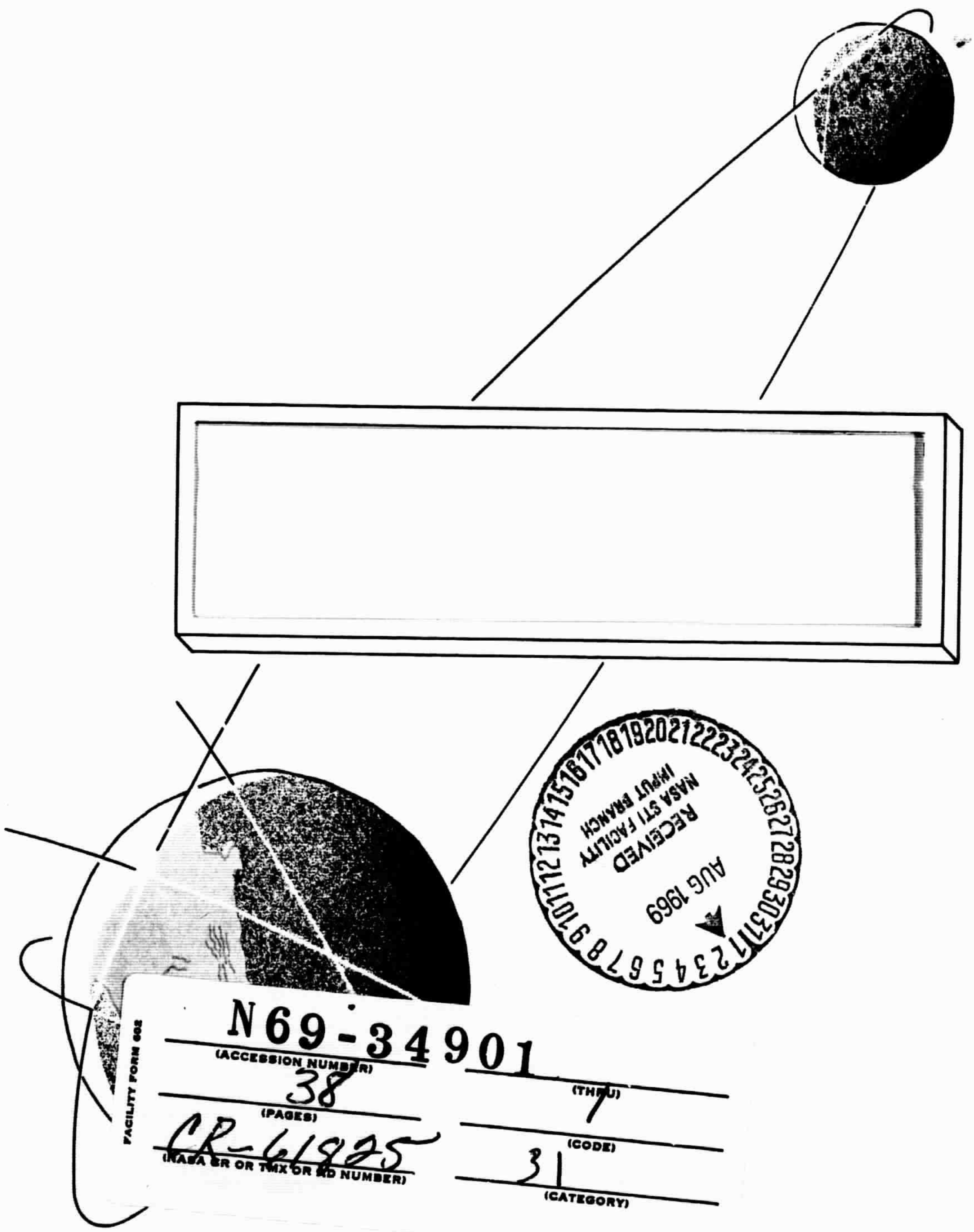

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SUBOPTIMAL COMPENSATION OF GYROSCOPIC COUPLING FOR INERTIA-WHEEL ATTITUDE CONTROL[1]

by
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Abstract

Two procedures are developed for the synthesis of inertia-wheel systems for three-dimensional attitude control. Both techniques compensate for inter-axis coupling due to the angular momentum of the inertia wheels. In addition, both procedures approximately minimize the integral of a quadratic function of system error and control effort and are suboptimal in a mathematical sense. The techniques developed in this paper are applied to the design of an attitude control system for a large axially symmetric spacecraft similar to the proposed Orbiting Astronomical Observatory. In a computer simulation, the suboptimal systems were shown to respond faster and more accurately than those designed by conventional transform techniques or by optimization procedures based on time-invariant approximations of the equations of motion. The suboptimal systems also had lower peak torque and peak power requirements than system designed by standard transform techniques.

Nomenclature

I_i = moment of inertia of the spacecraft about the B_i axis

J_i = moment of inertia of the inertia wheel rotating about the B_i axis

ω_i = total angular velocity of the spacecraft about the B_i axis

Ω_{B_i} = angular velocity of the inertia wheel rotating about the B_i axis relative to the spacecraft

T_{D_i} = sum of the disturbance torques about the B_i axis

ω = total angular velocity in terms of the body axes

ω_B = angular velocity of the spacecraft relative to the reference axes in terms of the body axes

Ω_R = angular velocity of the reference axes in terms of the reference axes

L = a 3×3 orthogonal matrix relating the reference axes to the body axes

x = the angular position and velocity of the spacecraft relative to the reference axes

A = matrix of the coefficients of the time-invariant coupling terms

C(t) = matrix of the coefficients of the time-varying coupling terms

u = control torque available from the inertia wheels

g(t) = summation of internal and external disturbance torques

SUBOPTIMAL COMPENSATION OF GYROSCOPIC COUPLING

FOR INERTIA-WHEEL ATTITUDE CONTROL

1. Introduction

In many attitude control situations, motor-driven inertia wheels may be preferable to gas jets as the primary source of control torque [1-3]. The most serious problem in the synthesis of such inertia-wheel systems arises from inter-axis coupling due to the angular momentum of the wheels. In many cases such coupling is considered to be negligible [1, 4-7],[4] and design procedures are based on time-invariant approximations of the equations of motion. In some cases this assumption is justifiable; however, in many cases inter-axis coupling can adversely affect system performance if ignored [3].

In the present study, two procedures are presented for the synthesis of inertia-wheel systems for three-dimensional attitude control. Both techniques compensate for inter-axis coupling due to the angular momentum of the inertia wheels. In addition, both procedures approximately minimize the integral of a quadratic function of system error and control effort and are suboptimal in a mathematical sense. The methods proposed are applicable to the design of attitude control systems for fine control (the correction of small errors).

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[4] Numbers in brackets indicate references listed at the end of the paper.

The procedures developed in this paper are applied to the design of an attitude control system for a large, axially symmetric spacecraft similar to the proposed Orbiting Astronomical Observatory. The resulting control laws are in physically realizable, feedback form. In a computer simulation, systems designed on the basis of the procedures outlined in this study are shown to respond faster and more accurately than those designed by conventional transform techniques or by optimization procedures based on time-invariant approximations of the equations of motion. Also the suboptimal systems have lower peak torque and peak power requirements than systems designed by use of standard transform techniques; consequently, the suboptimal systems could probably be built from smaller and lighter components.

2. Preliminary Considerations

Attitude control consists of applying torque to a spacecraft in such a way as to place and hold it in a specific angular orientation with respect to a three-dimensional frame of reference. In this study, the reference frame is assumed to have small angular velocity and acceleration. The flexibility of the spacecraft is considered negligible, and control torque is available about the three principle axes. The inertia wheels are assumed to be mounted at the center of mass of the spacecraft. Many important attitude control situations fall within the context of the above restrictions.

In Fig. 1, the axes denoted as B_1 , B_2 , and B_3 are assumed to be the principle axes of the spacecraft and are called the body axes. The axes labeled R_1 , R_2 , and R_3 represent the desired orientation of the spacecraft and are called the reference axes.

