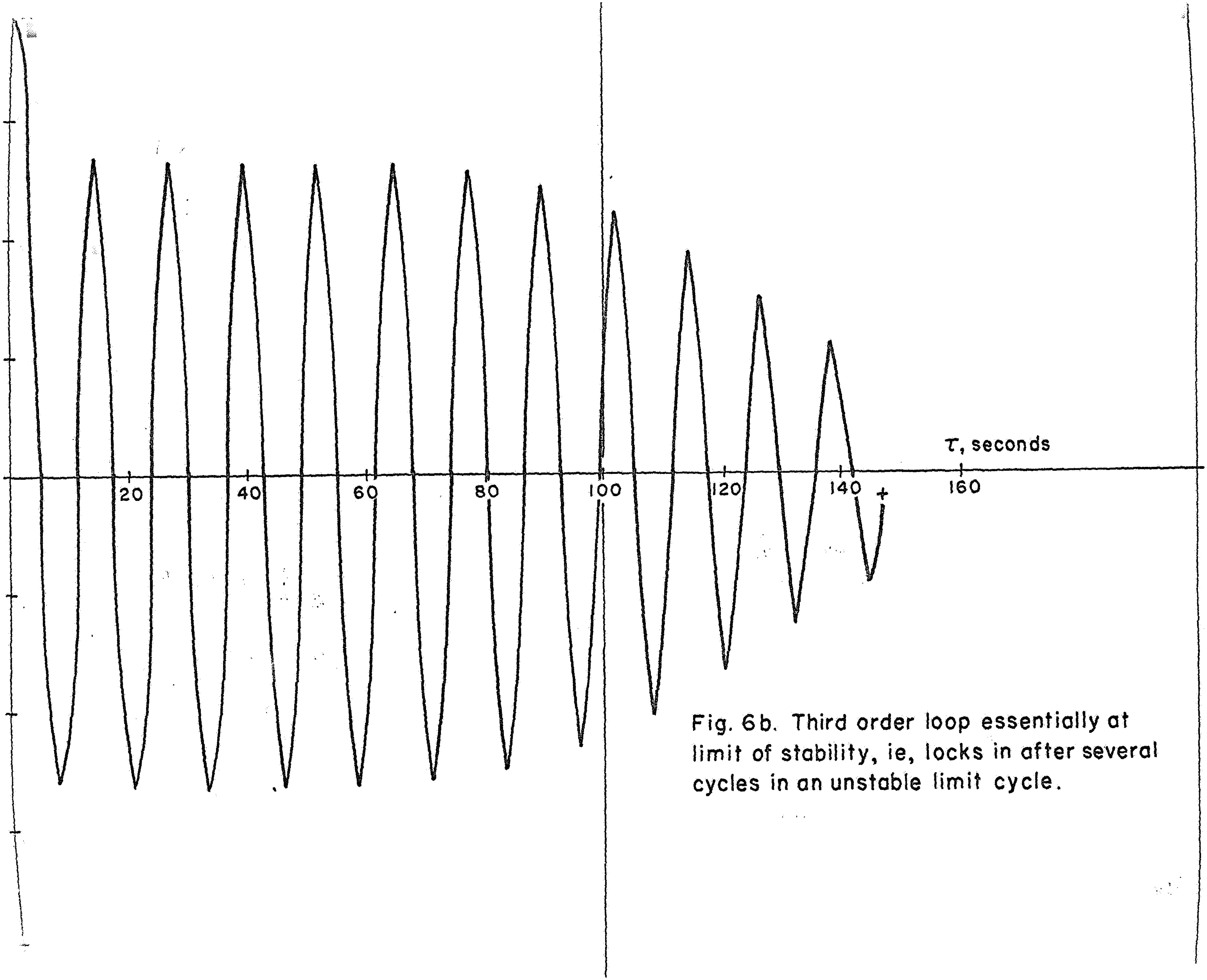


Fig. 6b. Third order loop essentially at limit of stability, ie, locks in after several cycles in an unstable limit cycle.

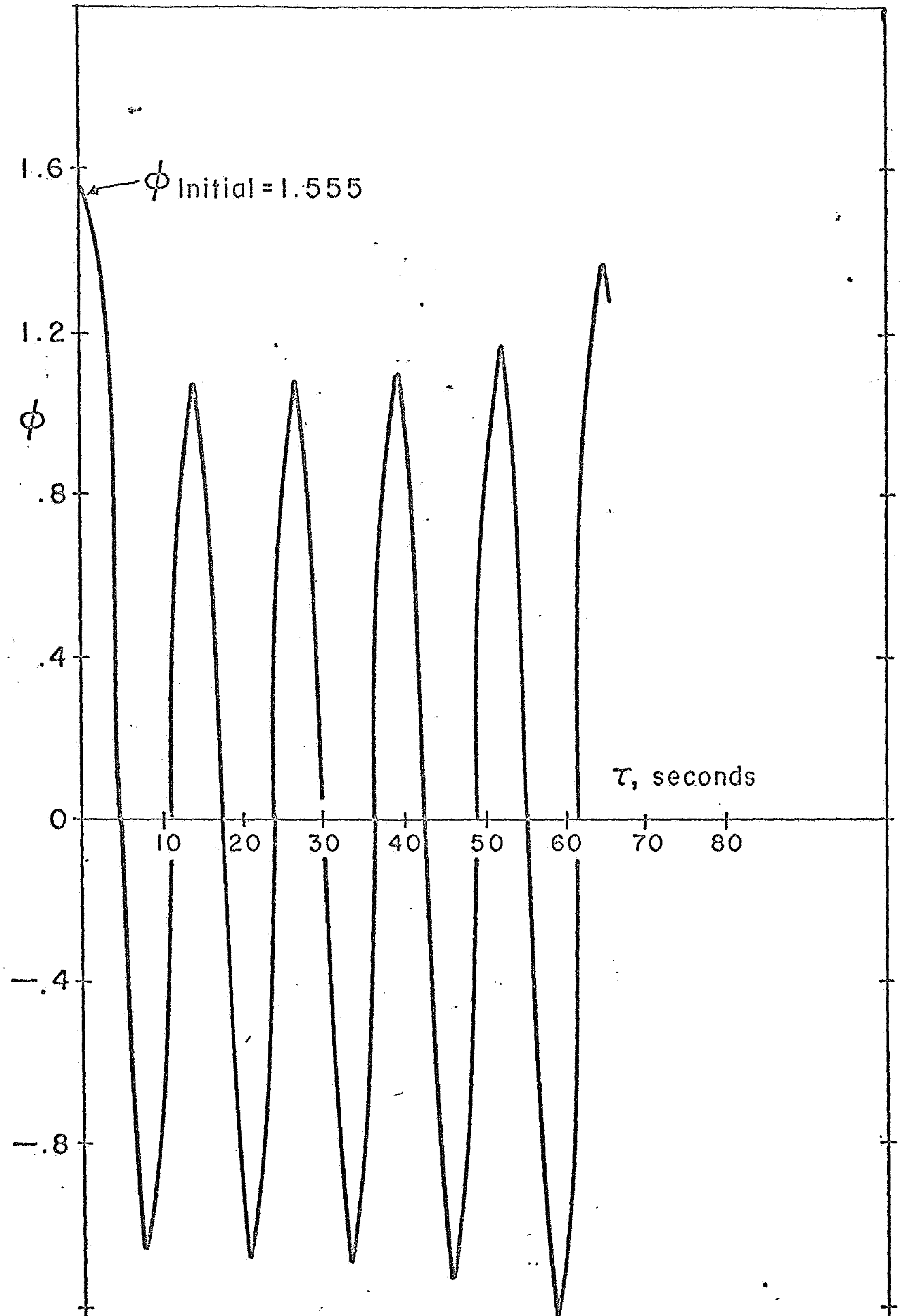


Fig. 6c. Third order loop fails to lock.

1.555

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Absolute stability and parameter sensitivity†

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This paper extends the notion of absolute stability to include the parameter variations of the linear part of the system. A simple analytic procedure is proposed to calculate the regions of absolute stability in the parameter space. Then, a parallelepiped of maximum volume is embedded in the region to interpret its boundaries and obtain readily the information about parameter variations which do not affect the system stability.

1. Introduction

Stability and sensitivity are two essential properties of dynamic control systems. While stability assures a proper functioning of the system, the sensitivity indicates the ability of the system to retain required performance characteristics despite changes in the operating conditions. These changes may occur due to the fact that the parameters of physical systems deviate from their nominal values either because of inaccuracies in the system components (time-invariant case), or because the system parameters vary in time (time-varying case). Therefore, a simultaneous consideration of stability and parameter sensitivity in system analysis is desired.

The Lur'e (1957) absolute stability concept and the related criterion of Popov (1963) are significant contributions to stability analysis of dynamic systems. This is mostly because the absolute stability concept is meaningful in a large class of closed-loop control systems, and the Popov criterion provides a simple procedure to conclude that kind of stability.

In the absolute stability analysis, the non-linear characteristic is not completely specified and it should only belong to a certain defined class of functions. On the other hand, the parameters of the linear part are specified numerically. This paper proposes an absolute stability definition which will relax the conditions on the linear part and allow system parameters to deviate from their nominal values. Then, a simple analytical procedure based upon the Popov criterion is presented to determine in the parameter space the region of parameter deviations which do not violate the absolute stability.

A graphical procedure for evaluation of the absolute stability regions in the parameter plane was given in Šiljak (1967). Under certain conditions that technique which is based upon the *envelope criterion* can also be extended to considerations in the parameter space.

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2. Absolute stability in the parameter space

The problem of Lur'e (1957) is formulated for a class of closed-loop control systems described by the equations:

$$\dot{x} = Px + q\phi(\sigma), \quad \sigma = r^T x, \tag{1}$$

where x, q, r are real n vectors, P is a real $n \times n$ matrix, the pair (P, q) is completely controllable, and $\phi(\sigma)$ is a real continuous scalar function of the real scalar σ such that it belongs to the class $A_\kappa: \phi(0) = 0,\ 0 < \sigma\phi(\sigma) < \kappa\sigma^2$. One asks: *Is the equilibrium state $x = 0$ of the system (1) asymptotically stable in the large for any $\phi(\sigma) \in A_\kappa$, i.e. is the system absolutely stable?*

The most important solution of the problem of Lur'e was given by Popov (1963) in terms of the frequency characteristic:

$$\chi(\lambda) = r^T(P - \lambda I)^{-1}q, \tag{2}$$

which is the transfer function of the linear part of the system (1) from the input ϕ to the output $(-\sigma)$, and $\lambda = \delta + j\omega$ is the complex variable. Yakobovich (1964) generalized the results of Popov and proved that if $\phi(\sigma) \in A_\kappa$ and all the roots of $|P - \lambda I| = 0$ are in the half-plane $\operatorname{Re}\lambda < \delta \le 0$, and if there is a real number v such that a Popov type inequality:

$$\pi(\delta, \omega) \equiv \frac{1}{\kappa} + \operatorname{Re}(1 + j\omega v)\,\chi(\delta + j\omega) > 0, \quad \forall \omega \ge 0, \tag{3}$$

is satisfied, then there exist positive constants ρ and ϵ such that, for any solution $x(t)$ of (1) and any $t \ge t_0$, one has $|x(t)| \le \rho |x(t_0)| \exp[(\delta - \epsilon)(t - t_0)]$.

Yakubovich (1964) also treated the forced system:

$$\dot{x} = Px + q\phi(\sigma) + f(t), \quad \sigma = r^T x, \tag{4}$$

where $\phi(\sigma) \in A_\kappa: \phi(0) = 0,\ 0 < \sigma\phi(\sigma) < \kappa\sigma^2,\ 0 < \sigma\phi'(\sigma) < \kappa\sigma^2, f(t)$ is a bounded n vector function on the interval $(-\infty, +\infty)$, and showed that a modification of (3):

$$\pi(\delta, \omega) \equiv \frac{1}{\kappa} + \operatorname{Re}\chi(\delta + j\omega) > 0, \quad \forall \omega \ge 0, \tag{5}$$

assures that there is a unique bounded solution $x_0(t)$ of (1) on $(-\infty, +\infty)$ and that for any $x(t)$ and $t \ge t_0$, one has $|x(t) - x_0(t)| \le \rho |x(t_0) - x_0(t_0)| \exp[(\delta - \epsilon)(t - t_0)]$. In the same paper, Yakubovich (1964) treated the discontinuous functions $\phi(\sigma)$ and showed that the absolute stability is based upon the same inequalities (3) or (5).

In application of the system (1), the linear part of the system contains parameters which may deviate from their nominal values. Then, it is necessary to relax the conditions on the linear part of the system and allow these parameters to vary in some neighbourhood of their nominal values while preserving the absolute stability of the system.

Let us assume that the transfer function $\chi(\lambda, p_1, p_2, \ldots, p_l)$ is a function of λ and l parameters $(p_1, p_2, \ldots, p_l)$, and let us suppose that the solution $x(t, p_1, p_2, \ldots, p_l)$ of (1) is well-defined (Petrovski 1966) in the l-dimensional euclidian space $(p_1, p_2, \ldots, p_l)$. Then, the definition of absolute stability for system (1) can be reformulated to include the parameter variations.

The equilibrium state $x = 0$ of the system (1) is said to be absolutely stable if it is asymptotically stable in the large for any $\phi(\sigma) \in A_\kappa$ and any set $(p_1, p_2, \ldots, p_l) \in R$.

When the system (1) is specified, one is interested to find: (*a*) the greatest value of κ and the largest region R; (*b*) a value of κ is given and the largest region R is to be determined. A graphical solution of these problems was given in Šiljak (1967) where the region R was determined by the envelope criterion as the largest set $\{(p_1, p_2, \ldots, p_l) \in R | \pi > 0, \forall \omega \geq 0\}$.

In this paper, a simple analytical solution is presented which first yields the region R in terms of a set of algebraic inequalities involving parameters. Then, a rectangular parallelepiped of maximum volume is embedded in the region to yield a convenient interpretation of the absolute stability region in the parameter space (this interpretation technique was proposed by George (1966, 1967) for approximation of finite regions of asymptotic stability and linear system analysis).

