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AN APPLICATION OF STATISTICS IN MIXTURE OF EXPONENTIAL DISTRIBUTIONS

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16. ABSTRACT

This paper is concerned with a distribution of an order statistic when a sample of n is composed of n_1 observations coming from a population with density

$$f_1(t) = \theta_1 e^{-t\theta_1}, \; 0 \leq t, \; 0 < \theta_1$$

and the remaining $n_2 (= n - n_1)$ observations coming from a population with density

$$f_2(t) = \theta_2 e^{-t\theta_2}, \; 0 \leq t, \; 0 < \theta_2.$$

In particular the expected value of the nth order statistic has been obtained when the sample is composed of two types of components and when it is composed of three types of components. The distribution of the nth order statistic in the most general situation has been obtained. A small application has been mentioned at the end.

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AN APPLICATION OF ORDER STATISTICS

IN MIXTURE OF EXPONENTIAL DISTRIBUTIONS

H. I. Patel

SUMMARY

This paper is concerned with a distribution of an order statistic when a sample of n is composed of n_1 observations coming from a population with density

$$f_1(t) = \theta_1 e^{-t\theta_1}, \quad o \leq t, \quad o < \theta_1$$

and the remaining $n_2 (= n - n_1)$ observations coming from a population with density

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In particular the expected value of the nth order statistic has been obtained when the sample is composed of two types of components and when it is composed of three types of components. The distribution of the nth order statistic in the most general situation has been obtained. A small application has been mentioned at the end.

AN APPLICATION OF ORDER STATISTICS IN MIXTURE OF EXPONENTIAL DISTRIBUTIONS

By

H. I. Patel

The University of Georgia

I. Introduction

Numerous writers including Pearson [10], Charlier [2,3], Wicksell [3], Rider [11], Blischke [1], Cohen [4,5,6,7,8], Hasselblad [9] and others have dealt with the problem of estimating parameters in mixture of distributions of various types. In this paper, we are concerned not with parameter-estimation, but rather with the expected value of order statistics in a mixture of two exponential distributions. The problem involved arises from a consideration of the "time to restore" service in a system composed of various types of components, wherein the failure of any component renders the system inoperative. As an example, the range support equipment at a missile test facility might constitute such a system. Let the system under consideration be composed of two types of components which we designate simply as type-A and type-B. Let the time to repair (i.e., restore to service) components of type-A be a random variable with density.

$$f_1(t) = \theta_1 e^{-t\theta_1} \quad \ldots\ldots.(1)$$

and let the corresponding density for type-B components be

$$f_2(t) = \theta_2 e^{-t\theta_2} \quad \ldots\ldots(2)$$

where the parameters θ_1 and θ_2 are known at a given moment, but suppose that n_1 components of type-A and n_2 components of type-B are in need of repair in order to restore the system to operating condition. Let us further assume that repair can be started simultaneously on all failed components and that the several repair times involved are independently distributed. The expected time required to "restore service" is then the expected value of the nth order statistic in a random mixed sample consisting of n_1 observations from (1) and n_2 observations from (2). Let T_n designate this order statistic and we proceed to determine $E(T_n)$, where $n = n_1 + n_2$.

II. Distribution of kth order statistic, when the number of observations coming from each of the populations is known.

Let us consider two independent sets of observations. Suppose the observations in each set are distributed independently with probability density functions given by (1) and (2).

Let n_i be the number of observations in the ith set, such that $n_1 + n_2 = n$.

We shall obtain the distribution of the kth order statistic, $x_{(k)}$, in the combined sample of $n_1 + n_2$ observations. $x_{(k)}$ is obviously the observation from either of the two sets.

(a) Suppose $x_{(k)}$ is the observation from the first set. Let k_1 be the number of observations from the first set below $x_{(k)}$, $(k_1 \leq k-1)$. Then in this case the distribution of $Y = X_{(k)}$ is given by

$$dG_1 = \sum_{k_1=o}^{k-1} \frac{n_1!n_2!F_1^{k_1}(y)F_2^{k-k_1-1}(y)\,[1-F_1(y)]^{n_1-k_1-1}[1-F_2(y)]^{n_2-k+k_1+1}\,dF_1(y)}{k_1!(n_1-k_1-1)!(k-k_1-1)!(n_2-k+k_1+1)!}$$

$$o \leq y \leq \infty,\ k_1 < \min(k,n_1),\ n_1-1-k_1 < \min(n-k+1,n_1). \quad \ldots\ldots\ (3)$$