The exact dynamical equations for inertia wheel control are

$$T_{D_1} - J_1(\dot{\Omega}_{B_1} + \dot{\omega}_{B_1}) = (I_1 + J_{2/2} + J_{3/2})\dot{\omega}_1 + J_3\Omega_{B_3}\omega_2 - J_2\Omega_{B_2}\omega_3 + (I_3 + J_{3/2} - (I_2 + J_{2/2}))\omega_2\omega_3$$

$$T_{D_2} - J_2(\dot{\Omega}_{B_2} + \dot{\omega}_2) = (I_2 + J_{1/2} + J_{3/2})\dot{\omega}_2 + J_1\Omega_{B_1}\omega_3 - J_3\Omega_{B_3}\omega_1 + (I_1 + J_{1/2} - (I_3 + J_{3/2}))\omega_1\omega_2 \tag{1}$$

$$T_{D_3} - J_3(\dot{\Omega}_{B_3} + \dot{\omega}_3) = (I_3 + J_{2/2} + J_{1/2})\dot{\omega}_3 + J_2\Omega_{B_2}\omega_1 - J_1\Omega_{B_1}\omega_2 + (I_2 + J_{2/2} - (I_1 + J_{1/2}))\omega_1\omega_2 \,.$$

Terms of the form $J_i(\dot{\Omega}_{B_i} + \dot{\omega}_i)$ represent electrically induced torques and are used for attitude control. The total angular velocity of the spacecraft is

$$\omega = \omega_B + L\Omega_R \quad . \tag{2}$$

Since the control system is to be used to correct small attitude errors, the equations of motion may be simplified by use of standard small angle assumptions. Terms of second order and higher involving the angular position and velocity errors may be neglected. Furthermore, it is not uncommon for the moments of inertia of the spacecraft to be more than a thousand times as large as those of the control wheels [4,5]; consequently, the moments of inertia of the control wheels may be ignored when summed with the moments of inertia of the spacecraft. Under the above assumptions, (1) and (2) may be written as

$$\dot{x} = Ax + C(t)x + B\,u + g(t) \quad . \tag{3}$$

where B is a 6×6 matrix such that $b_{ij} = 0$ except $b_{22} = 1/I_1$, $b_{44} = 1/I_2$, $b_{66} = 1/I_3$. The functional form of the elements of matrix C(t) are not known a priori.

The value of the integral of a quadratic function of the system error plus a quadratic function of the control effort has been widely used as a measure of control system performance. Such a performance index is often analytically attractive, and for inertia-wheel attitude control, a quadratic cost functional also makes sense from a physical standpoint.

The suboptimal systems developed in this study reduce the angular position and velocity errors to zero rapidly, and also approximately minimize,

$$J(x_o, u, t_o) = \int_{t_o}^{\infty} (x' Q x + u' R u)dt \qquad (4)$$

where Q and R are positive-definite, diagonal, constant matrices.[5] A small value of this integral indicates that both the error and control effort are kept small during most of the control interval. The error should not be large since excessive overshoot is to be avoided. The control effort should remain small to prevent saturation of the inertia wheels (the amount of control torque available from the wheels is limited) and also to conserve the amount of energy used (the torque output of the inertia wheels, u_1, is proportional to the current supplied to the electric motors driving the wheels).

3. Development of Suboptimal Techniques

The optimization problem considered is the determination of the control, u, which transfers any initial state, x_o, to the origin for the system governed by (3) and also minimizes the integral performance index (4).

Three techniques for the solution of the above problem are the calculus of variations [8], Pontryagins minimum (or maximum) principle [9], and the Hamilton-Jacobi theory [10]. Each of these approaches gives enough information to determine the mathematically optimal control for the problem defined by (3) and (4); however, the Hamilton-Jacobi theory gives the

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[5] A prime denotes the transpose of a vector or matrix.

most direct approach to the determination of both optimal and suboptimal control laws in feedback form. This approach depends upon the minimization with respect to u of a scalar function (H) defined by

$$H(x,p,u,t) = x'Qx + u'Ru + p'Ax + p'C(t)x + p'g(t) + p'Bu \quad (5)$$

where p(t) is a vector of the same dimensionality as x . The optimal control

$$u^*(t) = -\frac{1}{2} R^{-1}B'p \quad (6)$$

minimizes H . A scalar functional, called the Hamiltonian (H^*), is obtained by the substitution of (6) into (5) and is given by

$$H^* = x'Qx - \frac{1}{4}p'BR^{-1}B'p + p'Ax + p'C(t)x + p'g(t) \quad (7)$$

In the Hamilton-Jacobi approach, p(t) is set equal to the gradient of a scalar function of state and time; that is[6], $p(t) = V_x(x,t)$, where V(x,t) is a twice-continuously differentiable function satisfying the partial differential equation

$$V_t + H^*(x,V_x,t) = 0, \qquad V(0,t) = 0 \quad (8)$$

Equation (8) is known as the Hamilton-Jacobi equation, and its solution, V(x,t), evaluated at x_o and t_o is the minimum

[6] $V_x(x,t) = \text{grad } V(x,t)$

value of the integral performance index (4).

Method I

In the attitude control problem, the analytical solution of (8) appears impossible. Thus it is necessary to develop procedures for generating control laws which are suboptimal (approximately optimal). The first procedure for suboptimal control consists of using the control system to eliminate the most substantial time-varying terms in the equations of motion. The resulting system is then treated as linear for purposes of optimization. Cannon [2] suggests a similar procedure but mathematical optimization is not attempted.

A portion of the control, denoted by u_c, is used to eliminate all time-varying terms in (3). From (3)

$$B\, u_c = - C\,(t)x - g(t) \tag{9}$$

The remainder of the control is denoted by u_L. Applying u_c to (3) yields $\dot{x} = Ax + B\, u_L$.

The control u_L is to be selected in such a way as to minimize

$$J = \int_{t_o}^{\infty} (x'Qx + u_L' R u_L)\,dt \quad .$$

Using (9), the Hamilton-Jacobi equation is

$$- \tfrac{1}{4}\, V_x'\, B\, R^{-1}\, B' V_x + V_x' A x + x' Q x = 0 \tag{10}$$

and $u_L = - \frac{1}{2} R^{-1} B' V_x$.

Kalman [10] has demonstrated that the quadratic function, $V(x) = x'Px$, is a solution to (10) provided P is a symmetric positive definite matrix such that

$$PA + A'P - P B'R^{-1}B P + Q = 0 \tag{11}$$

The control, u, is the sum of u_L and u_c.

The attitude response of the spacecraft is determined by the values selected for the weighting matrices Q and R. In the remainder of this work it is assumed that identical response is desired about each axis of the spacecraft. This is the case in many attitude control situations. Also, the above assumption considerably simplifies algebraic manipulations; although it is not a condition which must be satisfied before the suboptimal procedures developed in this study can be applied.

Method II

The second method for suboptimal control is an extension of a technique developed for nonlinear systems by Garrard, McClamroch, and Clark [11] and is based on a procedure for obtaining approximate solutions to the Hamilton-Jacobi equation. The control, u, is divided into two components, u_D and u_s, where $Bu_D = - g(t)$. Using u_D, (3) reduces to[7] $\dot{x} = Ax + \varepsilon C(t) x + Bu_s$. The control u_s is chosen to approximately minimize $J = \int_{t_0}^{\infty} (x'Qx + u_s'Ru_s)dt$. The

[7] The parameter ε has been introduced for notational convenience.