Assume the transfer function of the linear part to be a rational function of the complex variable λ:

$$\chi(\lambda, p_1, p_2, \ldots, p_l) = \frac{\sum\limits_{k=0}^{m} c_k \lambda^k}{\sum\limits_{k=0}^{n} b_k \lambda^k}, \quad n > m, \tag{6}$$

in which the coefficients b_k and c_k are real functions of the parameters p_i $(i = 1, 2, \ldots, l)$. Then, let us express:

$$\lambda^k = X_k + jY_k, \tag{7}$$

where $\lambda = \delta + j\omega$, and

$$\left.\begin{aligned} X_k &= \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{2\nu} \delta^{k-2\nu} \omega^{2\nu}, \\ Y_k &= \sum_{\nu=1}^{k} (-1)^{\nu-1} \binom{k}{2\nu-1} \delta^{k-2\nu+1} \omega^{2\nu-1}. \end{aligned}\right\} \tag{8}$$

Functions X_k and Y_k can be easily calculated using the recurrence formulas: $X_{k+1} - 2X_1X_k + (X_1^2 + Y_1^2)X_{k-1} = 0$, $Y_{k+1} - 2X_1Y_k + (X_1^2 + Y_1^2)Y_{k-1} = 0$, $X_0 \equiv 1$, $X_1 \equiv \delta$, $Y_0 \equiv 0$, $Y_1 \equiv \omega$.

When δ is specified in an absolute stability problem, and (7), (8) are substituted in (3) or (5), one obtains:

$$\bar{\pi}(\omega, p_1, p_2, \ldots, p_l) \equiv \sum_{k=0}^{2n} a_k \omega^k > 0, \quad \forall \omega \geq 0, \tag{9}$$

where the coefficients $a_k = a_k(p_1, p_2, \ldots, p_l)$ are real functions of the parameters. For convenience, in (9), $1/\kappa$ and v of (3) are considered as parameters. Note that v is not a physical parameter and only its existence is required such that $\pi > 0$, $\forall \omega \geq 0$.

From (9), one can readily conclude that the system (1), or (3), is absolutely stable if the corresponding polynomial $\bar{\pi}$ has no positive real roots. For this to take place, it is sufficient that the following set of algebraic inequalities:

$$a_0 > 0, \quad a_k \geq 0, \quad (k = 2, 4, \ldots, 2n) \tag{10}$$

is satisfied.

For example, if the transfer function:

$$\chi(\lambda, p_1, p_2, p_3) = \frac{\lambda^2 + p_2\lambda + p_3}{p_1(\lambda+1)(\lambda+2)(\lambda+3)}, \tag{11}$$

$\kappa = 1$, and $\delta = 0$ ($\lambda = j\omega$) are specified, one obtains (9) as:

$$\bar{\pi}(\omega, p_1, p_2, p_3) \equiv p_1\omega^6 + (14p_1 - p_2 + 6)\omega^4 + (49p_1 + 11p_2 - 6p_3 - 6)\omega^2 + 36p_1 + 6p_3. \tag{12}$$

Inequalities (10) are:

$$\left.\begin{aligned} 6p_1 + p_3 > 0, \\ 49p_1 + 11p_2 - 6p_3 - 6 \geq 0, \\ 14p_1 - p_2 + 6 \geq 0, \\ p_1 \geq 0, \end{aligned}\right\} \tag{13}$$

which determine the boundaries of $\bar{R}$.

Inequalities (10) specify a region $\bar{R}(\bar{R} \subset R)$ of absolute stability in the parameter space which may appear to be an overly strict region since (10) are only sufficient conditions for $\pi > 0$, $\forall \omega \geq 0$. Conditions (10), however, lead to a convenient interpretation of the stability regions.

3. Interpretation procedure

After the inequalities (10) are specified, the problem of using them in practical problems is essentially one of interpretation. Since the practical problems may involve more than two parameters, an interpretation procedure for multi-parameter analysis is desired.

In general, to interpret the absolute stability region, let us imbed a parallelepiped Π into the convex region $\bar{R}$ determined by inequalities (10) which has sides perpendicular to the coordinate axes of the parameter space $(p_1, p_2, \ldots, p_l)$ and centre at the known stable point $\bar{M}(\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_l)$. Let the volume v of Π be defined as:

$$v = 2^l(p_1 - \bar{p}_2)(p_2 - \bar{p}_2) \ldots (p_l - \bar{p}_l). \tag{14}$$

Now, the function v should be maximized with respect to each inequality (10) separately considered as a constraint. Thus, a constraint:

$$a_k(p_1, p_2, \ldots, p_l) = 0, \tag{15}$$

may be represented as:

$$p_1 = p_1(p_2, p_3, \ldots, p_l). \tag{16}$$

Substituting (16) into (14) and extremizing, a necessary condition for $(p_2{}^0, p_2{}^0, \ldots, p_l{}^0)$ to occur at a maximum of v is that it be a solution to:

$$\frac{\partial v}{\partial p_i} = 0, \quad (i = 2, 3, \ldots, l). \tag{17}$$

Standard sufficient conditions for this solution to be maximal are given in (Goffman 1965).

Let the solutions $(p_1{}^0, p_2{}^0, \ldots, p_l{}^0)_k$, $(k=0, 2, \ldots, 2n)$ occur at maximum value of v subject to constraints (10), then the desired parallelepiped Π is given as:

$$\Pi = \left\{ (p_1, p_2, \ldots, p_l) \in R \;\middle|\; |p_i - \bar{p}_i| \le \min_k |p_i - p_i{}^0|_k, \quad (i = 1, 2, \ldots, l) \right\}. \tag{18}$$

Since each vertex point of Π is located in R containing the point $\bar{M}(\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_l)$, it follows that the parallelepiped Π is completely embedded in $\bar{R}$, i.e. $\Pi \subset \bar{R}$.

In case of the above specific example, let us choose the stable point $\bar{M}(0{\cdot}2\,;0\,;0)$. The volume to be maximized is:

$$v = 8(p_1 - 0{\cdot}2)p_2 p_3. \tag{19}$$

Maximization of v with respect to the constraint:

$$49p_1 + 11p_2 - 6p_3 - 6 = 0 \tag{20}$$

yields: $p_1{}^0 = 0{\cdot}178$, $p_2{}^0 = -0{\cdot}123$, $p_3{}^0 = 0{\cdot}226$. According to these values of parameters, the parallelepiped Π is determined by:

$$|p_1 - 0{\cdot}2| \le 0{\cdot}022, \; |p_2| \le 0{\cdot}123, \; |p_3| \le 0{\cdot}226. \tag{21}$$

One can readily check that all the vertex points of Π satisfy the rest of the constraints of (13). Therefore, (21) is the solution of the interpretation problem under consideration.

It should be noted that some of the constraint in (10) may not contain all the parameters, as it is clear from inequalities (13). Then, some of the parameters in incomplete inequalities are arbitrary and to make the maximization of v meaningful, one should consider the arbitrary parameters as constants.

For example, the optimization of v in (19) with respect to the constraint $14p_1 - p_2 + 6 \ge 0$ of (13) should be performed with $p_3 = c$ $(c \ne 0)$. Then, the maximization of $v = 8c(p_1 - 0{\cdot}2)p_2$ gives $p_1{}^0 = -0{\cdot}122$, $p_2{}^0 = 4{\cdot}292$. In applying eqn. (18) to determine the parallelepiped Π, these values are discarded and Π is given by (21).

In case of time-varying parameters, by the arguments of Yakubovich (1964) one can use the inequality (5) and prove the stability of either system (1) or (3). Then, as long as the parameters are varied inside the determined region $\bar{R}$ (or Π) the system is absolutely stable.

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PARAMETER ANALYSIS OF ABSOLUTE STABILITY*

D. Šiljak**

A reformulation of absolute stability is used which includes variations of the system parameters. The envelope method is developed to determine the stability regions in the parameter space. In addition, a simple analytic test for absolute stability is presented.

INTRODUCTION

Absolute stability has the useful property that a considerable freedom is left to the form of the system nonlinear characteristic. It is required that the characteristic belongs to a class of functions limited only by a certain sector condition. On the other hand, the parameters in the linear part of the system are specified numerically, which is quite an unrealistic constraint. The parameters always deviate from their nominal values and stability of the system may be destroyed.

This paper uses the reformulation of absolute stability [1,2] which relaxes the conditions on the linear part of the system and allows parameter variations. Then, the <u>envelope method</u> is developed which can be used either to determine graphically the region of parameter variations where the stability behavior is preserved, or to calculate the parameter

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values which yield the largest sector of nonlinear characteristics.

In addition, analytic test of absolute stability is presented which is based upon the Routh criterion. The test requires no graphical construction and is convenient for computer applications.

ABSOLUTE STABILITY IN THE PARAMETER SPACE

Free dynamic systems of the Lur'e class are described by equations

$$\dot{x} = Px + q\phi(\sigma), \quad \sigma = r^T x , \tag{1}$$

where x, q, r are real vectors, P is a real $n \times n$ matrix, and $\phi(\sigma)$ is a real-valued, continuous function of a real scalar σ which belongs to the class Φ_κ: $\phi(0) = 0$, $0 \leq \sigma\phi(\sigma) \leq \kappa\sigma^2$. Lur'e introduced the definition of absolute stability:

System (1) is said to be absolutely stable if the equilibrium $x = 0$ of (1) is globally asymptotically stable for any $\phi \in \Phi_\kappa$. Then, he formulated the problem to find conditions on P, q, r, and κ for absolute stability of (1). [1]

A useful solution of the Lur'e problem was given by Popov in terms of the frequency characteristic

$$\chi(\lambda) = r^T(P - \lambda I)^{-1}q \tag{2}$$

which is the transfer function of the linear part of (1) from the input ϕ to the output $-\sigma$, and $\lambda = \delta + j\omega$ is the complex variable. Assume that P is Hurwitz, that is, all zeros of $\Delta(\lambda) = \det(P - \lambda I)$ are in the left half plane $\mathrm{Re}\,\lambda < 0$ and χ is nondegenerate. Then:

System (1) is absolutely stable if the Popov inequality

$$\pi(\omega) = \kappa^{-1} + \mathrm{Re}(1 + j\omega\upsilon)\chi(j\omega) > 0 \quad \text{for all real} \quad \omega \geq 0 \tag{3}$$

holds for some real number υ. [1]

In application of the system (1), the linear part of the system contains parameters which may deviate from their nominal values. Then, it is necessary to relax the conditions on the linear part of the system and allow these parameters to vary in some neighborhood of their nominal values while preserving the absolute stability of the system.

Let us assume that the transfer function $\chi(\lambda, p)$ is a function of λ and parameter ℓ vector p, and let us suppose that the solution $x(t, p)$ of (1) is well-defined for parameter values in a certain region R of the ℓ-dimensional euclidian space $\{p\}$. Then, the definition of absolute stability for system (1) can be reformulated to include the parameter variations:

<u>System (1) is said to be absolutely stable if the equilibrium</u> $x = 0$ <u>of (1) is globally asymptotically stable for any</u> $\phi \in \Phi_\kappa$ <u>and any</u> $p \in R$.

When the system (1) is specified, one is interested to find: (a) The greatest value of κ and the largest region R; (b) A value of κ is given and the largest region R is to be determined.

Inequality (3) can be rewritten as

$$\Pi(\omega) \equiv |\Delta(j\omega)|^2\{\kappa^{-1} + \mathrm{Re}(1 + j\omega\upsilon)\chi(j\omega)\} > 0,$$

where Π is an even polynomial in ω. Therefore, the Popov inequality (3) is equivalent to

$$\Pi(\omega) \equiv \sum_{k=0}^{n} a_{2k}\omega^{2k} > 0 \quad \text{for all real} \quad \omega \geq 0, \tag{4}$$

where the coefficients $a_{2k} = a_{2k}(p)$, $(k = 0, 1, \cdots, n)$ of the polynomial Π are real functions of the parameter vector p. For convenience,

$1/\kappa$ and υ of (3) are considered as parameters. Note that υ is not a physical parameters and only its existence is required for $\pi > 0$.

From (4), we immediately conclude:

<u>Popov inequality (3) is satisfied if</u>

$$a_0 > 0, \; a_{2k} \geq 0, \; (k = 1, 2, \ldots , n). \tag{5}$$

Inequalities (5) lead to a convenient interpretation of the absolute stability region $R = \{p: \pi > 0, \forall\omega > 0\}$ in the parameter space. However, the inequalities provide only sufficient conditions for $\pi > 0$, $\forall\omega > 0$, and quite conservative estimates of R may be obtained. [2]

Since the same inequality (4) appears in the analysis of <u>exponential</u> absolute stability [1-3], the results obtained here can be used to investigate exponential stability regions in the parameter space.

ENVELOPE METHOD

To describe the envelope method, let us take equality in (4) so that

$$\Pi(\omega, p) = 0. \tag{6}$$

Equation (6) may represent a one-parameter family of hypersurfaces in the ℓ-dimensional parameter space, ω being the parameter of the family. For a fixed value of ω , the corresponding surface S determined by (6) divides the parameter space into regions with $\Pi > 0$ and $\Pi < 0$. Now, let us assume that the surface S intersects every surface corresponding to a value $\omega + \Delta\omega$ ($|\Delta\omega|$ sufficiently small). This intersection C can be represented by

$$\Pi(\omega, p) = 0, \qquad \Pi(\omega + \Delta\omega, p) = 0 . \tag{7}$$

Since C lies also on the surface $\Pi(\omega + \Delta\omega, p) - \Pi(\omega, p) = 0$, we may replace second equation of (7) by

$$\frac{1}{\Delta\omega}\{\Pi(\omega + \Delta\omega, p) - \Pi(\omega, p)\} = 0 .$$

As the increment $\Delta\omega$ tends to zero, we assume that the intersection C tends to a limiting position determined by equation

$$\Pi(\omega, p) = 0, \quad \frac{\partial \Pi}{\partial \omega} = 0 \ . \tag{8}$$

This limiting intersection represents the so-called characteristic curve C of the family of hypersurfaces (6). Geometrically, it is the curve on the hypersurface which contains every point of the set to which the points of intersection given by (7) tend when $\Delta\omega \to 0$. However, C will contain, in general, also other points. It may even be that neighboring surfaces of the family do not intersect at all but nevertheless determines a characteristic on each of the surfaces.

If the characteristics of the family (6) exist and if their totality obtained by letting ω assume all possible values generates a surface $E(p) = 0$, then that surface represents the envelope E of the family. It should be noted, however, that equations (8) may yield also singular loci such as node-loci, cusp-loci, etc. [4]

If the envelope E exists and is plotted for all positive values of ω, then it is clear from above that the envelope E will "envelope" the convex absolute stability region $R = \{p : \pi > 0, \forall \ \omega > 0\}$ (if such a region exists.)

