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The Inversion Theorem and Plancherel's Theorem in Infinite Dimensions

by

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The Inversion Theorem and Plancherel's Theorem in a Banach Space

1. Introduction

Let G be a locally compact abelian group with Haar measure μ, and let X be a Banach space and C be the set of complex numbers. A classic theorem due to Plancherel ([1], [2]) states that the Fourier transform maps $L_1(G,C) \cap L_2(G,C)^+$ onto a dense subset of $L_2(\hat{G},C)$ ($\hat{G}$ is the dual group of G and has Haar measure m) in such a way that $\int_G \alpha(g)\overline{\beta(g)}\mu(dg) = \int_{\hat{G}} \hat{\alpha}(\gamma)\overline{\hat{\beta}(\gamma)}m(d\gamma)$ for all α,β in $L_1(G,C) \cap L_2(G,C)$ where $\hat{\alpha}$ is the Fourier transform of α, given by $\hat{\alpha}(\gamma) = \int_G \overline{(g,\gamma)}\alpha(g)\mu(dg)$ for all γ in $\hat{G}$. Here (g,γ) denotes the action of the character γ on g in G. In this paper we extend this result to functions taking values in an inner product subspace of a Banach algebra.

Another well-known theorem ([1], [2]) states that if α is a positive definite element of $L_1(G,C) \cap L_\infty(G,C)$ then $\hat{\alpha}$ is in $L_1(\hat{G},C)$ and

$$\alpha(g) = \int_{\hat{G}} (g,\gamma)\hat{\alpha}(\gamma)m(d\gamma) \tag{1.1}$$

$^+$For $1 \leq p \leq \infty$ $L_p(G,X)$ is the space of μ-measurable functions f mapping G into X. For $1 \leq p < \infty$ we use the norm $\|\cdot\|_p$, where $\|f\|_p = \{\int_G \|f(g)\|^p \mu(dg)\}^{1/p}$, and for $p = \infty$ we use the norm $\|f\|_\infty$ which is the (μ) essential supremum of $\|f(g)\|$ on G. $\|\cdot\|$ denotes the norm in X.

for (almost) all g in G. This inversion theorem is also generalized to functions assuming values in certain admissible Banach spaces.

Our work relies heavily on an extension of Bochner's theorem established in [3]. We show that if p is in $L_1(G,X) \cap L_\infty(G,X)$, if p is positive definite (positivity is defined with respect to a particular cone in X), and if $p(0)$ satisfies a certain finiteness condition, then $\hat{p}$, the Fourier transform of p, is in $L_1(\hat{G},X)$ and the inversion formula 1.1 given for α holds for p. A sharper theorem states that if p is in $L_1(G,X) \cap L_\infty(G,X)$, if p is positive definite, and if there is a real, finite, regular Borel measure λ such that $\|\int_G \alpha(g)p(g)\mu(dg)\| \leq \int_{\hat{G}} |\hat{\alpha}(\gamma)| \lambda(d\gamma)$ for all α in $L_1(G,C)$, then $\hat{p}$ is in $L_1(\hat{G},X)$ and 1.1 is satisfied by p.

Using this theory we give new proofs of some results due to Hewitt and Wigner ([4]).

Now assume G is compact and $\mathscr{N}$ is the set of Hilbert-Schmidt operators on a separable Hilbert space H. Then we show that the closed maximal ideals of the algebra $L_2(G,\mathscr{N})$ are in a one to one correspondence with $\hat{G}$. The same result holds for $L_2(G,A)$ where A is any separable simple H^*-algebra.

Finally we prove existence and uniqueness theorems for equations of the form

$$(1.2) \qquad q(g) = r(g) + \int_G p(g-g')q(g')\mu(dg')$$

where r is in $L_2(G, \mathcal{N})$, p is in $L_1(G, \mathcal{L}(H,H))$, H is a separable Hilbert space and $\mathcal{L}(H,H)$ is the space of continuous linear operators mapping H into H (so $\mathcal{N} \subset \mathcal{L}(H,H)$). Solutions q are to be elements of $L_2(G, \mathcal{N})$.