Hamilton-Jacobi equation is

$$V_t + V_x'[Ax + \varepsilon\, C(t)\, x] - \frac{1}{4} V_x' BR^{-1}B'V_x + x'Qx = 0, \quad V(0,t) = 0 \tag{12}$$

An approximate solution to the above equation may be obtained by treating $C(t)$ as a constant matrix. If $C(t)$ is assumed constant, V is a function of the state alone. The scalar function $V(x)$ is assumed to be represented by a power series expansion in x of the form

$$V(x) = \sum_{n=2}^{\infty} \varepsilon^{(n-2)} V_n(x) \tag{13}$$

Substituting (13) into (12) and equating powers of ε to zero gives

$$\begin{aligned}
&\frac{\partial V_2'}{\partial x} Ax - \frac{1}{4} \frac{\partial V_2'}{\partial x} BR^{-1}B' \frac{\partial V_2}{\partial x} + x'Qx = 0 \\
&\frac{\partial V_3'}{\partial x} Ax + \frac{\partial V_2'}{\partial x}\, C\, x - \frac{1}{2} \frac{\partial V_2}{\partial x} BR^{-1}B' \frac{\partial V_3}{\partial x} = 0 \\
&\qquad \vdots \\
&\frac{\partial V_n'}{\partial x} Ax + \frac{\partial V_{n-1}}{\partial x} Cx - \frac{1}{4} \sum_{\substack{k \geq 2 \\ \ell \geq 2}}^{n+2} \frac{\partial V_k}{\partial x} BR^{-1}B' \frac{\partial V_\ell}{\partial x} = 0
\end{aligned} \tag{14}$$

where $k + \ell = n+2$.

In order to determine $V_2, V_3, \ldots V_n$, the above equations must be solved successively. The first equation of (14) is identical to (10), and the solution is given by (11). The remaining equations in (14) are linear, partial differential equations; however, an exact, analytical solution appears impossible. Approximate solutions can be obtained by assuming $V_n = x'M_n x$ for $n \geq 3$, where M_n is treated as a constant, symmetric matrix. Substituting into (14) yields the following set of linear algebraic equations

$$
\begin{aligned}
M_3(A-BR^{-1}B'P) + (A' - PBR^{-1}B')M_3 &= PC - C'P \\
&\vdots \\
M_n(A-BR^{-1}B'P) + (A'-PBR^{-1}B')M_n &= \sum_{\substack{k\geq 3 \\ \ell\geq 3}}^{n+2} M_k BR^{-1}B'M_\ell - M_{n-1}C - C'M_{n-1}
\end{aligned}
\tag{15}
$$

where $k + \ell = n + 2$. These equations may be solved for M_n, and the control, u_s, is given as

$$u_s = - R^{-1}B'[P+ \sum_{n=3}^{\infty} \varepsilon^{(n-2)} M_n]x \quad . \tag{16}$$

M_n is a function of C, and C is a function of time. Consequently u_s contains linear terms with variable coefficients. The complete control, u, is the summation of its two components, u_s and u_D.

In deriving control law (16), a number of assumptions have been made, and convergence to the optimal control is questionable. In actual practice, however, only a few terms can be used in chcosing a suboptimal control law; consequently, convergence is not of primary importance.

4. Suboptimal Control of a Large, Axially Symmetric Spacecraft

In Fig. 1, the B_1 axis is to be aligned with the R_1 axis. The B_1 axis is fixed in the spacecraft, and the R_1 axis is non-rotating ($\Omega_R = 0$). The complement of the angle between the R_1 and B_3 axes, α_c, is called the pitch angle, and the angular velocity about the B_2 axis is the pitch rate. Similarly, the complement of the angle between the R_1 and B_2 axes, γ_c, is called the yaw angle, and the angular velocity of the spacecraft about the B_3 axis is the yaw rate. The angular velocity about the B_1 axis is the roll rate. The B_1, B_2, and B_3 axes are the roll, pitch, and yaw axes respectively.

Gas jets are used for coarse control, and inertia wheels are used for fine control. Gas-jet control is begun when the spacecraft leaves the last stage of its booster vehicle. The gas-jet system must reduce the pitch and yaw angles to 0.15 radians and must drive the roll, pitch, and yaw rates to 0.0005 radians/sec.

After the gas-jet system has sufficiently reduced the attitude error, inertia wheels are used to further stabilize the spacecraft. The inertia-wheel system must reduce the pitch and yaw angles to 0.003 radians and must drive the pitch and yaw rates to zero within 10 minutes following the end of gas-jet control. The roll rate must be held to about 0.0005 radians/sec. The moments of inertia of the spacecraft are 1000 slug-ft^2 about the pitch and yaw axes and 900 slug-ft^2

about the roll axis. The performance specifications outlined above are almost identical to those for the proposed Orbiting Astronomical Observatory [5].

A functional block diagram of the combined gas-jet and inertia-wheel pitch axis control system is presented in Fig. 2. During gas-jet control, the angular velocity about the pitch axis is sensed by a rate gyro, and the attitude error is determined by optical sensors [5]. The attitude error signal is limited in magnitude and added to the pitch rate signal. This signal is used to actuate the gas-jet. After the gas-jet system reduces the pitch and yaw angles to 0.15 radians and the pitch, yaw and roll rates to 0.0005 radians/sec., the mode control logic switches to inertia-wheel control. The inertia-wheel control logic regulates the torque output of the inertia-wheel and motor combination, and this torque is used to control the pitch angle and the angular velocity about the pitch axis. The motor is assumed to provide torque proportional to the actuating signal [12].

In the inertia-wheel control mode the angular velocity of the spacecraft becomes very small and cannot be measured by the rate gyro; therefore, the attitude error must be differentiated in order to obtain an approximate value for the angular velocity about the pitch axis. Differentiation may be accomplished by analog or digital means, and the value of the differentiated attitude error approaches the actual angular velocity as the attitude error approaches zero.

The gas-jets are used to desaturate the inertia wheel when its angular momentum becomes large. When the angular velocity of the inertia wheel reaches a given value, the gas jet fires and creates an angular velocity error about the pitch axis. The angular velocity error created by firing the gas jet is in the opposite direction to the angular velocity of the inertia wheel, and as the inertia-wheel system corrects this error, the angular velocity of the wheel is reduced. Further consideration of desaturation is beyond the scope of this study.

A block diagram of the yaw-axis control system is nearly identical to Fig. 2 except the limited attitude error signal is subtracted from rather than added to the yaw-rate signal. A block diagram of the roll-axis control system is presented in Fig. 3. This system regulates the roll rate, and in the gas-jet phase, the control system is actuated by the output signal from a rate gyro which measures the angular velocity about the roll axis.

Gas-jet control could be used exclusively for control about the roll axis; however, in order to conserve fuel, an inertia wheel is used to regulate small errors. The inertia wheel system exerts full torque to oppose roll rate errors larger than 0.0005 radians/sec, but no effort is made to control smaller errors.

The value of the roll rate is used in implementing control logic for both methods for suboptimal, inertia-wheel stabilization. Since small values of this angular velocity cannot be measured, the roll rate is calculated by integrating the equation of motion about the roll axis. Since the spacecraft is axially symmetric, $I_2 = I_3 = I$. Under the assumptions outlined in Section 2, and neglecting disturbance torques, the approximate equations of motion are

$$\dot{\omega}_1 = -\frac{J_2 \Omega_{B_2}}{I_1}\dot{\gamma}_c - \frac{J_3 \Omega_{3_3}}{I_1}\dot{\alpha}_c + \frac{u_1}{I_1}$$

$$\ddot{\alpha}_c = -\frac{J_1 \Omega_{B_1}}{I}\dot{\gamma}_c + \frac{J_3 \Omega_{B_3}}{I}\omega_1 + \frac{u_2}{I} \qquad (17)$$

$$\ddot{\gamma}_c = \frac{J_2 \Omega_{B_2}}{I}\omega_1 - \frac{J_1 \Omega_{B_1}}{I}\dot{\alpha}_c + \frac{u_3}{I_1}$$

The suboptimal, inertia-wheel attitude control systems developed in this study are designed using these approximate equations of motion. Terms of the form $J_i \Omega_{B_i}$ represent the angular momentum of the inertia wheels and vary with time. These terms are often ignored [4-7] or treated as constant; however, if the control system does not compensate for the effects of these terms, the attitude response of the spacecraft may be adversely affected. The procedures for suboptimal control developed in this study take into account the time varying nature of these inter-axis coupling terms, and consequently yield better response than systems designed on the basis of a time-invariant approximation of the equations of motion.

