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SENSITIVITY GUIDANCE FOR ENTRY INTO

AN UNCERTAIN MARTIAN ATMOSPHERE

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ABSTRACT

The dynamics of entry into a planetary atmosphere, sensitivity analysis for control and for updating, and efficient implementation of the proposed guidance techniques are studied. To simplify the description of the guidance requirements, the trajectory equations are written with the altitude as the independent variable. Retro-propulsion is the assumed method of control.

In the main part of this thesis, the Martian entry guidance problem is found to be complicated by the great uncertainty in the present definition of the Martian atmospheric parameters. A guidance scheme that will produce a reference terminal condition whatever the actual atmosphere encountered on Mars entry is suggested. The approach is analytical. Sensitivity Analysis is applied to the entry dynamics in order to compute the effects of both density parameter deviations and control changes. After the atmospheric parameters are tracked, the control is determined on-board by using the sensitivity coefficients previously compiled. Control updating is provided by introducing a new sensitivity equation which reduces the on-board computation since all the required terminal sensitivity coefficients are now produced by the solution of one

equation. Numerical simulation assuming a VM-2 reference density and VM-1 actual density showed that the terminal velocity and range angle errors were reduced by at least 90% in comparison with those resulting from the uncontrolled VM-1 trajectory. The effects of delays in obtaining information describing the actual atmosphere and of inaccuracies in that information were also investigated.

Second order sensitivity functions are investigated with a view towards improving guidance system performance in the case of large deviations in the atmospheric parameters. Previous workers have derived higher order sensitivity equations using a single n-th order differential equation to model the physical system. However, the state vector described by n first order equations gives a more general approach for dynamical systems. A new vector-matrix differential equation for the second order sensitivity coefficients of a general system is obtained. It is found that the second order sensitivity forcing function depends on the present altitude in a planetary entry problem in contrast to the first order sensitivity forcing function which is independent of the present altitude. This point is important in the calculation of the terminal values of the second order sensitivity coefficients. With the first order coefficients, it was possible to describe all the terminal values by using the adjoint sensitivity equation. For the second order coefficients, this procedure is only possible for a certain approximation to the second order sensitivity forcing function.

PART I

INTRODUCTION

A. Historical Review

1. Atmospheric Flight Mechanics

Before reviewing specific efforts in the field of atmospheric flight mechanics, the relationship between the flight mechanics problem and the problem of guiding a capsule-lander entering an uncertain planetary atmosphere is considered. The essence of the guidance philosophy developed herein makes considerable use of certain partial derivatives of the state variables. These derivatives, called sensitivity coefficients, multiply the parameter or initial condition deviations to give deviations in the state variables. These sensitivity coefficients are usually found by numerical solution of a set of linear differential equations with variable coefficients. However, suppose that analytic, closed-form solutions for the entry dynamics were available. Then the sensitivity coefficients could be obtained as algebraic functions by performing the indicated partial differentiation. The difficulty with this approach is the lack of a general solution of the dynamical equations uniformly valid for Mars entry over all the regions of interest: Keplerian, Aero-gravity Perturbed, Aerodynamically Dominated, and Sonic Velocity. Specific contributions towards analytic solutions of the entry dynamics are now discussed.

Chapman[1] contributed a certain set of transformations which enabled him to approximate the entry dynamics (see Appendix A) by one nonlinear differential equation:

$$u \frac{d^2}{du^2} Z - \frac{d}{du} Z + \frac{Z}{u} = \frac{1-u^2}{uZ} \cos^4\theta - \sqrt{\beta R_0}\, \frac{L}{D} \cos^3\theta \tag{1-1}$$

where the variables, Z and u, are given by

$$u = \frac{V \cos\theta}{\sqrt{g R_0}} \tag{1-2}$$

$$Z = \left[\frac{1}{2\left(\frac{m}{C_D A}\right)}\right] \sqrt{\frac{R_0}{\beta}}\; \rho V \cos\theta \tag{1-3}$$

It should be noted that the following assumptions were made in developing Chapman's Equation:

(i) The percentage change in distance from the center of the planet is small compared to the percentage change in velocity.

(ii) $\left| (L/D) \tan\theta \right| << 1$

(iii) Non-rotating, spherical planet with exponential atmospheric density distribution.

From the practical point of view, this effort does not constitute an analytic solution of the entry dynamics. However, it does show that combining the physical state variables in certain different groups may lead to a simplification of the equations of motion.

Loh's work[2,3,4,5] constitutes an early approach toward analytic solutions of the entry dynamics.

In addition to the assumptions made in developing the time domain equations of motion (A-22,23,24,25), the density profile is written explicitly as

$$\rho(y) = \rho_o e^{-\beta y} \tag{1-4}$$

where ρ_o = surface density

β = inverse scale height

Noting that time does not appear in the equations of motion, the density is taken as the new independent variable resulting in

$$\frac{d \cos\theta}{d\rho} + \left(\frac{1}{\beta R_o}\right)\left(\frac{\cos\theta}{\rho}\right)\left(\frac{g R_o}{V^2} - 1\right) = \frac{1}{2}\left(\frac{L}{D}\right)\left(\frac{C_D A}{m\beta}\right) \tag{1-5}$$

$$\frac{d}{d\rho}\left[V^2/gR_o\right] + \left(\frac{C_D A}{m\beta}\right)\frac{\left[V^2/gR_o\right]}{\sin\theta} = \left(\frac{2}{\beta R_o}\right)\frac{1}{\rho} \tag{1-6}$$

Loh then claims that the quantity

$$\left(\frac{1}{\beta R_o}\right)\frac{\cos\theta}{\rho}\left(\frac{gR_o}{V^2} - 1\right)$$

"....must be insensitive to ρ or θ integration."[2] This statement is supported by references to first order theories and numerical simulation. Norman[6] also discusses this assumption with specific reference to ballistic entry. By the previous discussion, the following results

$$\cos\theta = \cos\theta_f + \left[\frac{1}{2}\left(\frac{L}{D}\right)\left(\frac{C_D A}{m\beta}\right) - \left(\frac{1}{\beta R_o}\right) \frac{\cos\theta}{\rho}\left(\frac{gR_o}{V^2} - 1\right)\right](\rho - \rho_f) \quad (1\text{-}7)$$

Next, the gravity component parallel to the velocity vector is neglected. Together with Equation (1-7), this assumption enables Loh to obtain

$$\ln\left[\frac{V^2/(gR_o)}{V_f^2/(gR_o)}\right] = \frac{[(C_D A)/(m\beta)](\theta - \theta_f)}{\frac{1}{2}\left(\frac{L}{D}\right)\left(\frac{C_D A}{m\beta}\right) - \frac{1}{\beta R_o}\frac{\cos\theta}{\rho}\left(\frac{gR_o}{V^2} - 1\right)} \quad (1\text{-}8)$$

Equations (1-7) and (1-8) are Loh's approximate solutions to the entry dynamics. One important point should be noted. These solutions are most valid during the initial phases of entry where the velocity is still of the same order of magnitude as the circular satellite velocity. Willes[7] and Citron and Meir[8] further discuss and evaluate Loh's work.

Willes[7] applied the Method of Matched Asymptotic Expansions to the entry dynamics problem with several objectives:

(i) defining the 'regions' in which specific approximate analytic solutions such as Loh's are valid,

(ii) extending the solutions to higher accuracy, and

(iii) combining the solutions to obtain composite solutions valid over several flight regimes.

From the point of view of the Martian entry guidance problem, the third objective is the most interesting. Intuition and experience have led the present investigator to divide the aerodynamic braking phase of Mars entry into two flight regimes: (i) a drag dominated phase characterized by velocities of the order of the circular velocity and slowly changing flight path angles, and (ii) a "near sonic" regime characterized by much smaller velocities and a more rapidly changing flight path angle. In terms of forces the "near sonic" regime is one in which drag and gravity are of the same order of magnitude. Analytic solutions for this second flight regime have not been obtained except for some special cases[9]. General analytic solutions for the "near sonic" regime and composite solutions valid over both regimes will be needed if one is to obtain closed form expressions for the sensitivity coefficients used in the guidance equations. Other works which contributed to the author's understanding of the atmospheric flight mechanics problem are

Martin[10], Platte[11], Hanin[12,13], Allen and Eggers[9] and Jungmann[14]. Related literature in the field of Matched Asymptotic Expansions is given in References 15 to 25 where the works of Van Dyke[15], Erdelyi[20], Wasow[24], and Cole[25] perhaps provide the most general coverage.

2. Present Knowledge of the Mars Atmosphere

Before beginning the discussion, it is necessary to relate knowledge of the Martian atmosphere to the problem of soft landing an unmanned capsule on the surface of Mars. First, newer estimates of the Martian atmospheric parameters have greatly influenced the design of a reference trajectory for such a landing. Early estimates of the Martian surface pressure ranged from 40 to over 100 millibars[26,27,28]. For these 'thick' atmospheres, Wingrove[29] shows that the aerodynamic braking provided by a vertical entry trajectory would be sufficient. For thinner atmospheres,[30,31,32] Wingrove suggested that 10 millibars is a reasonable lower limit on the surface pressure necessary for vertical entry trajectories. He also suggested that a surface pressure of 7 millibars is about the lowest limit for safe direct entry trajectories using terminal phase deceleration devices[29].

However, the latest estimates of the Martian atmospheric density parameters[33,34,35,36,37] obtained from Mariner IV experiments caused further revision of plans for soft landing. These studies of the Mariner IV data indicated that the Martian surface pressure is around 4 to 6 millibars, though this result is not conclusive. To compensate for the lessened atmospheric drag, two techniques have been suggested[38]. These are (1) lowering the capsule ballistic coefficient and (2) lengthening the flight path in the atmosphere. More will be said about these proposed strategies in the Statement of Problem section of this thesis. To conclude the review of data pertaining to the Martian atmosphere, it is noted that the most recent sources of data are References 39 and 40. For a detailed description of the atmospheric density models used in this thesis, see Appendix B.

3. Sensitivity Analysis

Before the development of analog and digital computers, the study of a dynamical system usually was to find an analytical closed-form solution for the differential equations describing that system. However, in the analysis of practical dynamic systems, along with the obtaining of solutions as functions of the independent variables, it is also important to have a knowledge of the variations of the solutions with respect to the parameters of the problem. If the solution of the dynamic system has been found in analytic closed form, then the problem of describing the variation with the parameters is quite simple. It is only necessary to perform the indicated partial differentiation in order to obtain the coefficients of a Taylor series expansion for the perturbed solution. For complicated dynamic systems such as the entry dynamics where analytical closed-form solutions are hard to find and only numerical techniques seem applicable, the approximation of the effects of parameter variation poses a greater problem. This is the problem to which Sensitivity Analysis is directed. Now we review specific contributions to Sensitivity Theory.

The analysis of Miller and Murray[41] constitutes an early development of a workable form of Sensitivity Analysis.

Though the motivation for these studies was quite different (Miller and Murray were interested in obtaining a mathematical basis for a general error analysis of solutions of systems of ordinary differential equations by analog computer methods), there is a general discussion which was useful to the present investigator. In particular, the relationship between modern sensitivity analysis and the more classical theory of differential equations is noted. Theorems for the dependence of systems of ordinary differential equations on parameters are also discussed. That perturbations of complicated nonlinear systems may be described by linear differential equations without "linearizing" or simplifying the given system is also shown.

As was mentioned, Sensitivity Analysis is closely related to some well known theorems of differential equations. Hartman[42] provides a concise review of these theorems in his general work on ordinary differential equations. The following system is considered

$$y' = f(t, y, z), \quad y(t_0) = y_0 \tag{1-9}$$

where $z = (z^1, \cdots, z^e)$ is a set of parameters and where, for each fixed z, the system has a unique solution

$$y = \eta(t, t_0, y_0; z) \tag{1-10}$$

That the assumption of uniqueness implies the continuity of the general solution is noted and a theorem for this is given and proved. Finally, the question of the differentiability of the general solution is discussed. A Theorem (due to Peano) is given that provides sufficient conditions for the existence of the first partial derivatives of the general solution.[42] This same theorem also specifies another differential equation whose solutions are the first partial derivatives of the general solution[42]. Since these first derivatives are equivalent to the first order sensitivity coefficients, any study of Sensitivity Analysis should include this work.

The work of Tomovic[43] perhaps contributed the most to the understanding of the theory of the sensitivity analysis of dynamic systems. The discussion begins with the following mathematical model:

$$F(\ddot{x}, \dot{x}, x, t, g_0) = 0 \tag{1-11}$$

Next the sensitivity coefficient $u(t, g_0)$ is defined by

$$u(t, g_0) = \lim_{\Delta g \to 0} \left[\frac{x(t, g_0 + \Delta g) - x(t, g_0)}{\Delta g} \right] = \frac{\partial}{\partial g_0} x(t, g_0) \tag{1-12}$$

where $x(t, g_0 + \Delta g)$ is the perturbed solution that results when $g_0 + \Delta g$ is substituted for g_0 in the original mathematical model. The sensitivity equation is then obtained by taking the partial derivative of equation of the model with respect to g_0.

The result is

$$\frac{\partial F}{\partial \ddot{x}}\ddot{u} + \frac{\partial F}{\partial \dot{x}}\dot{u} + \frac{\partial F}{\partial x}u = -\frac{\partial F}{\partial g_0} \qquad (1\text{-}13)$$

A second order sensitivity equation is also derived and that result will be discussed in another portion of this thesis.

Thompson and Kohr[44] extend Tomovic's analysis by considering an n-th order system modeled by a single scalar equation:

$$F(t, x, px, \ldots, p^n x, g_1, \cdots, g_m) = 0 \qquad (1\text{-}14)$$

where x is the solution,

$$p^k x = \frac{d^k x}{dt^k} \qquad (1\text{-}15)$$

and $g_1, \cdots, g_m$ are parameters. Again, first and second order sensitivity equations are derived.

The book Sensitivity Methods in Control[45] in the Proceedings of the International Symposium on Sensitivity Analysis, sponsored by IFAC, held in Dubrovnik, Yugoslavia in

1964. Papers presented were divided into the following categories: Basic Approaches, Sensitivity Functions, Compensation of Parameter Variations, Synthesis of Insensitive Systems, and Sensitivity and Optimality. The paper by Kukhtenko and Shevelev[45] dealing with the design of insensitive control systems was of particular interest since it gave a further application of Tomovic's concept of Second order sensitivity.

The next three papers are strictly applications oriented. Staffanson[46] deals with a kind of inverse sensitivity problem in which the object is to obtain improved system parameters from recorded text range data. Watson[47] applied Sensitivity Analysis in a somewhat novel manner. To design optimal reference trajectories insensitive to parameter variations for advanced ballistic missile systems, he added sensitivity functions directly into the cost functional. Hull and Gunckel[48] present sensitivity data for the Mars entry problem although it is not clear whether this data was obtained by solving for the actual sensitivity functions or by using the method of divided differences.

Finally, the survey of Kokotovic and Rutman[49] provides a comprehensive introduction as well as an extensive bibliography in the field of Sensitivity Analysis.

4. Entry Guidance

The function of the entry guidance system is to control the braking of a space vehicle in the vicinity of the planet's atmosphere. The final objective is usually the soft landing of the vehicle on the surface of the planet. In this section, both theoretical guidance schemes and various control techniques are reviewed. In addition, some guidance and control techniques suggested specifically for Mars are considered. Finally, some attempts to apply modern control theory to the entry guidance problem are mentioned.

In the past, methods for guiding the flight of an entry vehicle have been divided into the following categories:[50]

(i) linear perturbation guidance employing on-board calculation of future trajectories using approximate expressions.

(ii) linear perturbation guidance employing storage of reference trajectories and optimal feedback control.

(iii) guidance based on on-board fast-time integration of future trajectories.

The fast-time integration technique[1,50,51] uses a prediction of the range performance that would be obtained if the

the current control is held constant. The prediction is obtained by numerical integration of the equations of motion. After comparing the predicted range with the desired range, a second prediction is made using a "modified control" designed to reduce the range error. Since the prediction time must be small in comparison with the time of flight, Chapman's reduction of the entry dynamics[1] to a single non-linear differential equation is really the basis of this technique.

Linear perturbation guidance employing on-board calculation of future trajectories using approximate expressions is perhaps the most promising entry guidance technique since it has the ability to adapt to a wide range of terminal objectives.[50] In this technique, the control is formed by a summation of terms linearly proportional to the deviations of the actual trajectory from a reference trajectory. The proportionality factors are the functions which multiply the state variable deviations to give the control. These factors depend on the particular approximate solution of the trajectory dynamics. Approximate solutions are available for the following flight modes:[50] the equilibrium glide, constant path angle, constant altitude rate constant aerodynamic load factor and constant rate of change of load factor with velocity. Unfortunately, these solutions are applicable only to Apollo-type lifting entry into the Earth's atmosphere. As mentioned previously in this thesis, further approximate solutions will be needed before this technique can be fully applied to the ballistic entry planned for Mars landing. Examples of present applications of this guidance technique are given in reference 52, 53, 54, and 55.

Perturbation guidance using stored reference trajectories and optimal feedback control is possible when the steering objective and entry conditions are known with some certainty beforehand.[50] Such techniques have been developed by Bryson and Denham.[56]

While most of the above schemes use the lift to drag ratio as the control, variation of the ballistic coefficient above has also been suggested as a control technique by Phillips and Cohen[57] and Warden[58]. Control of the ballistic coefficient is particularly applicable to Mars entry.

Preliminary studies of the Mars entry guidance have been given by Hull and Gunckel[48], Woestemeyer[59], and Moore and Cork[60].

Finally, attempts to apply modern control system theory to the re-entry of aerospace vehicles have been made by Kishi et al[61,62] with moderate success.

B. Statement of the Problem

The Mars entry guidance problem is significantly more difficult than any previously considered entry problem for the following reason. In Earth re-entry the atmosphere is considered to be a well known quantity. The major disturbances are entry condition variations such as entry angle and/or entry velocity errors. In Mars entry, however, the atmosphere itself is a source of great uncertainty. The guidance system must be able to compensate for variations in the atmosphere as well as in the entry conditions.

Trajectory errors due to deviations between the actual and reference values of the Mars atmospheric parameters are important for two reasons. First, the trajectories obtained by applying aerodynamic braking are particularly sensitive to deviations in the parameters of the Martian atmosphere.[29,38,48] Second, the parameter deviations probably will be large. Present estimates of the parameters cover a wide range with surface pressures ranging from 4 to 10 millibars and scale heights ranging from 3 to 9 miles in the VM-1 to VM-8 atmospheres.[33,39]

The effect of trajectory sensitivity to parameter deviations is exemplified by the terminal altitude at the end of the aerodynamic braking phase. Variation in the surface pressure from 10 millibars to 4 millibars may lower the terminal altitude (where the vehicle has slowed to 1000 feet per second) from slightly over 40,000 feet to approximately 14,000 feet. Such a wide variation would be a problem in designing a terminal phase landing system. In addition, the terminal range angle is sensitive to density parameter deviations. This will affect the accuracy with which the capsule may reach a pre-specified landing point.

Thus guidance and control will be necessary to compensate for atmospheric parameter deviations if the entry capsule system is designed to operate for all known atmospheric possibilities. The objective of the proposed guidance schemes will be to produce a pre-specified terminal condition whatever the atmosphere encountered on Mars entry. More specifically, the problem of producing a desired velocity and range angle at some given terminal altitude above the planet's surface will be considered. The various approaches to this problem investigated in this thesis are outlined in the next paragraphs.

In the first part of this thesis, a first order sensitivity guidance scheme is suggested. Sensitivity analysis is applied to the entry dynamics in order to compute the first order effects of both density parameter deviations and control changes. After the atmospheric parameters are tracked, the controls are determined on-board by using the sensitivity coefficients previously compiled. Control updating is provided by introducing the adjoint sensitivity equations. This reduces the on-board computation since all the required terminal sensitivity coefficients are now produced by the real time solution of one differential equation.

