MODERN PROBLEMS OF CELESTIAL MECHANICS

G. N. Duboshin

Byulleten' Instituta Teoreticheskoy Astronomii
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The article is a presentation of a survey report, given at the first session of the plenum of the Committee on celestial mechanics attached to the astronomical council of the USSR Academy of Sciences on 5 October, 1965, in tbilisi.

The survey is accompanied by several comments of the author on the modern state of celestial mechanics and on the possible directions of its development in the near future.

1. The Objects and Goals of Research in Celestial Mechanics

The author has many times had occasion to comment that celestial mechanics should be understood as a part of astronomy concerned with the study of the motion of all celestial bodies (or of celestial objects), regardless of what forces they may be effected by.

Until quite recently, objects of study in celestial mechanics were or could be only natural celestial bodies - large and small planets of the solar system, their satellites, comets and meteors, stars and star systems etc.

The characteristic feature of all these objects is that their existence and motion is completely independent from the will of man, who can only observe celestial bodies from a distance, studying their nature and properties and following their motions through the heavens.

Thus, astronomy of the pre-cosmic era, and along with it its inseparable part as well celestial mechanics had a purely observational, one can even say, a contemplative character, and the experimental part until quite recently simply did not exist in astronomy. Of course, experiment here is implied to be the sum total of some sort of manipulations of celestial objects, carried out at the will and the desire of man.

At the present time, we must include in the family of celestial bodies also artifical celestial bodies - artificial satellites, cosmic stations, interplanetary vehicles, etc.

Therefore, in modern celestial mechanics among all possible celestial bodies we must distinguish natural and artificial bodies, implying under the latter for the time being all those which have been manufactured on earth, and which are then sent into space by man.

In the future, larger artificial celestial bodies will be fabricated on earth in parts, then will proceed into the cosmos and be assembled, so to speak on the wing by brigades of cosmic engineers, and technicians. It is quite possible that in the future artificial celestial bodies will also be constructed on the other planets (for example on the moon, Mars or Venus) from local materials or even in interplanetary space itself, from cosmic material.

However, the characteristic feature of artificial celestial bodies is not only that they are manufactured and sent into space by the human hand, but also that, in many instances, man can in one way or another control their movements, changing in the necessary manner the rate and direction of acceleration or changing the orientation of the body in space.

We know that there has never been in reality any control over the motion of a natural celestial body (but we have been able to read about this in science-fiction novels, for example, those of Jules Verne), but there is no doubt that in the distant future man will partially gain even this possibility.

Thus, modern celestial mechanics must study the motions of natural as well as those of artificial celestial bodies.

But both natural and artificial celestial bodies must be divided into separate groups, depending on their origin, on the possibility of observing them, or, finally, on their significance for human activity.

It is considered feasible to divide all celestial bodies first of all into the three following fundamental groups: 1)natural celestial bodies belonging to the solar system, 2) artificial celestial bodies, and 3) celestial bodies outside the solar system.

Clearly, such a division is quite conditional and no precise boundaries can be erected between these groups, so that the same celestial body may sometimes belong to two, or even three groups simultaneously.

In fact, on the one hand, it is difficult to establish limits for the solar system. On the other hand, it may happen sometime that a space ship will leave the solar system and make a journey into other stellar worlds. Finally, it is quite probable that in the far reaches of cosmic space other earths exist, populated by some kind of intelligent beings who also know how to build their own artificial celestial bodies. But in any case we can consider that a celestial body being considered belongs to only one of the three groups indicated, at least for a certain time interval.

It is obvious that each of these three groups, in its turn, may be subdivided into subgroups, and the latter even into individual parts, and the motion of the heavenly body belonging to this or that subgroup (or to its part) must often be studied by one of its own individual methods. Thus, we can distinguish among the bodies of the first group the large planets, their satellites, especially the moon, the small planets, comets, meteors, etc.

We know that these subgroups can be broken down into even smaller parts. For example, the large planets can be broken down into internal and external, satellites can be broken down into large and small or into near and far, etc. Artificial celestial bodies are divided into artificial earth satellites (or satellites of some other celestial body), interplanetary vehicles, carrier rockets etc. Artificial satellites are divided into or on partial groups, depending on the mission assigned them, or on which planet they orbit. In exactly the same way, interplanetary vehicles can be divided into types, depending on the planet of destination or on some other purpose.

Finally, we must include among bodies of the third group individual stars, then systems consisting of two or of several stars, systems consisting of large or of very large numbers of stars, and finally, galaxies, galaxy systems etc.

It is also worth noting that each of the groups or subgroups of real (that is, actually existing or actually observed) celestial bodies must include bodies not yet discovered, but guessed or conjectured, bodies which do not yet exist, but, for example, are being designed, and also fictitious or imagined bodies.

In reality, new small planets and comets are constantly being discovered, and sometimes new or natural satellites or even new large planets are being discovered. Moreover we are constantly building ever newer and newer artificial celestial bodies, which will then proceed into cosmic space with a previously established program. New types and forms of artificial celestial bodies are being designed which, perhaps, will never reach cosmic space, or will appear there very slowly, but the possible motions of which it will also be necessary, and useful or interesting to study beforehand.