Inertia-wheel control about the roll axis is given by

$$u_1 = \begin{cases} -0.025 \text{ sgn } (\omega_1) \text{ ft-lbs} & \text{for } 5\times10^{-4} \leq |\omega_1| \text{ radians/sec} \\ 0 & \text{for } |\omega_1| < 5\times10^{-4} \text{ radians/sec} \end{cases} \quad (18)$$

Laws for control about the pitch and yaw axes are synthesized using the two suboptimal techniques developed in the previous section. The results obtained are compared to those given by ignoring all coupling in the equations of motion (time-invariant linearization). Letting $\omega_1 = x_1$, $\alpha_c = x_2$, $\dot{\alpha}_c = x_3$, $\gamma_c = x_4$, and $\dot{\gamma}_c = x_5$, the approximate equations of motion are given by (3) where $a_{ij} = 0$ except $a_{34} = a_{55} = 1$,

$$C(t) = \begin{bmatrix} 0 & 0 & -\frac{J_3{}^{\Omega}B_3}{I_1} & 0 & -\frac{J_2{}^{\Omega}B_2}{I_1} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{J_3{}^{\Omega}B_3}{I} & 0 & 0 & 0 & \frac{J_1{}^{\Omega}B_1}{I} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{J_2{}^{\Omega}B_2}{I} & 0 & -\frac{J_1{}^{\Omega}B_1}{I} & 0 & 0 \end{bmatrix}$$

$$g(t) = 0, \qquad \text{and} \quad u' = [u_1, 0, u_2, 0, u_3]$$

The roll-axis control law is given by (18) and the logic for control about the pitch and yaw axis is selected to approximately minimize a performance index of the form $J = \int_{t_o}^{\infty} (y'Qy + u_y' Ru_y)dt$

where $y' = [x_2, x_3, x_4, x_5]$ and $u_y' = [0, u_2, 0, u_3]$.

The Hamilton-Jacobi equation (8) is

$$V_t + y'Qy - \frac{1}{4} V_y B'R^{-1}B V_y + V_x Ax + V_x C(t)x = 0 \qquad (19)$$

where B_y is a 4×4 matrix such that $b_{y_{ij}} = 0$

except $b_{y_{22}} = b_{y_{44}} = 1/I$.

If all time varying terms, $C(t)x$, are ignored; $q_{11} = q_{33} = q_1$ and $q_{22} = q_{44} = q_2$; and $r_{11} = r_{33} = 1$ and $r_{22} = r_{44} = \frac{1}{I^2}$; then $V(x) = x'Px$ where

p is a 5×5 matrix such that $p_{ij} = 0$ except,

$p_{22} = p_{44} = q_1(q_2 + 2(q_1)^{1/2})^{1/2}$, $p_{33} = p_{55} = p_{22}/q_1$,

$p_{32} = p_{23} = p_{54} = p_{45} = (q_1)^{1/2}$. From (6), the suboptimal control based on the above time-invariant approximations of the equations of motion is

$$\begin{aligned} u_2 &= - I\, p_{22}\, x_3 - I\, p_{12}\, x_2 \\ u_3 &= - I\, p_{22}\, x_5 - I\, p_{12}\, x_4 \end{aligned} \qquad (20)$$

If time-varying terms are ignored, values of $I\, p_{22} = 15.5$ ft-lbs-sec and $I\, p_{12} = .09$ ft-lbs yield adequate transient response.

Using Method I for suboptimal control, the control about the pitch and yaw axes is

$$u_2 = -15.5x_3 - .09x_2 - J_3\Omega_{B_3}x_1 - J_1\Omega_{B_1}x_5$$

$$u_3 = -15.5x_5 - .09x_4 - J_2\Omega_{B_2}x_1 + J_1\Omega_{B_1}x_3 \qquad (21)$$

and from Method II, a first-order approximation of (16) gives the following suboptimal control law

$$u_2 = -15.5x_3 - .09x_2 - J_3\Omega_{B_3}x_1 + .006J_1\Omega_{B_1}x_4$$

$$u_3 = -15.5x_5 - .09x_4 - J_2\Omega_{B_2}x_1 - .006J_1\Omega_{B_1}x_2 \qquad (22)$$

Results obtained from a digital computer simulation of the spacecraft and inertia-wheel control system are presented in Tables IA - IC. Control logic was generated by time-invariant linearization (20) and Methods I (21) and II (22) for suboptimal control. Computer simulation was carried out for several sets of initial conditions and for each set, values of zero and 4 slug - ft^2/sec (80% of the maximum) were used for the initial angular momentum of the inertia wheels. In implementing the various control laws approximate values for angular velocities about the pitch, roll, and yaw axes were used. However, system dynamics were simulated using the exact equations of motion.

For an initial inertia-wheel angular momentum of 4 slug-ft^2/sec about each axis, the control system designed on the basis of time-invariant linearization failed to drive the attitude error to within the necessary bounds; consequently, this system would be unacceptable unless the inertia wheels were desaturated frequently. The systems synthesized by use of both suboptimal control methods reduced the attitude error to within 0.003 radians within 10 minutes or less for all sets of initial conditions.

The results presented in Tables IA - IC may be summarized as follows:

(1) Method I gives a system which appears to have slightly lower peak power and torque requirements than the system designed on the basis of Method II.

(2) The system synthesized by using Method II appears to consume less energy than the system designed by use of Method I.[8]

(3) In general, Method II yields a slightly smaller value to performance index (4) than does Method I.

(4) Both methods give nearly the same values for two widely used indices of performance, the integral of the sum of the squares of the attitude and angular velocity errors about the pitch and yaw axes and the integral of the sum of the squares of the control effort about the pitch and yaw axes.

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[8] It is assumed that the inertia-wheels can restore energy to the system; hence negative values for energy consumption appear in Tables IB and IC.

In Fig. 4-7, the response of the spacecraft and inertia-wheel system about the pitch axis is illustrated for an initial inertia-wheel angular momentum of 4 slug-ft^2/sec. The unacceptable pitch angle and pitch rate response of the system designed on the basis of linearization is shown in Fig. 4 and Fig. 5. The remainder of the results presented in Fig. 4-8 may be briefly summarized as follows:

(1) Method II yields a control which drives the pitch angle to a small value faster than the control given by Method I; however, the system designed by Method I exhibits less overshoot (Fig. 4).

(2) Linearization is shown to yield a system which gives slightly higher inertia wheel angular momentum than either of the suboptimal designs (Fig. 6).

(3) The control torques generated by use of the two suboptimal procedures are so close to one another as to be indistinguishable; however, linearization yields a system which uses less torque over most of the control interval (Fig. 7).

As illustrated in Tables IA - IC, the performance characteristics of systems designed by time-invariant linearization and the two suboptimal control procedures are much the same for an initial, inertia-wheel angular momentum of zero. This is not surprising since the suboptimal control laws (21) and (22) approach the control law obtained by time-invariant linearization (20) as the angular momentum of the inertia wheels approaches zero.

In Fig. 8 and Fig. 9, the results obtained by application of the two suboptimal procedures developed in this study are compared to those obtained by investigators using standard transform techniques for control system design [5]. Conventional procedures yield a system which is much more oscillatory and which overshoots the position of zero error much more than the systems designed by use of the two suboptimal techniques. The results presented in Fig. 8 and Fig. 9 are for an initial, inertia-wheel angular momentum of zero. Since conventional transform techniques do not take into account time-varying, coupling terms which depend in magnitude on the angular momentum of the inertia wheels, even less accurate response may be expected from the conventional system as the angular momentum of the inertia wheels increases. Such inaccurate response is noted by Cook and Fleisig [5]. As shown in Table IA, the conventional control system requires almost ten times more torque and fifty time more power than the suboptimal systems;[9] consequently, the suboptimal systems could probably be constructed from small and lighter components.