In the second part of this thesis, the terminal error prediction equations are improved by adding terms involving the second order sensitivity coefficients, for which a new vector-matrix differential equation is derived. This formulation should be more useful than a previously obtained second order sensitivity equation[43,44] for a single higher order scalar differential equation. Complications in the implementation of second order sensitivity guidance are discussed. These include a difference in form between the first and second order sensitivity equations. For the first order case, it was possible to describe all the terminal values by using the adjoint sensitivity equation. For the second order coefficients, this procedure is only possible for a certain approximation to the second order sensitivity forcing function. Finally, second order sensitivity guidance is numerically simulated assuming a VM-2 reference atmosphere with a 7 mb surface pressure and VM-4 actual atmosphere with 10 mb surface pressure. It is noted that the density parameter deviations are large in this case.

PART II

First Order Sensitivity Guidance for Mars Entry

A. Outline

Estimates of the Martian atmospheric parameters and entry conditions are used to compute a reference trajectory. Sensitivity analysis is then used to compute the changes in the terminal values of the state variables (flight path angle, velocity, range angle) caused by deviations in the atmospheric parameters. Likewise the effects of changes in the control variables on the terminal values of the state variables are also computed by sensitivity analysis. All this information is stored in the guidance computer on-board the lander. During entry an adaptive model of the atmosphere gives information on the actual values of the atmospheric parameters. Since a specific structure of the model atmosphere has been assumed, as shown in Appendix B, knowledge of the parameters is equivalent to predicting the density at all altitudes. The atmospheric parameter deviations are computed and the controls adjusted so as to produce the desired terminal condition. It should be noted that the controls may be updated several times during entry to improve performance. The value of the control is constant during each interval. Modified sensitivity equations are developed which greatly simplify control updating. These are based on the adjoint sensitivity system.

B. Entry Dynamics

For two dimensional flight, assuming a spherically symmetric planet surrounded by a non-rotating atmosphere, the dynamical equations can be written as follows (see Appendix A).

$$\frac{dy}{dt} = -V \sin\theta$$

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{1}{V}\left[g(y) - \frac{V^2}{R_o + y}\right]\cos\theta \\ &\quad - \frac{C_L(\alpha)}{C_D}\frac{1}{2}\rho V \frac{C_D A}{m} + \frac{T}{mV}\sin(\alpha - \delta)\end{aligned} \qquad (2\text{-}1)$$

$$\begin{aligned}\frac{dV}{dt} &= -\frac{1}{2}\rho(y) V^2 \frac{C_D A}{m} + g(y)\sin\theta \\ &\quad - \frac{T}{m}\cos(\alpha - \delta)\end{aligned}$$

$$\frac{d\Omega}{dt} = \frac{V\cos\theta}{R_o + y}$$

where y = altitude

θ = flight path angle

V = magnitude of velocity

Ω = range angle

ρ = local density

g = local gravitational acceleration

T = magnitude of thrust

R_o = radius of Mars

$C_L(\alpha)$ = lift coefficient

α = angle of attack

δ = thrust angle

C_D = drag coefficient

m = mass of entry capsule

A = reference surface area

It is convenient to make the following changes of variable:

$$x = \left(\frac{h - y}{h}\right) N \qquad (2\text{-}2)$$

$$v = \frac{V}{\sqrt{g_0 R_0}}, \qquad f = \frac{T}{m}$$

where h = reference altitude

N = scale factor

g_0 = surface gravitational acceleration

The controls are assumed to be two variable thrust engines, one of which is fixed normal to the flight path direction and the other in the tangential direction. The magnitudes of the control accelerations are f_2 and f_1 respectively. In addition, ballistic entry without lift is assumed, i.e. $C_L(\alpha) = 0$ and $\alpha = 0$. With these assumptions, the entry dynamics (2-1) can be rewritten:

$$\frac{d\theta}{dx} = F_1(x, \theta, v, f_2) = \frac{\sigma_1}{\sigma_2 v \sin\theta}\left\{\left[\frac{g(x)}{\sigma_2 v} - \frac{v\,\sigma_2}{R_0 + h(1 - x/N)}\right]\cos\theta - \frac{f_2}{\sigma_2 v}\right\}$$

$$\frac{dv}{dx} = F_2(x, \theta, v, \rho(x; a, b, c), f_1)$$
$$= \frac{\sigma_1}{\sigma_3 v \sin\theta}\left\{-\sigma_4\, e^{a}\,[1 + c(1 - x)]^{b}\, v^2 + g(x)\sin\theta - f_1\right\} \qquad (2\text{-}3)$$

$$\frac{d\Omega}{dx} = F_3(x, \theta) = \frac{\sigma_1 \cot\theta}{R_0 + h(1 - x/N)}$$

where $\sigma_1 = h/N$

$\sigma_2 = \sqrt{g_0 R_0}$

$\sigma_3 = g_0 R_0$

$\sigma_4 = \frac{1}{2}\left(\frac{C_D A}{m}\right) g_0 R_0$

C. Control at the Beginning of Entry into the Mars Atmosphere

This method assumes that one has reference estimates, $\hat{a}$, $\hat{b}$, $\hat{c}$, for the atmospheric parameters, a, b, c before the mission is begun. The outputs of the adaptive atmospheric density model at the entry altitude are then assumed to be accurate enough to represent the actual Martian atmospheric parameters.

These "actual" parameters are different from their reference values. Two questions immediately arise. First, what is the effect of these deviations on the terminal values of the state variables? Second, how can the controls be adjusted so as to achieve the desired terminal condition? We are, in effect, looking for a method of predicting the terminal deviation in the state variables on the basis of information obtained at entry. To solve this problem we use sensitivity coefficients[43,45,49] which give the ratio of the deviation of the particular state variable from its reference value (as a function of altitude) to the parameter deviation at entry.

Deviations in the state variables caused by atmospheric density variations may now be expressed in the following manner. First let

$$z_1 = \theta, \quad z_2 = v, \quad z_3 = \Omega \tag{2-4a}$$

$$\eta_1 = a, \eta_2 = b, \eta_3 = e, \eta_4 = f_1, \eta_5 = f_2 \qquad (2\text{-}4b)$$

Then we obtain

$$\frac{dz_i}{dx} = F_i(\underline{z}, x, \hat{\eta}), \; i = 1, 2, 3 \qquad (2\text{-}4c)$$

Before one can proceed to the mathematical definitions of the parameter sensitivity function, it is necessary to describe the reference and perturbed trajectories used in that definition. The reference state variables will be given by $z_i(x, \hat{\eta}, x_E)$. The normalized altitude x is the independent variable and x_E denotes the entry altitude. This function results from solving the trajectory dynamics with the reference values of the atmospheric parameters and the constant initial conditions $z_i(x_E)$. The perturbed trajectory is given by $z_i(x, \hat{\eta}_l, \hat{\eta}_j + \Delta\hat{\eta}_j, x_E)$ for any $\Delta\hat{\eta}_j$ where $\hat{\eta}_l$ is a vector without the η_j term. It can be obtained from the substitution of the perturbed parameter $\hat{\eta}_j + \Delta\hat{\eta}_j$ into the equations of motion and the definition of the initial condition

$$z_i(x_E, \hat{\eta}_l, \hat{\eta}_j + \Delta\hat{\eta}_j, x_E) = z_i(x_E) = \text{constant} \qquad (2\text{-}5)$$

for all $\Delta\hat{\eta}_j$. Physically, this last definition can be interpreted as meaning that the reference and perturbed state variables are equal at the entry altitude x_E (before any effects of the atmosphere are felt). We now define the sensitivity function by

$$\begin{aligned} u_{ij}(x,\hat{\eta},x_E) &= \lim_{\Delta\hat{\eta}_j \to 0} \left\{ \frac{z_i(x,\hat{\eta}_l,\hat{\eta}_j+\Delta\hat{\eta}_j,x_E)}{\Delta\hat{\eta}_j} - \frac{z_i(x,\hat{\eta},x_E)}{\Delta\hat{\eta}_j} \right\} \\ &= \frac{\partial z_i(x,\hat{\eta},x_E)}{\partial\hat{\eta}_j} \end{aligned} \tag{2-6}$$

The perturbed state variable can be approximated at altitude x by $z_i(x,\hat{\eta}_l,\hat{\eta}_j+\Delta\hat{\eta}_j,x_E)$

$$= z_i(x,\hat{\eta},x_E) + u_{ij}(x,\hat{\eta},x_E)\,\Delta\eta_j(x_E) \tag{2-7}$$

The quantity $\Delta\eta_j(x_E)$ is a step function applied at $x = x_E$. Equation (2-7) shows that the effects of parameter deviations on the state variables at altitude x can be estimated (at x_E) once the sensitivity coefficients are known.

The procedure for calculating the sensitivity coefficients $u_{ij}(x,\hat{\eta},x_E)$ is now developed. First take the partial derivative of Equation (2-4c) with respect to $\hat{\eta}_j$ and set all the parameters equal to their reference values[49]. This results in

$$\begin{aligned} \frac{d}{dx} u_{1j}(x,\hat{\eta},x_E) &= \frac{\partial F_1}{\partial z_1} u_{1j}(x,\hat{\eta},x_E) \\ &+ \frac{\partial F_1}{\partial z_2} u_{2j}(x,\hat{\eta},x_E) + \frac{\partial F_1}{\partial \hat{\eta}_j} \end{aligned} \tag{2-8}$$

$$\frac{d}{dx} u_{2j}(x,\hat{\eta},x_E) = \frac{\partial F_2}{\partial z_1} u_{1j}(x,\hat{\eta},x_E) + \frac{\partial F_2}{\partial z_2} u_{2j}(x,\hat{\eta},x_E) + \frac{\partial F_2}{\partial \hat{\eta}_j}$$

$$\frac{d}{dx} u_{3j}(x,\hat{\eta},x_E) = \frac{\partial F_3}{\partial z_1} u_{1j}(x,\hat{\eta},x_E)$$

where $j = 1,\ldots, 5$ and F_1, F_2, and F_3 are given by Equations (2-3) and the partial derivatives $\partial F_k/\partial z_i$ are evaluated along the reference trajectory. The initial conditions for the $u_{ij}(x,\hat{\eta},x_E)$ can be obtained directly from the reference trajectory by means of Equations (2-6) and (2-5)

$$u_{ij}(x_E,\hat{\eta},x_E) = \frac{\partial z_i(x_E)}{\partial \hat{\eta}_j} = 0 \tag{2-9}$$

The important thing to realize is that the $u_{ij}(x,\hat{\eta},x_E)$ may be computed if the partial derivatives of F_i, $i = 1,2,3$ are known. If the index j is allowed to take on values 1..., 5 in Equation (2-8) there are 15 first order sensitivity equations. These together with the three equations of motion (total of 18 equations) are solved simultaneously. Then the resulting sensitivity functions will describe all possible effects of atmospheric parameter variations and control variable changes on the state variables. The values of the sensitivity functions at the terminal altitude x_T, $u_{ij}(x_T,\hat{\eta},x_E)$ can then be stored in the on-board computer. When the vehicle enters the Mars atmosphere and the actual values of $a(x_E)$, $b(x_E)$, and $c(x_E)$ become available from the adaptive model[63], we may immediately estimate the total change (at altitude x_T) in the ith state variable produced by atmospheric variations and control variable changes at x_E by

$$
\begin{aligned}
& z_i[x_T, a(x_E), b(x_E), c(x_E), f_1, f_2, x_E] \\
& - z_i(x_T, \hat{a}, \hat{b}, \hat{c}, 0, 0, x_E) \\
& = u_{i1}(x_T, \hat{q}, x_E)[a(x_E) - \hat{a}] \\
& \quad + u_{i2}(x_T, \hat{q}, x_E)[b(x_E) - \hat{b}] \\
& \quad + u_{i3}(x_T, \hat{q}, x_E)[c(x_E) - \hat{c}] \\
& \quad + u_{i4}(x_T, \hat{q}, x_E) f_1 \\
& \quad + u_{i5}(x_T, \hat{q}, x_E) f_2
\end{aligned}
\tag{2-10}
$$

As before, the $\hat{a}$, $\hat{b}$, $\hat{c}$ are the reference values of the atmospheric parameters. Note that the entry condition state variable errors have been assumed to be zero. If f_1 and f_2 are set equal to zero, Equation (2-10) predicts the uncontrolled terminal deviations in the ith state variable. The desired terminal condition is described by

$$
\begin{aligned}
& v[x_T, a(x_E), b(x_E), c(x_E), f_1, f_2, x_E] \\
& \qquad - v[x_T, \hat{a}, \hat{b}, \hat{c}, 0, 0, x_E] = 0 \\
& \Omega[x_T, a(x_E), b(x_E), c(x_E), f_1, f_2, x_E] \\
& \qquad - \Omega[x_T, \hat{a}, \hat{b}, \hat{c}, 0, 0, x_E] = 0
\end{aligned}
\tag{2-11}
$$

which constrains the terminal velocity and range angle. Then the controls f_1 and f_2 are obtained from the simple inverse sensitivity problem[43].

$$0 = u_{21}\,\Delta a + u_{22}\,\Delta b + u_{23}\,\Delta c + u_{24} f_1 + u_{25} f_2$$

$$0 = u_{31}\,\Delta a + u_{32}\,\Delta b + u_{33}\,\Delta c + u_{34} f_1 + u_{35} f_2 \qquad (2\text{-}12)$$

Because it is likely that the Martian atmospheric parameters will turn out to be slowly varying functions of altitude, the problem of updating the control during entry must be investigated.

D. Sensitivity Analysis and Updating the Control

Before considering the process of updating the controls it is convenient to introduce vector and matrix notation. First we define the sensitivity vector by

$$\underline{u}_j(x,\hat{\eta},x_E) = [u_{1j}(x,\hat{\eta},x_E),\, u_{2j}(x,\hat{\eta},x_E),\, u_{3j}(x,\hat{\eta},x_E)]^* \tag{2-13a}$$

and the sensitivity forcing vector by

$$\underline{h}_j(x,\hat{\eta}) = \left[\frac{\partial F_1}{\partial \hat{\eta}_j}(\underline{z}(x,\hat{\eta},x_E),x,\hat{\eta}),\ \frac{\partial F_2}{\partial \hat{\eta}_j}(\underline{z}(x,\hat{\eta},x_E),x,\hat{\eta}),\ 0\right]^* \tag{2-13b}$$

where * denotes the transpose operation. The sensitivity system matrix is given by

$$A(x,\hat{\eta}) = \begin{bmatrix} \dfrac{\partial F_1}{\partial z_1} & \dfrac{\partial F_1}{\partial z_2} & 0 \\ \dfrac{\partial F_2}{\partial z_1} & \dfrac{\partial F_2}{\partial z_2} & 0 \\ \dfrac{\partial F_3}{\partial z_1} & 0 & 0 \end{bmatrix} \tag{2-14}$$

Note that the z_i and $\hat{\eta}_j$ are still defined by Equation (2-4) and that the partial derivatives in $A(x,\hat{\eta})$ and $\underline{h}_j(x,\hat{\eta})$ are evaluated along the reference trajectory. Using Equations (2-13) and (2-14), Equation (2-8) may now be expected in more concise form:

$$\frac{d}{dx}\underline{u}_j(x,\hat{\eta},x_E) = A(x,\hat{\eta})\underline{u}_j(x,\hat{\eta},x_E) + \underline{h}_j(x,\hat{\eta}) \tag{2-15}$$
$$\underline{u}_j(x_E,\hat{\eta},x_E) = \underline{0}$$

Now suppose at some normalized altitude S $(x_E \leq s \leq x \leq x_T)$ new values for the atmospheric density parameters a, b, and c are obtained the adaptive density model[63]. The question is how to incorporate this updated atmospheric parameter information. We might replace x_E with S in Equation (2-15). This in effect resets the sensitivities to zero at altitude S. Mathematically, it means that we are dealing with a new perturbed solution which coincides with the reference trajectory at altitude S (instead of at altitude x_E). The sensitivity functions which describe the new perturbed solution are given by

$$\frac{d}{dx}\underline{u}_j(x,\hat{\underline{\eta}},s) = A(x,\hat{\underline{\eta}})\underline{u}_j(x,\hat{\underline{\eta}},s) + \underline{h}_j(x,\hat{\underline{\eta}})$$
$$\underline{u}_j(s,\hat{\underline{\eta}},s) \tag{2-16}$$

It is noted that $\underline{h}_j(x,\hat{\underline{\eta}})$ in Equation (2-16) does not depend on S. This point is not obvious and is discussed in Appendix C due to its importance to the rest of the analysis. The method of predicting the actual state variables at x_T is similar to that of Section C with one important exception. Before the assumption that the perturbed and reference trajectories were identical at x_E (Equation (2-5)) actually was true in a physical sense since the dynamics are very nearly Keplerian at the entry altitude. However, it is very unlikely that the actual trajectory will be identical to the reference trajectory at some lower altitude S . For this reason, the effects of state variable deviations $\Delta z_i(s), i=1,2,3$ at altitude S must be included in the estimate of the state variable error at the terminal altitude (the mathematics of this process is explained in Appendix D). With this complication noted, let us again consider Equation (2-16). This problem could be solved numerically. The values of the sensitivities at the terminal altitude, $u_j(x_T, \hat{q}, s)$ can be substituted in expressions similar to Equation (2-10) with the estimates of the atmospheric parameters a, b, c at altitude x_E replaced by the new estimates at altitude S : $a(s)$, $b(s)$, and $c(s)$. We would then re-adjust f_1 and f_2 to obtain zero deviation in the terminal velocity and range angle.

Numerical solutions of the combined system of (2-3) and (2-16) with x_E and s as initial altitude have been required in order to obtain the control applied at entry altitude x_E and the first update of control at altitude S. For many such updates the amount of on-board computation would be excessive. It is also clear that only the values of the sensitivities at the terminal altitude, x_T, are required. The variable of interest is the present altitude at which the parameter sensitivities are set equal to zero.

We now find new sensitivity equations whose solution will contain all the necessary terminal sensitivities with the present altitude appearing as the independent variable. That is, we obtain an analytical description for $\underline{u}_j(x_T,\hat{\underline{q}},s)$ as a function of s. To do this, Equation (2-16) is solved using the transition matrix $\Phi(x,s)$ for the homogeneous portion of Equation (2-16). This matrix is governed[64,65] by

$$\frac{d}{dx}\Phi(x,s) = A(x,\hat{\underline{q}})\Phi(x,s)$$
$$\Phi(s,s) = I\ , \quad I = \text{Identity Matrix} \tag{2-17}$$

It is noted that $\Phi(x,s)$ is also a function of the reference parameters but this is not shown for the purposes of simplicity. We also need the adjoint transition matrix which is described by

$$\frac{d}{dx}\psi^*(x_T,x) = -\psi^*(x_T,x)A(x,\hat{\underline{q}})$$
$$\psi^*(x_T,x_T) = I, \quad I = \text{Identity Matrix} \tag{2-18}$$

From these, the useful identity

$$\psi^*(x_T,s) = \Phi(x_T,s) \tag{2-19}$$

is obtained (see Appendix E for details). The solution of Equation (2-16) can then be shown to have the integral representation

$$\underline{u}_j(x,\hat{\underline{q}},s) = \int_s^x \Phi(x,\lambda)\underline{h}_j(\lambda,\hat{\underline{q}})\,d\lambda \tag{2-20}$$

Setting $x = x_T$ in Equation (2-20) yields

$$\underline{u}_j(x_T, \hat{q}, s) = \int_s^{x_T} \Phi(x_T, \lambda)\, \underline{h}_j(\lambda, \hat{q})\, d\lambda \qquad (2\text{-}21)$$

Substituting Equation (2-19) into (2-21) and then differentiating the result with respect to s gives

$$\frac{d}{ds}\, \underline{u}_j(x_T, \hat{q}, s) = -\psi^*(x_T, s)\, \underline{h}_j(s, \hat{q}) \qquad (2\text{-}22)$$

$$x_E \leq s \leq x_T$$

where $\underline{u}_j(x_T, \hat{q}, x_E)$ is used as the initial condition. Thus we may pre-compute and store $\Phi(x_T, x_E)$ and then solve the following (similar to Equation (2-18)).

$$\frac{d}{ds}\, \psi^*(x_T, s) = -\psi^*(x_T, s)\, A(s, \hat{q}) \qquad (2\text{-}23)$$

subject to $\psi^*(x_T, x_E) = \Phi(x_T, x_E)$ (similar to Equation (2-19). Equation (2-22) has the following initial condition:

$$\underline{u}_j(x_T, \hat{q}, s)\Big|_{s = x_E} = \underline{u}_j(x_T, \hat{q}, x_E) \qquad (2\text{-}24)$$

Equations (2-22) and (2-23) are solved simultaneously with the equations of motion to obtain the proper sensitivity coefficients for the updated atmospheric information. This is advantageous because it is only necessary to solve the whole system (of Equations (2-3), (2-23), and (2-22) once as the vehicle enters the atmosphere. Thus an efficient scheme for obtaining the required terminal sensitivities as functions of the actual altitude has been developed. Its application to the control updating problem is investigated in the next section.