To illustrate the envelope method, let us consider inequality (3) where

$$\chi(\lambda, p_2) = \frac{\lambda^2 + p_2}{(\lambda + 1)(\lambda + 2)(\lambda + 3)} ,$$

p_2 is an adjustable parameter, $p_1 = 1/\kappa$, and $\upsilon = 0$.

The polynomial is

$$\Pi(\omega, p_1, p_2) = p_1\omega^6 + (14p_1 + 6)\omega^4 + (49p_1 - 6p_2 - 6)\omega^2 + 36p_1 + 6p_2 \tag{10}$$

By using this $\Pi(\omega)$ and equations (8), we obtain the envelope shown in Figure 1. For $\omega = 0$, the second equation (8) vanishes identically and the corresponding part of the envelope is a locus of singular points which is the straight line $\omega = 0$. As ω increases, the envelope E is represented by the curves AB and CD. Since p_1 and p_2 enter linearily in (10), $\Pi(\omega, p_1, p_2) = 0$ represents in the parameter p_1p_2 plane a family of straight lines tangent to the envelope E.. All the tangents to the part CD of E are situated between the dashed lines on Figure 1 and do not enter the shaded convex region R. Therefore, $R = \{(p_1p_2) : \pi > 0 , \forall \omega \geq 0\}$ is the desired region of absolute stability in the parameter plane.

It should be noted that the introduction of an adjustable parameter p_2 in the linear part of the systems makes it possible to assure absolute stability for $\kappa = \infty$. A considerably smaller value of κ is obtained if $p_2 = 0$ which is far from the optimal value $p_2 = 0.84$.

ABSOLUTE STABILITY TEST

Let us find the necessary and sufficient conditions on the coefficients a_{2k} for the reformulated Popov inequality (4) to be satisfied. From (4), we have obviously:

Popov inequality (3) is satisfied if, and only if, the polynomial Π has no real zeros and $a_0 > 0$.

Since Π is an even polynomial, its zeros are distributed symmetrically with respect to both the real and the imaginary axis of the ω plane. By rotating the zeros ninety degrees around the origin, the nonexistence of real (positive) zeros of Π is equivalent to the condition that the polynomial

$$\Pi(j\omega) \equiv \sum_{k=0}^{n} (-1)^k a_{2k} \omega^{2k}$$

has n zeros with positive real parts. The Routh algorithm can be used immediately.

Note that $\Pi(j\omega)$ is also even and second row of the Routh array is treated in the usual way by forming the second row from the derivative of the polynomial $\Pi(j\omega)$. Thus, the Routh array is

$$\begin{array}{l|lllll}
\omega^{2n} & (-1)^n a_{2n} & (-1)^{n-1} a_{2n-2} & \cdots & -a_2 & a_0 \\
\omega^{2n-1} & (-1)^n 2n a_{2n} & (-1)^{n-1}(2n-2) a_{2n-2} & \cdots & -2a_2 & \\
\vdots & \vdots & \vdots & & & \\
\omega^0 & a_0 & & & &
\end{array}$$

and the following is relevant:

<u>Popov inequality (3) is satisfied if, and only if, the polynomial</u> Π <u>produces a Routh sequence</u>

$$(-1)^n a_{2n},\ (-1)^n 2n a_{2n},\ \ldots\ ,\ a_0$$

<u>with exactly</u> n <u>sign changes and</u> $a_0 > 0$.

Note that this result is useful when κ and υ in (3) are specified, which is inherent in the Routh test. However, the test requires no graphical construction and is convenient for computer application.

CONCLUSION

The proposed envelope method applies to the extended version of absolute stability, which permits the parameter variations. Therefore, the method has obvious advantage over the conventional frequency technique. The envelope method, however, is convenient to use in easily visualized two-parameter problems. In case of several variable parameters, the interpretation procedure of reference 2 can be more useful.

The absolute stability test is convenient when the parameters are specified. However, no graphical construction is necessary and the computer application is straightforward. It can be extended to test positive realness [5].

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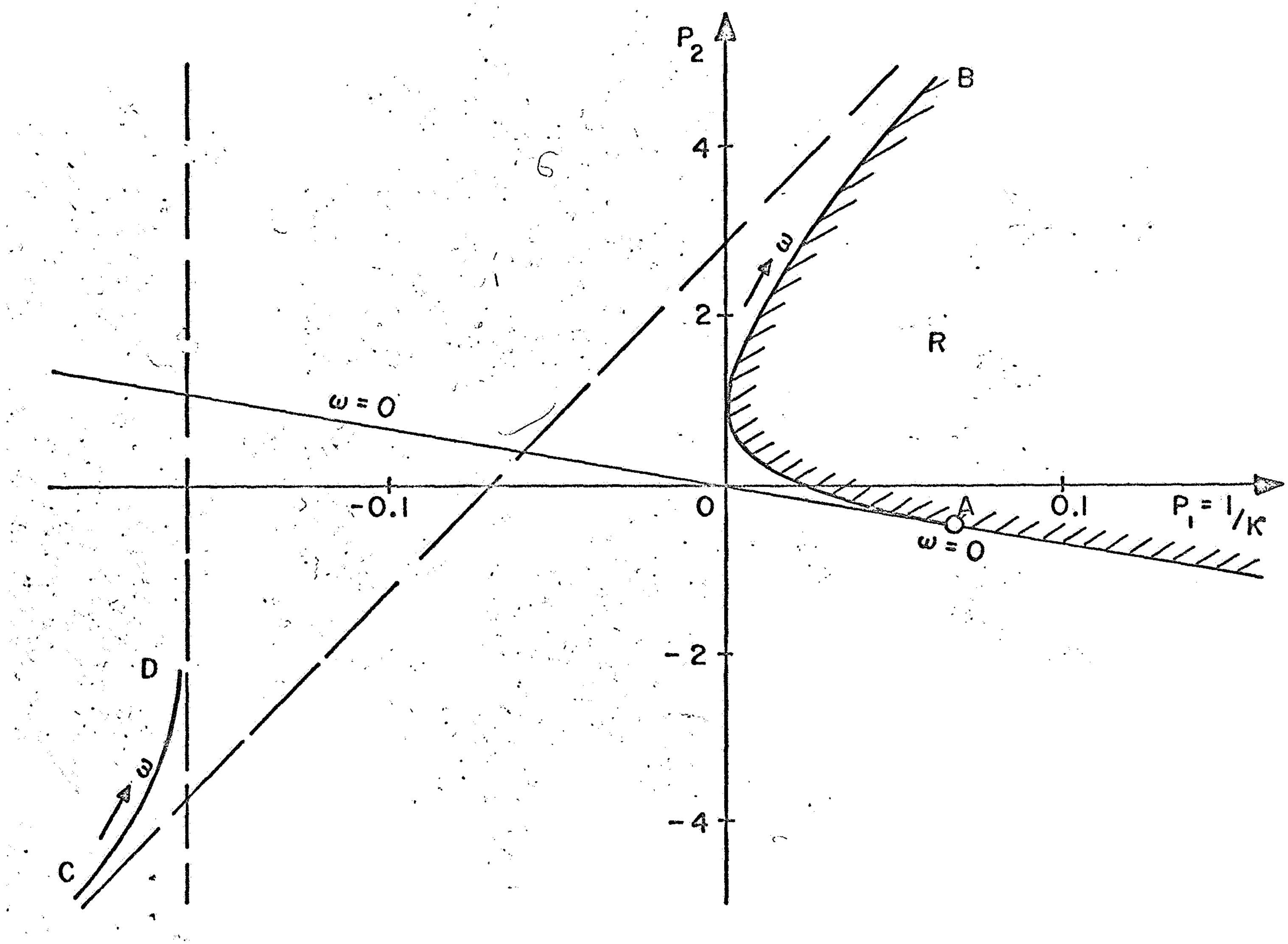
P2
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Conference Record of Second Asilomar Conference on Circuits and Systems, Pacific Grove, California, October, 1968.

AN ANALYTIC TEST FOR ABSOLUTE STABILITY AND POSITIVE REALNESS*

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Abstract: A simple numerical procedure based on the Routh algorithm is proposed for testing absolute stability and positive realness.

1. Free dynamic system of the Lur'e type are described by equations

$$\dot{x} = Px + q\phi(\sigma), \quad \sigma = r^T x, \qquad (1)$$

where x, q, r are real n-vectors, P is a real $n \times n$ matrix, and $\phi(\sigma)$ is a real-valued, continuous function of a real scalar σ which belongs to the class Φ_κ: $\phi(0) = 0$, $0 \leq \sigma\phi(\sigma) \leq \kappa\sigma^2$.[1,2]

We ask: Is the system (1) absolutely stable, i.e., is the equilibrium of (1) globally asymptotically stable for any $\phi(\sigma) \in \Phi_\kappa$.

2. Transfer function $\chi(\lambda)$ of the linear part in (1) is defined as

$$\chi(\lambda) = r^T(P - \lambda I)^{-1}q, \qquad (2)$$

where $\lambda = \delta + j\omega$ is the complex variable. Assume that χ is nondegenerate and P is Hurwitz, that is, all zeros of $\Delta(\lambda) = \det(P - \lambda I)$ are in the left half plane Re $\lambda < 0$. Then:

System (1) is absolutely stable if the Popov inequality

$$\pi(\omega) \equiv \kappa^{-1} + \mathrm{Re}(1 + j\omega\upsilon)\chi(j\omega) > 0 \qquad (3)$$
$$\text{for all real } \omega \geq 0$$

holds for some real number υ.[1,2]

3. Inequality (3) can be rewritten as

$$\Pi(\omega) \equiv |\Delta(j\omega)|^2[\kappa^{-1} + \mathrm{Re}(1 + j\omega\upsilon)\chi(j\omega)] > 0,$$

where Π is an even polynomial in ω. Therefore, the Popov inequality (3) is equivalent to

$$\Pi(\omega) \equiv \sum_{k=0}^{n} a_{2k}\,\omega^{2k} > 0 \qquad (4)$$
$$\text{for all real } \omega \geq 0,$$

and we conclude:

Popov inequality (3) is satisfied if, and only if, Π has no real zeros and $a_0 > 0$.

This result was used in the envelope criterion [2,3] which provides a graphical technique for absolute stability analysis in the parameter space.

4. Since Π is an even polynomial, its zeros are distributed symmetrically with respect to both the real and the imaginary axis of the ω-plane. By rotating the zeros ninety degrees around the origin, the nonexistence of real (positive) zeros of Π is equivalent to the condition that the polynomial

$$\Pi(j\omega) \equiv \sum_{k=0}^{n} (-1)^k a_{2k}\,\omega^{2k}$$

has n zeros with positive real parts. The Routh algorithm can be used immediately.

Note that $\Pi(j\omega)$ is also even and the second row of the Routh array is identically zero. This is a special case in the Routh algorithm and is treated in the usual way by forming the second row from the derivative of the polynomial $\Pi(j\omega)$. Thus, the Routh array is

$$\begin{array}{c|llll}
\omega^{2n} & (-1)^n a_{2n} & (-1)^{n-1}a_{2n-2} & \cdots -a_2 & a_0 \\
\omega^{2n-1} & (-1)^n 2na_{2n} & (-1)^{n-1}(2n-2)a_{2n-2} & \cdots -2a_2 & \\
\vdots & \vdots & \vdots & & \\
\omega^0 & a_0 & & &
\end{array}$$

and the following is relevant:

Popov inequality (3) is satisfied if, and only if, the polynomial Π produces a Routh sequence

$$(-1)^n a_{2n},\ (-1)^n 2na_{2n},\ \ldots,\ a_0$$

with exactly n sign changes and $a_0 > 0$.

Note that this result is useful when κ and υ in (3) are specified, which is inherent in the Routh test.

5. In view of (4), the following simple proposition can be used to test absolute stability:

Popov inequality (3) is satisfied if

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$$a_0 > 0,\ a_{2k} \geq 0,\ (k = 1, 2, \ldots, n). \quad (5)$$

Inequalities (5) lead to a convenient interpretation of absolute stability regions in the parameter space[4].

6. Exponential absolute stability[2] requires that an extended version of Popov's inequality

$$\pi(\omega) \equiv \kappa^{-1} + \mathrm{Re}(1 + j\omega\upsilon)\chi(\delta + j\omega) > 0 \quad (6)$$
$$\text{for all real } \omega \geq 0$$

be satisfied for a given δ and some υ.

Inequality (6) can be transformed into inequality (4). Express

$$\chi(\lambda) = \frac{\Gamma(\lambda)}{\Delta(\lambda)} \quad (7)$$

where Γ and Δ are real polynomials

$$\Gamma \equiv \sum_{k=0}^{m} b_k\lambda^k, \quad \Delta \equiv \sum_{k=0}^{n} c_k\lambda^k \quad (8)$$

and $m < n$. By substituting $\lambda = \delta + j\omega$ in (8), we obtain

$$\Gamma = \Gamma_1 + j\Gamma_2, \quad \Delta = \Delta_1 + j\Delta_2 \quad (9)$$

where

$$\Gamma_1 = \sum_{k=0}^{m} b_k X_k, \quad \Delta_1 = \sum_{k=0}^{n} c_k X_k$$

$$\Gamma_2 = \sum_{k=0}^{m} b_k Y_k, \quad \Delta_2 = \sum_{k=0}^{n} c_k Y_k.$$

Functions $X_k(\delta,\omega)$ and $Y_k(\delta,\omega)$ are defined [2] as

$$X_k = \sum_{i=0}^{k} (-1)^i \binom{k}{2i} \delta^{k-2i}\omega^{2i}$$

$$Y_k = \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{2i-1} \delta^{k-2i+1}\omega^{2i-1}$$

and can be obtained by recurrence formulas

$$X_{k+1} - 2X_1X_k + (X_1^2 + Y_1^2)X_{k-1} = 0$$

$$Y_{k+1} - 2X_1Y_k + (X_1^2 + Y_1^2)Y_{k-1} = 0$$

where $X_0 = 1$, $X_1 = \delta$, $Y_0 = 0$, $Y_1 = \omega$.

By using (7) and (9) in (6), we derive

$$\Pi(\omega) \equiv (\Delta_1^2 + \Delta_2^2)\kappa^{-1} + (\Delta_2\Gamma_1 - \Delta_1\Gamma_2)\omega\upsilon +$$
$$+ \Delta_1\Gamma_2 + \Delta_2\Gamma_1. \quad (10)$$

When κ, υ, δ are specified numerically, we can apply the Routh test of part 4 to prove exponential stability. Note that the introduction of X_k and Y_k functions makes the proposed test easy for computer application.