(b) Suppose $x_{(k)}$ is the observation from the second set. Then the distribution of $x_{(k)}$ is given by

$$dG_2 = \sum_{k_1=o}^{k-1} \frac{n_1!n_2!F_1^{k_1}(y)F_2^{k-k_1-1}(y)\,[1-F_1(y)]^{n_1-k_1}[1-F_2(y)]^{n_2-k+k_1}}{k_1!(k-k_1-1)!(n_1-k_1)!(n_2-k+k_1)!}dF_2(y)$$

$$0 \leq y \leq \infty,\ k_1 < \min(k,n_1+1) \text{ and } n_1-k_1 < \min(n-k+1,n_1+1) \quad \ldots\ldots(4)$$

In (3) and (4) we have

$$F_i(y) = \int_o^y \theta_i e^{-\theta_i x}dx = 1-e^{-\theta_i y} \quad \text{for } i = 1,2.$$

Writing $dG_1 = g_1(y)dy$ and $dG_2 = g_2(y)dy$, the density function of $Y = X_{(k)}$ will be given by

$$h(y) = g_1 + g_2$$

$$= \text{coefficient of } \theta^{k-1} \text{ in}$$

$$n_1 \sum_{k_1=0}^{k-1} \binom{n_1-1}{k_1}\binom{n_2}{k-k_1-1}\theta^{k-1} p_1^{k_1} p_2^{k-k_1-1} q_1^{n_1-k_1-1} q_2^{n_2-k+k_1+1} f_1(y)$$

$$+ n_2 \sum_{k_1=0}^{k-1} \binom{n_1}{k_1}\binom{n_2-1}{k-k_1-1}\theta^{k-1} p_1^{k_1} p_2^{k-k_1-1} q_1^{n_1-k_1} q_2^{n_2-k+k_1} f_2(y), \quad 0 \le y \le \infty$$

$= \text{coefficient of } \theta^{k-1}$ in

$$n_1 f_1(y)[\theta p_1+1-p_1]^{n_1-1}[\theta p_2+1-p_2]^{n_2} + n_2 f_2(y)[\theta p_1+q_1]^{n_1}[\theta p_2+q_2]^{n_2-1}, \quad \ldots.(5)$$

where $p_1 = F_1(y)$, $p_2 = F_2(y)$, $q_1 = 1-p_1, q_2 = 1-p_2$

and $f_i(y) = \theta_i e^{-\theta_i y}$, $i = 1,2$.

It should be shown that $h(y)dy$, $0 \le y$, stands for the distribution of Y, by showing that

$$\int_0^\infty h(y)dy = 1.$$

Now $\int_0^\infty h(y)dy$ = coefficient of θ^{k-1} in

$$\int_0^\infty n_1 f_1(y)[\theta p_1+q_1]^{n_1-1}[\theta p_2+q_2]^{n_2} dy + \int_0^\infty n_2 f_2(y)[\theta p_1+q_1]^{n_1}[\theta p_2+q_2]^{n_2-1} dy$$

= coefficient of θ^{k-1} in

$$\frac{1}{\theta-1}\int_0^\infty [\theta p_2+q_2]^{n_2} d[\theta p_1+q_1]^{n_1} + \frac{1}{\theta-1}\int_0^\infty [\theta p_1+q_1]^{n_1} d[\theta p_2+q_2]^{n_2}$$

$$= \text{coefficient of } \theta^{k-1} \text{ in } \frac{1}{\theta-1}\left[(\theta p_1+q_1)^{n_1}(\theta p_2+q_2)^{n_2}\right]_0^\infty$$

$$= \text{coefficient of } \theta^{k-1} \text{ in } (\theta^{n_1+n_2}-1)/(\theta-1)$$

$$= 1$$

III. Distribution of nth order statistic

Writing $k_i = n_i - 1$ and $k = n$ in (3) and $k_1 = n_1$ and $k = n$ in (4), we shall get

$$dH(y) = [n_1 F_1^{n_1-1}(y) F_2^{n_2}(y) f_1(y) + n_2 F_1^{n_1}(y) F_2^{n_2-1}(y) f_2(y)]\, dy,$$

$$o \le y \le \infty \qquad \ldots\ldots\ldots(6)$$

IV. Expected value of $Y = T_n$

$$E(Y) = \int_0^\infty y\,dH(y)$$

$$= \int_0^\infty y F_2^{n_2}(y)\, dF_1^{n_1}(y) + \int_0^\infty y F_1^{n_1}(y)\, dF_2^{n_2}(y)$$