5. Conclusions

The suboptimal control techniques developed in this

[9] Increasing the coefficients of the control law given by linearization by a factor of ten gives results similar to those obtained from a conventional control system.

work provide effective methods for synthesizing inertia-wheel attitude control systems. The two procedures for designing suboptimal, inertia-wheel systems take inter-axis coupling into account. If this coupling is ignored, unacceptable system response may result. Both suboptimal techniques yield physically realizable control laws in feedback form, and as demonstrated the previous section the suboptimal systems developed in this study give considerably more accurate control than is provided by suboptimal systems based on time-invariant approximations of the equations of motion.

Method I is computationally easier to use and gives simpler control laws than Method II. However, both of the methods developed in the present study appear to be applicable, in a practical sense, to the design of attitude control systems for a large class of spacecraft.

Better results might be obtained from Method II if more terms were used in the approximate solution of the Hamilton-Jacobi equation. However, before the tedious calculations necessary to determine these terms are carried out, the convergence of Method II to the optimal solution should be investigated. It is felt that modifications of the methods proposed by McClamroch [13] and others for sensitivity analysis might yield information on the convergence properties of Method II.

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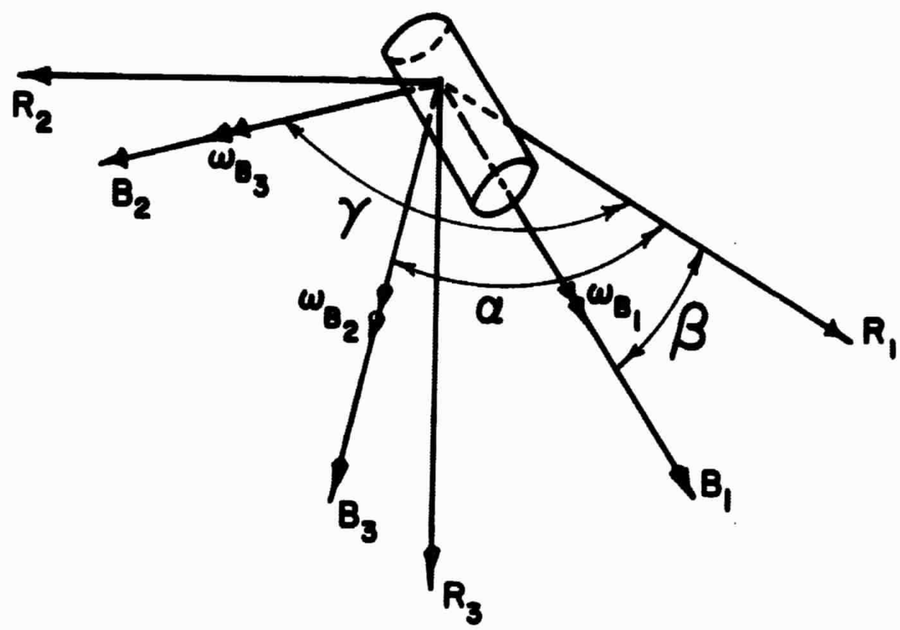

Fig. 1 Coordinate System for Example I

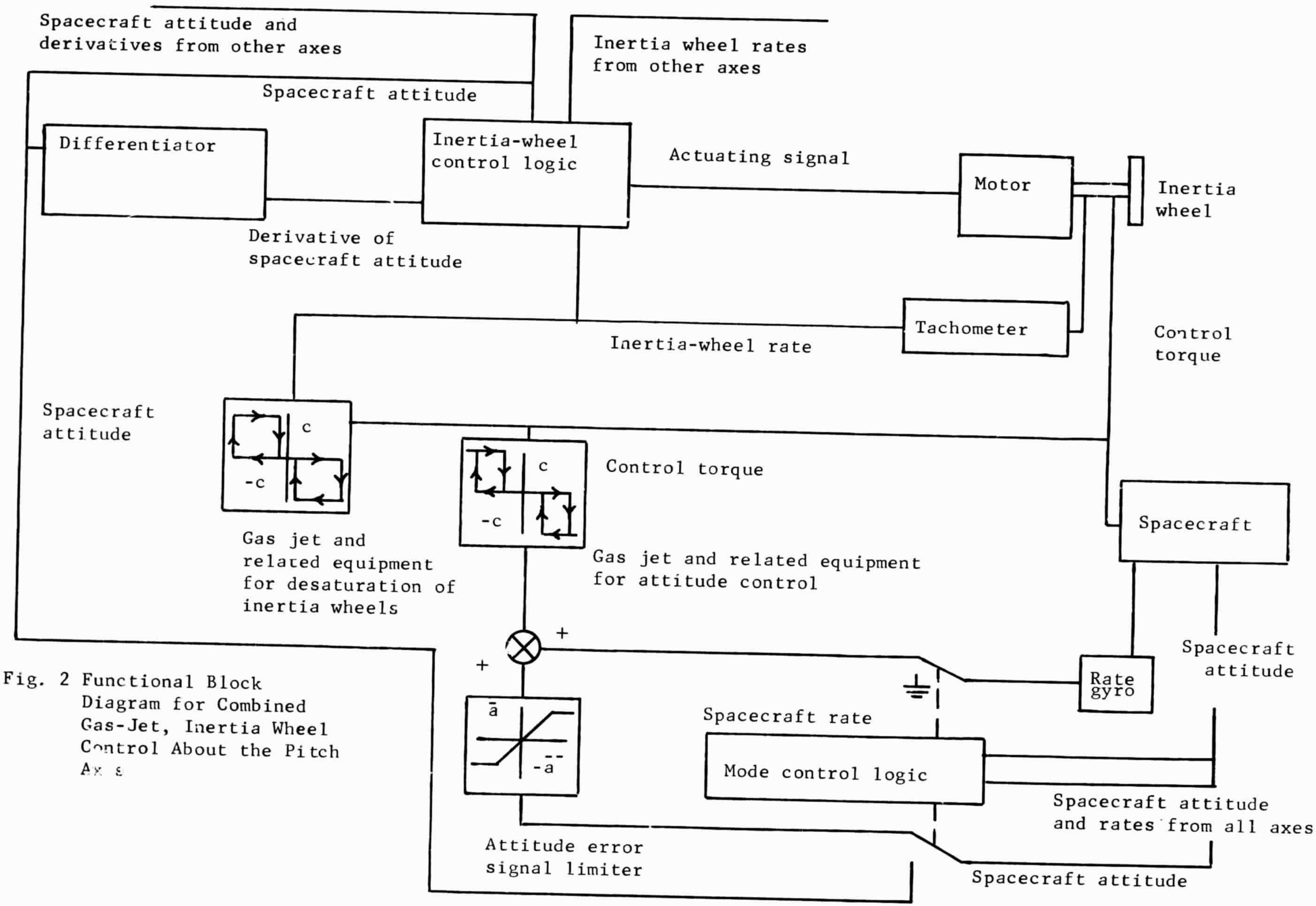

Fig. 2 Functional Block Diagram for Combined Gas-Jet, Inertia Wheel Control About the Pitch Axis