7. Note that Δ is Hurwitz and $\chi(\lambda)$ is positive real if

$$\chi(j\omega) \geq 0 \quad \text{for all real } \omega. \quad (11)$$

From (10),

$$\Pi(\omega) \equiv \Delta_1\Gamma_2 + \Delta_2\Gamma_1, \quad (12)$$

and we can apply the proposed test to (12) and verify (11). Since Π is even, the statement "for all real ω" is equivalent to "for all real $\omega \geq 0$".

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Regions of exponential stability for the problem of Lur'e*)

Bereiche der exponentiellen Stabilität beim Problem von Lurje

By D. ŠILJAK**) and S. WEISSENBERGER, Santa Clara, California (USA)

This paper proposes a method for computing finite regions of exponential stability from solutions of the absolute stability problem. A best quadratic estimate of the regions is found for a class of nonlinear characteristics which belong to a modified Lur'e sector. For a class of nonautonomous differential equations, a bound on the forcing function is found which guarantees that all the corresponding solutions remain inside the computed region.

Es wird ein Verfahren vorgeschlagen, mit dem endliche Bereiche der exponentiellen Stabilität aus der Lösung des Problems der absoluten Stabilität berechnet werden können. Für eine Klasse von nichtlinearen Kennlinien, die zu einem modifizierten Lurje-Sektor gehören, wird die beste quadratische Schätzung der Bereiche bestimmt. Bei einer Klasse von nichtautonomen Differentialgleichungen wird eine Beschränkung der Eingangsgröße berechnet, die sicherstellt, daß alle zugehörigen Lösungen in dem ausgerechneten Bereich bleiben.

1. Introduction

[Th]e *Lur'e* problem [1], [2] consists in finding a class of nonlinear characteristics for which the equilibrium point of otherwise linear differential equations is globally asymptotically stable. In posing this problem, however, often unrealistic assumptions are made regarding the structure of the differential equations in order to achieve the analytical simplicity of global stability. It is clear that physical networks and systems are not globally stable, nor is there any practical reason to make them so: it is desirable only to make the region of asymptotic stability sufficiently large. Consequently, a modification of the *Lur'e* problem was proposed [3], [4] to include cases with finite regions of asymptotic stability.

This paper proposes an extension of results obtained in [2] to [4] to estimate finite regions of exponential stability, thus providing additional information about how fast the solutions approach the equilibrium point. Computational aspects of constructing appropriate *Liapunov* functions are also discussed. Furthermore, the property of exponential stability makes it possible to consider a class of nonautonomous differential equations and determine a bound on the forcing function which guarantees that all the solutions remain bounded inside the computed region.

2. Basic equations

Let us consider the *Lur'e* class of differential equations [1]

$$\dot{x} = Px + q\varphi(\sigma), \sigma = r^T x \qquad (1)^{***)}$$

where x, q, r are real ν-vectors, P is a real $\nu \times \nu$ matrix, the pair (P, q) is completely controllable, (r^T, P) is completely observable, and $\varphi(\sigma)$ is a real continuous scalar function of the real scalar σ such that the sector conditions

$$\begin{gathered} 0 < \sigma\varphi(\sigma) < \varkappa\sigma^2, \sigma \neq 0 \\ \varphi(0) = 0 \end{gathered} \qquad (2)$$

are satisfied. Functions $\varphi(\sigma)$ with properties (2) are said to belong to the class $\Phi_\varkappa$.

On the basis of *Popov's* results [1], *Yakubovich* [2] showed that if $\varkappa \neq \infty$,

$$\pi(\omega) = \frac{1}{\varkappa} + \mathrm{Re}\,\chi(-\delta + j\omega) > 0, \ \forall \omega \geq 0 \qquad (3)$$

where

$$\chi(\lambda) = r^T (P - \lambda I)^{-1} q, \qquad (4)$$

$\lambda = -\delta + j\omega$, and the roots of $|P - \lambda I| = 0$ are all in the halfplane $\mathrm{Re}\,\lambda < -\delta \leq 0$, then there exist two positive constants ϱ and ε such that for any solution $x(t)$ of (1) and any $t \geq t_0$, we have

$$|x(t)| \leq \varrho\,|x(t_0)| \exp[-(\delta + \varepsilon)(t - t_0)]; \qquad (5)$$

that is, the equilibrium point $x = 0$ of (1) is globally exponentially stable with degree of stability δ for all $\varphi \in \Phi_\varkappa$.

If $\varkappa = \infty$, condition (3) should be supplemented by

$$\lim_{\omega \to \infty} \omega^2 \mathrm{Re}\,\chi(-\delta + j\omega) > 0. \qquad (6)$$

According to the *Yakubovich-Kalman* lemma [1], conditions (3) and (6) are necessary and sufficient for the existence of a *Liapunov* function

$$V(x) = x^T H x \qquad (7)$$

having the derivative along solutions of (1) as

$$\begin{aligned} -\dot{V} = {} & [x^T G_0 x + 2x^T g\varphi + \varkappa^{-1}\varphi^2] \\ & + (\sigma - \varphi\varkappa^{-1})\varphi + 2\delta V \end{aligned} \qquad (8)$$

where

$$-G_0 = HP_0 + P_0^T H, \ -g = Hq + \frac{1}{2} r \qquad (9)$$

and $P_0 = P + \delta I$ is *Hurwitz* [2]. The matrix $H > 0$ satisfies the following matrix inequalities:

$$\begin{aligned} & G_0 - \varkappa g g^T > 0, \text{ for } \varkappa \neq \infty, \\ & G_0 > 0, g = 0, \quad \text{for } \varkappa = \infty. \end{aligned} \qquad (10)$$

3. Regions of stability

As noted above, the satisfaction of condition (3) implies that the equilibrium point of (1) is globally exponentially stable for $\varphi \in \Phi_\varkappa$. If, however, the nonlinearity leaves the sector (2) for $|\sigma| \geq \alpha$, that is, $\varphi \in \Phi_{\varkappa,\alpha}$ where the class $\Phi_{\varkappa,\alpha}$ is defined by

$$\begin{gathered} 0 < \sigma\varphi(\sigma) < \varkappa\sigma^2, |\sigma| < \alpha, \sigma \neq 0, \\ \varphi(0) = 0 \end{gathered} \qquad (11)$$

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***) Capital Roman letters will denote matrices, lower case Roman letters will denote vectors, capital Greek letters will denote sets, and lower case Greek letters will denote scalars. The letter t will be used only for the time, and the letter V only for a *Liapunov* function. Vectors will be considered as column matrices, and the superscript T will denote the transpose. The notation $H > 0$ will mean that H is a positive definite real symmetric matrix. I is the identity matrix.

then the property of exponential stability will be of finite extent in the state space. A *region Ω_δ of exponential stability* is defined as the set of all points $x_0 = x(t_0)$ for which solutions of (1) starting at x_0 are exponentially stable with the degree δ.

We assume that (3) is satisfied for some δ and $\varkappa$, and that $\varphi \in \Phi_{\varkappa,\alpha}$ for some α. Then one is interested to find the largest region $\bar{\Omega}_\delta$ ($\bar{\Omega}_\delta \subseteq \Omega_\delta$) defined by

$$\bar{\Omega}_\delta = \{x \mid |\sigma| < \alpha;\ V(x) < \beta\} \tag{12}$$

where the constant β is determined from

$$\beta = \min_{|\sigma(x)|=\alpha} V(x). \tag{13}$$

Since $\sigma = r^T x$ and $V(x) = x^T H x$, it can be shown from (13) that

$$\beta = \alpha^2 (r^T H^{-1} r)^{-1}. \tag{14}$$

Therefore, with (14), (13) reduces to

$$\bar{\Omega}_\delta = \{x \mid x^T H x < \beta\}. \tag{15}$$

Now the problem is to find the matrix $H > 0$ in (15).

A search for an appropriate matrix H can proceed in two different directions based upon the results obtained in [3] and [4]. A *specific* matrix H can be found from

$$H P_0 + P_0^T H = -u u^T \tag{16}$$

if $\pi(\omega)$ of (3) is rewritten as

$$\pi(\omega) = \frac{\theta(j\omega)\,\theta(-j\omega)}{|P_0 - j\omega I|\,|P_0 + j\omega I|} \tag{17}$$

and u is chosen such that

$$u^T (P_0 - \lambda I)^{-1} q = \frac{\theta(\lambda)}{|P_0 - \lambda I|} - \varkappa^{-1/2} \tag{18}$$

as suggested in [3].

Another approach is to generate a *set* of appropriate H matrices directly from the matrix inequalities (10) [4]. By applying the *Sylvester* inequalities, we reduce (10) to a system of algebraic inequalities involving the elements of H. This approach allows a certain freedom in choosing the corresponding *Liapunov* function (7). In general, each *Liapunov* function produces a different region $\bar{\Omega}_\delta$ with respect to extent and orientation, and it is desirable to select one which produces in some sense the best estimate of the region $\bar{\Omega}_\delta$. A way to find a "best" estimate $\tilde{\Omega}_\delta$ is to maximize the volume of $\bar{\Omega}_\delta$ (a simple matter for quadratic regions) on the set of generated H matrices as suggested in [5].

To illustrate the analysis of the second approach, consider the equation

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -4 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} - \begin{bmatrix}0\\ 1\end{bmatrix}\varphi(x_1 + x_2). \tag{19}$$

The *Popov* condition (3) is satisfied for $\varkappa = \infty$ and $0 \le \delta \le 1$. From (10), $g = 0$,

$$H = \begin{bmatrix}h_{11} & 1/2\\ 1/2 & 1/2\end{bmatrix} \tag{20}$$

and the region $\bar{\Omega}_\delta$ is

$$2 h_{11} x_1^2 + 2 x_1 x_2 + x_2^2 < \alpha^2 \tag{21}$$

for all $\varphi \in \Phi_{\infty,\alpha}$.

The condition $G_0 > 0$ in (10), yields the *Sylvester* inequalities

$$h_{11} < 2/\delta,$$
$$|h_{11} - (\delta-2)^2| < [(\delta-2)^4 - \delta^2 + 4\delta + 4]^{1/2}. \tag{22}$$

The area of $\bar{\Omega}_\delta$ is $\pi\alpha^2 (2 h_{11} - 1)^{-1/2}$ and the maximum-area region is produced by the least h_{11} satisfying (22). Apparently, the extent of the region $\tilde{\Omega}_\delta$ is a function of δ. For instance, the area of $\tilde{\Omega}_0$ is more than three times the area of $\tilde{\Omega}_1$.

4. Boundedness

Consider now the equation

$$\dot{x} = P x + q\varphi(\sigma) + f(t),\ \sigma = r^T x \tag{23}$$

with the forcing function $f(t)$. In [2], it is shown that if condition (3) holds, then for every $f(t)$ bounded on $(-\infty, +\infty)$ and $\varphi \in \Phi_\varkappa$, there exists a bounded region Π such that the solutions $x(t)$ of (23) which start in Π remain there for all future time. In addition, there exists on $(-\infty, +\infty)$ a unique solution $x^0(t) \in \Pi$ which is exponentially stable in the region Π with the degree δ, that is, there are two positive constants ϱ and ε such that for any $t \ge t_0$ and any solution $x(t) \in \Pi$, we have

$$\begin{aligned}&|x(t) - x^0(t)| \\ &\quad \le \varrho\,|x(t_0) - x^0(t_0)| \exp[-(\delta+\varepsilon)(t - t_0)].\end{aligned} \tag{24}$$

We now proceed to compute a particular bound on $f(t)$ which will guarantee that $\Pi \subseteq \bar{\Omega}_\delta$ for all $\varphi \in \Phi_{\varkappa,\alpha}$.

We seek a bound ξ on $f(t)$,

$$|f(t)| < \xi,\ t \in (-\infty, +\infty) \tag{25}$$

which guarantees that no solution of (23) leaves the region $\bar{\Omega}_\delta$. This property of solutions is assured by requiring that the derivative of V along the solutions of (23), which is denoted by $\dot{V}_{(23)}$ be negative on $V = \beta$. According to (23)

$$\dot{V}_{(23)} = \dot{V}_{(1)} + 2 x^T H f \tag{26}$$

where $\dot{V}_{(1)}$ is given in (8). For $x \in \bar{\Omega}_\delta$ and $\varphi \in \Phi_{\varkappa,\alpha}$,

$$\dot{V}_{(1)} < -2\delta V. \tag{27}$$

Therefore, from (26) and (27), we get

$$\dot{V}_{(23)} < -2\delta V + 2 x^T H f. \tag{28}$$

Also,

$$V = x^T H x \le \eta\,|x|^2 \tag{29}$$

and $x^T H f \le \xi\eta\,|x|$ where

$$\eta = \max_\mu \{\lambda_\mu(H)\} \tag{30}$$

where λ_μ ($\mu = 1, 2, \ldots, \nu$) are the eigenvalues of the matrix H. Combining (28) and (29), we conclude that $\dot{V}_{(23)} < 0$ for all $|x| > \xi/\delta$ and $x \in \bar{\Omega}_\delta$. Consequently, if the sphere $|x| = \xi/\delta$ is contained inside $\bar{\Omega}_\delta$, then all solutions which start inside $\bar{\Omega}_\delta$ remain there for all future time. The largest value of ξ which assures this property of $\bar{\Omega}_\delta$ is given by

$$\xi = \delta(\beta/\eta)^{1/2}. \tag{31}$$

It is of interest to note that the bound ξ on the forcing function $f(t)$ in (31) depends directly on the degree δ of exponential stability.

5. Conclusion

Is has been shown how finite regions of exponential stability can be estimated for a *Lur'e* class of differential equations. Two methods of computing the corresponding *Liapunov* functions were indicated. When the differential equations contain a forcing term, the *Liapunov* function can be used to give information about the boundedness of solutions.
A significant extension of the proposed method would be to consider *Liapunov* functions of the complete *Lur'e* type, that is, "a quadratic form plus an integral of the nonlinearity". The additional term in the *Liapunov* function will allow improvement of the estimates of the stability regions. Future work should also be devoted to the computational aspect of the problem.
The presented results can be extended to cases when the differential equations have discontinuous nonlinear characteristics. Moreover, forcing terms which depend on both the states and time can be considered.