$$= [y\{F_2^{n_2}(y)F_1^{n_1}(y)-1\}]_0^\infty - \int_0^\infty [\{\sum_{i=0}^{n_1}(-1)^i \binom{n_1}{i} v^i\}\{\sum_{j=0}^{n_2}(-1)^j\binom{n_2}{j} w^j\}-1]dy$$

$$= -\int_0^\infty [\sum_{i=1}^{n_1}(-1)^i\binom{n_1}{i}v^i + \sum_{j=1}^{n_2}(-1)^j\binom{n_2}{j}w^j + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(-1)^{i+j}\binom{n_1}{i}\binom{n_2}{j}v^i w^j]dy$$

$$= -\frac{1}{\theta_1}\sum_{i=1}^{n_1}(-1)^i\frac{1}{i}\binom{n_1}{i} - \frac{1}{\theta_2}\sum_{j=1}^{n_2}(-1)^j\frac{1}{j}\binom{n_2}{j} - \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(-1)^{i+j}\frac{1}{i\theta_1+j\theta_2}\binom{n_1}{i}\binom{n_2}{j}$$

$$= \frac{1}{\theta_1}\sum_{i=1}^{n_1}\frac{1}{i} + \frac{1}{\theta_2}\sum_{i=1}^{n_2}\frac{1}{i} - \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(-1)^{i+j}\frac{1}{i\theta_1+j\theta_2}\binom{n_1}{i}\binom{n_2}{j}, \quad \ldots\ldots(7)$$

Where $v = e^{-y\theta_1}$ and $w = e^{-y\theta_2}$

If $|\theta_1-\theta_2|$ is not large, writing $1/(i\theta_1+j\theta_2)$ as approximately equal to $2/(\theta_1+\theta_2)(i+j)$, we shall get the last sum in (7) as

$$-\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(-1)^{i+j}\,2\binom{n_1}{i}\binom{n_2}{j}\Big/(i+j)(\theta_1+\theta_2)$$

$$=-\frac{2}{\theta_1+\theta_2}\int_0^1\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(-1)^{i+j}\binom{n_1}{i}\binom{n_2}{j}x^{i+j-1}dx$$

$$=-\frac{2}{\theta_1+\theta_2}\int_0^1\frac{(1-x)^n-(1-x)^{n_1}-(1-x)^{n_2}+1}{x}\,dx$$

$$=-\frac{2}{\theta_1+\theta_2}\Big[\sum_{i=1}^{n_1}\frac{1}{i}+\sum_{i=1}^{n_2}\frac{1}{i}-\sum_{i=1}^{n}\frac{1}{i}\Big] \qquad \ldots\ldots\ldots\ldots(8)$$

V. Example

The table for $E(T_n)$ for all possible combinations of n_1 and n_2, such that $n_1+n_2 = n = 10$, with $\theta_1=2$ and $\theta_2=3$ is given below.

n_1	n_2	$\frac{1}{\theta_1}\sum_1^{n_1}\frac{1}{i}$	$\frac{1}{\theta_2}\sum_1^{n_2}\frac{1}{i}$	$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(-1)^{i+j}\frac{1}{i\theta_1+j\theta_2}\binom{n_1}{i}\binom{n_2}{j}$	$E(T_n)$
0	10	--	0.9663	--	0.9663
1	9	0.5000	0.9330	0.4017	1.0413
2	8	0.7500	0.8959	0.5543	1.1016
3	7	0.9166	0.8543	0.6231	1.1578
4	6	1.0416	0.8067	0.6481	1.2102
5	5	1.1266	0.7511	0.6435	1.2592
6	4	1.2100	0.6944	0.6141	1.3053
7	3	1.2814	0.6111	0.5589	1.3486
8	2	1.3439	0.5000	0.4695	1.3894
9	1	1.3994	0.3333	0.3198	1.4279
10	0	1.4494	--	--	1.4494

Remarks:

(i) If $Y_n^{(i)}$ is the nth order statistic in a random sample of size n from the population $F_i(x) = \frac{1}{\theta_i} e^{-\theta_i x} dx$, $x \geq o$, $\theta_1 > 0$, then

$$E(Y_n^{(i)}) = \frac{1}{\theta_i} \sum_{j=1}^{n} \frac{1}{j}, \ i = 1,2.$$

In this example $E(Y_n^{(1)}) = 1.4494$ and $E(Y_n^{(2)}) = 0.9663$

(ii) Min. $[E(y_n^{(1)}), E(Y_n^{(2)})] \leq E(T_n) \leq$ max. $[E(Y_n^{(1)}), E(Y_n^{(2)})]$.