E. Implementation of Control Updating

The ith state variable deviation at terminal altitude x_T may now be estimated while the entry capsule is at actual altitude s by

$$\begin{aligned} &z_i[x_T, a(s), b(s), c(s), f_1, f_2, s] - z_i[x_T, \hat{a}, \hat{b}, \hat{c}, 0, 0, s] \\ &= u_{i1}(x_T, \hat{q}, s)[a(s) - \hat{a}] + u_{i2}(x_T, \hat{q}, s)[b(s) - \hat{b}] \\ &\quad + u_{i3}(x_T, \hat{q}, s)[c(s) - \hat{c}] + w_{i1}(x_T, \hat{q}, s)[\theta(s) - \hat{\theta}(s)] \\ &\quad + w_{i2}(x_T, \hat{q}, s)[v(s) - \hat{v}(s)] + w_{i3}(x_T, \hat{q}, s)[\Omega(s) - \hat{\Omega}(s)] \\ &\quad + u_{i4}(x_T, \hat{q}, s) f_1 + u_{i5}(x_T, \hat{q}, s) f_2 \end{aligned} \tag{2-25}$$

This is similar to Equation (2-10) with the exception that we have taken into account state variable deviations at the actual altitude s. In Equation (2-25), the $w_{ij}(x_T, \hat{q}, s)$ are obtained from $\psi^*(x_T, s)$ as in Appendix D. The $a(s)$, $b(s)$, $c(s)$ are the updated atmospheric parameters available at altitude s and $\theta(s)$, $v(s)$, and $\Omega(s)$ are obtained by measuring the actual path angle, velocity, and range angle at that altitude. If f_1 and f_2 are set equal to zero, Equation (2-25) gives an estimate of the uncontrolled terminal deviation. Then the controls f_1 and f_2 can be applied to produce a result similar to Equation (2-12) thus reducing the terminal velocity and range angle errors.

The computation of the sensitivity coefficients used in Equation (2-25) and their role in the guidance scheme are shown in Figure 1 where the following shortened notation is used:

$z_i(x,\hat{\eta},x_E)$	is replaced by	$\hat{z}_i(x)$
$u_{ij}(x,\hat{\eta},x_E)$	is replaced by	$\hat{u}_{ij}(x,x_E)$
$A(x,\hat{\eta})$	is replaced by	$\hat{A}(x)$
$h_j(x,\hat{\eta})$	is replaced by	$\hat{h}_j(x)$.

The quantity $\hat{\underline{z}}(x)$ is the solution of the entry dynamics based on the entry conditions (known only after de-orbit) and reference values assigned to the density parameters. The definition of the reference state variables $\hat{z}_i(x)$, and parameters $\hat{\eta}_j$, are given by Equation (2-4) and used to compute the sensitivity system matrix, $\hat{A}(x)$ and forcing function, $\hat{\underline{h}}_j(x)$. The sensitivity dynamics given by Equation (2-15) are solved and the terminal sensitivity, $\hat{\underline{u}}_j(x_T,x_E)$ is stored. In addition, the final value $\Phi(x_T,x_E)$ of the state transition matrix is obtained by solving the homogeneous sensitivity system. The reference trajectory $\hat{\underline{z}}(x)$, the parameter sensitivities $\hat{\underline{u}}_j(x_T,x_E)$, and the state transition matrix $\Phi(x_T,x_E)$ should be solved for simultaneously after de-orbit and before entry into the sensible atmosphere as shown in Figure 1.

During entry $\Phi(x_T,x_E)$ and $\hat{\underline{u}}_j(x_T,x_E)$ will be used as initial conditions in solving the adjoint sensitivity system for $\psi^*(x_T,s)$ and in solving for the $\hat{\underline{u}}_j(x_T,s)$. It is suggested that this simultaneous solution for $\psi^*(x_T,s)$ and $\underline{u}_j(x_T,s)$

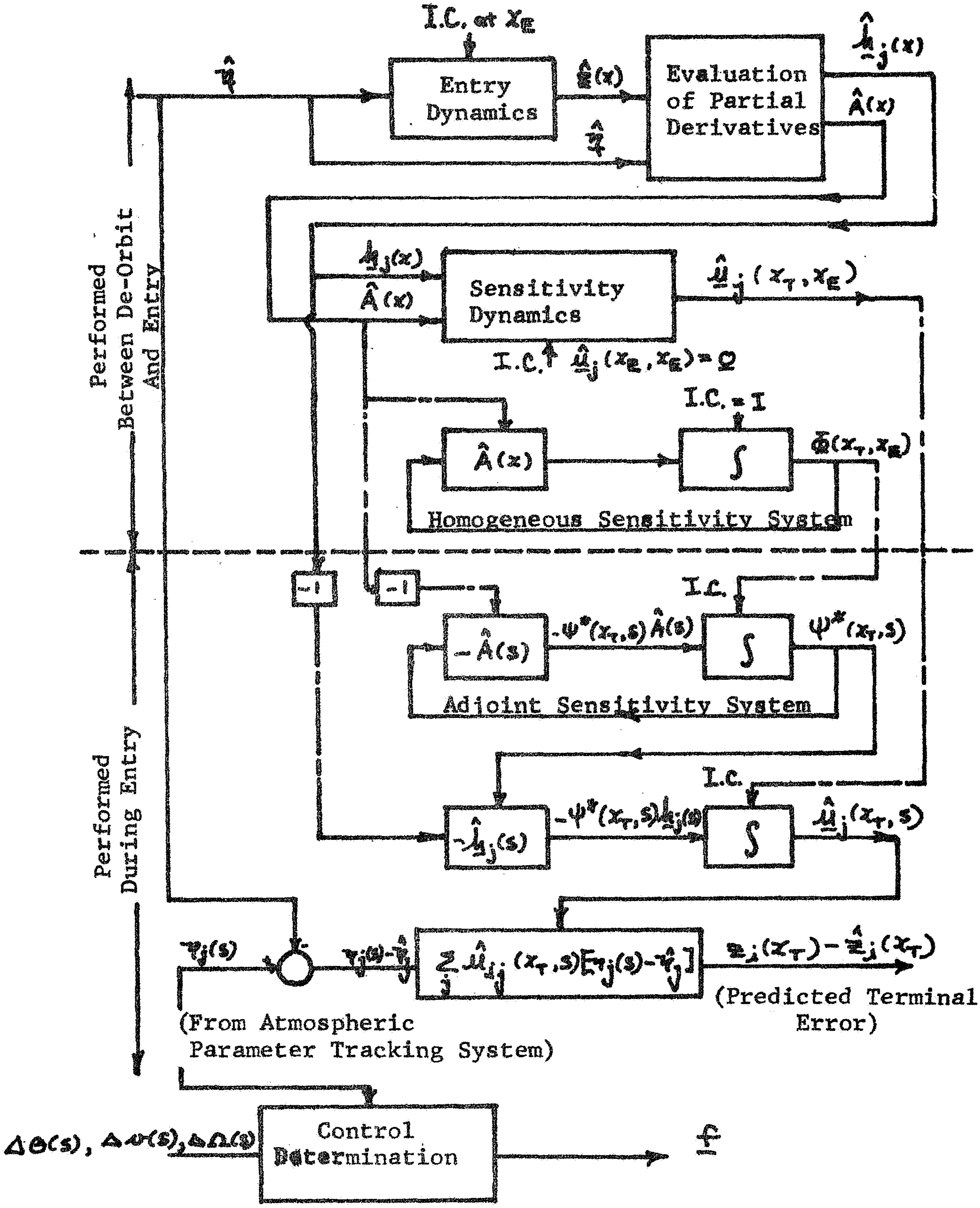


FIGURE 1 Sensitivity Guidance Schematic

be carried out in real "time", that is, the values of the sensitivities at S be obtained at real altitude S. This will reduce the computational capability required.

F. Numerical Simulation of First Order Sensitivity Guidance

We now examine a numerical example which will illustrate the usefulness of the proposed guidance scheme. In this example, the VM-2 model is taken as the reference atmosphere and VM-1 model[39] is taken as the actual atmosphere. Before discussing the guidance computations performed, straightforward numerical solutions for the uncontrolled trajectories are presented in Table 1. It is noted that the same initial conditions were assumed with both atmospheres. It is seen that not applying control with the VM-1 atmosphere results in the entry capsule reaching the terminal altitude 60 miles short of the target with a velocity 118 feet/sec faster than the same vehicle entering the VM-2 reference atmosphere. It is also seen that the atmospheric density parameter deviations are not reflected by deviations in the state variables until the capsule reaches an altitude well below the 800,000 foot altitude where Mars entry is usually considered to begin. This leads us to assume that the state variable errors are zero at some lower altitude. To simplify the guidance equations it is also assumed that adiabatic model (see Appendix B) may be used to represent the density throughout the controlled portion of the flight. The

	VM-1			VM-2		
ALTITUDE ft	Θ, degrees	V, fps	$R_o\Omega$, miles	Θ, degrees	V, fps	$R_o\Omega$, miles
800,000	16	15,000.	0	16	15,000	0
702,500	15.2	15,070.	66.	15.2	15,070	66.
605,000	14.4	15,140.	135.	14.4	15,140	135.
507,500	13.5	15,209.	207.	13.5	15,211	207.
410,000	12.6	15,260.	285.	12.6	15,284	285.
293,000	11.3	15,048.	386.	11.3	15,372	386.
195,500	10.4	12,777.	479.	10.1	15,443	480.
156,500	10.4	9,880.	518.	9.6	15,436	521.
98,000	14.2	3,651.	569.	8.8	14,025.	587.
59,000	29.6	1,371.	588.	9.4	5,218.	632.
20,000	58.6	778.	594.	46.0	660.	654.

Terminal Path Angle Deviation = 12.6 Degrees

Terminal Velocity Deviation = +118. Ft/Sec.

Terminal Range Deviation = -60. Miles

where 1 mile = 5280 feet.

density parameters a, b, and c for the VM-1 and VM-2 atmospheres are given in Appendix B. Further, we assume exact tracking of the actual density parameters once the capsule enters the Mars atmosphere.

Two computer programs were written to assist in evaluating the performance of the guidance scheme. The first program computes the quantities $\underline{u}_j(x_T, \hat{y}, x_E)$ and $\Phi(x_T, x_E)$ between de-orbit and entry by solving differential equations (2-3), (2-15), and (2-17) (with $S = x_E$) simultaneously. The following numerical data was used for the VM-2 reference trajectory.

y_E = 89,400 ft.

Θ_E = .15287 radians

V_E = 12,300 ft/sec.

Ω_E = 0

$\hat{a}$ = -10.2348

$\hat{b}$ = 2.7027

$\hat{c}$ = -.501556

N = 1

h = 61,000 ft.

$m/C_D A$ = .30 slugs/ft^2

g_0 = 12.3 ft/sec^2 for Mars

R_0 = 10.86 $(10)^6$ ft for Mars

$$y_T = 20{,}000 \text{ ft}$$

where y_E, Θ_E, V_E, and Ω_E, were taken from the VM-2 trajectory simulation previously discussed. The following values were obtained for $\hat{\underline{z}}(x_T)$, $\underline{u}_j(x_T, \hat{\underline{\eta}}, x_E)$ and $\Phi(x_T, x_E)$:

$$\hat{\Theta}(x_T) = .8116 \text{ radians}$$

$$\hat{v}(x_T) = .05726$$

$$\hat{\Omega}(x_T) = .02852 \text{ radians}$$

$$\underline{u}_1(x_T, \hat{\underline{\eta}}, x_E) = \begin{pmatrix} .7032 \\ .04360 \\ .009212 \end{pmatrix}$$

$$\underline{u}_2(x_T, \hat{\underline{\eta}}, x_E) = \begin{pmatrix} .3341 \\ .01328 \\ .006733 \end{pmatrix}$$

$$\underline{u}_3(x_T, \hat{\underline{\eta}}, x_E) = \begin{pmatrix} 2.436 \\ .08486 \\ .05660 \end{pmatrix}$$

$$\underline{u}_4(x_T, \hat{\underline{\eta}}, x_E) = \begin{pmatrix} .01605 \\ .001817 \\ .00007226 \end{pmatrix} \qquad \text{sec}^2/\text{ft}$$

$$\underline{u}_5(x_T, \hat{\underline{\eta}}, x_E) = \begin{pmatrix} .01472 \\ .001497 \\ .0003275 \end{pmatrix} \qquad \text{sec}^2/\text{ft}$$

$$\Phi(x_T, x_E) = \begin{pmatrix} -1.537 & -.05888 & 0.0 \\ .03511 & -.001105 & 0.0 \\ -.09704 & .006605 & 1.0 \end{pmatrix}$$

The above values of $\underline{\mu}_j(x_T, \hat{\eta}, x_E)$ and $\Phi(x_T, x_E)$ were then used as initial conditions (see Figure 1) for the second numerical program which solved the adjoint matrix for $\underline{\mu}_j(x_T, \hat{\eta}, s)$ and $\Psi^*(x_T, s)$ as functions of the present altitude subject to the same initial conditions on the state variables. In addition, this program computed the controls for each interval as a function of the density parameter deviation tracked and the state variable deviation at the beginning of the interval (see Equation (2-25)). The actual controlled VM-1 trajectory was generated by the following data

$$y_E = 89{,}400 \text{ ft}$$
$$\theta_E = .15287 \text{ radians}$$
$$V_E = 12{,}300 \text{ ft/sec}$$
$$\Omega_E = 0$$
$$a = -10.8977$$
$$b = 2.6316$$
$$c = -.2625$$

plus the values of the controls f_1 and f_2 implemented in each interval.

Figures 2 and 3 show velocity and range angle deviations from the reference trajectory for the controlled case (controls obtained from Equation 2-25) and the uncontrolled case (f_1 and f_2 set equal to zero.) Figure 2 shows

FIGURE 2 Velocity Deviation vs. Altitude
*Martian Circular Velocity = 11,558 Ft/Sec

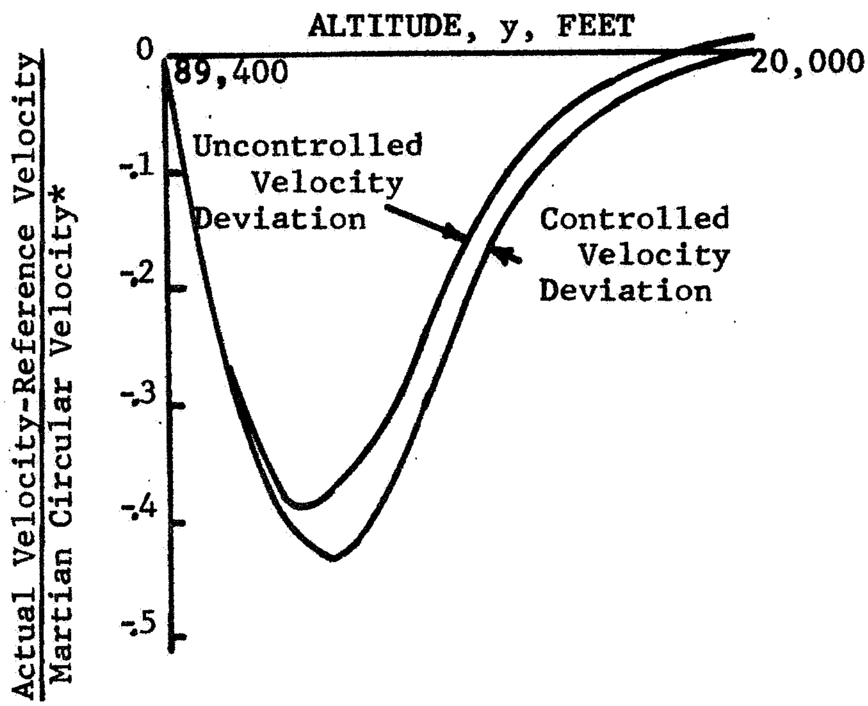


FIGURE 3

Range Angle Deviation vs. Altitude

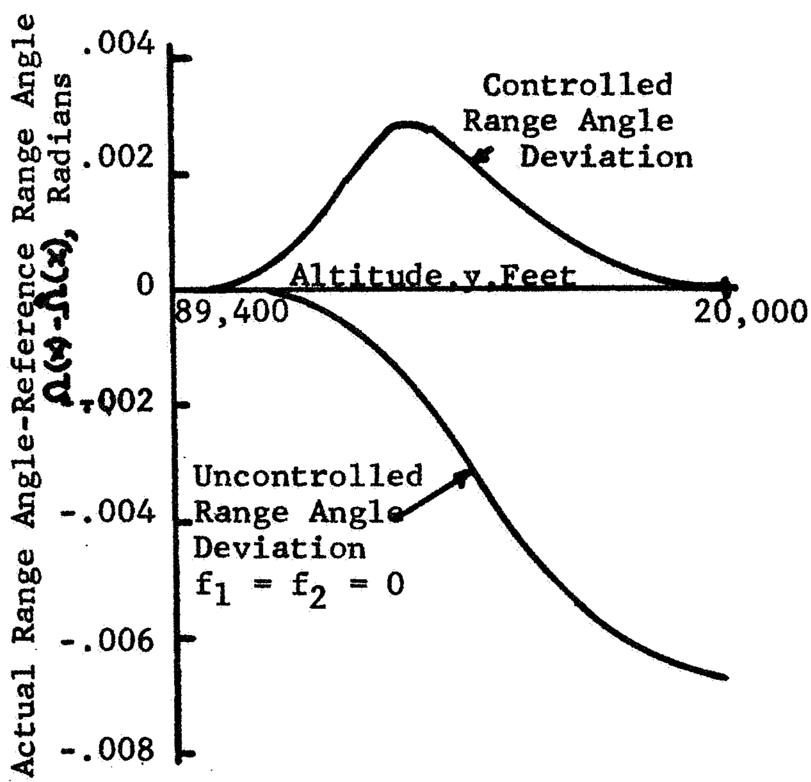
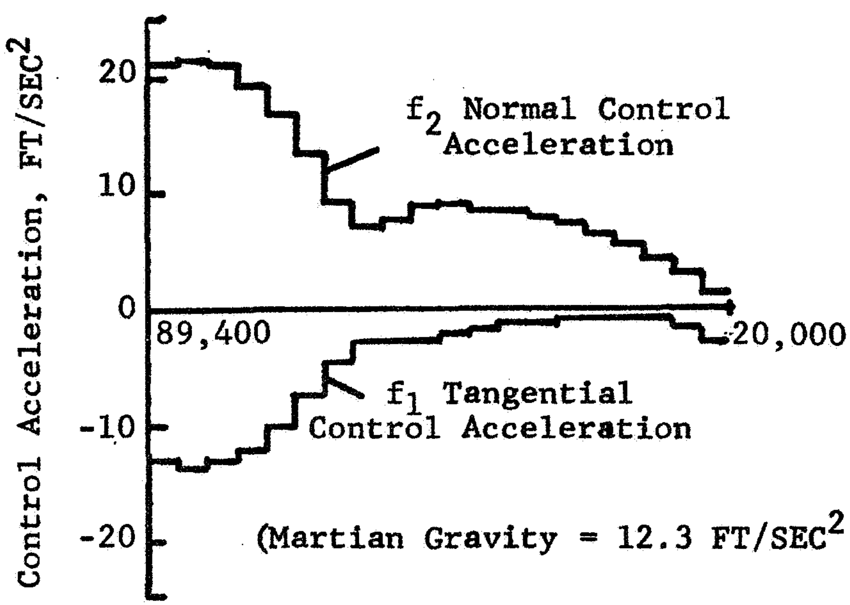


FIGURE 4

Control Accelerations vs. Altitude



that the velocity deviation at the terminal altitude is reduced by about 90% by the application of control. In Figure 3 range angle deviation is plotted as a function of altitude. Here the effect of the control is much more immediate with the uncontrolled range angle deviation always increasing. The terminal range angle deviation for the controlled trajectory is very small in comparison with the uncontrolled trajectory. In Figure 4 the control accelerations f_1 and f_2 are plotted as functions of altitude. The controls are seen to vary slowly over a major portion of the flight.