In general, celestial mechanics must be prepared beforehand, and perhaps, a long time before the time when human practice (understood, of course, in the sense of this word) will require all information possible on the motion of one or another celestial body.

Imagined celestial bodies actually do not exist, that is, bodies do not exist to which we, for one or another reason, assign arbitrary initial conditions or which are ideal bodies, having, to one degree or another, mechanical properties far removed from reality (for example, material and points, idea spears or ellipsoids, absolutely solid bodies, ideal liquids etc.)

Practically speaking, we sometimes study in celestial mechanics motions of only such fictitious bodies, because we do not have the opportunity not to define absolutely precisely initial conditions (even those corresponding to a singular moment of time!), which are the basis of every calculation of motion, and we do not have the opportunity not to consider in the statement of a problem the entire diversity of physical and mechanical features, inherent in every actually existing celestial body.

But, without forgetting for a moment these circumstances and bearing in mind that all astronomical observations, measurements and calculations are always to one degree or another approximations, we can usually consciously identify such imagined celestial bodies with realities which we observe and many of which we can call by name or distinguish from others by some features.

There is another category of imagined celestial bodies (both natural and artificial), not related to any actually existing objects of nature, but simply thought up for the convenience of solving a proposed problem, for convenience in working out a new method, or in general for investigating all possible cases, which, perhaps, might sometime be encountered in infinite space. Such, for example, are the material points which are objects for various mathematical studies in a problem of three or more bodies, such are the ideal liquids in the three of equilibrium forms, and such is a certain continuous medium, which is neither a gas, a liquid, nor dust, but which exerts some effect on bodies moving in it or itself undergoes some effect from other objects, etc.

It seems to me that the actually existing celestial bodies, which are objects of celestial mechanics, are many times less than imagined bodies (simply because we can think up whatever we like and as much as we like). Therefore, a very great amount of research in the field of celestial mechanics is related to these very imagined or fictitious bodies and, unfortunately, the greater part of them remain without practical application, being the results of solutions to mathematical problems and the development of techniques "for any eventuality." However, celestial mechanics as a part of astronomy is a science, destined from its very beginning for the solution of the most practical problems which are simply necessary for most aspects of the practical function of mankind.

But the actual solution of practical problems quite often, encounters significant, and sometimes insurmountable difficulties, the result of which we are always forced to take the path of simplifications or approximations or to find some kind of detour. In the process of finding a suitable, necessary solution of a practical problem, we often unwillingly clash with other problems, sometimes far removed from that with which we are directly concerned, but for which we can more or less find a simple solution.

We solve such simplified problems, and the result obtained is often unsuitable or poorly applicable in practice, and it must be used at best only for training and purely methodological purposes, or yielded to an archive and held in reserve, hoping that someday, perhaps, it will be useful.

And actually, as we all well know, many times in the history of science it has happened that some theory which at first was completely abstract and impractical has suddenly acquired an important new practical significance.

The most impressive example in celestial mechanics of a similar case is the classical problem of two stationary centers, which, from the time of Euler, has often been the subject of many detailed and comprehensive mathematical studies, but which in the course of two hundred years had absolutely no practical application.

Incidentally, it does not at all follow from what has been said that this problem could not earlier have had an astronomical application. In fact, it was often demonstrated earlier that certain solutions of this curious problem, integrated to termination in quadratures, could have been used as a first approximation if only in the theory of the motion of certain asteroids, considering the motion determined by the formulas of this theory as unperturbed, instead of the Keplerian,and then using the normal procedure of the variation method of arbitrary constance to calculate the perturbations. Needless to say, not a single attempt was ever made to use this mathematically well-developed theory in astronomy and the calculating formulas were never previously derived.

This circumstance can probably be explained, on the one hand, by the relative complexity of the formulas, which require the use of the nontheless cumbersome apparatus of the theory of elliptical functions, which are less suitable for astronomers than are trigonometric functions, and, on the other hand, by the known inadequacy and inaccuracy of observational material pertaining to the small planets and comets.

The situation in this field has changed sharply in very recent years, next to the research in both the Soviet Union and in the United States which showed a fine opportunity for actual application of the theory of the two stationary centers problem in astrodynamics for studying the motion of an artificial satelite in the gravitational field of the earth. Calculating formulas have now been derived which define "unperturbed" motion, and the methodology of calculating various types of perturbations has been worked out.

The new, or rather rejuvenated theory of the two stationary centers problem may, of course, also be used for studying the motions of satellites (both natural and artificial) of other planets, as well as for building a new analytical theory of the flights of space ships to the moon, to Mars, or to Venus.

It goes without saying that these new studies pertain only to fictitious artificial celestial bodies, to hypothetical artificial earth satellites or to hypothetical interplanetary vehicles, but the new formulas, apparently, have not yet been applied to the actual calculations of the motions of specific, already existing satellites.

Having enumerated the objects being studied in the field of celestial mechanics and having given them a certain classification (which is, of course, not at all obligatory and is introduced here only temporarily for greater ease of presentation), we shall now spend some time on the consideration of the goals of studying the motions of these objects in our science.