Proof:

Let $\theta_1 < \theta_2$. Then

$$\frac{1}{\theta_2} \left[\sum_{i=1}^{n_1} \frac{1}{i} + \sum_{i=1}^{n_2} \frac{1}{i} - \sum_{i=1}^{n} \frac{1}{i} \right] \leq \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (-1)^{i+j} \frac{1}{i\theta_1 + j\theta_2} \binom{n_1}{i} \binom{n_2}{j}$$

$$\leq \frac{1}{\theta_1} \left[\sum_{i=1}^{n_1} \frac{1}{i} + \sum_{i=1}^{n_2} \frac{1}{i} - \sum_{i=1}^{n} \frac{1}{i} \right]$$

Hence $\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (-1)^{i+j} \frac{1}{i\theta_1 + j\theta_2} \binom{n_1}{i} \binom{n_2}{j} - \frac{1}{\theta_2} \left[\sum_{i=1}^{n_1} \frac{1}{i} + \sum_{i=1}^{n_2} \frac{1}{i} - \sum_{i=1}^{n} \frac{1}{i} \right]$

$$\leq \left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right) \left[\sum_{i=1}^{n_1} \frac{1}{i} + \sum_{i=1}^{n_2} \frac{1}{i} - \sum_{i=1}^{n} \frac{1}{i} \right]$$

$$\leq \left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right) \left[\sum_{i=1}^{n_1} \frac{1}{i} \right], \quad \text{since } n_2 \leq n$$

From this we get

$$\frac{1}{\theta_2} \sum_{i=1}^{n} \frac{1}{i} \leq \frac{1}{\theta_2} \sum_{i=1}^{n_2} \frac{1}{i} + \frac{1}{\theta_1} \sum_{i=1}^{n_1} \frac{1}{i} - \sum_i \sum_j (-1)^{i+j} \frac{1}{i\theta_1 + j\theta_2} \binom{n_1}{i} \binom{n_2}{j}$$

or $E(Y_n^{(2)}) \leq E(T_n)$

Similarly it can be shown that

$E(Y_n^{(1)}) \geq E(T_n)$

(iii) If we want to compute $E(T_n)$ for a new set of parameters θ_1' and θ_2', such that $\frac{\theta_1'}{\theta_2'} = \frac{\theta_1}{\theta_2}$, then $E(T_n') = \frac{\theta_1'}{\theta_1} \cdot E(T_n)$, where T_n' is the nth order statistic in a mixed sample of the populations with the new set of parameters.

VI. General Case

Let there be k populations. Let $Y^{(i)}$ (i=1, 2, ...k) be independent random variables with the distribution

$$dF_i = \theta_i e^{-y\theta_i} dy, \quad 0 \leq y \leq \infty, \quad \theta_i > 0$$

Let $Y_{n_i}^{(i)}$ be the n_ith order statistic in a random sample of size n_i in the ith group (i=1,2,...,k). Let $n_1+n_2+..+n_k = n$.

Then the distribution of $X = T_n$, the nth order statistic in a mixed sample will be given by

$$dG(x) = P(Y_{n_i}^{(i)} < x,\ i=1,2,\ldots,k-1 \mid Y_{n_k}^{(k)} = x)\cdot h_k(x)dx$$

$$+P(Y_{n_i}^{(i)} < x, i=1,2,\ldots,k-2,k \mid Y_{n_{k-1}}^{(k-1)} = x)\ h_{k-1}(x)dx$$

. . . .

. . . .

$$+P(Y_{n_i}^{(i)} < x, i=2,3,\ldots,k \mid Y_{n_1}^{(1)} = x)\cdot h_1(x)dx,$$

where $h_i(x)dx = n_i[F_i(x)]^{n_i-1} f_i(x)dx$ stands for the distribution of the n_ith order statistic in the ith group (i=1,2,...,k).