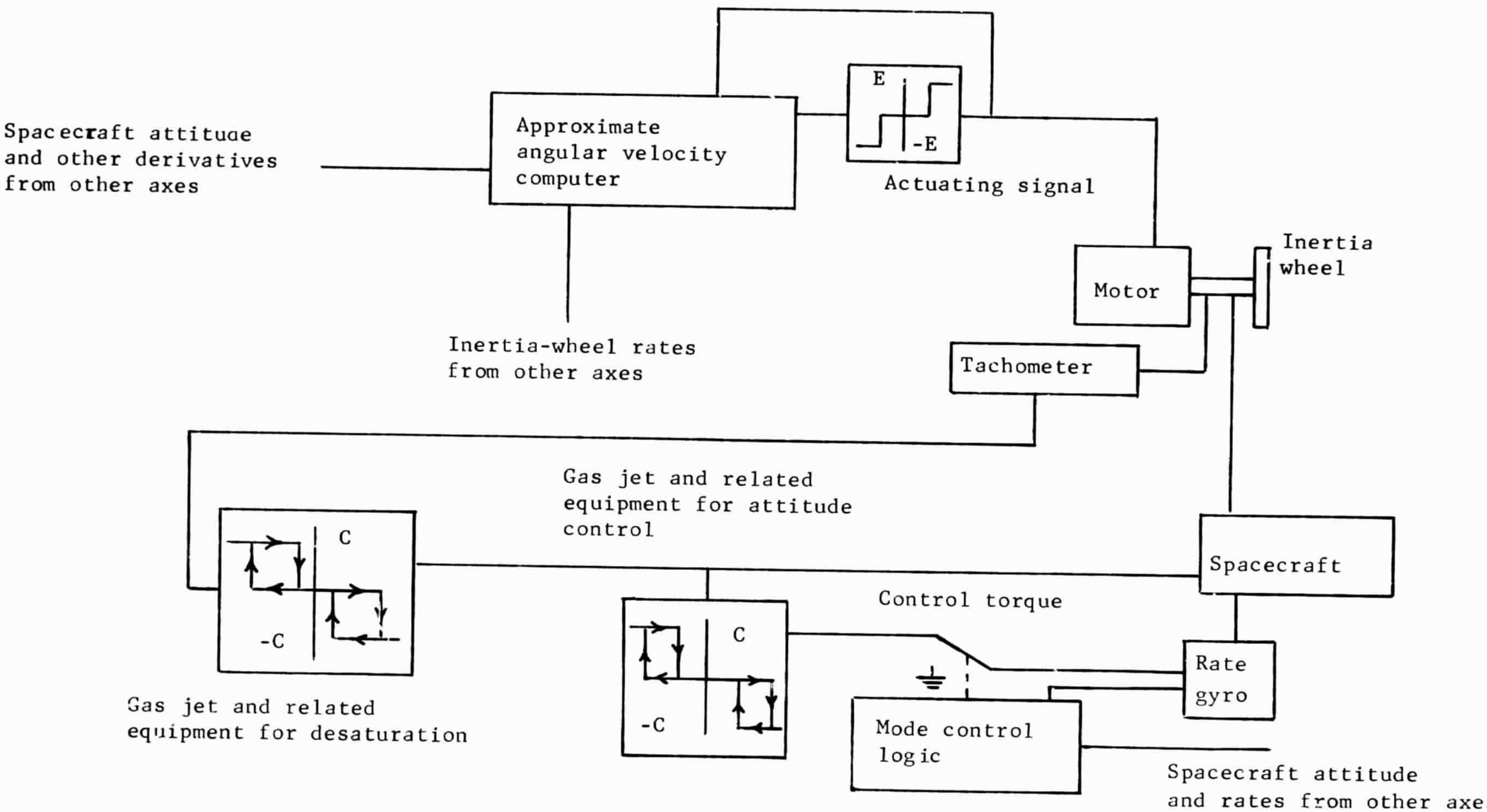

Fig. 3 Functional Block Diagram for Combined Gas-Jet, Inertia-Wheel Control About the Roll Axis

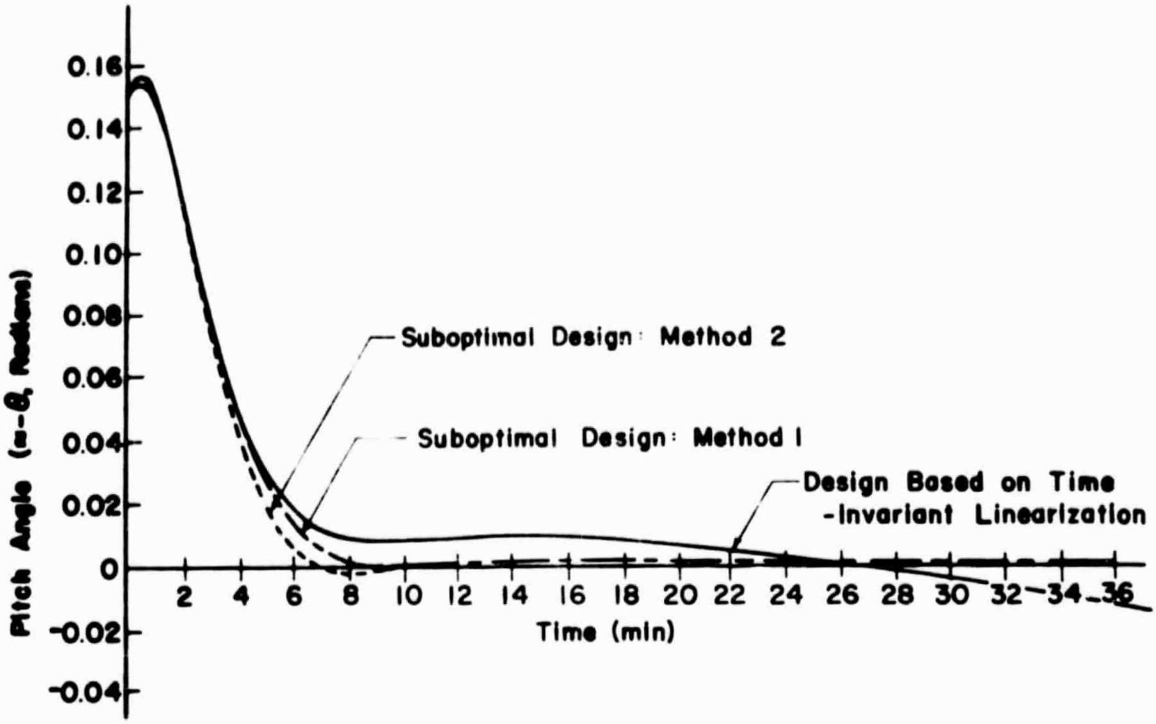

Fig. 4 Example I, Pitch Angle vs. Time; Initial Inertia Wheel Angular Momentum = 4 slug-ft^2/sec

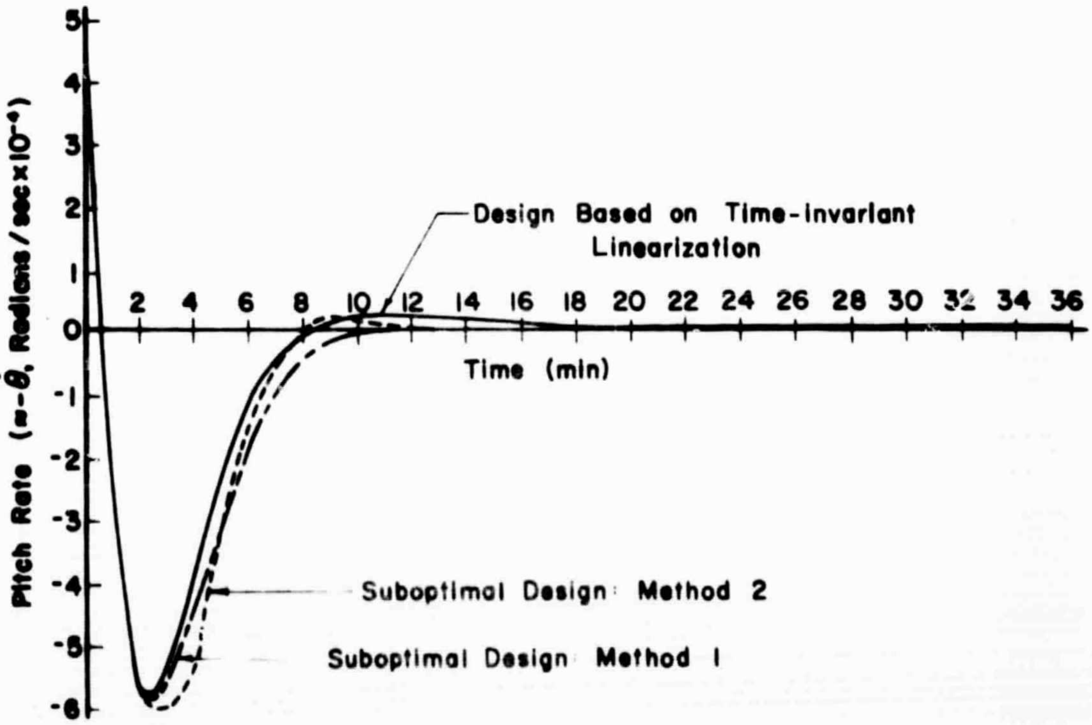

Fig. 5 Example I, Pitch Rate vs. Time; Initial Inertia-Wheel Angular Momentum = 4 slug-ft^2/sec