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Mitteilungen

IFAC-Symposium „Technical and Biological Problems of Control"

Von H. M. LIPP, Karlsruhe

Vom 24. bis 28. September 1968 wurde in Yerevan/Armenien (UdSSR) ein von der International Federation of Automatic Control (IFAC) veranstaltetes Symposium über „Technical and Biological Problems of Control" durchgeführt. Diese vom sowjetischen nationalen Komitee für Regelungstechnik organisierte Tagung ist mit dem Ziel veranstaltet worden, durch die Begrenzung auf einen relativ kleinen Bereich des Regelungs- und Steuerungsgebietes einen möglichst intensiven Informationsaustausch zwischen den Teilnehmern zu erreichen. Von den etwa 300 Teilnehmern dieses Symposiums (darunter 130 aus anderen Ländern als der Sowjetunion) waren 126 Vorträge eingereicht worden, die acht Themenbereiche ergaben:

I. General Problems of Physiological Mechanisms;
II. Models of Neuronal Structures;
III. Movements Control;
IV. Bioelectric Control and Artificial Organs;
V. Computer Use for Biological Information Processing;
VI. Man-Machine Interaction in Control Systems;
VII. Pattern Recognition;
VIII. Adaptive Systems.

Im Rahmen dieser Zeitschrift sind vor allem die Themengruppen VI bis VIII von Interesse. Die hier gegebene Übersicht beschränkt sich daher auf diese Bereiche.
Probleme des Zusammenwirkens von Mensch und Maschine in Regelsystemen wurden in drei Vorträgen behandelt. Zwei der Referate untersuchten technisch-physiologische bzw. -psychologische Aspekte, im dritten versuchten die Autoren die Hauptrichtungen dieses Forschungsgebietes systematisch darzustellen. Drei Punkte sind dabei wesentlich:
1. Optimale Verteilung von Funktionen zwischen Maschine und menschlichem Operator. 2. Optimale Koordinierung von Operator und den die Information präsentierenden Einheiten. 3. Auswahl und Schulung des Operators.
Die Gruppe Zeichenerkennung war mit zwölf Beiträgen am umfangreichsten, wobei die Vorträge thematisch sehr verschieden waren. Einer der Vortragenden berichtete über die Zeichenerkennungsfähigkeit bestimmter nichtlinearer Filterungen in Form sogenannter „polynomial machines". Weitere Arbeiten mathematischer Natur behandelten die Bestimmung von Wahrscheinlichkeiten bei Musterfolgen und Probleme bei der linearen Separierung von Mustern. Zwei der Referate befaßten sich mit der Generalisierung von Mustern und den entsprechenden Prozessen.
Die Klassifizierung optisch vorliegender Zeichen wurde in nur einem Vortrag untersucht. Zwei Arbeiten zeigten Probleme auf, die bei der Verarbeitung und Erkennung multidimensionaler Informationen, vor allem bei experimentellen Daten aus Biologie und Medizin, entstehen.
Adaptive Systeme waren Gegenstand von drei Vorträgen. Eines der adaptiven Modelle zur Beschreibung von Regelsystemen für zeitoptimale Regelung schien gute Ergebnisse bei der praktischen Anwendung zu zeigen. Die Approximation der optimalen Steuerfunktion konnte dabei erfolgen wahlweise über

a) stückweise lineare Darstellung mit konstanter Intervalllänge und adaptiver Veränderung der Steigung des Segments,
b) stückweise lineare Darstellung mit variabler Intervalllänge und fester Steigung des Segments und
c) nichtlineare Beschreibung.

Ein weiterer Beitrag dieser Gruppe behandelte die optimale Bestimmung von Systemzuständen durch Messungen mit minimalen Kosten.
Obwohl die relativ kleine Zahl von Teilnehmern und Vortragenden einen intensiven persönlichen Kontakt ermöglichte, wird der Nutzen der Vorträge selbst vom Rezensenten nicht zu hoch eingeschätzt. Die für jeden Vortragenden vorgesehene Zeit von 20 Minuten wurde durch die (ohnehin problematische) Satz-für-Satz-Übersetzung erheblich reduziert, so daß kaum die Möglichkeit bestand, tiefergehende Betrachtungen wiederzugeben. Da die Veröffentlichung der vollständigen Beiträge erst nach der Tagung erfolgen wird, konnten die Teilnehmer nur über persönlichen Kontakt Näheres zu den einzelnen Arbeiten erfahren.

Heinrich Toeller, Ehrensenator der Technischen Hochschule Darmstadt

Dr.-Ing. *Heinrich Toeller* wurde am 29. November 1968 zum Ehrensenator der Technischen Hochschule Darmstadt ernannt. Damit wird eine Persönlichkeit geehrt, deren Weitblick und Uneigennützigkeit wir in der wissenschaftlich-technischen Gemeinschaftsarbeit auf unserem Fachgebiet entscheidendes verdanken. So hat sich *H. Toeller* persönlich und mit Nachdruck in der Arbeit der Ingenieur-Verbände

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REGIONS OF EXPONENTIAL BOUNDEDNESS FOR THE PROBLEM OF LUR'E*

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Abstract

This paper considers forced systems of the Lur'e type in which the nonlinearity violates the sector condition in the neighborhood of the origin. It is shown that the satisfaction of a Popov condition and the boundedness of the forcing function and the nonlinearity (where the sector condition is violated) imply the exponential boundedness of the system motion. Quadratic Liapunov functions are used to obtain estimates of the region which system motions enter sooner than an exponential.

Introduction

The classical Lur'e problem consists of finding conditions under which the equilibrium of a system with a single nonlinearity, restricted to lie in a specified sector, is globally asymptotically stable. By modifying the Popov solution of this problem, Yakubovich[1] stated the conditions under which the stability is exponential, thus adding information about how fast the equilibrium is approached. Later, the class of Lur'e systems was enlarged by allowing the nonlinearity to ultimately leave the sector, at the expense of limiting exponential stability to a finite region.[2]

In this paper, the class of Lur'e systems is further broadened to include a number of practical situations in which the nonlinearity is outside the sector in the neighborhood of the origin but ultimately enters the sector and remains there.[3] Furthermore, bounded perturbations of the system are considered. A Popov condition is used to guarantee exponential boundedness, and estimates of the region which system motions ultimately enter are provided by means of quadratic Liapunov functions.

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Exponential Boundedness

Let us consider a forced system of the Lur'e type described by the nth-order equation

$$\dot{x} = Px + q\varphi(\sigma) + f(x,t), \quad \sigma = r^T x \qquad (1)^*$$

where the function $\varphi(\sigma)$ satisfies the conditions

$$|\varphi(\sigma)| \leq \beta \quad , \quad |\sigma| < \alpha \qquad (2a)$$

$$0 < \sigma\varphi(\sigma) < \varkappa\sigma^2 \quad , \quad |\sigma| \geq \alpha \qquad (2b)$$

and $\varkappa > 0$, α, $\beta \geq 0$.

For $|\sigma| > \alpha$, φ is assumed to be continuous, while for $|\sigma| < \alpha$, φ is allowed to be discontinuous and multivalued. The function $f(x,t)$ is bounded,

$$|f(x,t)| \leq \gamma \qquad (3)$$

where $\gamma \geq 0$.

Under the stated conditions, the system (1) may not be stable in the sense of Liapunov, but its motions can exhibit boundedness properties. Of particular interest are systems whose motions, besides being bounded, all enter at some time a certain bounded region Ω and stay there for all future time--the motions are <u>ultimately bounded</u>. Moreover, one would like to have a quantitative estimate of the rate at which the motions approach this region Ω. A common comparison function is the exponential and we arrive at the following:

*Lower case Roman letters denote vectors, capital Roman letters denote matrices, lower case Greek letters denote scalars, and capital Greek letters denote sets. Vectors will be considered as column matrices, and superscript T denotes the transpose. The notation $H > 0$ will mean that H is a positive definite real symmetric matrix. I is the identity matrix. The letter t is used for the time, and the letter V for a Liapunov function. The region Ω^c is the compliment of Ω.

Definition: Solutions $x(t,x_o,t_o)$ of the equation

$$\dot{x} = p(x,t) \qquad (4)$$

are <u>exponentially bounded</u> with respect to a compact region Ω if:

a) there exists a $t_1 \geq t_o$ such that $x(t,x_o,t_o) \in \Omega$ for all $x_o, t_o, t \geq t_1$; and

b) there exist two positive numbers ζ and δ such that, in Ω^c, $|x(t,x_o,t_o)| \leq \zeta |x_o| \exp[-\delta(t - t_o)]$ for all $x_o \in \Omega^c$ and all t_o.

Clearly, this definition implies ultimate boundedness.

By using well-known results[4] we can provide sufficient conditions for exponential boundedness of solutions of equation (4):

Theorem: Let Ω_ν be a compact region $\{x: V(x) \leq \nu\}$ where $\nu > 0$, $V(x)$ be a scalar function with continuous first partial derivatives, and

$$V > 0 \quad , \quad x \in \Omega_\nu^c \qquad (5a)$$

$$V \to \infty \quad , \quad |x| \to \infty \qquad (5b)$$

$$-\dot{V} - 2\delta V > 0, \quad x \in \Omega_\nu^c \qquad (5c)$$

where $\dot{V} = dV/dt = (\nabla V)^T p$ and $\delta > 0$. Then the solutions of equation (4) are exponentially bounded with respect to

$$\Omega_\nu = \{x: V(x) \leq \nu + \epsilon\} \qquad (6)$$

where $\epsilon > 0$ is arbitrary.

To guarantee exponential boundedness for equation (1), let us use the function

$$V = x^T H x. \qquad (7)$$

By the Yakubovich-Kalman lemma[6] the satisfaction of the Popov condition

$$\kappa^{-1} + \text{Re}\,\chi(-\delta + j\omega) > 0 \text{ for all} \qquad (8)$$

real $\omega \geq 0$ where $\chi(\lambda) = r^T(P - \lambda I)^{-1}q$, results in

$$-\dot{V} - 2\delta V = x^T(G - \kappa g g^T)\,x + \kappa^{-1}(\kappa g^T x + \varphi)^2$$
$$+(\sigma - \kappa^{-1}\varphi)\varphi - 2\,x^T\,Hf, \quad \kappa \neq \infty \quad (9a)$$

$$-\dot{V} - 2\delta V = x^T G x + \sigma\varphi - 2\,x^T\,Hf, \quad \kappa = \infty \quad (9b)$$

where

$$-G = HP_\delta + P_\delta^T H, \quad -g = Hq + \tfrac{1}{2}\,r,$$
$$P_\delta = P + \delta I, \qquad (10)$$

the matrix P_δ is assumed to be Hurwitz and a matrix $H > 0$ exists such that the inequalities

$$Q = G - \kappa g g^T > 0, \quad \kappa \neq \infty \qquad (11a)$$

$$Q = G > 0, \; g = 0, \quad \kappa = \infty, \qquad (11b)$$

are satisfied.

According to the lemma, the function V in (7) satisfies the condition (5a,b) above. To establish the remaining condition (5c) we make use of the assumptions (2) on φ. By using (2) and (3) in (9) and applying (7) and (10) we obtain

$$-\dot{V} - 2\delta V \geq x^T Q x - \lambda_H \gamma |x| - \mu, |\sigma| < \alpha \quad (12a)$$

$$-\dot{V} - 2\delta V \geq x^T Q x - \lambda_H \gamma |x|, \quad |\sigma| \geq \alpha \quad (12b)$$

where

$$\mu = (\alpha + \kappa^{-1}\beta)\,\beta, \qquad (13)$$

and λ_H is the largest eigenvalue of H. From (12), we conclude that there exists a region Ω_ν (and thus Ω) outside which the condition (5c) is satisfied. Thus, whenever the Popov condition (8) is satisfied, the motion of the system (1)-(3) is exponentially bounded.

To determine a region Ω_ν in (6), it is necessary to compute ν from

$$\nu = \max_{x \in \Lambda} V(x) \qquad (14)$$

where Λ is the boundary of any compact region outside which $-\dot{V} - 2\delta V > 0$ everywhere. Standard techniques of nonlinear programming can be used to compute ν.

The least conservative ν is obtained by choosing $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ where

$$\Lambda_1 = \{x: x^T Q x - \lambda_H \gamma |x| = \mu, |r^T x| \leq \alpha\} \quad (15a)$$

$$\Lambda_2 = \{x: |r^T x| = \alpha, \; 0 \leq x^T Q x - \lambda_H \gamma |x| \leq \mu\} \quad (15b)$$

$$\Lambda_3 = \{x: x^T Q x - \lambda_H \gamma |x| = 0, |r^T x| \geq \alpha\} \quad (15c)$$

Computation can be facilitated by accepting more conservative estimates of ν. For example, we may take as the constraint for maximization of V(x) in (14)

$$\Lambda = \{x: \lambda_Q |x|^2 - \lambda_H \gamma |x| = \mu\} \qquad (16)$$

which represents a spherical surface enclosing the sets Λ_1, Λ_2, and Λ_3. In constructing Λ, we made use of the inequality $\lambda_Q |x|^2 \leq x^T Q x$ where λ_Q is the smallest eigenvalue of Q. From (16), the radius ρ

is given by

$$\rho = \frac{\lambda_H \gamma}{2\lambda_Q} + \left[\left(\frac{\lambda_H \gamma}{2\lambda_Q}\right)^2 + \frac{\mu}{\lambda_Q} \right]^{1/2} \tag{17}$$

and (14) gives

$$\nu = \lambda_H \rho^2 \tag{18}$$

Expression (18) allows a solution of the inverse problem in which ν and δ are specified and a set of corresponding numbers α, β, γ should be determined. Note also that conditions (10) and (11) allow some freedom in the choice of H, thus permitting improvements in the sharpness of results. In fact, the elements of H could be chosen to minimize the volume of Ω_ν, subject to constraints (10) and (11), in the same way as volume is maximized in the problem of estimating finite regions of stability.[5]

Conclusions

A useful modification of the Lur'e problem has produced systems with exponentially bounded motions which ultimately enter a compact region. Estimates of the region were given using a quadratic Liapunov function.

Improved results could be obtained by including an integral of the nonlinearity in the Liapunov function. However, stronger restrictions on the nonlinearity would be necessary to guarantee the exponential property of the motion.[6] Finally, note that the obtained results apply directly to systems with discontinuous nonlinearities when motions are supplemented with the sliding regime.[1]

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ABSOLUTE STABILITY TEST FOR DISCRETE SYSTEMS*

A numerical absolute stability test for nonlinear discrete system is proposed. The test is derived from the similar test for continuous systems [1] by the use of the bilinear transformation.