In this regard, different scientists at different times have expressed different opinions. Poincare, for example, thought that the finite goal of celestial mechanics was only to verify the validity of Newton's universal law of gravitation. Others suggested that the only purpose of this field of astronomy is to calculate and to compile tables of the motions of the moon and the large planets. Still others saw the idea of celestial mechanics as the opportunity to resolve problems of a cosmogonic nature, and a fourth group in the development of mathematical methods of intergrading differential equations of the problem of three and more material points etc.

From our point of view, the object and purpose of celestial mechanics is the universal study of the motion of any celestial body, regardless of the forces which influence. We understand under study of motion the search for opportunities to obtain sufficiently complete information on the position of a body, its speed and its orientation relative to some other body or to a system of other bodies.

Thus, the study of the motion of any celestial body includes, in our opinion, first the determination of common properties of its forward and rotary motions (and also the deformations which change the form and distribution of the matter in the body), and then finding ways which permit us to determine numerical values of parameters which characterize these motions at some arbitrary moment of time, both finite and as far removed as one wishes either in the future or in the past.

The result of such study can be, for example, the compilation of tables of motion which permit us to rapidly and easily find numerical values of coordinates and velocities and to use the information obtained for solving diverse practical problems. Knowledge of the common properties of motion or, at least, some of them may create additional opportunities for more complete study of the nature in general, and in particular, for studying the past and future history of the earth and the solar system.

We note here that in the study of the motions of actually existing, and also of contemplated celestial bodies, we constantly use certain numerical characteristics, the knowledge of which is necessary in like manner both for establishing common laws of motion and for compiling tables. Such characteristics are, for example, the mass of the planets and the sun, average distances of the planets from the sun and the periods of their rotation, the dimensions, mass and moments of inertia of the earth and the moon etc. These characteristics are known only approximately, sometimes very roughly, and all inaccuracies in their values inevitably cause corresponding errors in the positions and velocities of celestial bodies, especially those far removed from the initial moment of time.

But, knowing how to calculate the motions of celestial bodies and to determine their position from observations, we have the opportunity to combine the variable parameters of the bodies with certain of the above-mentioned characteristics (which we consider or agree to consider constant) thus, we may, from the observed motions of celestial bodies, generally determine these characteristics or at least refine their numerical values and such a problem may be called, if one wishes, the reverse problem of celestial mechanics.

We should also include in the reverse problem of celestial mechanics the problem of finding and determining previously unknown forces, the effects of which may appear when comparing calculated coordinates with those observed. However, this latter problem has been barely defined and if it is ever resolved, it will be with the aid of a number of hypotheses stated previously on the laws of the effect of such unknown forces.

In concluding this section we shall make several more short comments pertinent to the subject of astrodynamics, under which is understood the usually new field of celestial mechanics, which, however, has no definitely established content relative to the motions of artificial celestial bodies.

Incidentally, this name, I have in mind the term "astrodynamics" is clearly inaccurate and always requires additional interpretation. Equally inaccurate are other names which find favor among specialists concerned with the study of the motions of artificial celestial bodies. Some of these names are: cosmodynamics, rocket dynamics, celestial ballistics, cosmic ballistics, the dynamics of satellites and so on. However, due to the lack of another, more accurate name, I prefer to use the term "astrodynamics" simply for a short representation of the theory of the motion of artificial celestial bodies.

I understand astrodynamics to be a subject of the comprehensive study of the motions of all possible types of artificial celestial bodies, both already existing, and either conjectured or imagined, and even those which have ceased to exist, but observations of which made in their times still remain.

The goals of astrodynamics and celestial mechanics partially coincide. In any case, this concerns already launched and then uncontrolled artificial celestial bodies, which, once they reach cosmic space, continue to move for a time under the influence of natural, cosmic forces. Moreover, other goals emerge in astrodynamics, including those in which a designed object will continue to move after its launch in a completely determined, fixed manner, that is, its motion has previously fixed properties. This requires additional calculations of initial conditions achieved at launch, as well as design characteristics of the object, which is a subject of concern for engineers and technicians. These additional calculations, sometimes called by specialists as planning of orbits, substantially distinguish the problem of astrodynamics from classical celestial mechanics, in which we do not have the opportunity (or, rather, which we do not yet have) neither to assign arbitrary initial conditions to real celestial bodies, nor to introduce any changes in their "design".

But this does not at all exhaust the goals of astrodynamics. In fact, a projected orbit or the generally planned motion (because not only forward but the rotary motions of artificial objects are planned), due to various inaccuracies, errors, and miscalculations, always deviate more or less from the desired program, as a result of which it is usually necessary to correct this motion "in flight," adding to natural, constantly active forces those additional and artificial ones which are active for a very short or limited time. Such correction is the first step on the way to realizing in the future freely controlled cosmic flight, when a future cosmonaut, like the modern pilot of an aircraft, will confidently fly his spaceship, using the engines installed on board, and, of course, always bearing in mind natural forces, intelligently using their compacities in the proper manner.

2. Statement of the Problems of Celestial Mechanics and Methods of Solving Them

We shall briefly outline how the problem is stated on the motion of any celestial body or any system of celestial bodies. First, the real problem changes more or less according to an artificial scheme, which we nevertheless try to make as close as possible to reality or, at least, to defend by more or less plausible, although rather nebulous, judgments.