Now since $Y_{n_1}, Y_{n_2}, \ldots, Y_{n_k}$ are independently distributed,

$$P(Y_{n_i}^{(i)} < x, i=1,2,\ldots j-1,j+1,\ldots k \mid Y_{n_j}^{(j)} = x)$$

$$= P(Y_{n_i}^{(i)} < x, i=1,2,\ldots,j-1,j+1,\ldots,k)$$

$$= \prod_{i=1}^{j-1} P(Y_{n_i}^{(i)} < x) \prod_{i=j+1}^{k} P(Y_{n_i}^{(i)} < x), \text{ for } j=1,2\ldots,k.$$

Thus $$dG(x) = \sum_{j=1}^{k} \prod_{i=1}^{j-1} [F_i(x)]^{n_i} \prod_{i=j+1}^{k} [F_i(x)^{n_i} d[F_j(x)]^{n_j}, \quad 0 \le x \le \infty \quad \ldots (9)$$

VII. Expected value T_n in case of 3 types of components.

Now we shall find the expected value of $X=T_n$ in case of 3 groups.

$$E(X) = \int_0^\infty x dG(x)$$

$$= \int_0^\infty x d(\prod_{i=1}^{3} [F_i(x)]^{n_i} - 1)$$

$$= -\int_0^\infty (\prod_{i=1}^{3} [F_i(x)]^{n_i} - 1) dx.$$

Writing $v = \exp(-\theta_1 x)$, $w = \exp(-\theta_2 x)$ and $z = \exp(-\theta_3 x)$, we get

$$E(X) = -\int_0^\infty [(1-v)^{n_1}(1-w)^{n_2}(1-z)^{n_3} - 1] dx$$

$$= -\int_0^\infty [(1 + \sum_{i=1}^{n_1} T_{1i})(1 + \sum_{i=1}^{n_2} T_{2i})(1 + \sum_{i=1}^{n_3} T_{3i}) - 1] dx$$

$$= -\int_0^\infty (\sum_{i=1}^{n_1} T_{1i} + \sum_{i=1}^{n_2} T_{2i} + \sum_{i=1}^{n_3} T_{3i} + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} T_{1i} T_{2j} + \sum_{i=1}^{n_1} \sum_{j=1}^{n_3} T_{1i} T_{3j}$$

$$+ \sum_{i=1}^{n_2} \sum_{j=1}^{n_3} T_{2i} T_{3j} + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_3} T_{1i} T_{2j} T_{3k}) \, dx$$

$$= - \sum_{j=1}^{3} \frac{1}{\theta_j} \sum_{i=1}^{n_j} (-1)^i \frac{1}{i} \binom{n_j}{i} - \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (-1)^{i+j} \frac{1}{i\theta_1+j\theta_2} \binom{n_1}{i}\binom{n_2}{j}$$

$$- \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (-1)^{i+j} \frac{1}{i\theta_1+j\theta_2} \binom{n_1}{i}\binom{n_3}{j} - \sum_{i=1}^{n_2} \sum_{j=1}^{n_3} (-1)^{i+j} \frac{1}{i\theta_1+j\theta_3} \binom{n_2}{i}\binom{n_3}{j}$$

$$- \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_3} (-1)^{i+j+k} \frac{1}{i\theta_1+j\theta_2+k\theta_3} \binom{n_1}{i}\binom{n_2}{j}\binom{n_3}{k}, \quad \ldots\ldots\ldots(10)$$

Where $T_{1i} = (-1)^i \binom{n_1}{i} v^i, T_{2i} = (-1)^i \binom{n_2}{i} w^i, T_{3i} = (-1)^i \binom{n_3}{i} z^i$.

VIII. Application

Suppose a mining company surveys n different points randomly selected in some area. A fixed amount of ore is collected from each location (corresponding to each point selected). Now the company's main interest is in analysing these samples of ores in the minumum period of time in order to make some important decisions. Suppose there are two laboratories working at different efficiencies. Let us further assume that in each laboratory the work of analysing n samples can commence simultaneously at n counters.

Let us define the following parameters:

c_i = Cost of analysing a sample in laboratory i, i=1, 2.

d = Cost per unit time that the company has to wait until all the results are known.

n_i = Number of samples sent to laboratory i, i=1, 2.

t_i = Random variable denoting the time required for analysing a sample in ith laboratory. The distribution of t_i is assumed to be exponential with parameter θ_i, i=1,2.

T_n = Number of units of time the company has to wait until the decision is taken.

Then the company would wish to send n_i samples to the ith laboratory such that the expected cost

$$\phi_{n_1,n_2} = E(dT_n + c_1n_1 + c_2n_2)$$

is minimum.

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