A free discrete system of the Lur'e type is described by the difference equations

$$x_{t+1} = Px_t + q\phi(\sigma_t) , \qquad \sigma_t = r^T x_t \tag{1}$$

where x , q , r are real vectors, P is a real $n \times n$ matrix, and $\phi(\sigma)$ is a continuous real scalar function of the real variable σ which belongs to the class $\Phi_\kappa : \phi(0) = 0,\ 0 \leq \sigma\phi(\sigma) \leq \kappa\sigma^2 ,\ \kappa < +\infty$.

Denote by $G(z) = r^T(P-zI)^{-1}q$, where z is the complex variable, the transfer function of the linear part of (1). Let P be Hurwitz, that is, all zeros of $\Delta(z) = \det(P-zI)$ are inside the unit circle $|z| = 1$. Then (Tsypkin [2]):

<u>Theorem 1</u>: <u>System (1) is absolutely stable if</u>

$$\kappa^{-1} + \mathrm{Re}\, G(z) > 0 , \qquad \forall z:|z| = 1 . \tag{2}$$

Instead of verifying (2) by the usual graphical construction, the system stability can be determined analytically by a simple transformation convenient for computer calculations.

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Let

$$G(z) = \frac{\sum_{i=0}^{m} b_i z^i}{\sum_{i=0}^{n} c_i z^i}, \quad n > m \,. \tag{3}$$

then, by using the bilinear transformation

$$z = \frac{1 + w}{1 - w} \tag{4}$$

(2) can be rewritten as

$$\kappa^{-1} + \mathrm{Re}\, G_1\,(j\nu) > 0 \quad , \quad \forall \text{ real } \nu \geq 0 \tag{5}$$

where

$$G_1(w) = \frac{B(w)}{C(w)} = \frac{\sum_{k=0}^{n} B_k w^k}{\sum_{k=0}^{n} C_k w^k} \tag{6}$$

and $w = j\nu$. Coefficients in (6) are

$$\begin{aligned} B_k &= \sum_{i=0}^{m} \sum_{j=0}^{i} (-1)^{j+k} \binom{n-i}{k-j} \binom{i}{j} b_i \\ C_k &= \sum_{i=0}^{n} \sum_{j=0}^{k} (-1)^{j+k} \binom{n-i}{k-j} \binom{i}{j} c_i \end{aligned} \tag{7}$$

Inequality (5) is equivalent to

$$\Pi(\nu) \equiv |C(j\nu)|^2[\kappa^{-1} + \mathrm{Re}\, G_1(j\nu)] > 0 \quad , \quad \forall \text{ real } \nu \geq 0 \tag{8}$$

where $\Pi(\nu)$ is an even polynomial

$$\Pi(\nu) \equiv \sum_{k=0}^{n} a_{2k}\, \nu^{2k} \; . \tag{9}$$

From (8) and (9) we conclude:

Theorem 2: Inequality (2) is satisfied if and only if $\Pi(\nu)$ has no real zeros and $a_0 > 0$.

To verify conditions in Theorem 2, we can use the Routh test as proposed in reference [1]:

Theorem 3: Inequality (2) is satisfied if and only if the polynomial Π produces a Routh sequence

$$(-1)^n a_{2n} \, , \; (-1)^n 2n a_{2n} \, , \; \ldots \, , \; a_0$$

with exactly n sign changes and $a_0 > 0$.

Note that this result is useful when κ is specified. Then the coefficients a_{2k} of Π in (9) are given numerically. Coefficients a_{2k} can be calculated from B_k and C_k by a recursive algorithm outlined in reference [1].

An immediate result of the above is:

Theorem 4: Inequality (2) is satisfied

a) if $a_0 > 0$, $a_{2k} \geq 0$, $k = 1,2,\ldots, n$; and

b) only if the number of sign changes in the coefficients a_{2k} is even.

This result can be used as a simple test to conclude absolute stability.

The proposed absolute stability criterion can be extended directly to systems with multiple nonlinearities. It is only necessary to note that the test should be applied as many times as there are principal minors to be checked for positive definiteness of a given matrix [3].

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Adaptation Using Singular Lines on the Parameter Plane.*

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Introduction. The coefficient plane and parameter plane as developed by Mitrovic[1], Siljak[2,3], and others[4,5], portrays the relationships between the roots of the characteristic equation and the parameters of the system. Its basis, therefore, is in linear theory. It has also been shown that when the parameters of the system vary the operation of such nonlinear systems may be represented on the parameter plane by letting the operating point (M-point) move. This technique successfully predicts limit cycles[3] and the transient response[6,7] of the nonlinear system, and seems to be applicable under the same conditions that constrain the use of describing functions. When the conditions that exist in a control application require adaptation of the feedback loop some method must be devised for identifying the change within the system, and a means of adjustment must be provided which is capable of counteracting the effects of the parameter variation.

This paper is concerned with a method for providing the needed adjustment. If the system can be designed so that its parameter plane equations provide "singular lines" at suitable root values, then an adjustment is obtained which permits the system to counteract a parameter variation and return its basic mode of operation to exactly the original dominant root condition.

Parameter Plane Theory - Singular Lines

Given a polynomial

$$F(s) = \sum_{0}^{k} a_k s^k = 0 \tag{1}$$

where the a_k may have any of the following forms.[4]

$$\begin{aligned}
&\text{Case I} && a_k = b_k\alpha + c_k\beta + d_k \\
&\text{Case II} && a_k = b_k\alpha + c_k\beta + h_k\alpha\beta + d_k \\
&\text{Case III} && a_k = b_{k2}\alpha^2 + b_{k1}\alpha + h_k\alpha\beta + c_{k1}\beta + c_{k2}\beta^2 + d_k \\
&\text{Case IV} && a_k = b_{kn}\alpha^n + b_{k(n-1)}\alpha^{n-1} + \cdots h_{k(n-1)}\alpha^{n-1}\beta \\
& && \quad + \cdots c_{k(n-1)}\beta^{n-1} + c_{kn}\beta^n + dk
\end{aligned} \tag{2}$$

Let $s = \omega_n(-\zeta + j\sqrt{1-\zeta^2}) = \omega_n(\cos\theta + j\sin\theta) = \omega_n e^{j\theta}$ (3)

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Then $s^k = \omega^k e^{jk\theta} = \omega^k(\cos k\theta + j \sin k\theta)$ (4)

$$T_k(-\zeta) = \cos k\theta \quad (5)$$

$$\bar{U}_k(-\zeta) = \sin k\theta/\sin\theta$$

where $T_k(-\zeta)$ and $\bar{U}_k(-\zeta)$ are the well known Chebyshev functions. It can be shown that

$$T_k(-\zeta) = (-1)^k \, T_k(\zeta) \quad (6)$$

$$\bar{U}_k(-\zeta) = (-1)^{k+1} \, U_k(\zeta)$$

Substituting in Eq. (1) and requiring summation reals and summation imaginaries to become zero independently results in

$$\sum_{k=0}^{n} a_k \omega_n^{\ k} (-1)^k \, T_k(\zeta) = 0$$

$$\sum_{k=0}^{n} a_k \omega_n^{\ k} (-1)^{k+1} \, \bar{U}_k(\zeta) = 0 \quad (8)$$

but $T_k(\zeta) = \zeta \, \bar{U}_k(\zeta) - \bar{U}_{k-1}(\zeta)$ (8)

which leads to

$$\sum_{k=0}^{n} (-1)^k \, a_k \omega_n^{\ k} \, \bar{U}_{k-1}(\zeta)$$

$$\sum_{k=0}^{n} (-1)^k \, a_k \omega_n^{\ k} \, \bar{U}_k(\zeta) \quad (9)$$

For a_k of the forms defined as Case I and Case II in Eq. (9) rearrange to

$$\text{Case I} \quad \alpha B_1 + \beta C_1 + D_1 = 0 \quad (10a)$$

$$\alpha B_2 + \beta C_2 + D_2 = 0$$

$$\text{Case II} \quad \alpha B_1 + \beta C_1 + \alpha\beta H_1 + D_1 = 0 \quad (10b)$$

$$\alpha B_2 + \beta C_2 + \alpha\beta H_2 + D_2 = 0$$

where

$$B_1 = \sum_{k=0}^{n} (-1)^k \, b_k \omega_n^{\ k} \, \bar{U}_{k-1}(\zeta) \qquad B_2 = \sum_{k=0}^{n} (-1)^k \, b_k \omega_n^{\ k} \, \bar{U}_k(\zeta)$$

$$C_1 = \sum_{k=0}^{n} (-1)^k \, c_k \omega_n^{\ k} \, \bar{U}_{k-1}(\zeta) \qquad C_2 = \sum_{k=0}^{n} (-1)^k \, c_k \omega_n^{\ k} \, \bar{U}_k(\zeta) \quad (11)$$

$$H_1 = \sum_{k=0}^{n} (-1)^k h_k \omega_n^k \overline{U}_{k-1}(\zeta) \qquad H_2 = \sum_{k=0}^{n} (-1)^k h_k \omega_n^k \overline{U}_k(\zeta)$$

$$D_1 = \sum_{k=0}^{n} (-1)^k d_k \omega_n^k \overline{U}_{k-1}(\zeta) \qquad D_2 = \sum_{k=0}^{n} (-1)^k d_k \omega_n^k \overline{U}_k(\zeta) \tag{11}$$

The parameter plane equations provide singular lines on the parameter plane if Eq. (10) are linearly dependent; i.e., if certain ζ_s, ω_{ns} pairs provide

$$\begin{aligned} B_1 &= LB_2 \\ C_1 &= LC_2 \\ H_1 &= LH_2 \\ D_1 &= LD_2 \end{aligned} \tag{12}$$

where L is a constant, then there exists on the α, β plane a locus of points defining ζ_s, ω_{ns}. Thus for the singular case there are an infinite number of (α, β) pairs which provide the same pair of complex roots. If the singular case exists, and if the available root pairs ζ_s, ω_{ns} are suitable for dominant root design, then the singular lines are added to the usual parameter plane plot and by inspection of the plot a segment of the singular line is located such that the desired dominance condition exists. (Note that in specific cases dominance may not obtain).

Some Systems Exhibiting Singular Lines

The usual, simple, feedback control loop does not provide singular line characteristics as functions of normally adjustable parameters. From Eq. 10, 11, 12, it is seen that a minimum of three simultaneous equations must be satisfied after parameters α and β have been chosen, thus a system must contain a minimum of five parameters to make the singular case possible. Experience suggests that more than five parameters are desirable, that a judicious selection of α and β is necessary, and that the existence of suitable singular lines may depend on the numerical value of any non-adjustable plant parameters.

The signal flow diagram of Fig. 1 represents an inertially stabilized space vehicle, and for certain combinations of transfer functions this system exhibits singular lines. The characteristic equation is obtained from

$$\Delta = 1 + 2\, a_{11}\, G_c\, G + (a_{11}{}^2 - a_{12}{}^2)(G_c\, G)^2 = 0 \tag{13}$$

and for $G_1 = G_2 = G = \frac{K}{s^2}$; $G_{c1} = G_{c2} = G_c = K_1(\frac{\tau s+1}{\gamma\tau s+1})$

$$a_{11} = a_{22}, \quad a_{12} = a_{21}$$

$$\alpha = 2a_{11}\,KK_1, \quad \beta = (a_{11}^2 - a_{12}^2)(KK_1)^2$$

the parameter plane diagram is shown on Fig. 2, including singular lines.

For $G_1 = G_2 = G_c = \frac{K}{s^2 + 2s + 2}$ and all other quantities as previously defined, the parameter plane diagram is shown on Fig. 3, including singular lines.

Basic Concept of Self-Adaptation Using Singular Lines.

Consider Fig. 4, which shows one singular line for $\zeta = \zeta_s$ and $\omega_n = \omega_{ns}$ on the α-β parameter plane. β is the varying plant parameter and α is an adjustable parameter in the controller. Initial operation is on the singular line at point M_1 so that ζ_s, ω_{ns} define dominant roots. When β varies an amount $\Delta\beta$ this change in the parameter must be identified using any suitable scheme. The adjustable parameter, α, is changed an amount $\Delta\alpha$, returning the system to operation on the singular line at point M_2. The system again has root at ζ_s, ω_{ns}. The other characteristic roots are changed, but for reasonable ranges of α and β dominance is retained by the roots at ζ_s, ω_{ns}.

A Specific Case of Self-Adaptation.

Fig. 5a shows a first order plant with variable pole. It is desired to incorporate this plant in a closed loop system and dominant roots at $\zeta = 0.5$, $\omega_n = 4$ are specified. Since the plant pole is variable, self adaptation is planned and two sections of compensation are inserted to provide suitable parameters for obtaining singular lines, and one of the zeros is designated as α.

The characteristic equation is

$$s^3+[(p_1+p_2+KK_1)+\beta]s^2+[p_1p_2+(p_1+p_2)\beta+KK_1\alpha+KK_1z]s+[p_1p_2\beta+KK_1z\alpha]=0 \tag{14}$$

applying Eq. (12) for L = 1.0,

$$KK_1z+KK_1\,\omega_n\cos\theta = KK_1\,\omega_n\sin\theta$$

$$p_1p_2+(p_1p_2)\omega_n(\cos\theta - \sin\theta) = \omega_n^2(\sin 2\theta - \cos 2\theta) \tag{15}$$

$$(p_1p_2+KK_1z)(\cos\theta - \sin\theta)+(p_1p_2+KK_1)(\cos 2\theta - \sin 2\theta) = \omega_n^2(\sin 3\theta - \cos 3\theta)$$

Substituting $\omega_n = 4$ and $\theta = 120^{\circ}$ (for $\zeta = \frac{1}{2}$) we obtain three equations in KK_1, p_1, $p_2 z$. Choosing $p_1 = 2.0$ and solving:

$p_1 = 2.0$; $p_2 = -1.465$; $KK_1 = 3.47$; $z = 5.464$

The equation of the singular line becomes

$$\alpha - \beta = 0 \tag{16}$$

From Equation 16 the singular line requires pole-zero cancellation, which is a trivial case. To show that the system does self-adapt the scheme of Fig. 5a was used. (Note: The identification method was developed by S. R. Parker and R. Desrosiers, and is to be published[9].) Fig. 5b shows the effectiveness of the self adaptation.