This initial stage on the path of scientific research is always completely obligatory, because we clearly acknowledge that it is not even possible to dream of any complete reflection of reality by the still rather primitive means at out disposal. The simplifications, theories, and substitutions which we make are stipulated by the unusual complexity of nature, which we always try to guess and comprehend, but from a complete knowledge of which we are still far removed.

There is simply very much that we still do not know, and, perhaps, much that we do not even suspect within the limits of our solar system or even of the earth. We do not know fully how celestial bodies are constructed, what form they have, and we also do not know all the forces acting in cosmic space and which are the causes of the motions of material objects located in it. Therefore, we are forced to substitute real celestial bodies with imaginary ones, suitable for systematic research, and of all active forces to consider and give attention to only certain ones (for example, gravitational) which we may, at least, measure in some way or those which we can somehow discover when they appear.

By replacing real celestial bodies with imaginary ones, and real forces of nature with simplified schemes, and by then applying common laws of mechanics (thought up, however, by ourselves or by our predecessors), without at the same time guaranteeing their absoluteness or immutability, we finally gained the opportunity to state a mathematical problem on the motion of bodies which interest us and to write, in accordance with accepted prerequisites, several differential equations for the problem of celestial mechanics which we are considering.

These equations are usually presented in the following form:

$$\frac{dx}{dt} = X(t \mid x \mid \mu), \qquad (1)$$

where x denotes the aggregate of a finite number of variable parameters, a knowledge of the numerical values of which determines in a certain, previously selected system the computation of the position, speed, and orientation of bodies being considered, and small μ is the aggregate of constant parameters (or of those which we agree to consider constant), which determine the established scheme of our system of celestial bodies.

By assigning the additional initial value $x^{0} = x(t_{0})$ of variable vector x, independent of the parameter of μ; we bring any basic (or direct) problem of celestial mechanics to functions satisfying the equations of (1) and which accept given initial values at a given initial period.

If we can somehow integrate the equations of (1), then we shall receive as a result a general solution (or a general integral) which is written analytically in the following manner:

$$F(t \mid x \mid \mu \mid C) = 0, \tag{2}$$

where C denotes the aggregate of the arbitrary constance of integration.

Determining these constants from the given system of initial conditions x^{0} and resolving, when this is possible, equations (2) relative to variables of x we obtain a general analytical solution of the form

$$x = f(t \mid x^{0} \mid \mu), \tag{3}$$

from which we may also find numerical values of our variables for certain, at least, moments of time and establish the properties of these variables as functions of time and of the initial values, that is, we can determine the properties of the motion being studied, generally speaking, in a certain interval of time and for a certain domain of values for values of x^{0}.

An ideal case is that in which a solution of type (3) is suitable (both theoretically and practically) for all moments of time and for all possible values of small x^{0}. Then, on the basis of the given initial values, we can follow a motion as far as possible in the future or in the past, and can also establish what the values of x^{0} must be, so that the motion possesses the assigned properties; for example, so that it is periodic, conditionally periodic or asymptotic, or so that it does not leave a given area, or, on the other hand, so that a moving body may recede into infinity etc.

Moreover, having the solution of (3) and letter form, we can also obtain the solution of the reverse problem, which, as we already noted, is included in the determination of numerical values of parameters (or, at least, of some of them), having from observations a number of numerical values of the variables of x, which correspond to a number of separate moments of time.

In certain problems of astrodynamics related to planning motions having the required properties, we must simultaneously determine part of the initial conditions and parameters or solve boundary-value problems, in which certain of the values of x are given for a single moment of time, and other values are given for another moment of time. It is also possible to solve such problems in principle, when an analytical solution of our equations in the form of (3) has been found.

Methods which permit us to obtain such an analytical solution in letter form are, at the present time, usually called analytical methods, and the aggregate of such methods is appropriately called analytical celestial mechanics.

It should be recalled that 50-60 years ago the term "analytical celestial mechanics" was used, which was understood to be the investigation of fundamental problems (in essence, almost exclusively problems of three material points) in the field of complex numbers and with application of the methods of the theory of functions of complex variables. However, this concept of analytical mechanics must now be considered obsolete and hardly accurate. In fact, the apparatus of the theory of analytical functions is now so well developed and has become so commonly used, that it is impossible or unsuitable to separate a class of problems in which it is used exclusively. Therefore, we should consider that analytical methods of celestial mechanics at present can and must use all the achievements of mathematics, setting as its goal to obtain, no matter what the cost, the solution of equations of (1) and letter form.

Unfortunately, celestial mechanics is still very far away from achieving this ideal goal (but towards which we must still always strive), because the precise and complete integration of differential equations of motion is almost always unattainable.

We must therefore usually be satisfied with approximate integration, which yields some approximate solution, for the variable values of x or certain more or less simple letter formulas, suitable for calculation, at least, in a certain time interval and in a certain domain of initial values and constance of perimeters.

The methods used in celestial mechanics for finding approximate letter solutions of differential equations of motion are quite numerous and diverse, but all of them may be divided into two basic groups, which we shall characterize in general terms in the following manner.