2. Bochner's Theorem and Dominated Functions

Let X be a Banach space, X^* the dual of X and X^{**} the dual of X^*. For φ in X^* we denote the action of φ on $x \in X$ by (x,φ). Given a subset of X^* we can define a cone of "positive" elements in X.

DEFINITION 2.1. *Let* Φ *be a subset of* X^*. *The subset* K_Φ *of* X *given by*

$$K_\Phi = \{x \in X: (x,\varphi) \geq 0 \ \text{for all} \ \varphi \in \Phi\} \tag{2.2}$$

is called the cone determined by Φ.

Sometimes we write simply K if Φ is fixed by the context. K_Φ is the set of "positive" elements.

Let G be a σ-finite locally compact abelian group with Haar measure μ and let $\hat{G}$ be its dual group with Haar measure m.

DEFINITION 2.3. *Let* p *be a measurable map of* G *into* X. *Then* p *is* Φ-*positive definite if*

$$\sum_{n=1}^{N} \sum_{m=1}^{N} c_n \bar{c}_m (p(g_n - g_m), \varphi) \geq 0 \tag{2.4}$$

for any integer N, *any* $c_1, \ldots, c_N$ *in* C, *any* $g_1, \ldots, g_N$ *in* G, *and all* φ *in* Φ. *If* p *is in* $L_\infty(G,X)$ *then* p *is integrally* Φ-*positive definite if*

$$\left(\int_G \int_G \alpha(g) \overline{\alpha(g')} p(g-g') d\mu d\mu, \varphi\right) \geq 0 \tag{2.5}$$

for all α *in* $L_1(G,C)$ *and all* φ *in* Φ.

Next we impose a condition which relates Φ to the topology of X.

DEFINITION 2.6. *The family* Φ *is full if there is a* $\rho > 0$ *such that*

$$\|x\| \leq \rho \sup_{\substack{\varphi \in \Phi \\ \varphi \neq 0}} \{|(x,\varphi)|/\|\varphi\|\} \tag{2.7}$$

for all x *in* X.

The following two propositions examine the relationship between the two notions of positive-definiteness.

PROPOSITION 2.8. *If* Φ *is full and* p *is* Φ-*positive definite then* p *is in* $L_\infty(G,X)$ *and* $p(0)$ *is in* K_Φ.

Proof: It is readily shown that for g in G, φ in Φ, $|(p(g),\varphi)| \leq (p(0),\varphi)$ so that $\|p(g)\| \leq \rho\|p(0)\|$.

PROPOSITION 2.9. *Let* p *be in* $L_\infty(G,X)$ *such that one version of* p *is* ωX-*continuous*[+]. *Then* p *is* Φ-*positive definite if and only if* p *is integrally* Φ-*positive definite.*

Proof: See [3] or [7].

We shall see shortly (corollary 2.15) that all those elements of $L_\infty(G,X)$ of interest to us have the continuity required in proposition 2.9.

Next we recall some results from measure theory. Let S be a locally compact topological space and let $\Sigma(S)$ be the Borel field of S (i.e. the smallest σ-field containing the closed sets of S).

DEFINITION 2.10. *A vector measure* ν *is a weakly countably additive set function defined on* $\Sigma(S)$ *and taking values in* X. ν *is weakly regular if the scalar measures* $(\nu(\cdot),\varphi)$ *are regular*[++] *for all* φ *in* X^*. ν *is* Φ-*positive if* $(\nu(E),\varphi) \geqq 0$ *for all* φ *in* Φ *and* E *in* $\Sigma(S)$.

DEFINITION 2.11. *A set function* ν^{**} *mapping* $\Sigma(S)$ *into* X^{**} *is weak-*-regular if* $(\varphi,\nu^{**}(\cdot))$ *is a regular scalar measure for all* φ *in* X^*. ν^{**} *is* Φ-*positive if* $(\varphi,\nu^{**}(E)) \geqq 0$ *for all* φ *in* Φ, E *in* $\Sigma(S)$.

[+] The mapping f of G into X is ωX-continuous if it is continuous when the weak topology is imposed on X. G retains its usual topology.