The system of Fig. 5 is Type Zero. A slight modification in design permits conversion to Type 1. Plant block diagram and responses are shown in Fig. 6.

Fig. 8 shows adaptive compensation for a variable pole using velocity feedback. This works for either dominant complex roots or a dominant real root as shown on Fig. 8.

<u>Conclusions:</u> A new concept in self-adaptive adjustment has been presented -- use of singular lines on the parameter plane to provide a convenient adjustable variable. An example has shown that the design ideas are feasible and simulation demonstrates that the method works.

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APPENDIX I. PARAMETER PLANE ADAPTATION FOR OVERDAMPED SYSTEMS.

In many applications, particularly in process control systems, an overdamped response is required, i.e., it is desired that a real root be dominant. On the parameter plane, for a system specified by Eq.(1) and Case I of Eq.(2), a real root is defined by a straight line. (Note that substitution of $s=-\sigma+j0$ into Eq.(1) when the a_k are

Case I leads to

$$\alpha B_R + \beta C_R + D_R = 0 \tag{19}$$

where $B_R = \sum_{k=0}^{n}(-1)^k b_k \sigma^k \quad C_R = \sum_{k=0}^{n}(-1)^k c_k \sigma^k \quad D_R = \sum_{k=0}^{n}(-1)^k d_k \sigma^k$

and Eq.(19) is the equation of a straight line on the α-β plane.)

Since every real root for such a system is described by a straight line on the parameter plane, the M-point may be located on the line defining a dominant real root. The equation of that line is then determined and inserted in the adaptive computer, and adaptation is effected exactly as explained for singular lines. Fig. 7a shows the block diagram of a type 1 system with variable plant parameter α and adjustable parameter β. A dominant real root is desired at $\sigma = -2$, for which the parameter line is

$$\beta = 0.40 + 0.8\alpha \tag{17}$$

The M-point is chosen at $\alpha_o = 4.0$, $\beta_o = 3.6$, at $t = 0.1$ sec. α is changed to $\alpha = 3.5$. The identification process determined the value of α to be 3.49597, and adjusted β accordingly. Fig. 7b shows the transient response of the adapted system and compares it with that of the unadapted system.

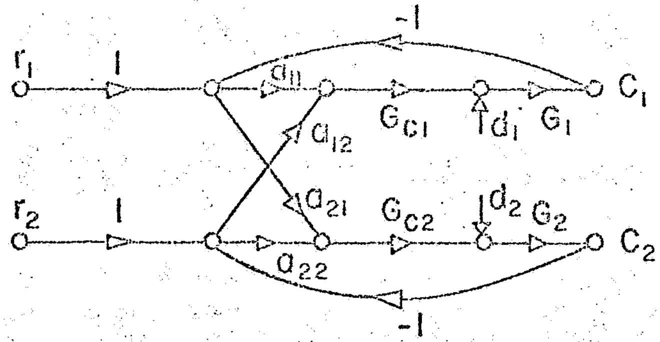
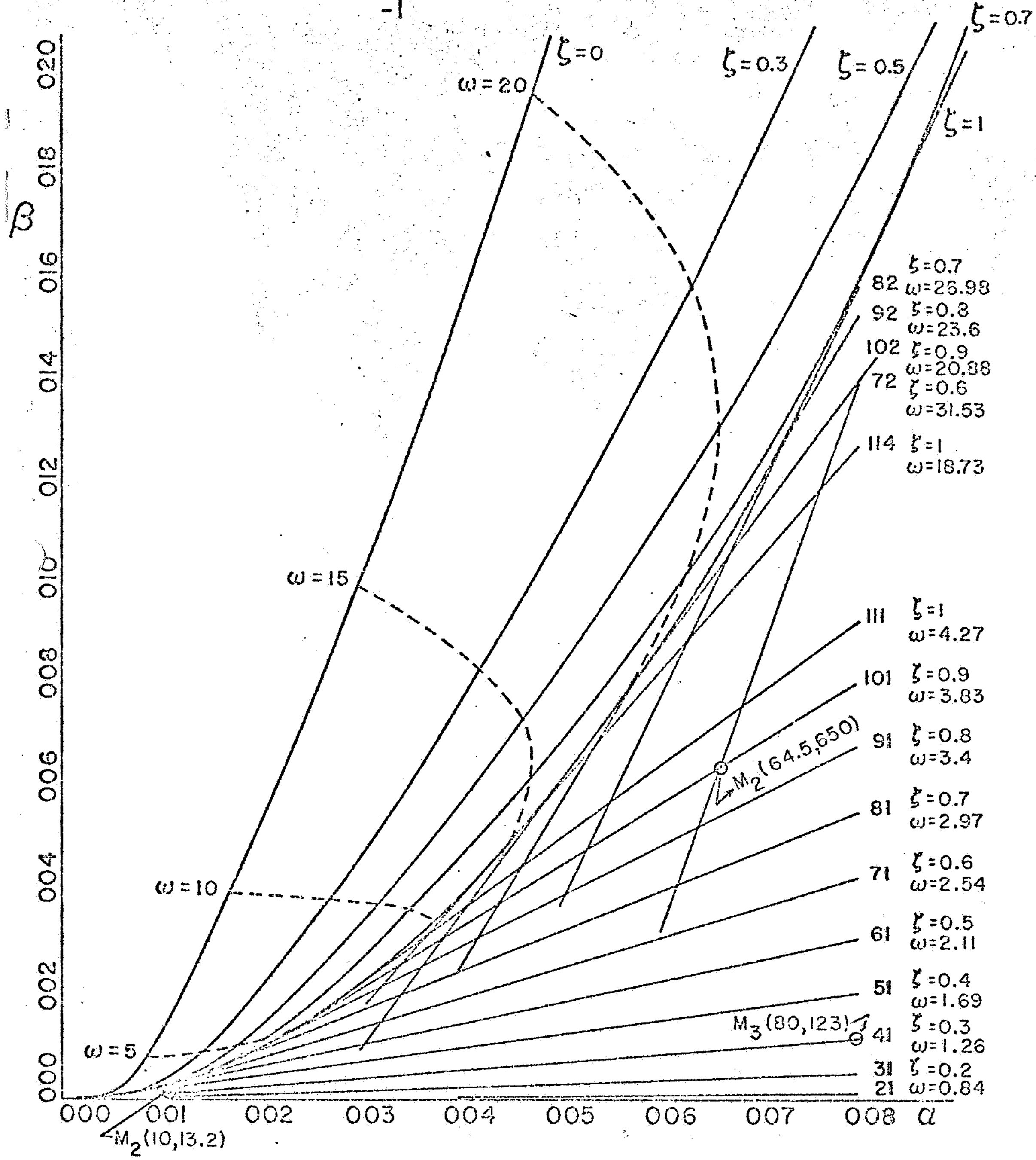


FIG.1 FLOWGRAPH OF ALINEAR MULTIVARIABLE SYSTEM.



X-SCALE=1.00E+01 UNITS/INCH.
Y-SCALE=2.00E+02 UNITS/INCH.

FIG.2 SINGULAR LINES IN THE PARAMETER PLANE FOR FIG.1 $G_1 = \frac{K}{s^2}$.