We shall include in the first groups all methods which give an approximate solution of the most basic equations of (1), which we shall temporarily call here precise equations (within the frame work of the stated problem and the selected scheme). Such an approximate solution

$$\bar{x} = f(t \mid x^0 \mid \mu), \qquad (4)$$

of course, does not satisfy the equations of (1), so that we have only

$$\frac{d\bar{x}}{dt} \approx X(t \mid \bar{x} \mid \mu), \qquad (5)$$

and, therefore, the question always arises about estimating the absolute value of difference between the accurate solution of x and the derived approximation of x, which is in itself a difficult mathematical problem.

The basic mathematical apparatus of the majority of methods of this first group is an infinite series of different form and structure, depending either on the problem being considered, or on the domain of initial conditions, which play the main role in the given problem, or on the values of constant parameters, which are included in the differential equations.

Thus, a trigonometric series which has secular and mixed members and which is disposed according to the extent of the masses of the planets has long been used for the problem on the motion of a system of material points similar to the solar system. A series also permits us to represent the motions of the large planets with a sufficient degree of accuracy and within a certain limited interval of time. However, all classical series of this type are divergent, and, therefore, are unsuitable for establishing the basic properties of the motion of the solar system. In particular, such series are inadequate to solve the problem on the stability of the solar system. All attempts to obtain a purely trigonometric series suitable for all values of time have been unsuccessful until quite recently.

The first important results in this field were achieved quite recently by the prominent Soviet mathematicians A. N. Kolmogorov and V. I. Arnol'd, who solved the problem on the motion of a system similar to our solar system, using a purely trigonometric series, suitable for every moment of time at sufficiently small values of the masses of the planets and under almost all initial conditions (of course, for those also under which constant energy remains negative). However, as the authors of this remarkable mathematical study themselves explained, the results which they obtained cannot solve the problem on the stability of the present solar system, because the higher limits of the masses of the planets, at which the series obtained would remain convergent, have not yet been found. Moreover, it would be extremely difficult to establish to which field (a convergent or problematical) the real initial conditions of the solar system would be pertinent. Therefore, the above-mentioned mathematical studies remain, for the time being, inapplicable for specific astronomical problems and still require a great deal of additional work.

Besides series with trigonometric members, series have for a long time been used in celestial mechanics which are arranged either directly according to rates of time or according to rates of some other, more convenient value, which plays the role of an independent variable. Such series have the advantage that, as they are being built up, the domain of initial values plays no substantial role, and, moreover, their convergence can always be easily established (even the absolute, at least, in a certain, if small, interval of time). However, such series are also hardly suitable to explain the common properties of motion and are usually used only for numerical calculations.

In many cases it is possible to construct a series arranged according to degrees of the independent variable and which are absolutely convergent at all values of time, such, for example, as the Zundman series in the problem of three bodies, but their convergence is so slow that it completely deprives them of any practical use.

Theories of motion are now being developed, based on the implication of a series of polynomials, and also on the use of integration processes. These trends are quite promising and permit us to expect many new useful and interesting results.

It should be noted that if an infinite series is used which satisfies differential equations and which is convergent even in a certain interval of time, then such series determine a precise solution of our equations. An approximate solution can be obtained by substituting an infinite series with certain aggregates of their selected members independently of the fact whether they converge or diverge, provided that the differential equations are satisfied formerly. The problem on the actual use of such approximate solutions is, of course, solved by comparing computed and observed values of the values of x, which represent these series.

We shall turn to a second group of methods which give an approximate solution of the problem of interest to us. The aim of these methods is to obtain a precise solution, but not of the same precise equations of (1), but of certain approximate or simplified equations obtained from the equation of (1) by replacing their right members with functions which are barely distinguishable from the given ones, but which make it possible to completely integrate the new equations. If we can build up such equations as

$$\frac{dx}{dt} = X^*(t \mid x \mid \mu), \qquad (6)$$

where

$$X^* \approx X,$$

then their precise solution

$$x^* = f^*(t \mid x^0 \mid \mu) \tag{7}$$

is a certain approximate solution of the equations of (1).

Such a solution may be compared with the approximate solution of precise equations, and also with observations (when they occur). And if such comparison produces favorable results, then they can be used along with the approximate solution of $\tilde{x}$ of the precise equations of (1). The advantage of the methods of the second group is that for the precise solution of x^* of the approximate equations of (6), it is sometimes possible to obtain greatly simplified formulas than those provided by the methods of the first group.

It may happen that solution of x^* defines the motion being studied with insufficient accuracy, especially at large time intervals. Then the solution of x^* may be considered simply as the first approximation, defining a certain unperturbed motion, and one may then search for subsequent approximations, for example, by applying to the formulas of (7) the classic method of the variation of arbitrary constance or some other method. A solution more accurate than x^* is obtained in this way, and which may be absolutely precise if the mathematical process of approximations can be brought to an end and if its convergence to the solution of system (1) can be proved.

Another advantage of the methods of the second group is that in a case when the solution of x^* is expressed by finite formulas, it is convenient to study these formulas, and, thus, to establish certain important properties of the motion being studied. For example, the second group includes methods based on the study of Keplerian motion as unperturbed, methods based on the study of the first approximation of the problem of two stationary centers, methods of averaging the equations of (1) etc.