[++] A scalar measure λ is regular if, given $\epsilon > 0$ and $E \in \Sigma(S)$ with $\|\lambda\|(E) < \infty$ (i.e. λ has finite variation on E), then there is a compact $K \subset E$ and an open $O \supset E$ such that $\|\lambda\|(O-K) < \epsilon$.

If ν is a vector measure we denote its variation on a measurable set E by $\|\nu\|(E)$ and its semi-variation by $|\nu|(E)$ ([5], [6]). The following theorem, an extension of Bochner's theorem, is essential to our work. The proof is given in [3]. We assume Φ is full.

THEOREM 2.12. (A) *If ν is a weakly regular Φ-positive vector measure defined on $\Sigma(\hat{G})$ and if*

$$p(g) = \int_{\hat{G}} (g, \gamma)\nu(d\gamma) \tag{2.13}$$

then p is an integrally Φ-positive definite element of $L_\infty(G,X)$.

(B) *If p is an integrally Φ-positive definite element of $L_\infty(G,X)$, then there is a set function ν^{**} mapping $\Sigma(\hat{G})$ into X^{**} such that* (i) *ν^{**} is weak-*-regular, Φ-positive with finite semi-variation and* (ii)

$$(p(g),\varphi) = \int_{\hat{G}} (g,\gamma)(\varphi,\nu^{**}(d\gamma)) \tag{2.14}$$

for all φ in X^* and almost all g in G.

COROLLARY 2.15. *If p is an integrally Φ-positive definite element of $L_\infty(G,X)$, then one version of p is ωX-continuous. If p is given by 2.13, where ν is a weakly regular Φ-positive vector measure defined on $\Sigma(\hat{G})$, then p is a continuous map of G into X.*

Proof: This follows from the relevant regularity. See also [7].

With the aid of theorem 2.12 we can prove a useful inversion theorem. However, a different version of Bochner's theorem will allow us to establish a sharper theorem. We require first the following

DEFINITION 2.16. p in $L_\infty(G,X)$ is dominated if there exists a finite, regular, positive Borel measure λ, such that

$$(2.17)\qquad \left\| \int_G \alpha(g)p(g)\mu(dg) \right\| \leq \int_{\hat{G}} |\hat{\alpha}(\gamma)|\,\lambda(d\gamma)$$

for all α in $L_1(G,C)$, where $\hat{\alpha}$ is the Fourier transform of α, i.e. $\hat{\alpha}(\gamma) = \int_G \overline{(g,\gamma)}\alpha(g)\mu(dg)$.

DEFINITION 2.18. Let Φ be a subset of X. Assume there is a function ϕ_o mapping K_Φ into $R^+ \cup \{\infty\}$* in a linear manner such that φ_o is uniformly positive on K_Φ, i.e. there exists $k > 0$ such that $k(x,\varphi_o) \geq \|x\|$ for all x in K_Φ. Furthermore assume there is an at most countable sequence $\{\varphi_i\}$ in Φ and a sequence $\{c_i\}$ in R^+ such that $(x,\varphi_o) = \sum_{i=1}^{\infty} c_i(x,\varphi_i)$ for all x in K_Φ. Then we say that the pair (Φ,X) is admissible. We let $K_o = \{x \in K_\Phi : (x,\varphi_o) < \infty\}$.

LEMMA 2.19. If (Φ,X) is admissible, if Φ is full, and if $p \in L_\infty(G,X)$ is integrally Φ-positive definite with $p(0)$ in K_o, then

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*R^+ is the set of non-negative real numbers.

p is dominated.

Proof: We note first that $p(0)$ is well defined by corollary 2.15. Let $\psi(\alpha) = \int_G \alpha(g)p(g)\mu(dg)$ for all α in $L_1(G,C)$, then $(\psi(\alpha),\varphi) = \int_{\hat{G}} \hat{\alpha}(\gamma)(\varphi,\nu^{**}(d\gamma))$ for some weak-*-regular, Φ-positive set function ν^{**} given by theorem 2.12. We can actually define $\hat{\psi}(f)$ mapping $C_o(\hat{G})^+$ into X by $(\hat{\psi}(f),\varphi) = \int_{\hat{G}} f(\gamma)(\varphi,\nu^{**}(d\gamma))$. Then $\hat{\psi}$ is a bounded linear map, $\|\hat{\psi}(f)\| \leq \|f\|_\infty |\nu^{**}|(\hat{G})$.