The data plotted in Figures 2 and 3 is summarized in the following Table 2:

State Variable	Terminal Error	
	no control	with control
Velocity, ft/sec	+ 87.5	-8.7
Horizontal Range, miles	- 14.1	+.003

TABLE 2

Another example assumed the VM-8 model with 5 mb surface pressure as the actual atmosphere. All the other data was the same as before. The results for that example are summarized in Table 3.

State Variable	Terminal Error	
	no control	with control
Velocity, ft/sec	+ 232.	+23.
Horizontal Range, miles	+ 6	+ .1

TABLE 3

For the above example (VM-2 reference, VM-8 actual), the dependence of sensitivity guidance performance on error-free parameter tracking was investigated. The assumption of error-free tracking really involved two separate assumptions. First, it was assumed that the parameter information was immediately available at the beginning of entry. Second, it was assumed that the parameter information[63] was error free. To simulate the effect of a delay in obtaining the actual density parameters an interval of altitude delay (where f_1 and f_2 are set equal to zero) was introduced in the application of the updated control. Digital simulation showing the effects of delays up to 20,000 ft. on the terminal velocity and range angle errors are plotted in Figures 5 and 6. It is seen that a delay of 10,000 feet increased the velocity error by about 25% and the range angle error by about 75%.

To simulate the effects of tracking errors, a

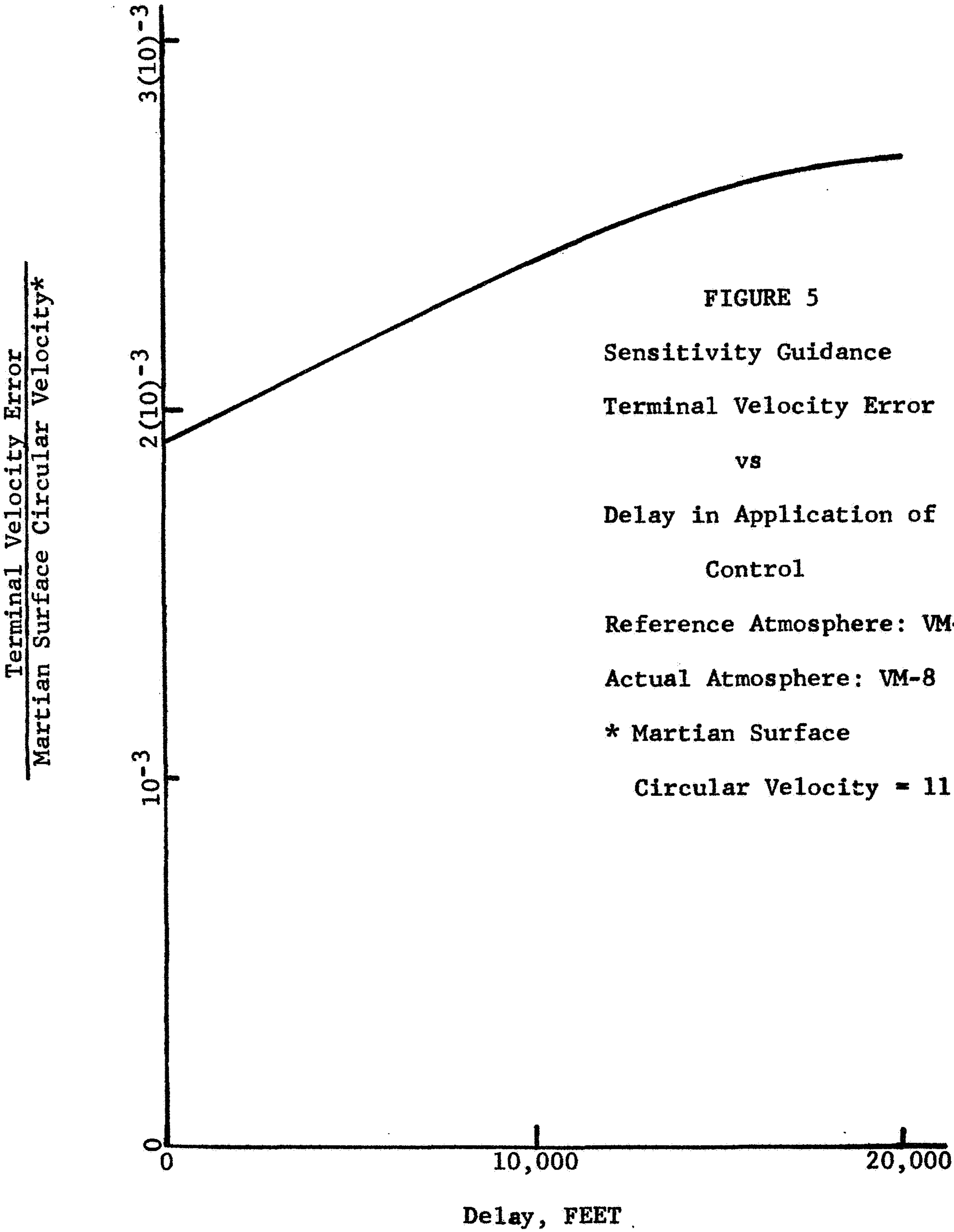


FIGURE 5

Sensitivity Guidance

Terminal Velocity Error

vs

Delay in Application of

Control

Reference Atmosphere: VM-2

Actual Atmosphere: VM-8

* Martian Surface

Circular Velocity = 11,558 F

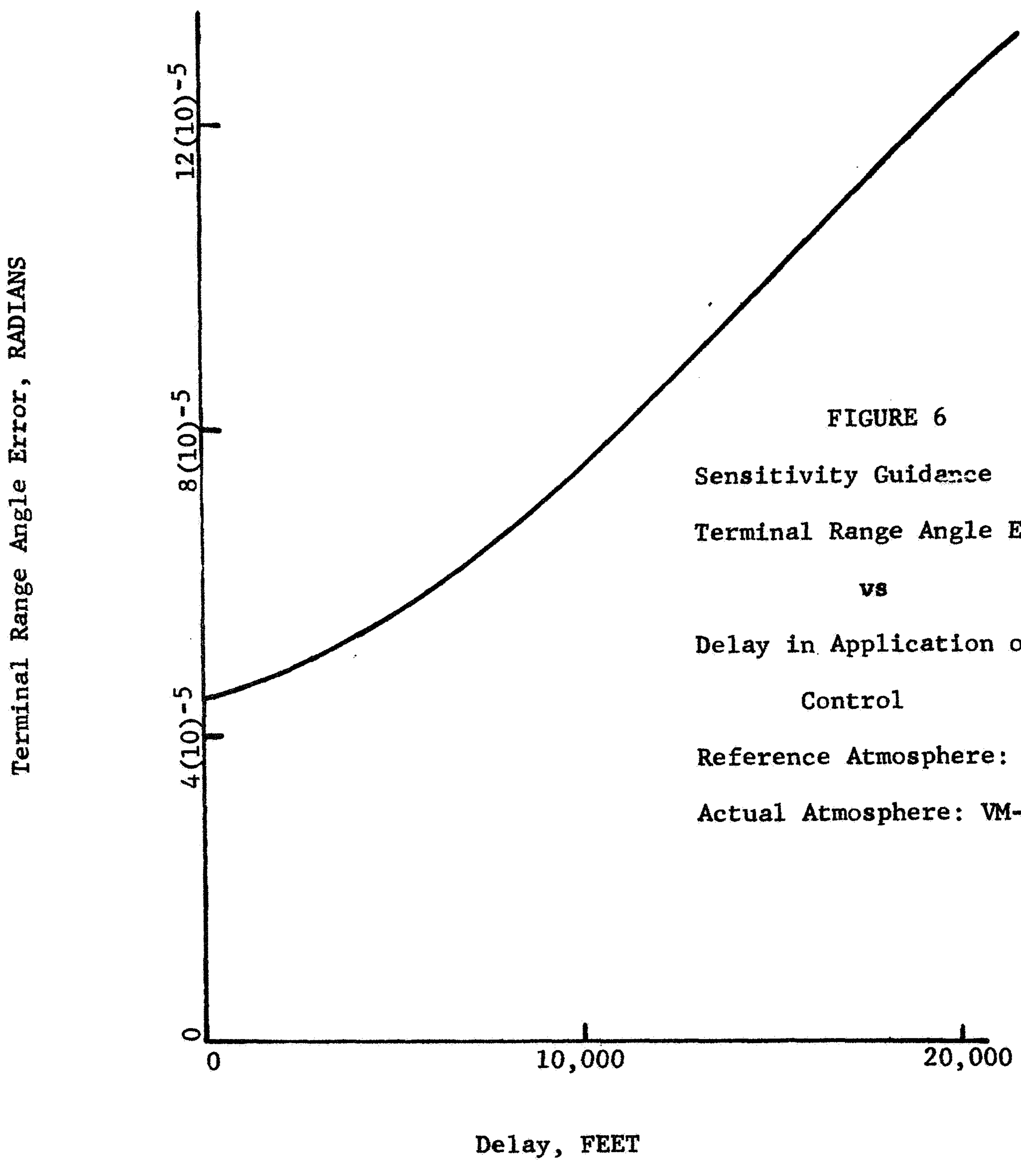


FIGURE 6

Sensitivity Guidance

Terminal Range Angle Erro

vs

Delay in Application of

Control

Reference Atmosphere: VM-

Actual Atmosphere: VM-8

constant error was introduced into the value of parameter 'a' produced by the adaptive model in the above example. The effects of this tracking error are shown in Figures 7 and 8.

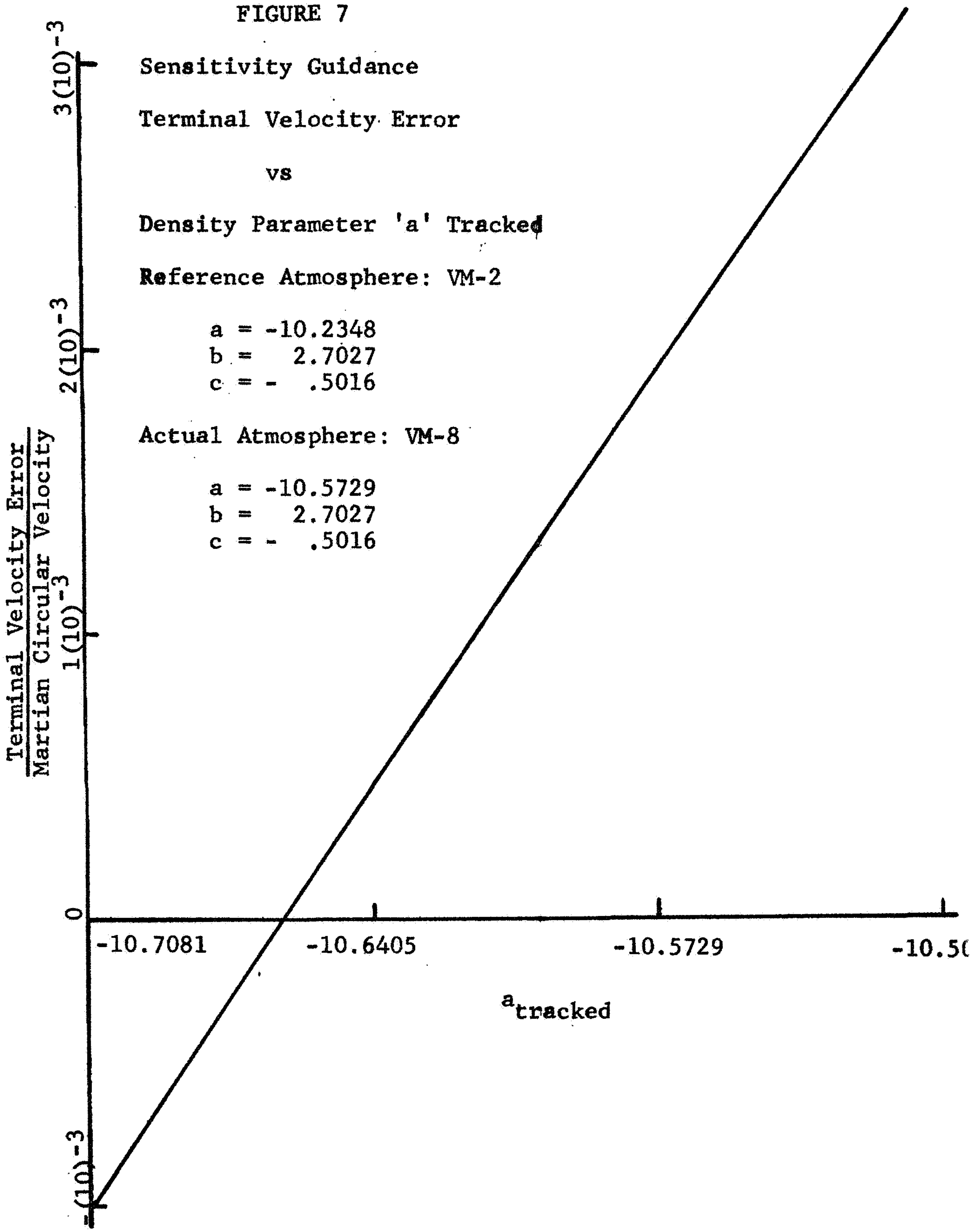


FIGURE 7

Sensitivity Guidance

Terminal Velocity Error

vs

Density Parameter 'a' Tracked

Reference Atmosphere: VM-2

a = -10.2348
b = 2.7027
c = - .5016

Actual Atmosphere: VM-8

a = -10.5729
b = 2.7027
c = - .5016

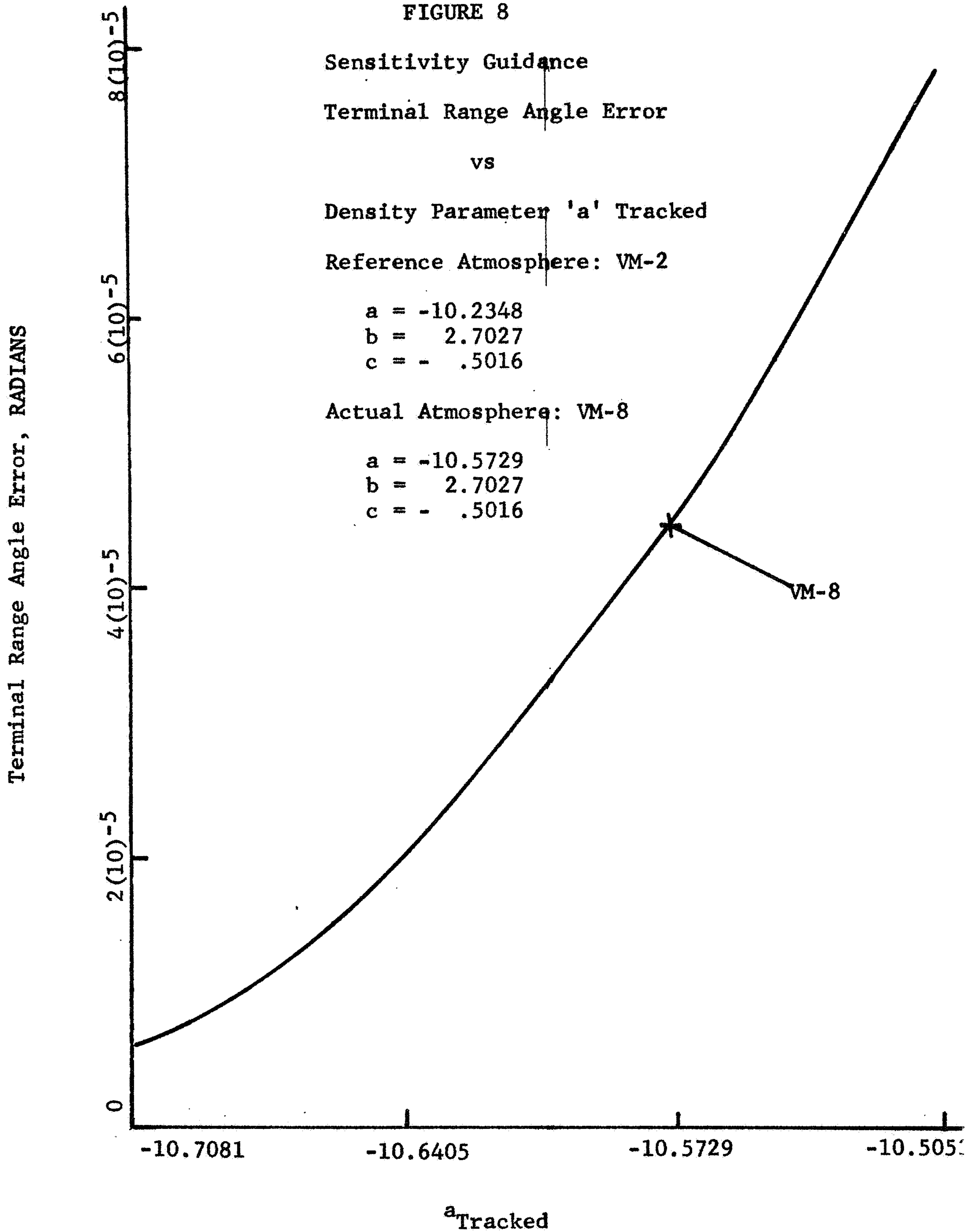


FIGURE 8

Sensitivity Guidance

Terminal Range Angle Error

vs

Density Parameter 'a' Tracked

Reference Atmosphere: VM-2

a = -10.2348
b = 2.7027
c = - .5016

Actual Atmosphere: VM-8

a = -10.5729
b = 2.7027
c = - .5016

PART III

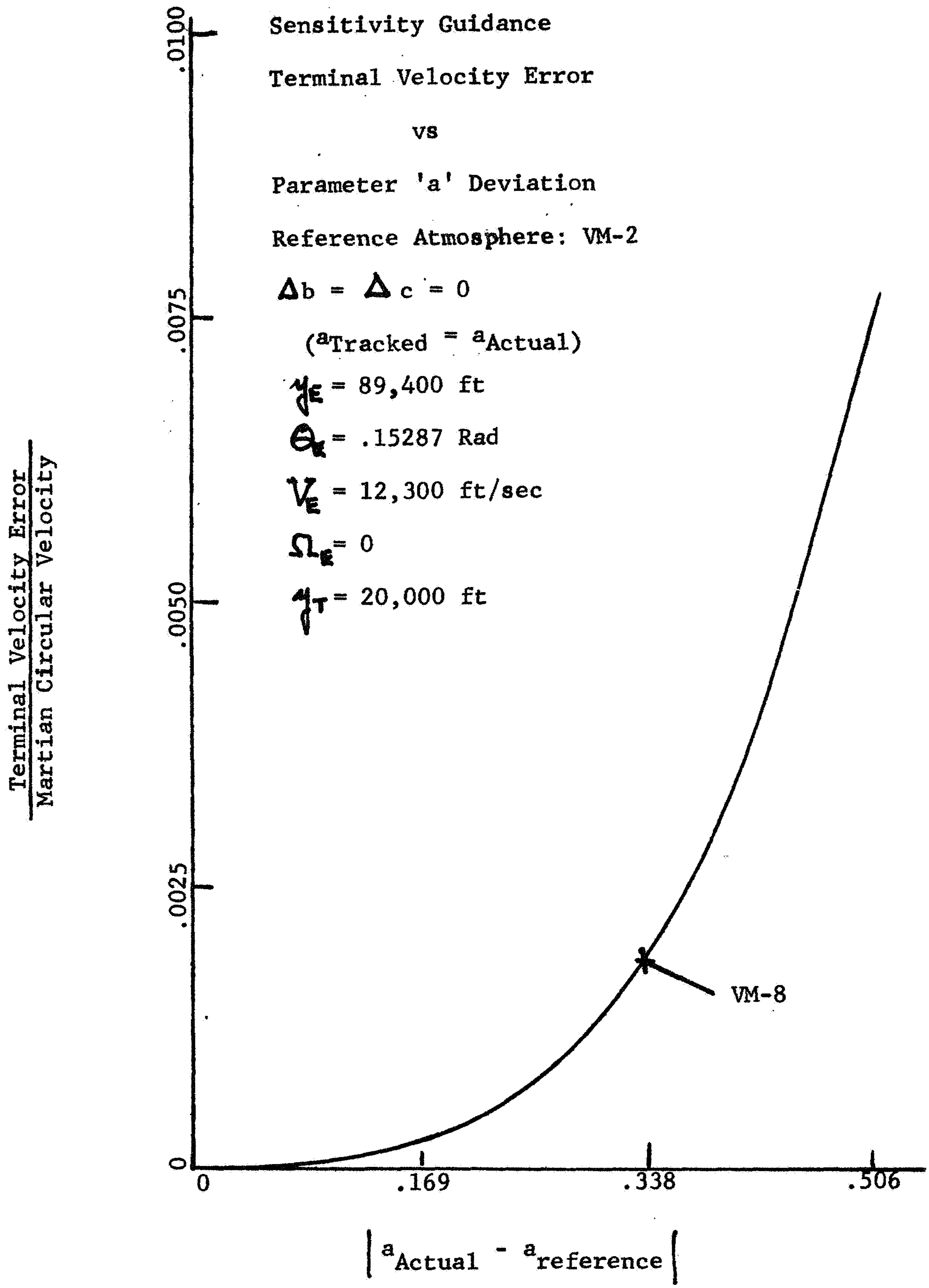
Second Order Sensitivity Guidance

A. Outline

In the previous portion of this thesis a variational guidance scheme based on sensitivity analysis for controlling entry into an uncertain Martian atmosphere was proposed. This approach produced satisfactory results if the atmospheric parameter deviations were not too large, for example with VM-2 (7mb) and VM-1 (7mb). Other satisfactory results were obtained with a VM-2 (7mb) reference atmosphere and VM-4 (10mb), VM-7 (5mb), and VM-8 (5mb) actual atmospheres. However, further simulation of sensitivity guidance using only first order error prediction shows a marked increase in terminal errors if the density parameter deviations are large (see Figure 9).