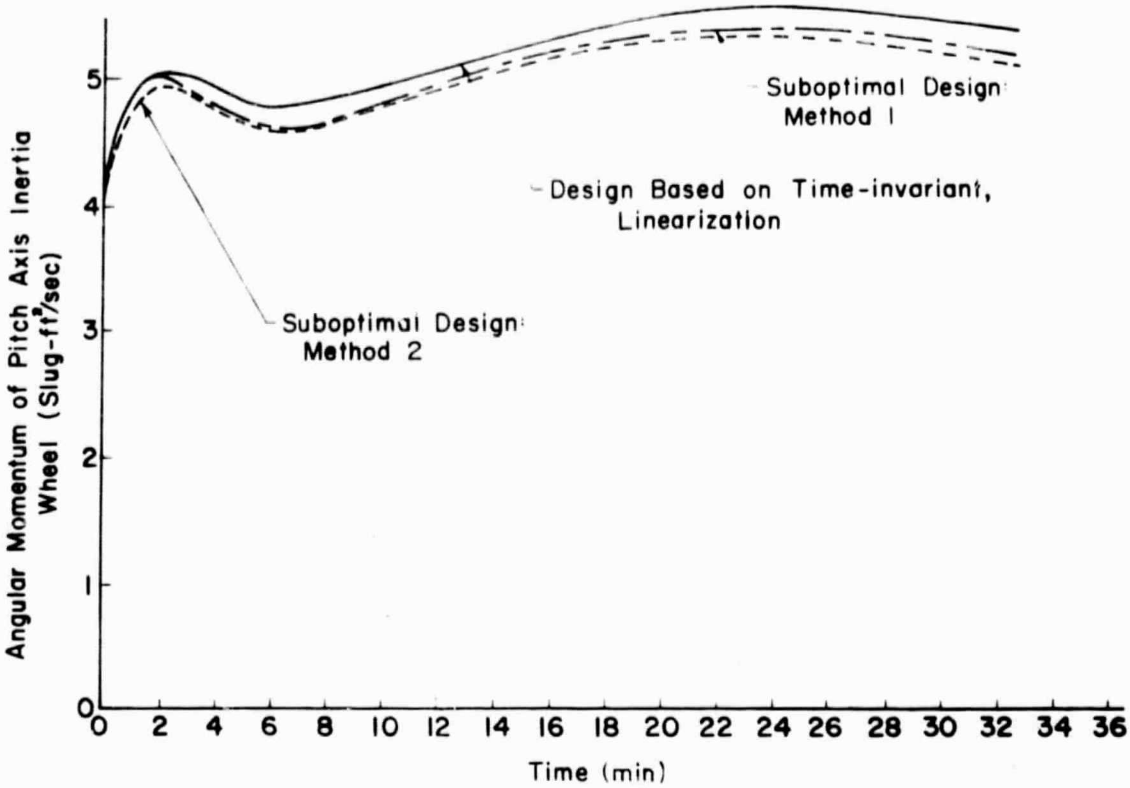

Fig. 6 Example I, Angular Momentum of Pitch Axis Inertia Wheel vs. Time; Initial Inertia-Wheel Angular Momentum = 4 slug-ft^2/sec

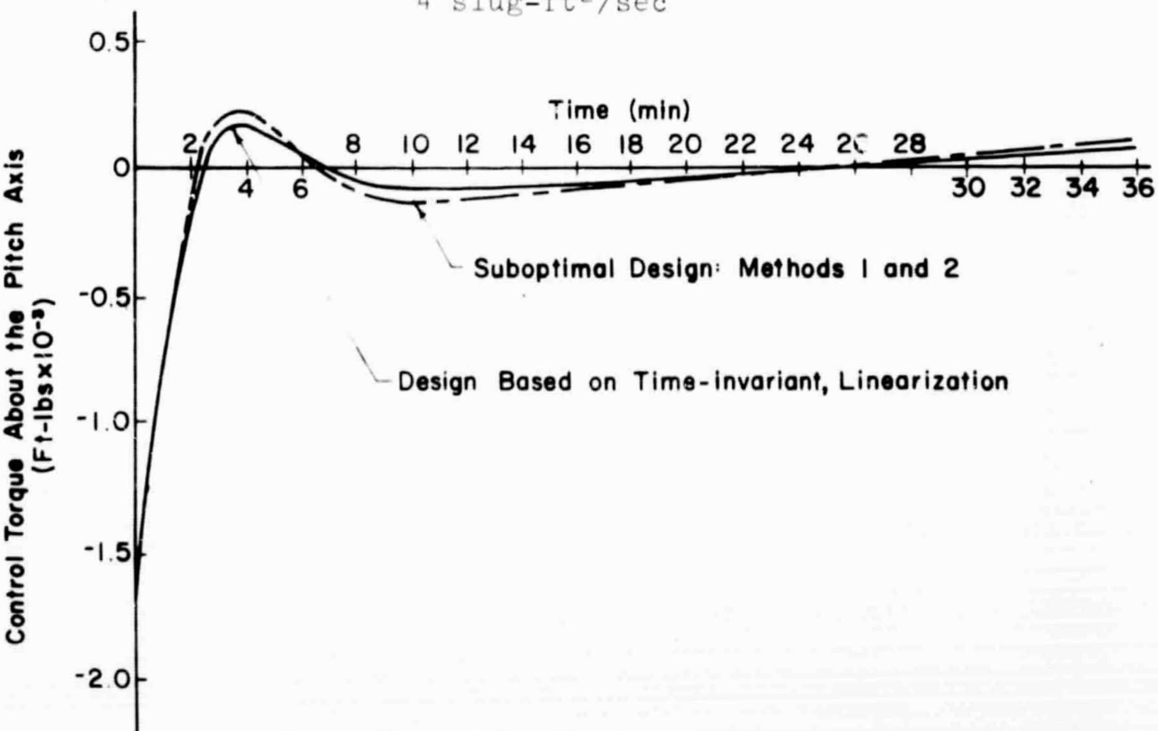

Fig. 7 Example I, Control Torque about Pitch Axis vs. Time; Initial Inertia-Wheel Angular Momentum = 4 slug-ft^2/sec