The presented division of analytical methods into two groups is, of course, quite conditional and it is quite difficult to draw a strict boundary between these two groups, because methods are possible which are not related to either the first or the second group.

The common goal of analytical methods of celestial mechanics is the possibly more complete study of the motion of a celestial body of interest to us from a quantitative, as well as from a qualitative aspect. However, as follows from what had been said above, analytical methods usually give only a certain approximate solution, suitable for numerical calculations inside a certain interval of time, but which are unsuitable to express the common properties of motion, especially over very large intervals of time.

Therefore, along with analytical methods in celestial mechanics, qualitative methods exist, the goal of which (as Poincare defined) is the study of common properties of motion by the investigation of differential equations of (1), but without applying any approximate methods of integration. The basic qualitative methods developed by Poincare, Lyapunov, and their numerous successors, proved quite effective for the solution of simple problems and rapidly penetrated from celestial mechanics to other fields of knowledge (physics and engineering), where they have been of substantial benefit.

As regards celestial mechanics itself, unfortunately, these methods have as yet given very little. This is explained by the complexity and awkwardness of the equations of (1) and by their high order. In any case, the problem of the stability of the solar system has also not been resolved by qualitative methods, and the theory of periodic solutions has also not yet found sufficiently broad application.

The methods of the theory of the stability of motion, created by the appointive and later developed, mainly, by scientists of Soviet schools, also remain hardly applicable in celestial mechanics and are used only in certain of the simplest model problems, hardly related to specific problems of the study of motions of real celestial bodies.

Finally, I shall again touch upon numerical methods of celestial mechanics, or, more precisely, methods of numerical integration of differential equations of (1), which we shall continue to call for the time being precise equations. We know that such methods originated in celestial mechanics in order to find isolated numerical values of functions satisfying the equations of motion. These methods have come into use also in other fields, where they have been widely disseminated, especially after the appearance of high-speed computers, which are now also used for solution of problems of celestial mechanics.

At first the results of numerical integration were compared either with the results of calculations according to precise or approximate analytical formulas, or directly with observed data. It was thus established that numerical integration yields reliable results, at the least, at rather small intervals of time, and it may be considered a quite suitable method of studying motions.

However, it should not be forgotten that numerical methods give only an approximate solution of the equations of (1), corresponding to completely define numerical values of initial data and constant parameters. It is true that, with the help of high-speed computers, numerical methods make it possible to solve even more complex problems, as, for example, to obtain a set (of final) solutions, corresponding to a certain domain of initial conditions and parameters, and the subsequent study of numerical results and graphs obtained. Such mass calculations are presently being used successfully in astronautics in order to calculate the motions of artificial celestial bodies and to resolve many special problems of astrodynamics. Such calculations are even being carried out in celestial mechanics by Stremgren and his students and followers in order to explain the types of motions in a limited problem of three bodies. These calculations were made manually, because no auxiliary calculating devices (except for the simplest calculating machines) existed at that time. These calculations developed, for example, several families of periodic solutions of the problem, which, unfortunately, remain without any implication in celestial mechanics.

Numerical methods have now become so wide spread that sometimes conviction is established in their universality and in the possibility of substituting them for analytical and qualitative methods. There has also arisen the tendency to call the result of numerical integration a precise solution and to verify the reliability of some analytical method by comparing its results with numerical integration. Without opposing such comparsion, I would simply like to note that it is also useful for numerical theory, which can in no way serve as a standard of absolute reliability and accuracy.

I would also like to note that in recent years a trend has begun to develop of combining analytical and numerical methods, which leads to the formation of a letter analytical theory and then to subsequent calculation using these formulas with the aid of high-speed computers. Such a trend should be recognized as the most effective and promising.

3. Unsolved Problems of Celestial Mechanics and a Look at the Future

We outlined above a statement of the problems of celestial mechanics and noted methods of solving them. We must say, however, that the majority of problems of our science remain unsolved or, in any case, poorly solved. In fact, even the simplest fundamental problems cannot always be considered worked out to the end, at least from the point of view of the possibilities of their practical application.

This state of affairs pertains most of all to the theory of unperturbed Keplerian motion, which still remains one of the most important and the results of which are widely used directly as the first approximation in any theory of perturbations. The fact is that motions quite close (at least over a long period of time) to circular, Keplerian motions are encountered almost exclusively in classical celestial mechanics. Therefore, it was sufficient to expand the coordinates of elliptical motion according to the order of eccentricity, and this method has been well developed. At the present time we must often deal with orbits of great eccentricity and with hyperbolic orbits. It is true that expansions of elliptical motion into a Fourier series converge (but not absolutely) for all values of eccentricity, although it is known that with increase of eccentricity, these series converge more and more slowly, which, of course, presents many difficulties when using them. The matter becomes even worse in the case of hyperbolic motion, because no expansions for this type of motion, to my knowledge, exists.

Thus, significant difficulties arise when developing a theory of perturbations for hyperbolic motions, when the osculating orbit is a hyperbola, if only with an eccentricity hardly distinguishable from unity.

Another difficulty is that classical series are not always suitable for elliptical motion and in a number of cases it would be desirable to have expansions which do not contain trigonometric functions, but those based on some other functions.