In this section of the thesis we improve the terminal error prediction equations by including the second order terms. These terms result from the multiplication of products of parameter and/or initial condition deviations by the second order sensitivity coefficients, for which a new vector-matrix differential equation is derived. This formulation should be more useful than a previously obtained second order sensitivity equation[43,44] for a single higher scalar

FIGURE 9


Sensitivity Guidance
Terminal Velocity Error
vs
Parameter 'a' Deviation
Reference Atmosphere: VM-2
$\Delta b = \Delta c = 0$
($a_{Tracked} = a_{Actual}$)
$y_E = 89{,}400$ ft
$\theta_E = .15287$ Rad
$V_E = 12{,}300$ ft/sec
$\Omega_E = 0$
$y_T = 20{,}000$ ft
VM-8
.0100
.0075
.0050
.0025
0
0
.169
.338
.506
Terminal Velocity Error / Martian Circular Velocity
$|a_{Actual} - a_{reference}|$

differential equation. The application of this analysis to the entry guidance problem is investigated and certain difficulties are noted. These include a difference in form between the first and second order sensitivity equations. For the first order case, it was possible to describe all the terminal values by using the adjoint sensitivity system. For the second order coefficients, this procedure is only possible for an approximation of the second order sensitivity forcing function.

B. Derivation of the Vector-Matrix Second Order Sensitivity Equation

Before developing the second order sensitivity equations from the first order sensitivity equations and the original system dynamics, the three different types of second order coefficients are defined. The Class 1 coefficients are the quantities which multiply the second order parameter deviations in a Taylor series expansion for the perturbed state variables. The vector for the Class 1 second order sensitivity coefficients is written as the partial derivative of the first order parameter sensitivity vector by

$$\underline{p}_{jk}^{(1)}(x,\hat{\gamma},s) = \frac{\partial}{\partial \hat{\gamma}_k} \underline{u}_j(x,\hat{\gamma},s) \tag{3-1}$$

The Class 2 coefficients give the effects of mixed parameter-initial condition deviations. The vector for the Class 2 coefficients is defined in terms of the first order sensitivity vectors either by

$$\underline{p}_{jk}^{(2)}(x,\hat{\gamma},s) = \frac{\partial}{\partial \hat{\gamma}_k} \underline{w}_j(x,\hat{\gamma},s) \tag{3-2}$$

or

$$\underline{p}_{jk}^{(2)}(x,\hat{\gamma},s) = \frac{\partial}{\partial z_j(s)} \underline{u}_k(x,\hat{\gamma},s) \tag{3-3}$$

where $\underline{w}_j(x,\hat{\gamma},s)$ is the first order state variable sensitivity vector given in Equation (D-2). The definitions given in Equations (3-2) and (3-3) are equivalent since one can be obtained from the other by interchanging the order of the partial differentiation. Finally, the Class 3 coefficients give the effects of second order initial condition deviations on the perturbed state variables. Mathematically, the vector of Class 3 coefficients can be expressed by

$$p_{jk}^{(3)}(x,\hat{\eta},s) = \frac{\partial}{\partial z_k(s)} w_j(x,\hat{\eta},s) \qquad (3\text{-}4)$$

Alternatively, all three kinds of second order coefficients could be expressed as second partial derivatives of the reference state vector $z(x,\hat{\eta},s)$ and this is shown in Table 4.

The differential equations describing the $p_{jk}^{(i)}$ are now obtained by differentiating the first order sensitivity equations with respect to either $\hat{\eta}_\ell$ or $z_m(s)$, $\ell = 1,\ldots,5$ and m = 1,...3. In particular, the differential equation for the $p_{jk}^{(1)}$ is obtained by differentiating Equation (2-16) with respect to $\hat{\eta}_k$.

A remark is necessary before this operation can be performed, however. Since the elements of $A(x,\hat{\eta})$ matrix and the $h_j(x,\hat{\eta})$ forcing vector are explicit functions of the state variables which themselves are functions of the parameters and the initial conditions, the chain-rule must be used in differentiating Equation (2-16). To the author's knowledge, this point has not been discussed previously though it is implied by the analysis of Reference 44 for a single higher order scalar differential equation. We now obtain the following for $\underline{p}_{jk}^{(1)}$

$$\frac{d}{dx}\underline{p}_{jk}^{(1)}(x,\hat{\underline{q}},s) = A(x,\hat{\underline{q}})\underline{p}_{jk}^{(1)}(x,\hat{\underline{q}},s) + [\underline{u}_k(x,\hat{\underline{q}},s)\cdot\nabla_{\underline{z}} + \frac{\partial}{\partial \hat{q}_k}]\underline{h}_j(x,\hat{\underline{q}})$$
$$+ [(\underline{u}_k(x,\hat{\underline{q}},s)\cdot\nabla_{\underline{z}} + \frac{\partial}{\partial \hat{q}_k})A(x,\hat{\underline{q}})]\underline{u}_j(x,\hat{\underline{q}},s) \qquad (3\text{-}5)$$

where the scalar operator $\underline{u}_k(x,\hat{\underline{q}},s)\cdot\nabla_{\underline{z}}$ can be written explicitly as

$$\underline{u}_k(x,\hat{\underline{q}},s)\cdot\nabla_{\underline{z}} = u_{1k}(x,\hat{\underline{q}},s)\frac{\partial}{\partial z_1} + u_{2k}(x,\hat{\underline{q}},s)\frac{\partial}{\partial z_2} + u_{3k}(x,\hat{\underline{q}},s)\frac{\partial}{\partial z_3} \qquad (3\text{-}6)$$

Further simplification is achieved by noting that

$$[\underline{u}_k(x,\hat{\underline{q}},s)\cdot\nabla_{\underline{z}}]\underline{h}_j(x,\hat{\underline{q}})$$

$$= \begin{pmatrix} u_{1k}\dfrac{\partial^2 F_1}{\partial \hat{q}_j \partial z_1} + u_{2k}\dfrac{\partial^2 F_1}{\partial \hat{q}_j \partial z_2} + u_{3k}\dfrac{\partial^2 F_1}{\partial \hat{q}_j \partial z_3} \\ u_{1k}\dfrac{\partial^2 F_2}{\partial \hat{q}_j \partial z_1} + u_{2k}\dfrac{\partial^2 F_2}{\partial \hat{q}_j \partial z_2} + u_{3k}\dfrac{\partial^2 F_2}{\partial \hat{q}_j \partial z_3} \\ u_{1k}\dfrac{\partial^2 F_3}{\partial \hat{q}_j \partial z_1} + u_{2k}\dfrac{\partial^2 F_3}{\partial \hat{q}_j \partial z_2} + u_{3k}\dfrac{\partial^2 F_3}{\partial \hat{q}_j \partial z_3} \end{pmatrix} \qquad (3\text{-}7)$$

$$= \begin{bmatrix} \frac{\partial^2 F_1}{\partial z_1 \partial \hat{u}_j} & \frac{\partial^2 F_1}{\partial z_2 \partial \hat{u}_j} & \frac{\partial^2 F_1}{\partial z_3 \partial \hat{u}_j} \\ \frac{\partial^2 F_2}{\partial z_1 \partial \hat{u}_j} & \frac{\partial^2 F_2}{\partial z_2 \partial \hat{u}_j} & \frac{\partial^2 F_2}{\partial z_3 \partial \hat{u}_j} \\ \frac{\partial^2 F_3}{\partial z_1 \partial \hat{u}_j} & \frac{\partial^2 F_3}{\partial z_2 \partial \hat{u}_j} & \frac{\partial^2 F_3}{\partial z_3 \partial \hat{u}_j} \end{bmatrix} \begin{bmatrix} u_{1k} \\ u_{2k} \\ u_{3k} \end{bmatrix}$$

$$= \left[\frac{\partial}{\partial \hat{u}_j} A(x, \hat{\underline{q}}) \right] \underline{u}_k(x, \hat{\underline{q}}, s)$$

where $\underline{h}_j(x, \hat{\underline{q}})$ is defined in Equation (2-13b) and $A(x, \hat{\underline{q}})$ is defined in Equation (2-14). The substitution of Equation (3-7) into Equation (3-5) then leads to

$$\begin{aligned} \frac{d}{dx} \underline{r}_{jk}^{(1)}(x, \hat{\underline{q}}, s) = A(x, \hat{\underline{q}}) \underline{r}_{jk}^{(1)}(x, \hat{\underline{q}}, s) \\ + \left[\frac{\partial}{\partial \hat{u}_k} A(x, \hat{\underline{q}}) \right] \underline{u}_j(x, \hat{\underline{q}}, s) \\ + \left[\frac{\partial}{\partial \hat{u}_j} A(x, \hat{\underline{q}}) \right] \underline{u}_k(x, \hat{\underline{q}}, s) \\ + \frac{\partial}{\partial \hat{u}_k} \underline{h}_j(x, \hat{\underline{q}}) \qquad \text{(3-8)} \\ + \left[\left(\underline{u}_k(x, \hat{\underline{q}}, s) \cdot \nabla_{\underline{z}} \right) A(x, \hat{\underline{q}}) \right] \underline{u}_j(x, \hat{\underline{q}}, s) \end{aligned}$$

To obtain the differential equation describing the

$p_{jk}^{(2)}(x,\hat{\eta},s)$ sensitivity vector, we differentiate Equation (D-3) with respect to $\hat{\eta}_k$. Following the same approach used to derive Equation (3-8), we obtain

$$\frac{d}{dx} p_{jk}^{(2)}(x,\hat{\eta},s) = A(x,\hat{\eta}) p_{jk}^{(2)}(x,\hat{\eta},s) + \left[\frac{\partial}{\partial \hat{\eta}_k} A(x,\hat{\eta})\right] w_j(x,\hat{\eta},s) + \left[(u_k(x,\hat{\eta},s)\cdot \nabla_z) A(x,\hat{\eta})\right] w_j(x, \tag{3-9}$$

For the $p_{jk}^{(3)}(x,\hat{\eta},s)$, Equation (D-3) is differentiated with respect to $z_k(s)$ leading to

$$\frac{d}{dx} p_{jk}^{(3)}(x,\hat{\eta},s) = A(x,\hat{\eta}) p_{jk}^{(3)}(x,\hat{\eta},s) + \left[(w_k(x,\hat{\eta},s)\cdot \nabla_z) A(x,\hat{\eta})\right] w_j(x,\hat{\eta},s) \tag{3-10}$$

To summarize, we note that all the second order sensitivity vectors can be described by the following vector-matrix differential equation

$$\frac{d}{dx} p_{jk}^{(i)}(x,\hat{\eta},s) = A(x,\hat{\eta}) p_{jk}^{(i)}(x,\hat{\eta},s) + g_{jk}^{(i)}(x,\hat{\eta},s) \tag{3-11}$$

where $p_{jk}^{(i)}$ and $g_{jk}^{(i)}$ are given in the following table:

TABLE 4

Class =i	Meaning of $p_{jk}^{(i)}$	Forcing Function $g_{jk}^{(i)}(x,\hat{\eta},s)$
1	$\dfrac{\partial^2}{\partial \hat{\eta}_j \partial \hat{\eta}_k} z(x,\hat{\eta},s)$	$\left[\frac{\partial}{\partial \hat{\eta}_k} A(x,\hat{\eta})\right] u_j(x,\hat{\eta},s)$ $+\left[(u_k(x,\hat{\eta},s)\cdot \nabla_z) A(x,\hat{\eta})\right] u_j(x,\hat{\eta},s)$ $+\left[\frac{\partial}{\partial \hat{\eta}_j} A(x,\hat{\eta})\right] u_k(x,\hat{\eta},s) + \frac{\partial}{\partial \hat{\eta}_k} h_j(x,\hat{\eta})$
2	$\dfrac{\partial^2}{\partial z_j(s) \partial \hat{\eta}_k} z(x,\hat{\eta},s)$	$\left[\frac{\partial}{\partial \hat{\eta}_k} A(x,\hat{\eta})\right] w_j(x,\hat{\eta},s)$ $+\left[(u_k(x,\hat{\eta},s)\cdot \nabla_z) A(x,\hat{\eta})\right] w_j(x,\hat{\eta},s)$

Class $= i$	Meaning of $v_{jk}^{(i)}$	Forcing Function $g_{jk}^{(i)}(x,\hat{q},s)$
3	$\dfrac{\partial^2}{\partial z_j(s)\,\partial z_k(s)}\, z(x,\hat{q},s)$	$[(\underline{w}_k(x,\hat{q},s)\cdot\nabla_z)A(x,\hat{q})]\,\underline{w}_j(x,\hat{q},s)$

C. Second Order Sensitivity Analysis for Updating the Control

The problem of mechanizing the first order sensitivity guidance scheme was solved by using the adjoint sensitivity Equation (2-22) to compute the required terminal sensitivities. This was advantageous since a separate 'fast-time' integration of Equation (2-16) was no longer needed for each control update. In this part of the thesis, we apply a similar 'adjoint' approach to second order sensitivity analysis in order to simplify the on-board computation of the required second order coefficients.

First, however, a remark is made about the structure of the second order sensitivity equations. Comparison of the First Order Sensitivity Equation (2-16) with the Second Order Sensitivity Equation (3-11) reveals a difference in form between the two relations. The $g_{jk}^{(i)}$ clearly are a function of S (see Table 4) while the h_j are not. It is shown that the h_j do not depend on S in Appendix C of this thesis. As might be expected, this difference in form complicates the application of the 'adjoint' approach to second order sensitivity. To reiterate, the aim of the following analysis is to produce a differential equation describing the $p_{jk}^{(i)}(x_T, \hat{q}, s)$ as functions of S, the present altitude.

Proceeding in a straightforward manner it is possible to write the solution of Equation (3-11) in terms of its state transition matrix as follows

$$p_{jk}^{(i)}(x,\hat{\eta},s) = \int_s^x \Phi(x,\lambda)\, g_{jk}^{(i)}(\lambda,\hat{\eta},s)\,d\lambda \tag{3-12}$$

Since we are interested in the terminal values of the second order coefficients we can substitute x_T, the normalized terminal altitude, for x in Equation (3-12) which gives

$$p_{jk}^{(i)}(x_T,\hat{\eta},s) = \int_s^{x_T} \Phi(x_T,\lambda)\, g_{jk}^{(i)}(\lambda,\hat{\eta},s)\,d\lambda \tag{3-13}$$

We now differentiate Equation (3-13) with respect to s which results in

$$\begin{aligned} \frac{d}{ds} p_{jk}^{(i)}(x_T,\hat{\eta},s) &= -\Phi(x_T,s)\, g_{jk}^{(i)}(s,\hat{\eta},s) \\ &\quad + \int_s^{x_T} \frac{\partial}{\partial s}\left[\Phi(x_T,\lambda)\, g_{jk}^{(i)}(\lambda,\hat{\eta},s)\right] d\lambda \end{aligned} \tag{3-14}$$

Since the state transition matrix $\Phi(x_T,\lambda)$ does not depend on s, Equation (3-14) can be rewritten

$$\begin{aligned} \frac{d}{ds} p_{jk}^{(i)}(x_T,\hat{\eta},s) &= -\Phi(x_T,s)\, g_{jk}^{(i)}(s,\hat{\eta},s) \\ &\quad + \int_s^{x_T} \Phi(x_T,\lambda)\left[\frac{\partial}{\partial s} g_{jk}^{(i)}(\lambda,\hat{\eta},s)\right] d\lambda \end{aligned} \tag{3-15}$$

The effect of the dependence of $g_{jk}^{(i)}(x,\hat{\eta},s)$ on s is now evident. It results in the integral term in the right hand side of Equation (3-15). This term was not present in the analogous first order equation which is shown here for comparison:

$$\frac{d}{ds}\underline{u}_j(x_T,\hat{\eta},s) = -\Phi(x_T,s)\underline{h}_j(s,\hat{\eta}) \tag{3-16}$$

The difficulties caused by this more complicated dependence will become clear as we try to simplify Equation (3-15). The evaluation of the $g_{jk}^{(i)}(s,\hat{\eta},s)$ follows directly from Table 4. The results are given below in Table 5.

i	$g_{jk}^{(i)}(s,\hat{\eta},s)$
1	$\frac{\partial}{\partial\hat{\eta}_k}\underline{h}_j(s,\hat{\eta})$
2	$[\frac{\partial}{\partial\hat{\eta}_k}A(s,\hat{\eta})]\underline{d}_j$
3	$[(\underline{d}_k\cdot\nabla_{\underline{z}})A(s,\hat{\eta})]\underline{d}_j$

TABLE 5

It is noted that $\underline{d}_k$ and $\underline{d}_j$ are given in Equation (D-5). The evaluation of the partial derivative $\frac{\partial}{\partial s}g_{jk}^{(i)}(\lambda,\hat{\eta},s)$ in Equation (3-15) is not quite as simple.