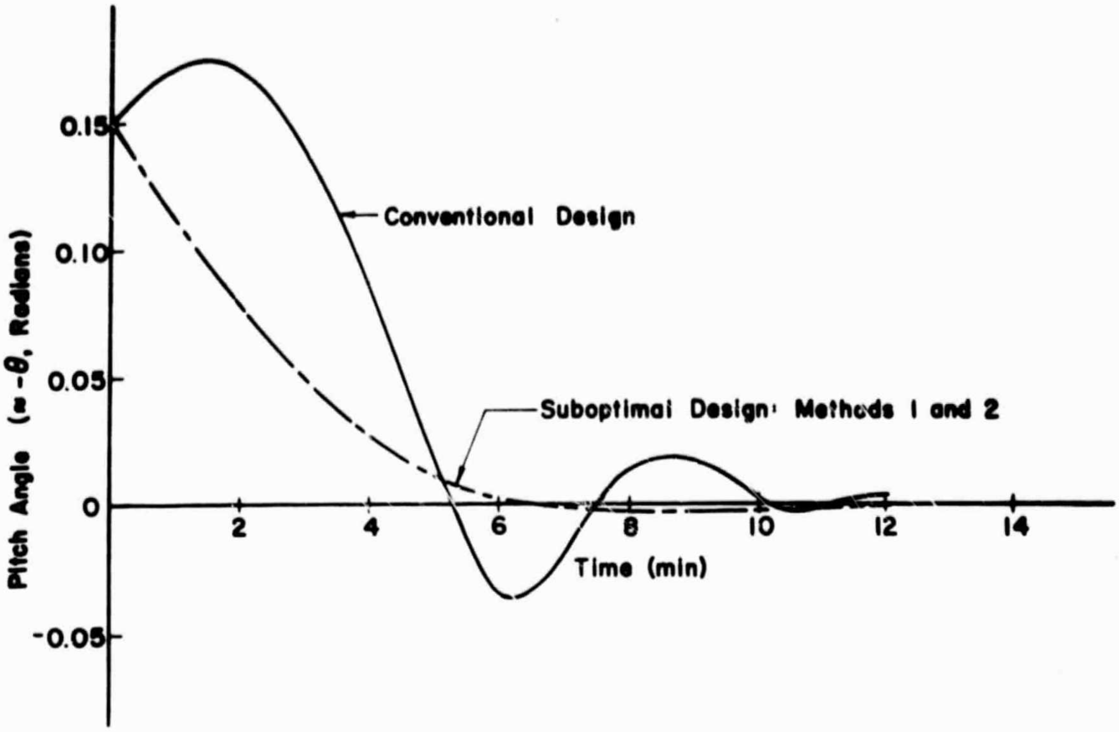

Fig. 8 Example I, Pitch Angle vs. Time, Conventional and Suboptimal Designs; Initial Inertia-Wheel Angular Momentum = 0

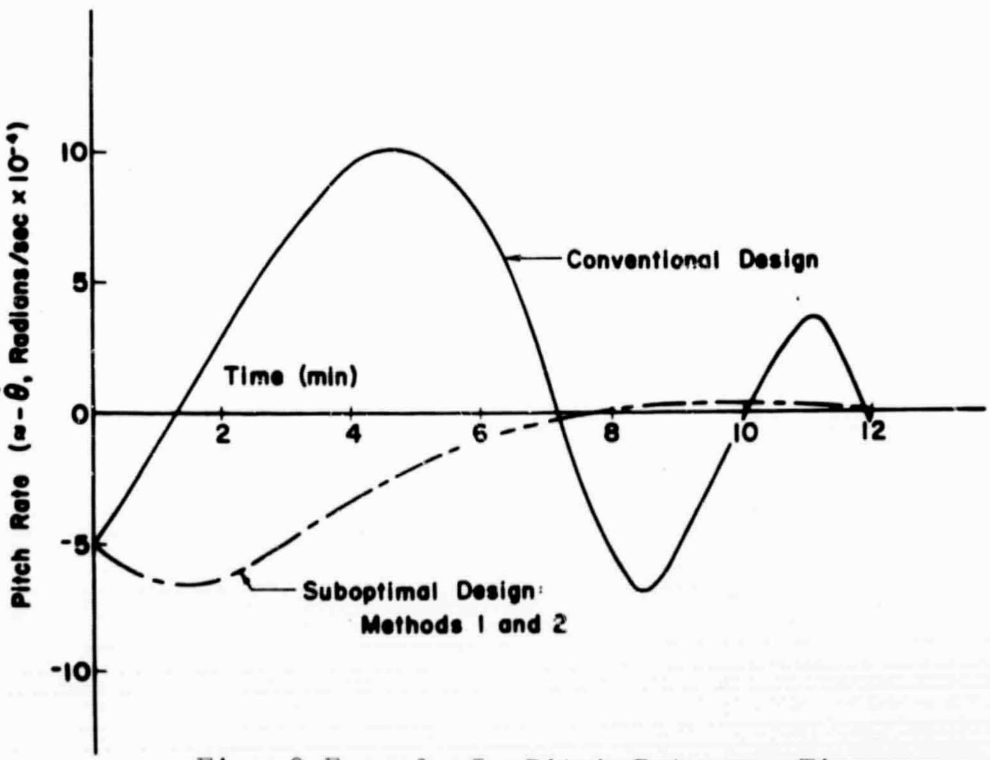

Fig. 9 Example I, Pitch Rate vs. Time, Conventional and Suboptimal Designs; Initial Inertia-Wheel Angular Momentum = 0

TABLE IA

Performance Characteristics for
Inertia-Wheel Control

Initial Conditions - Pitch and Yaw Angles = 0.15 radians
Pitch, Yaw and Roll Rates = 0.0005 radians/sec

Method of Control System Design	Time-Invariant Linearization		Suboptimal: Method I		Suboptimal: Method II		Conventional Transform Techniques	
Initial, Inertia-Wheel Angular Momentum (% of Maximum)	0	80	0	80	0	80	0	80
Response Time (Minutes)	8.0	Does not Stabilize	7.3	8.0	7.7	9.0	13.0	-
Peak Torque (ft-lbs x 10^{-3})	17.41	17.39	20.0	20.3	20.0	18.4	-	170[a]
Peak Power (ft-lbs/sec x 10^{-3})	13.41	145.6	13.4	163.0	13.4	147.8	-	4500[a]
Energy Consumed (ft-lbs x 10^{-3})	4.24	-	4.44	158.2	4.43	67.4	-	-
Quadratic Performance Index	694.8	988.1	694.7	703.6	714.7	695.0	-	-
Integral of Quadratic Function of Error	706.2	-	708.0	701.1	708.1	692.5	-	-
Integral of Quadratic Function of Control	154.3	-	153.9	168.0	154.0	190.9	-	-

[a] Maximum Available

TABLE IB

Performance Characteristics for
Inertia-Wheel Control

Initial Conditions - Pitch and Yaw Angles = 0.15 radians
Pitch, Yaw and Roll Rates = 0 Radians/sec

Method of Control System Design	Time-Invariant Lineartization		Suboptimal: Method I		Suboptimal: Method II		Conventional Transform Techniques	
Initial, Inertia-Wheel Angular Momentum (% of Maximum)	0	80	0	80	0	80	0	80
Response Time (Minutes)	6.8	Does not Stabilize	6.8	7.0	6.8	8.3	11.0	-
Peak Torque (ft-lbs x 10^{-3})	13.5	13.5	13.5	13.5	13.5	17.1	-	170[a]
Peak Power (ft-lbs/sec x 10^{-3})	5.31	108.0	5.31	108.0	5.31	136.8	-	4500[a]
Energy Consumed (ft-lbs x 10^{-3})	0.12	-	0.12	-44.0	0.12	-56.5	-	-
Quadratic Performance Index	631.2	-	631.2	653.5	631.2	649.0	-	-
Integral of Quadratic Function of Error	667.0	-	667.0	666.2	667.0	664.7	-	-
Integral of Quadratic Function of Control	118.8	-	118.8	142.0	118.8	139.5	-	-

[a] Maximum available

TABLE IC

Performance Characteristics for
Inertia-Wheel Control

Initial Conditions - Pitch and Yaw Angles = 0.15 radians
Pitch, Yaw and Roll Rates = -0.0005 radians/sec

Method of Control System Design	Time-Invariant Linearization		Suboptimal: Method I		Suboptimal: Method II		Conventional Transform Techniques	
Initial, Inertia-Wheel Angular Momentum (% of Maximum)	0	80	0	80	0	80	0	80
Response Time (Minutes)	7.9	Does not Stabilize	9.3	7.5	9.2	10.0	9.7	-
Peak Torque (ft-lbs x 10^{-3})	14.6	14.6	20.0	19.7	20.0	21.6	-	170[a]
Peak Power (ft-lbs x 10^{-3})	13.5	133.1	13.4	158.2	13.4	173.5	-	4500[a]
Energy Consumed (ft-lbs x 10^{-3})	4.12	-	3.9	-151.2	3.9	-149.2	-	-
Quadratic Performance Index	694.8	-	695.4	743.9	695.3	734.7	-	-
Integral of Quadratic Function of Error	706.2	-	708.4	711.3	708.4	714.3	-	-
Integral of Quadratic Function of Control	154.3	-	154.1	119.9	154.2	185.3	-	-

[a] Maximum available