At the present time, expansions are now beginning to be used which are coordinates of Keplerian motions and which are arranged on the order of a certain independent variable, performing the function of time. Such functions for which the type of motion is indistinguishable, that is, they are identically applicable both for elliptical and for hyperbolic motions, are especially advantageous. It is my opinion that such work should be continued and developed, especially if we keep in mind their application to development of new theories of perturbed motion.

Another of the simplest problems of celestial mechanics, still in a most primitive state, is the well known problem of two stationary centers, which was already mentioned above. It is now clear that the theory of this problem may have an important practical significance, in providing a better (in comparsion with the Keplerian) first approximation and having a greater collection of orbits, each of which can be unperturbed or intermediate for the corresponding theory of perturbed motion. In order to do this, we must derive from the complete system of first integrals of this problem formulas giving suitable expansions of coordinates of a moving point. Certain formulas of this type have already been worked out for a narrow domain of initial values, but they do not encompass all necessary cases, and, therefore, such work must be expanded and continued. Apparently, it is also appropriate here to use expansions on the order of some suitable variable, if one wishes to avoid the cumbersome apparatus of the theory of elliptical functions.

I shall note from the general, simplest problems of celestial mechanics and other limited problem of three bodies, which has long been the subject of the multitude of diverse studies, but which has no common solution to the present time. Almost all previous studies in this field were mainly of a qualitative nature and were related almost exclusively to plain and circular problems.

Meanwhile, theories of motion in the space elliptical and hyperbolic limited problem are needed both for applications to already existing celestial bodies (natural and artificial), and for purposes of the more remote future. In order to formulate such theories, new, as yet uninvented mathematical means are probably needed, because the methods of modern mathematics are, apparently, completely insufficient in this case.

It seems to me that such a more general, limited problem of three bodies, rather than a plain or circular problem, is one of the most important and needed problems of celestial mechanics at the present time, which requires the most rapid solution. Again, I have in mind the analytical solution of the problem, but because of the great difficulties which arise in this area, it is also necessary to conduct broad qualitative studies, without at the same time ruling out numerical surveys with the aid of high-speed computers.

The general problem of three bodies, when the masses have an approximately identical order, is of no special interest for the solar system, but, undoubtedly, is extremely important for the study of star systems. This field of celestial mechanics is an embryonic state, but I feel that the time has come (an opinion shared by many other specialists) to begin comprehensive and serious research.

It is now difficult to formulate any definite plans for developing this completely new field of celestial mechanics. It would be useful to arrange for this purpose a conference with the participation of mathematicians and specialists in celestial mechanics, and also of star astronomers, in any case those who are concerned with the problems of stellar dynamics[1].

Thus far, the theme of this article has been the general theories of motion, related to imaginary celestial bodies, but which could be applicable to some specific natural or artificial body, such as a natural satellite, asteroid, a comet, an artificial satellite, and interplanetary vehicle, moving under the influence of natural forces of nature. A multitude of different problems emerges here, for which we must have an analytical apparatus worked out in sufficient detail, because a single numerical integration is clearly incapable of providing everything necessary for their solution.

These problems become even more complicated when it becomes necessary to consider the effect of additional forces under our control, along with natural forces (for example, mutual forces of gravity). Although such problems have until now been foreign to celestial mechanics, the time has come to devote attention to them. I have in mind problems on motion in a gravitational field, with an additional small pool or with a solar sail etc.

Moreover, in connection with the constantly developing technology of interplanetary flights, one of the most important problems of modern celestial mechanics is the clarification or even a complete rearrangement of the classical theories of the motion of the moon, of certain satellites and of the entire family of the large planets. It is now possible to conduct such research within the framework of classical schemes, that is, by using differential equations of type (1), but by posing the problem of formulating and analytical theory, that is, obtaining such a letter solution of these equations which would make it possible to determine the positions and velocities of these natural bodies with accuracy, is already necessary for the best calculations of all kindsof interplanetary flights.

Astrodynamicists can indicate (and actually demonstrate) which theories of motion need to be developed first and what accuracy they should guarantee. The very formulation of an analytical theory is the affair of celestial mechanics and the corresponding specialists must concern themselves with this problem. Such work has already begun for the large planets in the Institute of Theoretical Astronomy, where a new analytical theory, apparently capable of satisfying modern requirments is being formulated. Awaiting its turn is the formulation of a theory of motion of the satellites of the large planets and first of all those

[1]The first conference of this type was held a year after presentation of this report (in September, 1966) at ITA(The Institute of Theoretical Astronomy).

of Mars, which will undoubtedly play an important role in the near future in calculations of flights to this planet and its mastery for the needs of mankind. Of course, while conducting this research, we should not forget the problems of celestial mechanics itself, in particular the problems on the motion of planets and satellites during very large periods of time.

Special attention must be given to the heretofore incompletely solved problem on the stability of the solar system, that is, of the real solar system with actual values of mass and accurately known initial conditions. It is known that certain theoretical astronomers and certain mathematicians at the present time are seriously concerning themselves with this problem and we may hope that in the near future, this problem, so brilliantly pushed forward by V. I. Arnol'd, will finally be completely resolved.