To indicate the problems involved, we consider the $i=1$ case in detail. Using the definition of $g_{jk}^{(1)}$ given in

Table 4, we obtain

$$\frac{\partial}{\partial s} g_{jk}^{(1)}(\lambda,\hat{q},s) = \left[\frac{\partial}{\partial \hat{q}_k} A(\lambda,\hat{q})\right]\frac{\partial}{\partial s}\underline{u}_j(\lambda,\hat{q},s) + \left[\frac{\partial}{\partial \hat{q}_j} A(\lambda,\hat{q})\right]\frac{\partial}{\partial s}\underline{u}_k(\lambda,\hat{q},s) + \frac{\partial}{\partial s}\left\{\left[(\underline{u}_k(\lambda,\hat{q},s)\cdot\nabla_{\underline{z}})A(\lambda,\hat{q})\right]\underline{u}_j(\lambda,\hat{q},s)\right\} \tag{3-17}$$

If we substitute Equation (3-16) into (3-17) and neglect terms containing products of two sensitivity coefficients, it follows that

$$\frac{\partial}{\partial s} g_{jk}^{(1)}(\lambda,\hat{q},s) = -\left[\frac{\partial}{\partial \hat{q}_k} A(\lambda,\hat{q})\right]\Phi(\lambda,s)\underline{h}_j(s,\hat{q}) - \left[\frac{\partial}{\partial \hat{q}_j} A(\lambda,\hat{q})\right]\Phi(\lambda,s)\underline{h}_k(s,\hat{q}) \tag{3-18}$$

The substitution of Equations (3-18) and the result for $g_{jk}^{(1)}(s,\hat{q},s)$ from Table 5 into Equation (3-15) lead to

$$\frac{d}{ds}\underline{p}_{jk}^{(1)}(x_T,\hat{q},s) = -\Phi(x_T,s)\frac{\partial}{\partial \hat{q}_k}\underline{h}_j(s,\hat{q}) - \int_s^{x_T}\Phi(x_T,\lambda)\frac{\partial}{\partial \hat{q}_k}A(\lambda,\hat{q})\Phi(\lambda,s)\underline{h}_j(s,\hat{q})d\lambda - \int_s^{x_T}\Phi(x_T,\lambda)\frac{\partial}{\partial \hat{q}_j}A(\lambda,\hat{q})\Phi(\lambda,s)\underline{h}_k(s,\hat{q})d\lambda \tag{3-19}$$

But, because Φ is a state transition matrix, it is possible to write

$$\Phi(\lambda,s) = \Phi(\lambda,x_T)\,\Phi(x_T,s) \tag{3-20}$$

Substitution of Equation (3-20) into (3-19) and some algebraic manipulation leads to

$$\frac{d}{ds} p^{(1)}_{jk}(x_T, \hat{q}, s) = -\Phi(x_T, s)\frac{\partial}{\partial \hat{q}_k} h_j(s, \hat{q})$$
$$-\left[\int_s^{x_T} \Phi(x_T, \lambda)\frac{\partial}{\partial \hat{q}_k} A(\lambda, \hat{q})\Phi(\lambda, x_T)d\lambda\right]\Phi(x_T, s) h_j(s, \hat{q}) \qquad (3\text{-}21)$$
$$-\left[\int_s^{x_T} \Phi(x_T, \lambda)\frac{\partial}{\partial \hat{q}_j} A(\lambda, \hat{q})\Phi(\lambda, x_T)d\lambda\right]\Phi(x_T, s) h_k(s, \hat{q})$$

Finally, we can write

$$\frac{d}{ds} p^{(1)}_{jk}(x_T, \hat{q}, s) = -\Phi(x_T, s)\frac{\partial}{\partial \hat{q}_k} h_j(s, \hat{q})$$
$$- I_k(s)\,\Phi(x_T, s) h_j(s, \hat{q}) \qquad (3\text{-}22)$$
$$- I_j(s)\,\Phi(x_T, s) h_k(s, \hat{q})$$

where

$$I_\ell(s) = \int_s^{x_T} \Phi(x_T, \lambda)\frac{\partial}{\partial \hat{q}_\ell} A(\lambda, \hat{q})\Phi(\lambda, x_T)d\lambda \qquad (3\text{-}23)$$

and

$$\frac{d}{ds} I_\ell(s) = -\Phi(x_T, s)\frac{\partial}{\partial \hat{q}_\ell} A(s, \hat{q})\,\Phi^{-1}(x_T, s) \qquad (3\text{-}24)$$

for $\ell = j, k$. Thus we have obtained (in an approximate sense) a system of differential equations for $p^{(1)}_{jk}(x_T, \hat{q}, s)$ in which s, the present altitude, appears as the independent variable.

To summarize, in this section we noted a difference in form between the first and second order sensitivity equations. Also we approximately obtained an adjoint description for the terminal values of the second order coefficients.

D. Control Equations for Second Order Sensitivity Guidance

In this section a general second order error prediction equation is developed for the terminal deviation in the i-th state variable as a function of the parameter deviations, the initial condition deviations, and arbitrary controls f_1 and f_2. Before beginning the analysis, a revised notation is introduced for the second order coefficients:

$$P1(i,j,k) = \frac{\partial^2}{\partial \hat{\eta}_j \partial \hat{\eta}_k} z_i(x,\hat{\eta},s)$$

$$P2(i,j,k) = \frac{\partial^2}{\partial z_j(s)\partial \hat{\eta}_k} z_i(x,\hat{\eta},s) \tag{3-25}$$

$$P3(i,j,k) = \frac{\partial^2}{\partial z_j(s)\partial z_k(s)} z_i(x,\hat{\eta},s)$$

Also, the first order error prediction Equation (2-25) for the i-th state is rewritten

$$\widetilde{\Delta z}_i(x_T) = \sum_{j=1}^{5} u_{ij}(x_T,\hat{\eta},s)\,\Delta\eta_j(s) + \sum_{j=1}^{3} w_{ij}(x_T,\hat{\eta},s)\,\Delta z_j(s) \tag{3-26}$$

With these preliminaries in mind, the second order error prediction equation can be written

$$\widetilde{\widetilde{\Delta z}}_i(x_T) = \widetilde{\Delta z}_i(x_T) + \frac{1}{2!}\left\{ \sum_{j,k=1}^{5} P1(i,j,k)\,\Delta\eta_j(s)\,\Delta\eta_k(s) + \sum_{j=1}^{3}\sum_{k=1}^{5} P2(i,j,k)\,\Delta z_j(s)\,\Delta\eta_k(s) + \sum_{j,k=1}^{3} P3(i,j,k)\,\Delta z_j(S)\,\Delta z_k(s) \right\} \tag{3-27}$$

Since the effects of the controls are of particular interest, the terms involving f_1 and f_2 are written explicitly giving

$$
\begin{aligned}
\Delta\hat{\tilde{z}}_i(x_T) = & \sum_{j=1}^{3} w_{ij}\,\Delta z_j(s) + \sum_{j=1}^{3} u_{ij}\,\Delta\eta_j(s) \\
& + u_{i4} f_1 + u_{i5} f_2 \\
& + \frac{1}{2}\Big\{ \sum_{j,k=1}^{3} P1(i,j,k)\,\Delta\eta_j(s)\,\Delta\eta_k(s) \\
& \qquad + \sum_{j,k=1}^{3} P2(i,j,k)\,\Delta z_j(s)\,\Delta\eta_k(s) \\
& \qquad + \sum_{j,k=1}^{3} P3(i,j,k)\,\Delta z_j(s)\,\Delta z_k(s) \\
& + \sum_{j=1}^{3} P1(i,j,4)\,\Delta\eta_j(s)\, f_1 \\
& + \sum_{j=1}^{3} P1(i,j,5)\,\Delta\eta_j(s)\, f_2 \\
& + \sum_{j=1}^{3} P1(i,4,j)\,\Delta\eta_j(s)\, f_1 \\
& + \sum_{j=1}^{3} P1(i,5,j)\,\Delta\eta_j(s)\, f_2 \\
& + \sum_{j=1}^{3} P2(i,j,4)\,\Delta z_j(s)\, f_1 \\
& + \sum_{j=1}^{3} P2(i,j,5)\,\Delta z_j(s)\, f_2 \\
& + P1(i,4,4) f_1^2 + P1(i,4,5) f_1 f_2 \\
& + P1(i,5,4) f_2 f_1 + P1(i,5,5) f_2^2 \Big\}
\end{aligned}
\qquad (3\text{-}28)
$$

Because of the symmetric property of $g_{jk}^{(1)}$ and $g_{jk}^{(3)}$ we can show

$$P1(i,j,k) = P1(i,k,j)$$
$$P3(i,j,k) = P3(i,k,j) \tag{3-29}$$

Substitution of Equation (3-29) into (3-28) and algebraic manipulation lead to

$$\Delta\tilde{z}_i(x_T) = E_i^{(0)} + E_i^{(1)} f_1 + E_i^{(2)} f_2 + \tfrac{1}{2} P1(i,4,4) f_1^2 + P1(i,4,5) f_1 f_2 + \tfrac{1}{2} P1(i,5,5) f_2^2 \tag{3-30}$$

where
$$E_i^{(0)} = \sum_{j=1}^{3} [w_{ij}\, \Delta z_j(s) + u_{ij}\, \Delta \eta_j(s)] + \tfrac{1}{2} \sum_{j,k=1}^{3} [P3(i,j,k)\, \Delta z_j(s)\, \Delta z_k(s) + P2(i,j,k)\, \Delta z_j(s)\, \Delta \eta_k(s) + P1(i,j,k)\, \Delta \eta_j(s)\, \Delta \eta_k(s) \tag{3-31}$$

$$E_i^{(1)} = u_{i4} + \tfrac{1}{2} \sum_{j=1}^{3} P2(i,j,4)\, \Delta z_j(s) + \sum_{j=1}^{3} P1(i,j,4)\, \Delta \eta_j(s) \tag{3-32}$$

$$E_i^{(2)} = u_{i5} + \tfrac{1}{2} \sum_{j=1}^{3} P2(i,j,5)\, \Delta z_j(s) + \sum_{j=1}^{3} P1(i,j,5)\, \Delta \eta_j(s) \tag{3-33}$$

In order to achieve zero terminal deviation in the range angle and velocity, we set

$$\Delta\tilde{z}_i(x_T) = 0, \quad i = 2,3 \tag{3-34}$$

This results in a set of two simultaneous quadratic equations for the controls f_1 and f_2. Techniques for solving these equations are discussed in the next section of this thesis.

E. Numerical Simulation of Second Order Sensitivity Guidance

We now examine a numerical example which will illustrate the effectiveness of the second order sensitivity guidance. This example will also give an indication of the complexity of the computational requirements. Three specific tasks are considered. The first is the solution of the second order sensitivity equations. The results of this task are the sensitivity coefficients required at the beginning of entry into the Martian atmosphere. The second task involves the solution of the simultaneous quadratic control Equations (3-34) for f_1 and f_2. Finally, these controls are substituted into a trajectory program and the terminal errors compared for first and second order guidance schemes. All three tasks are now discussed in detail.

As shown in Table 4, the solutions to the second order sensitivity Equations (3-11) depend on the first order sensitivity vectors $\underline{u}_j(x, \hat{\eta}, s)$ and $\underline{w}_j(x, \hat{\eta}, s)$. For this reason, a numerical program was formulated for the simultaneous solution of the reference states, the first order sensitivities, and the second order sensitivities. Specifically, the program (see Appendix G) solves Equation (2-3) for $\underline{z}(x_T, \hat{\eta}, x_E)$, Equation (2-15) for the $\underline{u}_j(x_T, \hat{\eta}, x_E)$, Equation (D-3) for the $w_j(x_T, \hat{q}, x_E)$, and Equations (3-11) (with i = 1,2,3) for the $p_{jk}^{(i)}(x_T, \hat{q}, x_E)$. To estimate the size of this task it is noted that there are 3 reference state equations, 24 first order sensitivity equations, and 108 second order sensitivity equations. This program was run assuming a VM-2 reference atmosphere and the following numerical data:

y_E = 89,400 ft.

θ_E = .15287 radians

V_E = 12,300 ft/sec

Ω = 0

$\hat{a}$ = -10.2348

$\hat{b}$ = 2.7027

$\hat{c}$ = -.501556

N = 1

h = 61,000 ft.

$m/C_D A$ = .30 slugs/ft^2

g_0 = 12.3 ft/sec^2 for Mars

R_0 = 10.86(10)^6 ft for Mars

y_T = 20,000 ft.

Since the above data is identical to that used in Section II.F of this thesis the results for the reference states and the first order coefficients are not repeated. The following data was obtained for the second order coefficients where the notation given in Equation (3-25) is used. The Class 1 coefficients are given in Table 6 followed by Tables 7 and 8 for the Class 2 and Class 3 coefficients. Note that $x = x_T$ and $s = x_E$ in Equation (3-25) for P1(i,j,k).

TABLE 6 - Class 1

i	j	k	P1 (i,j,k)
1	1	1	.1115
2	1	1	$.7189\ (10)^{-1}$
3	1	1	$-.6436\ (10)^{-3}$
1	1	2	$-.4476\ (10)^{-1}$
2	1	2	$-.2892\ (10)^{-1}$
3	1	2	$.1484\ (10)^{-2}$
1	1	3	.2821
2	1	3	.2020
3	1	3	$-.1520\ (10)^{-1}$
1	1	4	$.2217\ (10)^{-1}$
2	1	4	$.4574\ (10)^{-3}$
3	1	4	$-.4468\ (10)^{-4}$
1	1	5	$-.2691\ (10)^{-1}$
2	1	5	$.2317\ (10)^{-2}$
3	1	5	$-.9397\ (10)^{-4}$
1	2	2	$.6851\ (10)^{-1}$
2	2	2	$.1035\ (10)^{-1}$
3	2	2	$-.2371\ (10)^{-2}$
1	2	3	.2934
2	2	3	-.1023
3	2	3	$.3000\ (10)^{-2}$
1	2	4	$-.1075\ (10)^{-1}$
2	2	4	$.4013\ (10)^{-4}$
3	2	4	$.3365\ (10)^{-4}$
1	2	5	$.1382\ (10)^{-1}$
2	2	5	$-.9458\ (10)^{-3}$
3	2	5	$.4278\ (10)^{-4}$
1	3	3	$.1452\ (10)^{1}$

(Cont'd)

i	j	k	P 1 (i,j,k)
2	3	3	.5508
3	3	3	$-.8609\ (10)^{-1}$
1	3	4	$.7816\ (10)^{-1}$
2	3	4	$-.6140\ (10)^{-3}$
3	3	4	$-.2815\ (10)^{-3}$
1	3	5	-.1029
2	3	5	$.6558\ (10)^{-2}$
3	3	5	$-.2587\ (10)^{-3}$
1	4	4	$.1054\ (10)^{-2}$
2	4	4	$-.4813\ (10)^{-4}$
3	4	4	$-.1577\ (10)^{-5}$
1	4	5	$.3899\ (10)^{-3}$
2	4	5	$-.3954\ (10)^{-4}$
3	4	5	$-.3970\ (10)^{-5}$
1	5	5	$-.1389\ (10)^{-2}$
2	5	5	$-.5116\ (10)^{-4}$
3	5	5	$.2368\ (10)^{-4}$

TABLE 7 - Class 2

i	j	k	P 2 (i, j, k)
1	1	1	$.1124\ (10)^{1}$
1	2	1	$.9186\ (10)^{-1}$
1	3	1	0.0
2	1	1	-.1873
2	2	1	$-.1133\ (10)^{-2}$
2	3	1	0.0
3	1	1	$.5530\ (10)^{-1}$
3	2	1	$-.3153\ (10)^{-2}$
3	3	1	0.0
1	1	2	-.4897
1	2	2	$-.2215\ (10)^{-1}$
1	3	2	0.0
2	1	2	$.8058\ (10)^{-1}$
2	2	2	$.1172\ (10)^{-4}$
2	3	2	0.0
3	1	2	$-.5361\ (10)^{-1}$

Table 7 - Class 2 Cont'd....)

i	j	k	P 2 (i, j, k)
3	2	2	$.1604\ (10)^{-2}$
3	3	2	0.0
1	1	3	$.3882\ (10)^{1}$
1	2	3	.1118
1	3	3	0.0
2	1	3	-.5956
2	2	3	$.4712\ (10)^{-5}$
2	3	3	0.0
3	1	3	.4898
3	2	3	$-.1088\ (10)^{-1}$
3	3	3	0.0
1	1	4	$-.6879\ (10)^{-1}$
1	2	4	$-.1268\ (10)^{-2}$
1	3	4	0.0
2	1	4	$.1495\ (10)^{-2}$
2	2	4	$.5639\ (10)^{-4}$
2	3	4	0.0
3	1	4	$.6524\ (10)^{-3}$
3	2	4	$.2577\ (10)^{-4}$
3	3	4	0.0
1	1	5	$.2468\ (10)^{-1}$
1	2	5	$.2150\ (10)^{-2}$
1	3	5	0.0
2	1	5	$-.2884\ (10)^{-2}$
2	2	5	$.7596\ (10)^{-5}$
2	3	5	0.0
3	1	5	$-.2353\ (10)^{-2}$
3	2	5	$-.1149\ (10)^{-3}$
3	3	5	0.0

TABLE 8 - Class 3

i	j	k	P 3 (i, j, k)
1	1	1	$-.5183\ (10)^{1}$
2	1	1	.7398
3	1	1	.6772
1	1	2	$-.1097\ (10)^{1}$
2	1	2	$.2312\ (10)^{-1}$
3	1	2	$-.4885\ (10)^{-1}$
1	1	3	0.0
2	1	3	0.0
3	1	3	0.0
1	2	2	$.5077\ (10)^{-1}$
2	2	2	$.2272\ (10)^{-2}$
3	2	2	$-.8581\ (10)^{-2}$
1	2	3	0.0
2	2	3	0.0
3	2	3	0.0
1	3	3	0.0
2	3	3	0.0
3	3	3	0.0

Before attempting to solve the simultaneous quadratic control equations (3-34), two additional definitions are introduced. These are

$$H_2(f_1, f_2) = E_2^{(0)} + E_2^{(1)} f_1 + E_2^{(2)} f_2 + \tfrac{1}{2} P1(2,4,4) f_1^2 + P1(2,4,5) f_1 f_2 + \tfrac{1}{2} P1(2,5,5) f_2^2 \qquad (3\text{-}35)$$

and

$$H_3(f_1, f_2) = E_3^{(0)} + E_3^{(1)} f_1 + E_3^{(2)} f_2 + \tfrac{1}{2} P1(3,4,4) f_1^2 + P1(3,4,5) f_1 f_2 + \tfrac{1}{2} P1(3,5,5) f_2^2 \qquad (3\text{-}36)$$

where $x = x_T$ and $s = x_E$ in Equations (3-35), (3-36), (3-32), and (3-33).

The $E_{\lambda}^{(m)}$ are defined in Section III.D. Using Equations (3-35) and (3-36) the control problem stated by Equation (3-34) may be restated as finding f_1 and f_2 such that

$$H_2(f_1, f_2) = 0$$
$$H_3(f_1, f_2) = 0 \tag{3-37}$$

The solution of these equations is discussed assuming a VM-2 reference atmosphere and a VM-4 actual atmosphere where a,b,c for VM-4 are

$$a = -9.9075$$
$$b = 2.3256$$
$$c = -.5439$$

It also is assumed that there are no initial condition deviations at the entry altitude. Two different techniques were applied to determine the values of f_1 and f_2 that solve Equation (3-37). First, Newton's Method[69] was tried with the solutions of Equations (2-12) for f_1 and f_2 used as the starting values. The iterated solutions resulting from this approach failed to converge. The Steepest Descent Technique[70,71,72] was applied with much more success. Since Steepest Descent is usually applied to minimization problems, it was necessary to restate problem. To do this, we define a new function of f_1 and f_2 given by

$$G(f_1, f_2) = [H_2(f_1, f_2)]^2 + k^2 [H_3(f_1, f_2)]^2 \tag{3-38}$$

The control problem is to find the values of f_1 and f_2 that minimize $G(f_1, f_2)$. For the present problem (VM-4 actual atmosphere and VM-2 reference atmosphere), the sequences of $G(f_1, f_2)$, $H_2(f_1, f_2)$, $H_3(f_1, f_2)$, f_1, and f_2 resulting from the application of Steepest Descent are shown in Table 9 for k^2 equal to one. The Steepest Descent Computer Program is given in Appendix G.