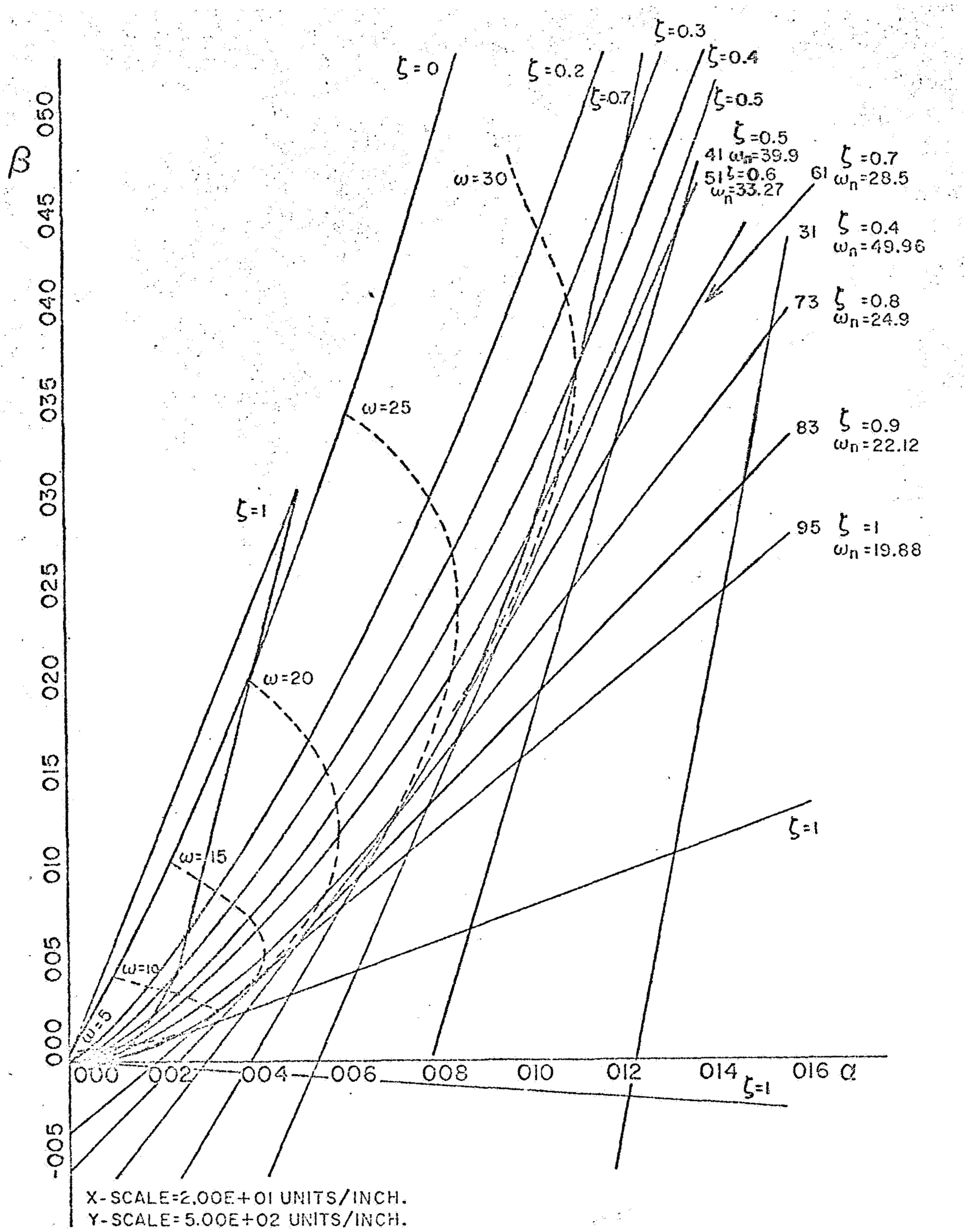


FIG.3 SINGULAR LINES ON THE PARAMETER PLANE OF FIG.1 $G_1 = \frac{K}{s^2+2s+2}$

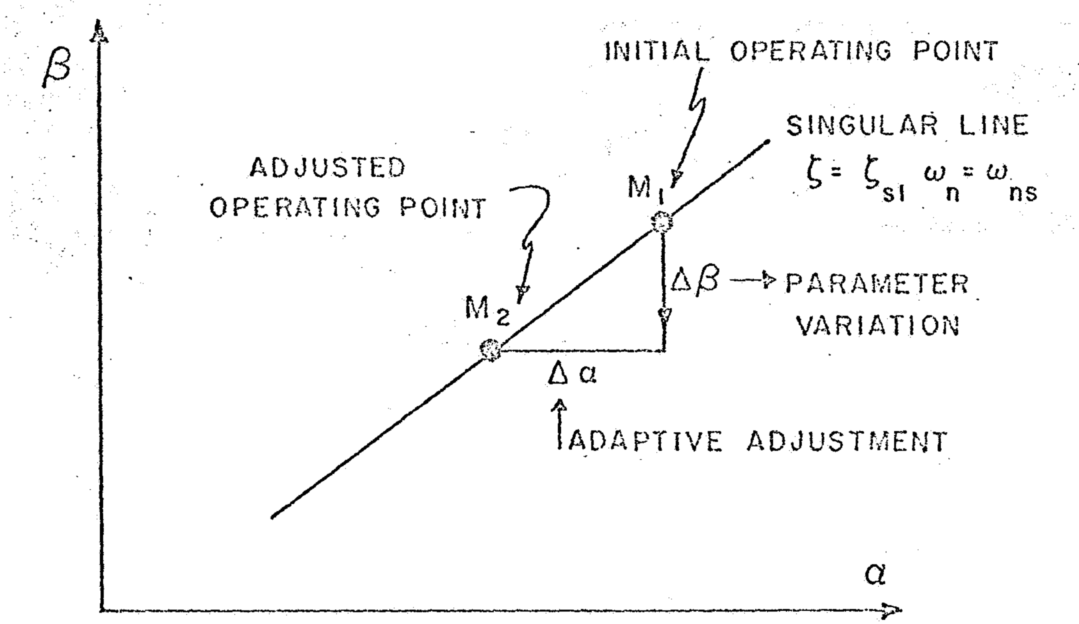
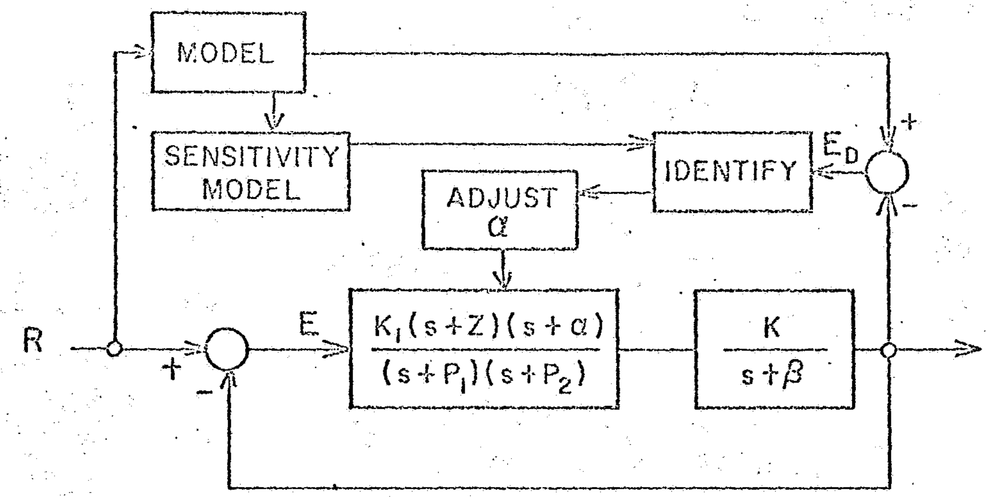
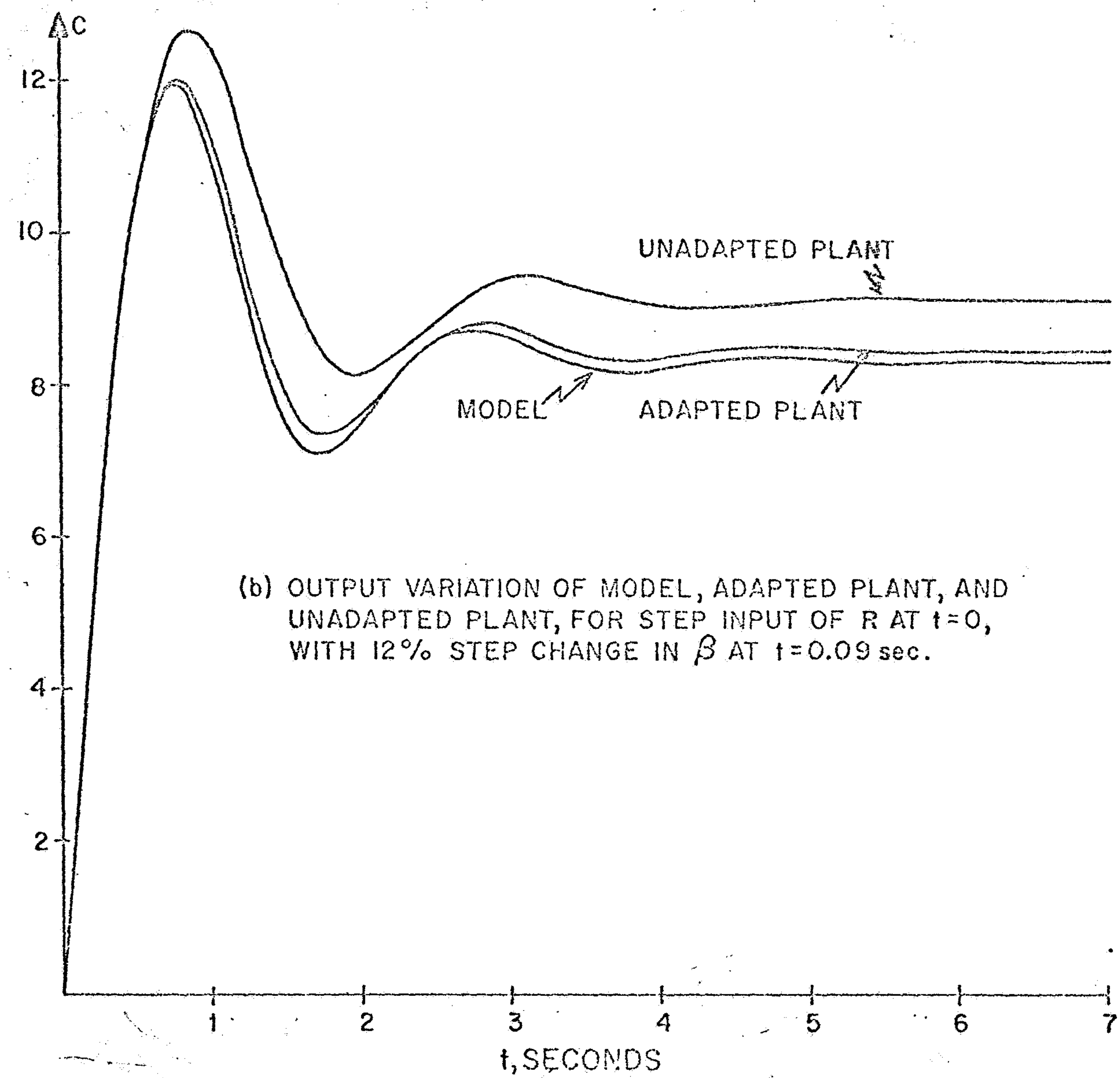


FIG. 4. BASIC CONCEPT OF SELF-ADAPTATION WITH SINGULAR LINES.



(a) BLOCK DIAGRAM

FIG. 5 SELF-ADAPTED RESULTS



(b) OUTPUT VARIATION OF MODEL, ADAPTED PLANT, AND UNADAPTED PLANT, FOR STEP INPUT OF R AT t=0, WITH 12% STEP CHANGE IN β AT t=0.09 sec.

FIG. 5 SELF-ADAPTED RESULTS.

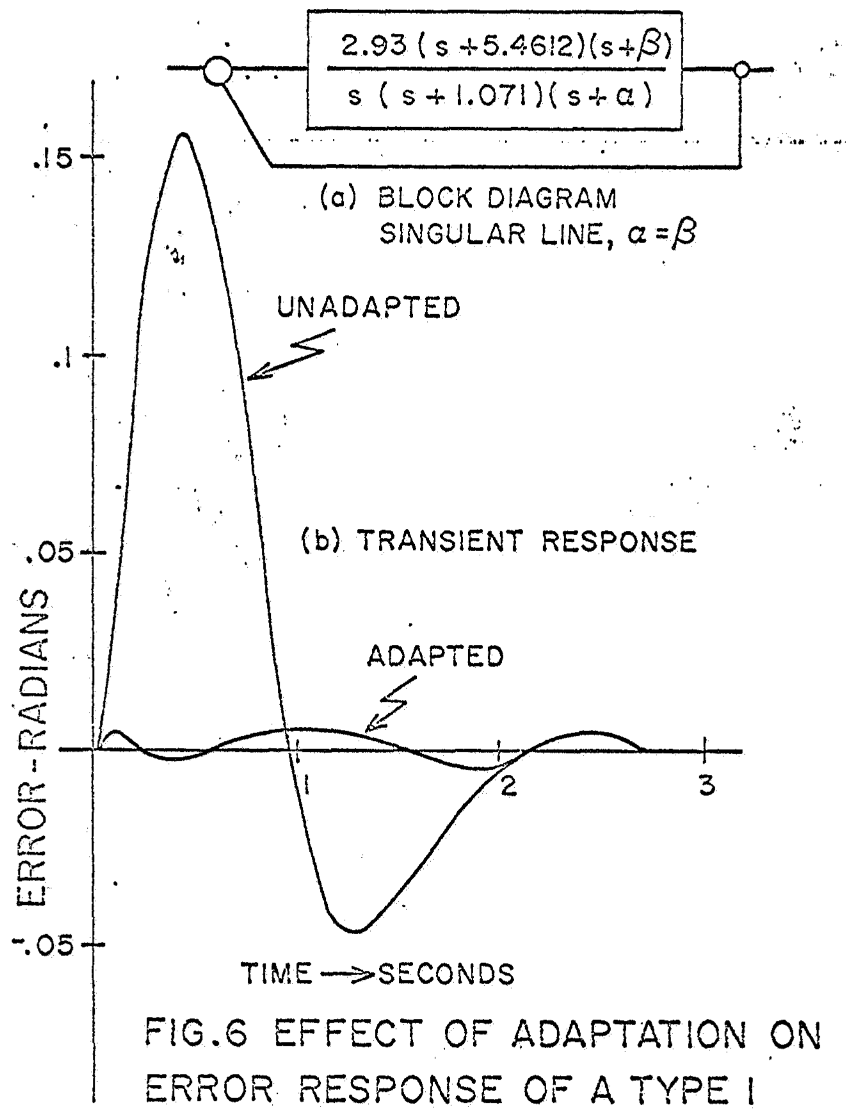


FIG.6 EFFECT OF ADAPTATION ON ERROR RESPONSE OF A TYPE I SYSTEM.

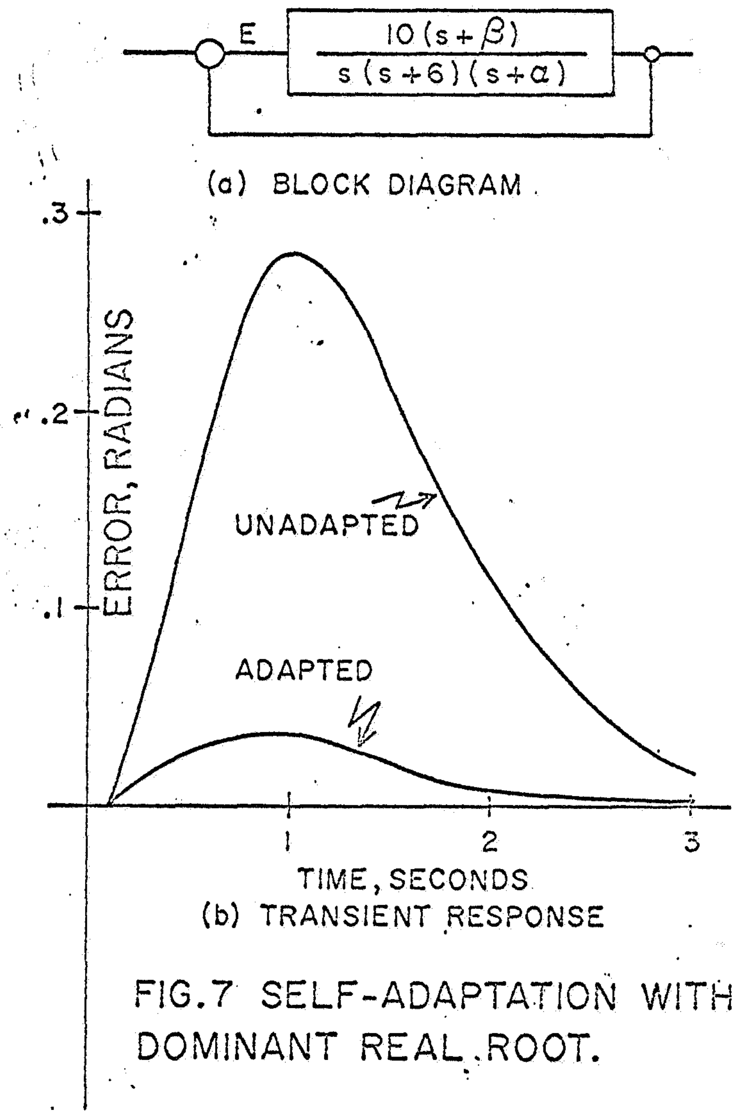


FIG.7 SELF-ADAPTATION WITH A DOMINANT REAL ROOT.

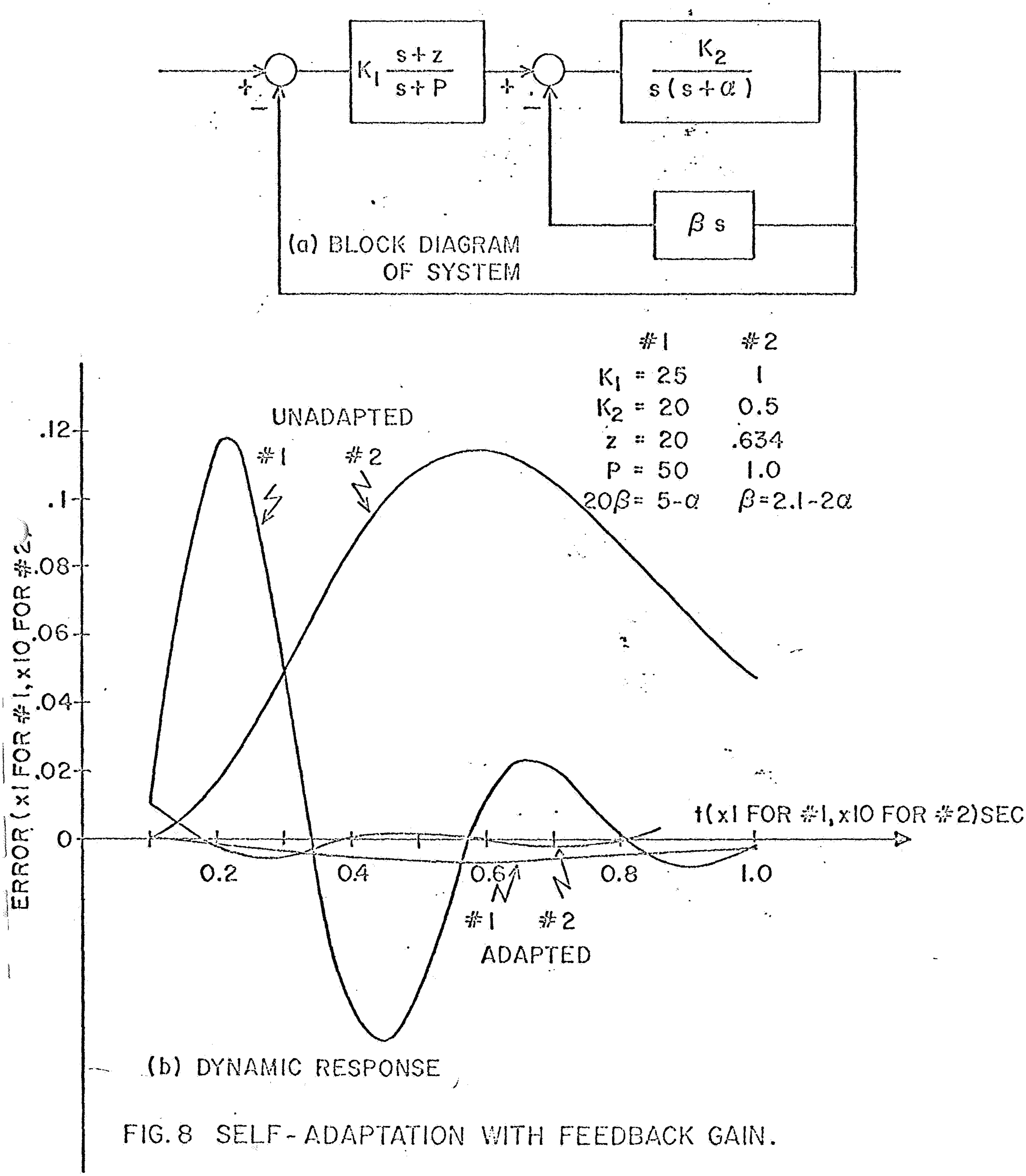


FIG. 8 SELF-ADAPTATION WITH FEEDBACK GAIN.