A more pertinent, but no less important problem for celestial mechanics is the formulation of an effective theory of periodic solutions of differential equations of celestial mechanics and first of all of equations of the limited problem of three bodies, both circular and elliptical. I have already noted above that this theory until now had been subjected almost exclusively to mathematical development and, in essence, had absolutely no practical applications. Several fortunate exceptions from what has been said are only implications of the theory of periodic solutions to the moon and to certain satellites of Jupiter. I shall not dwell on this important problem, I shall note only that the periodical solutions in the theories of Poincare and Lyapunov are solutions only close to other, already known periodic solutions. The problem itself on the search for periodic solutions of differential equations, when nothing else is unknown about them, remains completely untouched.

Along with the development of a theory of periodic solutions, which can have many implications even in astrodynamics, it is also necessary to continue development and improvement of the theory of the stability of motion, problems of which in the field of celestial mechanics are especially complex and difficult, and for the most part still remain unsolved even in the first approximation.

I shall make several additional comments on the statement of new problems. The equations of (1), which we agreed above to call precise equations of the motion of celestial bodies, are such equations only within the framework of an accepted scheme and previously made simplifying hypothesis. In reality, these equations are not precise and the actual motions of celestial bodies are controlled by some other equations, which we cannot even write and work out completely. In view of the ever increasing requirements for accuracy which celestial mechanics must guarantee, it is natural to oppose and even to try to solve a problem on those unequivocal errors which occur as a result of using the equations of type (1) and the problem on the effect of these errors on our calculations and the results obtained with their help.

As we have already said, we cannot work out absolutely precise equations, but we can pose the problem on the next approximation to reality, taking into account certain factors, which were rejected in classical statements of problems.

Thus, considering the theory of motion of the solar system, all bodies of which in the classical statement of the problem where replaced by material points, it is natural in the next step of approximation to consider these bodies, for example, as absolutely solid, having definite form and a definite internal structure. Thus, we come to a new problem, that of the forward-rotary motion of planets (it is also natural to pose such a problem for the moon), in which the differential equations of motion do not split up into equations only of forward and only of rotary motion, but determine both one and the other motion jointly in relation to each other. We can in the very same manner study and determine the effects of the shapes of celestial bodies on their forward motions and vice versa.

Considering a problem closer to reality than classical problems, described by equations of type (1), we will obtain new equations which can be written, similar to the equations of (1), in the following general form:

$$\left.\begin{aligned}\frac{dx}{dt}&=X(t\,|\,x\,|\,\mu)+R(t\,|\,x\,|\,\mu\,|\,\xi\,|\,\nu),\\ \frac{d\xi}{dt}&=\Xi(t\,|\,x\,|\,\mu\,|\,\xi\,|\,\nu),\end{aligned}\right\} \qquad (8)$$

where ξ denotes the aggregate of some other variables, and ν that of some other parameters. For example, in the problem already mentioned above on the forward-rotary motion of a system of absolutely solid bodies, ξ is the essence of variable, determining orientations of bodies in space, and ν is some of their dynamic characteristics.

If, as this is done in classical statements of problems, we throw out the small members of R in the equations of (8), we obtain the classical system of (1), which determines the aggregate of fundamental variables of x. Solving the classical problem by some method, we obtain its solution (precise or approximate), after which we can concern ourselves with finding the solution of the second system.

$$\frac{d\xi}{dt}=\Xi(t\,|\,x\,|\,\mu\,|\,\xi\,|\,\nu), \qquad (9)$$

in which the values of x, if they are actually included in equation (9), are already known functions of time. The solution obtained will be represented by the formulas of type

$$\left.\begin{aligned} x &= f(t \mid x^0 \mid \mu), \\ \xi &= \varphi(t \mid x^0 \mid \xi^0 \mid \mu \mid \nu), \end{aligned}\right\} \tag{10}$$

and the unknown, generally speaking, solution of system (8) can be presented in the form

$$\left.\begin{aligned} x &= F(t \mid x^0 \mid \xi^0 \mid \mu \mid \nu), \\ \xi &= \Phi(t \mid x^0 \mid \xi^0 \mid \mu \mid \nu). \end{aligned}\right\} \tag{11}$$

The fundamental problem of celestial mechanics at the next stage of its development is the search for a more general solution of (11) or at least finding estimates for the absolute values of the differences of F - f and Φ - ϕ.

Of course, this problem is not an easy one and the difficulties, with which we shall have to deal here, will undoubtedly significantly excede all difficulties of the classical problems. Therefore, the possibility of obtaining some valuable results in such a more general problem is undoubtedly still very remote, but nevertheless, this problem, even in certain of its particular varities, must even now, be correctly stated and methods for solving it must be thought out or at least contemplated.

We note that the problem of estimating the moduli of the differences F minus f and Φ minus ϕ or, in essence, a somewhat generalized and modified problem of the stability of motion, determined by the classical scheme relative to constantly active perturbations, that is, it is the same Lyapunov problem, only in a somewhat more general statement.

Therefore, the theory of the stability of motion is closely related to subsequent development of celestial mechanics and must also be developed and improved in a direction which can yield effective methods for solving its problem.