TABLE 9 - Steepest Descent
Control Determination

f_1	f_2	$H_2\ (f_1,f_2)$	$H_3\ (f_1,f_2)$	$G(f_1,f_2)$
-11.75**	5.43**	$.153\ (10)^{-2}$	$-.251\ (10)^{-3}$	$.239\ (10)^{-5}$
-11.67	5.49	$.136\ (10)^{-2}$	$-.230\ (10)^{-3}$	$.191\ (10)^{-5}$
-11.59	5.55	$.120\ (10)^{-2}$	$-.208\ (10)^{-3}$	$.147\ (10)^{-5}$
-11.51	5.60	$.103\ (10)^{-2}$	$-.187\ (10)^{-3}$	$.109\ (10)^{-5}$
-11.42	5.66	$.862\ (10)^{-3}$	$-.165\ (10)^{-3}$	$.770\ (10)^{-6}$
-11.34	5.72	$.693\ (10)^{-3}$	$-.143\ (10)^{-3}$	$.501\ (10)^{-6}$
-11.26	5.78	$.524\ (10)^{-3}$	$-.121\ (10)^{-3}$	$.289\ (10)^{-6}$
-11.18	5.84	$.354\ (10)^{-3}$	$-.985\ (10)^{-4}$	$.135\ (10)^{-6}$
-11.10	5.90	$.183\ (10)^{-3}$	$-.752\ (10)^{-4}$	$.392\ (10)^{-7}$
-11.03	5.97	$.126\ (10)^{-4}$	$-.494\ (10)^{-4}$	$.260\ (10)^{-8}$
.	.			
.	.			
.	.			
-11.075	6.06			

** Starting values determined by trial and error.

The performance of the First and Second Order Sensitivity Guidance schemes is compared in Table 10. First the uncontrolled terminal velocity and range angle deviations are given. Then the results of the First Order Sensitivity Guidance with updating are given. Next, First Order Guidance without updating is simulated. In this case, the controls determined at the beginning of entry are applied all the way down to 20,000 ft. Finally, Second Order Guidance without updating is implemented.

TABLE 10

Type of Control	Control Values		Actual Trajectory Deviations at the Terminal Altitude		Residues	
	f_1	f_2	Δv_T	$\Delta \Omega_T$	$H_2(f_1, f_2)$	$H_3(f_1, f_2)$
None	0	0	$-.1221(10)^{-1}$	$-.3343(10)^{-2}$		
1st Sens. Guid. <u>With</u> Updating	Vary with altitude		$-.5166(10)^{-3}$	$.648\ (10)^{-6}$		
1st Sens. Guid. <u>Without</u> Updating	-14.0476	6.57	$.177(10)^{-2}$	$.101\ (10)^{-2}$	$.364(10)^{-2}$	$.514(10)^{-3}$
2nd Sens. Guid. <u>Without</u> Updating	-11.075**	6.06**	$-.150(10)^{-2}$	$.352\ (10)^{-3}$	$-.507(10)^{-4}$	$.631(10)^{-5}$

Reference Atmosphere: VM-2
Actual Atmosphere: VM-4
No state variable errors at x_ϵ.
y_ϵ = 89,400 ft., $\hat{\theta}_\epsilon$ = .15287 radians, $\hat{V}_\epsilon$ = 12,300 ft/sec., $\hat{\Omega}_\epsilon = 0$.
** Obtained from Steepest Descent Computer Program.

PART IV

Conclusions

Trajectory dynamics, sensitivity analysis for control and for updating, and efficient implementation of the proposed guidance techniques are studied in this thesis.

The trajectory equations have been transformed from the time domain to a normalized altitude domain. This is advantageous since the density (the main source of uncertainty) now appears as a function of the independent variable. This transformation also simplifies the description of the guidance requirements.

The following atmospheric model is used

$$\rho(x) = e^{a}\left[1 + c(1-x)\right]^{-b}$$

assuming that the parameters a, b, c are slowly varying functions of altitude. The model will fit any of the published Martian atmospheres. The parameters, a,b, c may be tracked as functions of x, the normalized altitude. In this way we can predict the density in the future as the vehicle enters the Mars atmosphere.

The entry guidance schemes described consitute a new application and extension of sensitivity theory. Previous work[47] has assumed that the actual variations of the density parameters are unknown and minimized the terminal state variable sensitivities to such variations. Our approach is more specific in that sensitivity analysis is applied directly to the entry dynamics in order to compute the effects of both density parameter deviations and control changes.

In the process of obtaining an efficient procedure for updating the controls a new sensitivity equation was developed which describes the terminal sensitivities as functions of the altitude where the sensitivity coefficients are assumed to be zero. This result is important because it reduces the on-board computational requirements.

The on-board computational effort is divided into two phases. Between de-orbit and entry the reference trajectory and the first set of sensitivities are computed. These sensitivities are used as the initial conditions for the computations performed to update the control during entry. Numerical simulation of the first order guidance scheme for a specific case showed that the terminal velocity deviation was reduced by 90% in comparison to the uncontrolled trajectory. The terminal range angle deviation was reduced by an even greater amount with the application of the same control.

Second Order Sensitivity Analysis is introduced in Part III in order to improve the accuracy of the error prediction equations. A new vector-matrix differential equation for the second order sensitivity coefficients of a general system is derived. This formulation should be more useful than a previously obtained second order sensitivity equation[43,44] for a single higher order scalar differential equation. There is a difference in form between the first and second order sensitivity equations. For the first order case, it was possible to describe all the terminal values by using the adjoint sensitivity equation. For the second order coefficients, it is shown that this procedure is only possible for a certain approximation to the second order sensitivity forcing function. Numerical simulation of second sensitivity guidance including solution of the nonlinear control equations is performed and the results summarized in Table 10.

Some extensions of the work in this thesis are now discussed. For First Order Sensitivity Guidance, the singularity in Equation (2-3) at zero flight path angle has proved troublesome in certain cases. This problem is particularly evident with a 'thin' reference atmosphere and a much 'thicker' actual atmosphere. Using the range angle as the independent variable removes that singularity and might allow a wider range of controlled trajectories.

Other possible control variables should be investigated. For example, it may be possible to achieve sufficient trajectory control by implementing discrete changes in the vehicle ballistic coefficient, $\mathcal{M}/C_D A$. Such an approach would be advantageous since it would eliminate the fuel consumed by retro-propulsive control of the aerodynamic braking phase.

As mentioned in the Introduction, uniformly valid analytic solutions of the entry dynamics would lead to closed form algebraic expressions for the sensitivity coefficients. Such closed form expressions might facilitate a more flexible implementation of sensitivity guidance and certainly would reduce the complexity of the on-board calculations.

For Second Order Sensitivity Guidance, the problem of updating is still somewhat unresolved. It is not really clear whether the benefits warrant the additional complexity that updating may involve. Perhaps a hybrid approach utilizing approximate second order coefficients might be effective.

Finally, the nonlinear control equations (3-37) deserv further attention. Since these equations represent the intersections of a pair of conic sections, the possibility of multiple solutions would appear to exist.

PART V

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APPENDIX A

The Time Domain Equations of Atmospheric Flight Mechanics

Two dimensional entry into the non-rotating atmosphere of a spherical planet is considered. The geometry is given in Figure A-1 where

O = Center of Planet

P = Position of Vehicle

PH = Local Horizontal

Ω = Range Angle

$\underline{V}$ = Velocity

$\underline{r}$ = Vector from O to P

θ = Flight Path Angle

$\underline{e}_T$ = Unit Vector in the Direction of the Velocity

$\underline{e}_N$ = Unit Vector Perpendicular to $\underline{e}_T$

A moving coordinate system oriented parallel to the position vector is also used. The relationship between this coordinate system and the $\underline{e}_T, \underline{e}_N$ velocity coordinate system is given in Figure A-2 where

$$\underline{e}_R = \frac{\underline{r}}{|\underline{r}|} \qquad \text{(A-1)}$$

and $\underline{e}_\Omega$ is perpendicular to $\underline{e}_R$. Analytically, the relationship between the $\underline{e}_T$, $\underline{e}_N$ velocity coordinate system is given

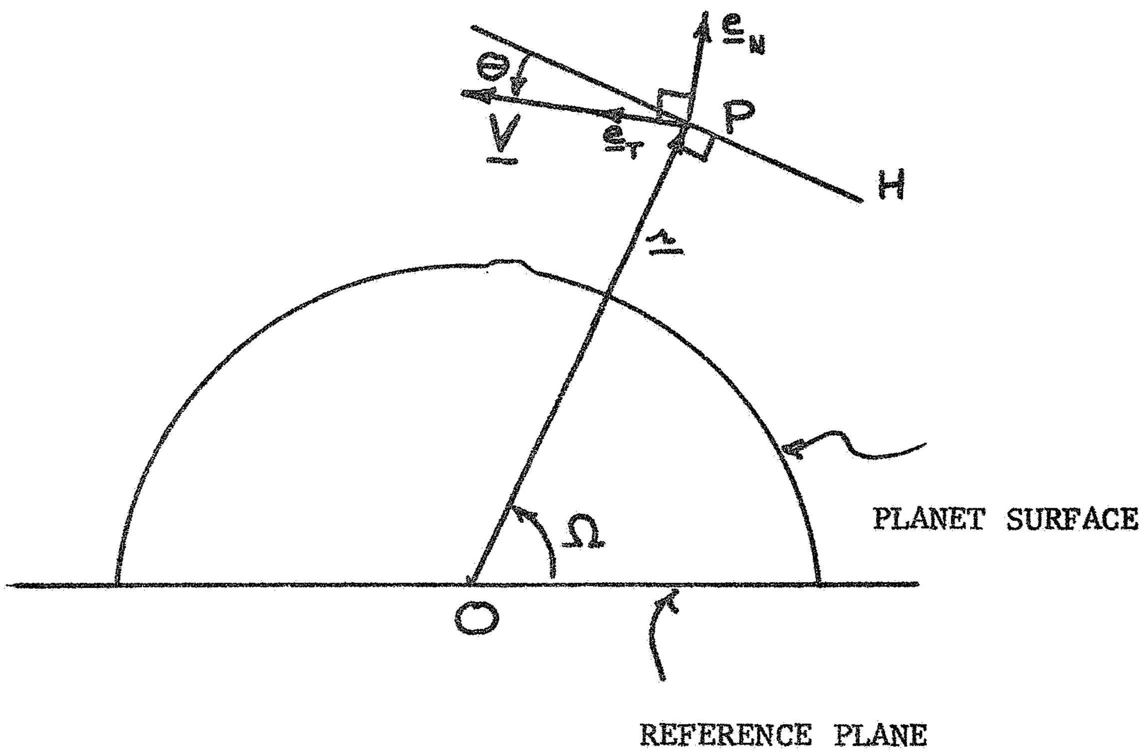


FIGURE A - 1. Entry Dynamics Geometry

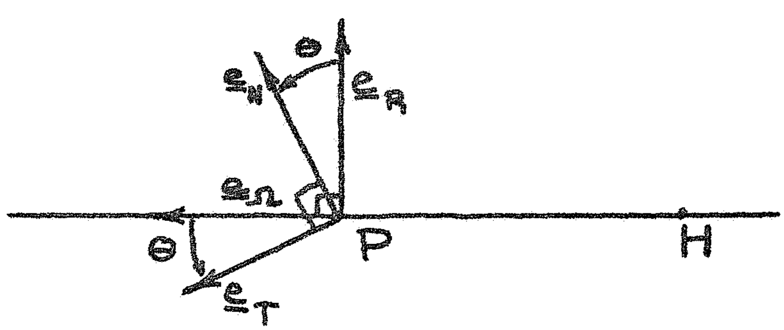


FIGURE A - 2. Entry Dynamics Geometry

by $$\underline{e}_N = \underline{e}_R \cos\Theta + \underline{e}_\Omega \sin\Theta$$
$$\underline{e}_T = -\underline{e}_R \sin\Theta + \underline{e}_\Omega \cos\Theta \qquad \text{(A-2)}$$

The vehicle position vector, $\underline{r}$, can be expressed in the $\underline{e}_R$, $\underline{e}_\Omega$ system as

$$\underline{r} = r\,\underline{e}_R \qquad \text{(A-3)}$$

If Equations (A-2) are solved for $\underline{e}_R$, and the result substituted into Equation (A-3), the position vector is obtained in terms of $\underline{e}_T$ and $\underline{e}_N$:

$$\underline{r} = r\cos\Theta\,\underline{e}_N - r\sin\Theta\,\underline{e}_T \qquad \text{(A-4)}$$

An expression for the velocity in the $\underline{e}_T$, $\underline{e}_N$ coordinates is found by differentiating Equation (A-4) with respect to time:

$$\dot{\underline{r}} = \frac{d}{dt}(r\cos\Theta)\,\underline{e}_N - \frac{d}{dt}(r\sin\Theta)\,\underline{e}_T$$
$$+ r\cos\Theta\,\dot{\underline{e}}_N - r\sin\Theta\,\dot{\underline{e}}_T \qquad \text{(A-5)}$$

But the angular motion of the $\underline{e}_T$, $\underline{e}_N$ frame with respect to the fixed reference plane (see Figure A-1) is given by

$$\dot{\underline{e}}_N = (\dot{\Omega} + \dot{\Theta})\,\underline{e}_T$$
$$\dot{\underline{e}}_T = -(\dot{\Omega} + \dot{\Theta})\,\underline{e}_N \qquad \text{(A-6*)}$$

Substitution of Equations (A-6) into Equation (A-5) leads

* Equations (A-6) can also be derived by differentiating Equations (A-2) and noting that $\dot{\underline{e}}_R = \dot{\Omega}\,\underline{e}_\Omega$ and $\dot{\underline{e}}_\Omega = -\dot{\Omega}\,\underline{e}_R$

to
$$\dot{\underline{r}} = \left[r(\dot{\Omega}+\dot{\theta})\cos\theta - \frac{d}{dt}(r\sin\theta)\right]\underline{e}_T + \left[r(\dot{\Omega}+\dot{\theta})\sin\theta + \frac{d}{dt}(r\cos\theta)\right]\underline{e}_N \quad \text{(A-7)}$$

which can be simplified to read
$$\dot{\underline{r}} = \left[r\dot{\Omega}\cos\theta - \dot{r}\sin\theta\right]\underline{e}_T + \left[r\dot{\Omega}\sin\theta + \dot{r}\cos\theta\right]\underline{e}_N \quad \text{(A-8)}$$

However, since $\underline{e}_T$ was defined to be in the direction of the velocity vector, it follows that

$$\underline{V} = V\underline{e}_T \quad \text{(A-9)}$$

By using the definition of the velocity and Equations (A-8) and (A-9), the following two scalar equations are obtained
$$r\dot{\Omega}\cos\theta - \dot{r}\sin\theta = V$$
$$r\dot{\Omega}\sin\theta + \dot{r}\cos\theta = 0 \quad \text{(A-10)}$$

The solution of Equations (A-10) for $\dot{r}$ and $\dot{\Omega}$ gives the following geometrical relations:

$$\dot{r} = -V\sin\theta \quad \text{(A-11)}$$
$$\dot{\Omega} = \frac{V\cos\theta}{r} \quad \text{(A-12)}$$

In order to obtain the remaining equations of motion, an expression for the vehicle acceleration is needed. This is obtained by differentiating Equation (A-9) with respect to time which gives

$$\ddot{\underline{r}} = \dot{V}\,\underline{e}_T + V\,\dot{\underline{e}}_T \tag{A-13}$$

Substitution of Equation (A-6) into (A-13) and substitution of Equation (A-12) into the result then gives

$$\ddot{\underline{r}} = \dot{V}\,\underline{e}_T - V\left[\dot{\Theta} + \frac{V \cos\Theta}{r}\right]\underline{e}_N \tag{A-14}$$

Next, the forces acting on the entry vehicle are considered. These include drag, lift, and gravity. Note that for the present only unpowered flight is included. The external force (excluding retro-propulsion) on the capsule can be expressed by

$$\underline{F}_{EXT} = -m g\,\underline{e}_R - D\,\underline{e}_T + L\,\underline{e}_N \tag{A-15}$$

where m = the mass of the entry vehicle

g = gravitational acceleration

D = drag force

L = lift force

The external force can be rewritten in terms of $\underline{e}_T$ and $\underline{e}_N$ only, resulting in

$$\underline{F}_{EXT} = (m g \sin\Theta - D)\,\underline{e}_T + (L - m g \cos\Theta)\,\underline{e}_N \tag{A-16}$$

Now, by using Newton's Law, the following is obtained:

$$\dot{V}\,\underline{e}_T - V\left[\dot{\theta} + \frac{V\cos\theta}{r}\right]\underline{e}_N$$
$$= \left(g\sin\theta - \frac{D}{m}\right)\underline{e}_T + \left(\frac{L}{m} - g\cos\theta\right)\underline{e}_N \qquad \text{(A-17)}$$

The equivalent scalar equations are

$$\dot{V} = g\sin\theta - \frac{D}{m} \qquad \text{(A-18)}$$

and

$$V\dot{\theta} = \left(g - \frac{V^2}{r}\right)\cos\theta - \frac{L}{m} \qquad \text{(A-19)}$$

Equations (A-11), (A-12), (A-18), and (A-19) constitute the time domain equations of motion for entry into a planetary atmosphere.

Two additional steps are usually introduced. First, the drag is approximated by

$$D = \frac{1}{2}\rho(y)\,V^2\,(C_D A) \qquad \text{(A-20)}$$

where ρ = atmospheric density

C_D = drag coefficient

A = reference area

y = altitude

Also, the variable r , which represents the distance from the center of the planet, is replaced by y , the altitude of the entry capsule above the planet's surface. The quantities r

and y are related by

$$r = R_0 + y \tag{A-21}$$

The substitution of Equations (A-20) and (A-21) into (A-11), (A-12), (A-18), and (A-19) leads to the following revised set of entry equations

$$\dot{y} = -V \sin\theta \tag{A-22}$$

$$V\dot{\theta} = \left(g - \frac{V^2}{r}\right)\cos\theta - \left(\frac{L}{D}\right)\frac{1}{2}\rho(y)V^2\left(\frac{C_D A}{m}\right) \tag{A-23}$$

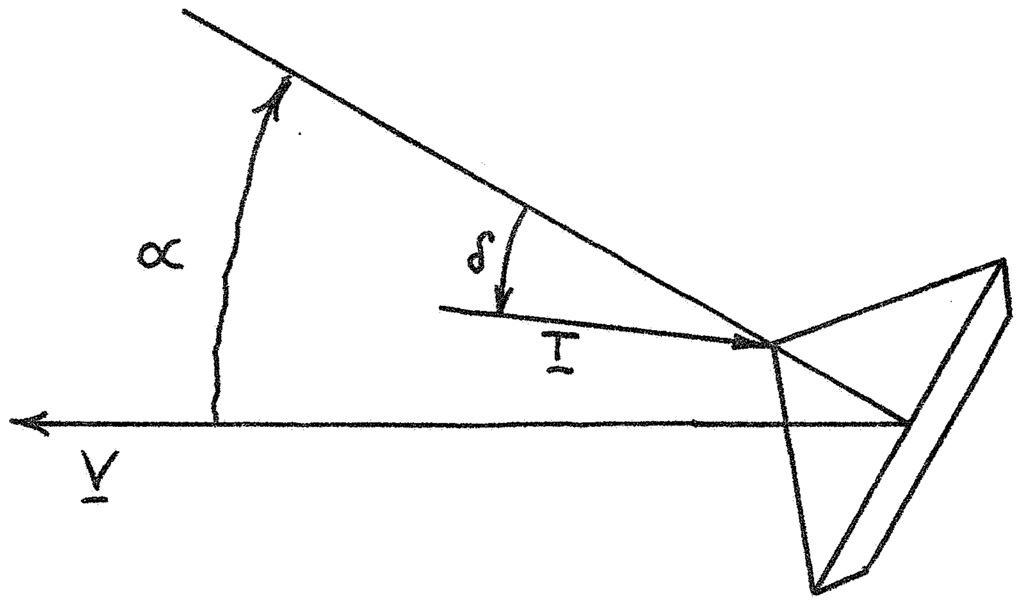
$$\dot{V} = -\frac{1}{2}\rho(y)V^2\left(\frac{C_D A}{m}\right) + g\sin\theta \tag{A-24}$$

$$\dot{\Omega} = \frac{V\cos\theta}{R_0 + y} \tag{A-25}$$

One final addition is now made. It is assumed that retro-propulsion is possible with the geometry of the thrust angles being shown in Figure A-3. Equation (A-16) is now replaced by

$$\begin{aligned} \underline{F}_{EXT} = &\left[mg\sin\theta - D - T\cos(\alpha-\delta)\right]\underline{e}_T \\ &+ \left[L - mg\cos\theta - T\sin(\alpha-\delta)\right]\underline{e}_N \end{aligned} \tag{A-26}$$

The retro-propulsion terms are shown in the revised entry equations:



α = Lift Angle

$\underline{V}$ = Velocity Vector

δ = Thrust Angle

$\underline{T}$ = Thrust Vector

FIGURE A - 3 Thrust Angle Geometry

$$\dot{y} = -V \sin\Theta$$

$$V\dot{\Theta} = \left(g - \frac{V^2}{r}\right)\cos\Theta - \frac{1}{2}\left(\frac{L}{D}\right)\rho(y)V^2\left(\frac{C_D A}{m}\right) + \frac{T}{m}\sin(\alpha - \delta) \tag{A-27}$$

$$\dot{V} = -\frac{1}{2}\rho(y)V^2\frac{C_D A}{m} + g\sin\Theta - \frac{T}{m}\cos(\alpha - \delta)$$

$$\dot{\Omega} = \frac{V\cos\Theta}{R_0 + y}$$

APPENDIX B

Martian Atmospheric Density Models

At present there are two density models for Mars being used by NASA[30]. For altitudes greater than the height of the tropopause, an isothermal model is used:

$$\rho = \rho_{ref}\, e^{-\beta y} \tag{B-1}$$

For altitudes less than the heigh of the tropopause an adiabatic density model is used:

$$\rho = \rho_o \left(1 + \frac{\Gamma}{T_o}\, y\right)^{\frac{1}{\gamma - 1}} \tag{B-2}$$

Where ρ = surface density (slugs/ft^3)

T_o = surface temperature (^{o}R)

β = inverse scale heigh (in stratosphere) (1/ft)

Γ = adiabatic lapse rate (^{o}R/ft)

γ = ratio of specific heats

y = height above surface (ft)

If the density is assumed to be a continuous function of altitude, then it can be shown that

$$\rho_{ref} = \rho_o\, e^{\beta H_{TROP}} \left(1 + \frac{\Gamma}{T_o} H_{TROP}\right)^{\frac{1}{\gamma - 1}} \tag{B-3}$$

where H_{TROP} = the height of the tropopause.

Because of the height of the tropopause varies from 56,000 feet to 63,000 feet depending upon the atmospheric model used and because the main part of the aerodynamic braking occurs in the lower portion of the atmosphere, the adiabatic density model is used exclusively in this thesis.

The uncertainty in the present definition of the parameters of the adiabatic model is indicated by the allowable ranges for three parameters for the VM-1 to VM-8 models:[33]

$$1.32(10)^{-5} < \rho_0 < 4.98(10)^{-5}$$

$$-.00321 < \Gamma < -.00213$$

$$360 < T_0 < 495$$

$$1.37 < \gamma < 1.43$$

$$.0043 < \frac{\Gamma}{T_0} < .0089$$

To obtain formulas for the atmospheric parameters a, b, and c used in this thesis, the following transformation is introduced

$$x = \left(\frac{h-y}{h}\right)N \qquad \text{(B-4)}$$

where x = normalized altitude

h = reference altitude

N = scale factor

To obtain an adiabatic model of the density as a function of x, Equation (B-4) is solved for y and the result substituted for y in Equation (B-2). This leads to

$$\rho(x) = \rho_0 \left[1 + \frac{\Gamma}{T_0} h \left(1 - \frac{x}{N}\right)\right]^{\frac{1}{\gamma-1}} \tag{B-5}$$

In terms of a, b, and c the density can be expressed as

$$\rho(x) = e^{a}\left[1 + c\left(1 - \frac{x}{N}\right)\right]^{b} \tag{B-6}$$

where

$$a = \ln \rho_0 \tag{B-7}$$

$$b = \frac{1}{\gamma - 1} \tag{B-8}$$

$$c = \frac{\Gamma h}{T_0} \tag{B-9}$$

Equation (B-6) is the density model used throughout this thesis.

An alternative model was used in some earlier studies by this investigator. That model is given by

$$\rho(x) = e^{A}(C + x)^{B} \tag{B-10}$$

The quantity χ is still defined by Equation (B-4) and the parameters A, B, and C are defined by

$$A = \ln \rho_0 + \frac{1}{\gamma-1} \ln \left(-\frac{\Gamma h}{T_0 N}\right) \tag{B-11}$$

$$B = \frac{1}{\gamma-1} \tag{B-12}$$

$$C = \frac{1 + \frac{\Gamma h}{T_0}}{\left(-\frac{\Gamma h}{T_0 N}\right)} \tag{B-13}$$

While the model given by Equation (B-10) seems to have a simpler form than that of Equation (B-6) this advantage proved only temporary. During simulation of the first order sensitivity guidance system with Equation (B-10) as the density model certain difficulties were encountered when the actual atmosphere differed from the reference atmosphere in all three parameters: A, B, and C. Such a case occurs when VM-2 is the reference atmosphere and VM-1 is assumed to be the actual atmosphere. Examination of the results showed that the difficulty was caused by significant differences between the predicted uncontrolled errors and the actual uncontrolled errors. Further study showed that because parameters A and C both depended on the physical parameter (Γ/T_0), deviations in (Γ/T_0) caused larger deviations in the model parameters A and C. The following tables illustrate this point for VM-2, VM-1 case.

	VM-2	VM-1
A	-12.0997	-14.4176
B	2.7027	2.6316
C	.9938	2.8097
a	-10.2348	-10.8977
b	2.7027	2.6316
c	- .501556	- .262485
	h = 61,000 feet N = 1.0	

Old Model Equation (B-10)	New Model Equation (B-6)
$\Delta A = -2.3179$ $\Delta B = -\ .0711$ $\Delta C = +1.8159$	$\Delta a = -.6629$ $\Delta b = -.0711$ $\Delta c = +.239071$

The first table gives the density parameters for the VM-2 and VM-1 models. Here A, B, C were computed using Equations (B-11), (B-12), and (B-13) and a, b, c were computed using Equations (B-7), (B-8), and (B-9). The second table compares the deviation of the parameters for the old and new models for a VM-2 reference atmosphere and VM-1 actual atmosphere. Clearly the parameter deviations are smaller for the model given by Equation (B-6). In order to improve the performance of the first order error prediction equations, the atmospheric density model given by Equation (B-6) was adopted.

Finally, a, b, and c for VM-1 through VM-10 are given in the table on this page.

Mars Atmospheric Data for N = 1 and h = 61,000			
VM-	a	b	c
1	-10.8977	2.6316	-.2625
2	-10.2348	2.7027	-.5016
3	-10.5383	2.6316	-.2625
4	- 9.9075	2.3256	-.5439
5	-10.2046	2.6316	-.2625
6	- 9.7262	2.2222	-.4778
7	-11.2353	2.6316	-.2625
8	-10.5729	2.7027	-.5016
9	- 9.8452	2.6316	-.2626
10	- 9.5061	2.4390	-.4033

APPENDIX C

Proof that the First Order Sensitivity Forcing Vector $\underline{h}_j(x,\hat{\eta})$ is Independent of Present Altitude $\underline{s}$

It is not obvious that the first order sensitivity system forcing vector, $\underline{h}_j(x,\hat{\eta})$ (in Equation (2-16) should not be written with $\underline{s}$ as an argument. This appendix will show that it is correct to write the $\underline{h}_j$ as functions of x and $\hat{\eta}$. This result is important in the derivation of Equation (2-22) which describes the terminal parameter sensitivities. The method of approach will be to explore a simple problem and then suggest a more general proof.

For a simplified model, we chose the following first order, constant coefficient system:

$$\frac{d}{dx} z(x,\hat{a}) = \hat{a}\, z(x,\hat{a}) \qquad \text{(C-1)}$$

where: $\hat{a} =$ reference value of parameter a

$x =$ independent variable

$z(x,\hat{a}) =$ reference state variable

This model is desirable because an analytical solution for the reference trajectory is easily obtained. If the initial condition is given by

$$z(x_{s},\hat{a}) = z_{s} = \text{constant} \qquad \text{(C-2)}$$

the solution of Equation (C-1) follows

$$z(x,\hat{a}) = z_E e^{\hat{a}(x-x_E)} \tag{C-3}$$

The sensitivity coefficient $\mu(x,\hat{a},x_E)$ is defined by

$$\mu(x,\hat{a},x_E) = \frac{\partial}{\partial \hat{a}} z(x,\hat{a}) \tag{C-4}$$

Differentiating Equation (C-1) with respect to $\hat{a}$ gives the following first order sensitivity equation

$$\frac{d}{dx}\mu(x,\hat{a},x_E) = \hat{a}\,\mu(x,\hat{a},x_E) + z_E e^{\hat{a}(x-x_E)} \tag{C-5}$$

Now consider a new perturbed trajectory which coincides with the reference trajectory at S instead of x_E. The sensitivity coefficient describing this new perturbed trajectory is governed by

$$\frac{d}{dx}\mu(x,\hat{a},s) = \hat{a}\,\mu(x,\hat{a},s) + z_s e^{\hat{a}(x-s)} \tag{C-6}$$

The quantity $z_s e^{\hat{a}(x-s)}$ can be identified as the sensitivity forcing function. That is

$$h(x,\hat{a}) = z_s e^{\hat{a}(x-s)} \tag{C-7}$$

In order to show that $h(x,a)$ is not a function of S, we must prove that $z_s e^{\hat{a}(x-s)}$ reduces to $z_E e^{\hat{a}(x-x_E)}$ (see Equation (C-5). Note that the point (s, z_s) is on the reference trajectory (given by Equation (C-3). Thus

$$z_S = z_E e^{\hat{a}(S - x_E)} \tag{C-8}$$

Substitution of Equation (C-8) in Equation (C-7) gives

$$\begin{aligned} h(x,\hat{a}) &= z_E e^{\hat{a}(S-x_E)} e^{\hat{a}(x-S)} \\ &= z_E e^{\hat{a}(x-x_E)} \end{aligned} \tag{C-9}$$

which is the desired result. The important concept is that the sensitivity coefficient $u(x,\hat{a},S)$ always describes perturbations from the same reference trajectory. The variable is the point at which the perturbed trajectory coincides with the reference trajectory.

So far we have shown that the h_j are not functions of S for a specific case. We now consider a general system given by

$$\frac{d}{dx} z(x,\hat{a}) = F\left[z(x,\hat{a}), x, \hat{a}\right] \tag{C-10}$$

In particular, we consider two different solutions of Equations (C-10). These are given by the following integral equations:

$$z_1(x,\hat{a}) = z_E + \int_{x_E}^{x} F[z_1(\lambda,\hat{a}),\lambda,\hat{a}]\,d\lambda \tag{C-11}$$

$$z_2(x,\hat{a}) = z_S + \int_{S}^{x} F[z_2(\lambda,\hat{a}),\lambda,\hat{a}]\,d\lambda \tag{C-12}$$

where z_E and z_S are constants. If we define the respective sensitivity coefficients by

$$\mu(x,\hat{a},x_E) = \frac{\partial}{\partial \hat{a}} z_1(x,\hat{a}) \tag{C-13}$$

$$\mu(x,\hat{a},s) = \frac{\partial}{\partial \hat{a}} z_2(x,\hat{a}) \tag{C-14}$$

the following sensitivity equations result:

$$\begin{aligned}\frac{d}{dx}\mu(x,\hat{a},x_E) = {} & \frac{\partial}{\partial z_1(x,\hat{a})} F[z_1(x,\hat{a}),x,\hat{a}]\,\mu(x,\hat{a},x_E) \\ & + \frac{\partial}{\partial \hat{a}} F[z_1(x,\hat{a}),x,\hat{a}]\end{aligned} \tag{C-15}$$

$$\begin{aligned}\frac{d}{dx}\mu(x,\hat{a},s) = {} & \frac{\partial}{\partial z_2(x,\hat{a})} F[z_2(x,\hat{a}),x,\hat{a}]\,\mu(x,\hat{a},s) \\ & + \frac{\partial}{\partial \hat{a}} F[z_2(x,\hat{a}),x,\hat{a}]\end{aligned} \tag{C-16}$$

Clearly, if we can show that

$$z_2(x,\hat{a}) = z_1(x,\hat{a}), \quad x > s \tag{C-17}$$

then the sensitivity forcing term $\partial F[z_2(x,\hat{a}),x,\hat{a}]/\partial \hat{a}$ will not depend on s and the present formulation (Equation (2-16) is correct. To prove Equation (C-17), recall that the point (s, z_s) is on the reference trajectory. Therefore,

$$z_s = z_E + \int_{x_E}^{s} F[z_1(\lambda,\hat{a}),\lambda,\hat{a}]\,d\lambda \tag{C-18}$$

Substituting Equation (C-18) into Equation (C-12) gives

$$\begin{aligned}z_2(x,\hat{a}) = {} & z_E + \int_{x_E}^{s} F[z_1(\lambda,\hat{a}),\lambda,\hat{a}]\,d\lambda \\ & + \int_{s}^{x} F[z_2(\lambda,\hat{a}),\lambda,\hat{a}]\,d\lambda\end{aligned} \tag{C-19}$$

Algebraic manipulation of Equation (C-19) leads to

$$z_2(x,\hat{a}) = z_E + \int_{x_E}^{x} F[z_2(\lambda,\hat{a}),\lambda,\hat{a}]$$
$$+ \int_{x_E}^{s} \{F[z_1(\lambda,\hat{a}),\lambda,\hat{a}] - F[z_2(\lambda,\hat{a}),\lambda,\hat{a}]\}d\lambda \quad \text{(C-20)}$$

Subtracting Equation (C-11) from (C-20) leads to

$$z_2(x,\hat{a}) - z_1(x,\hat{a})$$
$$= \int_{s}^{x} \{F[z_2(\lambda,\hat{a}),\lambda,\hat{a}] - F[z_1(\lambda,\hat{a}),\lambda,\hat{a}]\}d\lambda \quad \text{(C-21)}$$

Note that Equation (C-21) contains the following initial condition:

$$z_2(s,\hat{a}) - z_1(s,\hat{a}) = 0 \quad \text{(C-22)}$$

Differentiating Equation (C-21) with respect to x gives

$$\frac{d}{dx}[z_2(x,\hat{a}) - z_1(x,\hat{a})]$$
$$= F[z_2(x,\hat{a}),x,\hat{a}] - F[z_1(x,\hat{a}),x,\hat{a}] \quad \text{(C-23)}$$

Since $z_2(x,\hat{a}) - z_1(x,\hat{a})$ is small, Equation (C-23) can be replaced by $\frac{d}{dx}[z_2(x,\hat{a}) - z_1(x,\hat{a})]$

$$= \frac{\partial}{\partial z_1(x,\hat{a})} F[z_1(x,\hat{a}),x,\hat{a}]\,[z_2(x,\hat{a}) - z_1(x,\hat{a})] \quad \text{(C24)}$$

If the quantity $\partial F[z_1(x,\hat{a}),x,\hat{a}]/\partial z_1(x,\hat{a})$ can be shown to satisfy a Lipschitz Condition[66], then Equation (C-24) will have a unique solution. However, because of the initial condition given by Equation (C-22), every member of a Picard sequence of approximate solutions of Equation (C-24) will be zero. Therefore the solution of Equation (C-24) must be

$$Z_2(x, \hat{a}) - Z_1(x, \hat{a}) = 0 \qquad \text{(C-25)}$$

Thus (see Equation (C-17)) the first order sensitivity forcing function does not depend on $\dot{S}$ for a general dynamical system.

APPENDIX D

State Variable Sensitivity

After the initiation of entry, the assumption that the entry capsule is on the reference trajectory will no longer be valid. Thus the effects of present state variable deviations must be included in the estimate of the terminal state variable deviation. The first order state variable sensitivities $w_{ij}(x,\hat{\eta},s)$ are given by

$$w_{ij}(x,\hat{\eta},s) = \frac{\partial}{\partial z_j(s)} z_i(x,\hat{\eta},s) \qquad \text{(D-1)}$$

where the z_i are defined by Equation (2-4a). The analysis is made more concise by defining

$$\underline{w}_j(x,\hat{\eta},s) = [w_{1j}(x,\hat{\eta},s), w_{2j}(x,\hat{\eta},s), w_{3j}(x,\hat{\eta},s)]^* \qquad \text{(D-2)}$$

where * denotes transpose. The state variable sensitivity equations

$$\frac{d}{dx}\underline{w}_j(x,\hat{\eta},s) = A(x,\hat{\eta})\,\underline{w}_j(x,\hat{\eta},s) \qquad \text{(D-3)}$$

are obtained by the same process used to obtain the parameter sensitivity equations. The system matrix $A(x,\hat{\eta})$ is the same as that given in Equation (2-14). The initial conditions for (D-3) are given by

$$\underline{w}_j(s,\hat{\eta},s) = \underline{d}_j \qquad \text{(D-4)}$$

where $\underline{d}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\underline{d}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\underline{d}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(D-5)

It is simple to show that the solution of Equation (D-3) subject to (D-4) is

$$\underline{w}_j(x, \hat{\eta}, s) = \Phi(x, s)\, \underline{d}_j \tag{D-6}$$

where $\Phi(x,s)$ is given by Equation (2-17). Using Equation (2-19) we may write

$$\underline{w}_j(x_T, \hat{\eta}, s) = \Psi^*(x_T, s)\, \underline{d}_j \tag{D-7}$$

where $\Psi^*(x_T, s)$ is the solution of Equation (2-23). The use of the $\underline{w}_j(x_T,\hat{\eta},s)$ in computing terminal trajectory errors caused by state variable deviations is discussed in Part II.E of this thesis.

APPENDIX E

Derviation of Equation (2-19)

Recall that the state transition matrix of the homogeneous senstivity system, $\Phi(x,s)$ is described by

$$\frac{d}{dx}\Phi(x,s) = A(x,\hat{q})\,\Phi(x,s) \tag{E-1}$$

$$\Phi(s,s) = I \tag{E-2}$$

and its adjoint, $\psi(x_T,x)$ described by

$$\frac{d}{dx}\psi^*(x_T,x) = -\psi^*(x_T,x)A(x,\hat{q}) \tag{E-3}$$

$$\psi^*(x_T,x_T) = I \tag{E-4}$$

Then premultiplying (E-1) by $\psi^*(x_T,x)$ gives

$$\psi^*(x_T,x)\frac{d}{dx}\Phi(x,s) = \psi^*(x_T,x)A(x,\hat{q})\Phi(x,s) \tag{E-5}$$

Postmultiplying (E-3) by $\Phi(x,s)$ gives

$$\left[\frac{d}{dx}\psi^*(x_T,x)\right]\Phi(x,s) = -\psi^*(x_T,x)A(x,\hat{q})\Phi(x,s) \tag{E-6}$$

Adding Equations (E-5) and (E-6) gives

$$\psi^*(x_T,x)\frac{d}{dx}\Phi(x,s) + \left[\frac{d}{dx}\psi^*(x_T,x)\right]\Phi(x,s) = 0 \tag{E-7}$$

which reduces to

$$\frac{d}{dx}\left[\psi^*(x_T,x)\,\Phi(x,s)\right] = 0 \tag{E-8}$$

Integrating gives

$$\Psi^*(x_T, x)\,\Phi(x, s) = C \tag{E-9}$$

where C is a constant matrix. Substituting Equation (E-2) into (E-9) gives

$$\Psi^*(x_T, s) = C \tag{E-10}$$

Substituting (E-4) into (E-9) gives

$$\Phi(x_T, s) = C \tag{E-11}$$

Finally, Equations (E-10) and (E-11) may be combined to give the required identity

$$\Psi^*(x_T, s) = \Phi(x_T, s) \tag{E-12}$$