If f is in $C_o(\hat{G})$ then $f = f_1 - f_2 + if_3 - if_4$ where f_i is in $C_o(\hat{G})$, $f_i(\gamma) \geq 0$, and each pair of functions (f_1,f_2), (f_3,f_4) has disjoint support. Hence $f_i(\gamma) \leq |f(\gamma)|$, and $\hat{\psi}(f_i)$ is in K_Φ so that $\|\hat{\psi}(f_i)\| \leq k(\hat{\psi}(f_i),\varphi_o) = k\sum_{j=1}^{\infty} c_j(\hat{\psi}(f_i),\varphi_j) = k\sum_j c_j \int_{\hat{G}} f_i(\gamma)$ $(\varphi_j,\nu^{**}(d\gamma))$. Consider now the set function λ given by $\lambda(E) = \sum_{i=1}^{\infty} c_i(\varphi_i,\nu^{**}(E))$, $E \in \Sigma(\hat{G})$. Then $\lambda(E) \geq 0$ for all E in $\Sigma(\hat{G})$, and also λ is additive. Moreover $\lambda(E) \leq (p(0),\varphi_o) < \infty$ as $p(0)$ is in K_o.

λ is countably additive because $\lambda(\bigcup_j E_j) = \sum_i \sum_j c_i(\varphi_i, \nu^{**}(E_j)) = \sum_j \sum_i c_i(\varphi_i,\nu^{**}(E_j)) = \sum_j \lambda(E_j)$, if the E_j are disjoint (note that $c_i(\varphi_i,\nu^{**}(E_j)) \geq 0$ for all i,j). Also λ is regular, for given $\epsilon > 0$ and E in $\Sigma(\hat{G})$, there is a number N such that $\sum_{N+1}^{\infty} c_i(\varphi_i,\nu^{**}(\hat{G})) < \epsilon/2$ and there is a compact $K \subset E$ and an open $O \supset E$ such that $(\varphi_i,\nu^{**}(O-K)) < \epsilon/2Nc_i$, $i = 1,2,\ldots,N$. Hence $\lambda(O-K) < \epsilon$.

$^+C_o(\hat{G})$ is the space of continuous functions mapping $\hat{G}$ into C, which vanish at ∞ if $\hat{G}$ is only locally compact.

Then $\|\hat{\psi}(f)\| \leq \sum_{i=1}^{4} \|\hat{\psi}(f_i)\| \leq k \sum_i \int_{\hat{G}} f_i(\gamma) d\lambda \leq 4k \int_{\hat{G}} |f(\gamma)| d\lambda$. It follows that if $\lambda' = 4k\lambda$ then $\|\psi(\alpha)\| \leq \int_{\hat{G}} |\hat{\alpha}(\gamma)| d\lambda'$. This establishes the lemma.

We can now state the alternate version of Bochner's theorem. Assume Φ is full and countable.

THEOREM 2.20. *p is a dominated, integrally Φ-positive definite element of $L_\infty(G,X)$ if and only if there is a weakly regular Φ-positive vector measure ν mapping $\Sigma(\hat{G})$ into X such that ν has finite variation, i.e. $\|\nu\|(\hat{G}) < \infty$, and such that*

(2.21) $$p(g) = \int_{\hat{G}} (g,\gamma)\nu(d\gamma).$$

For the proof see [3]. In this case, of course, p is continuous by corollary 2.15.

3. Inversion Theorems

If $p \in L_1(G,X)$ we recall that the Fourier transform of p is given by

(3.1) $$\hat{p}(\gamma) = \int_G (g,\gamma)p(g)\mu(dg).$$

For convenience we let $\mathscr{P} = \{p \in L_\infty(G,X)\colon p$ is integrally Φ-positive definite$\}$ and $\mathscr{P}_d = \{p \in \mathscr{P}\colon p$ is dominated$\}$. We recall that if $p \in \mathscr{P}$ then p is ωX-continuous (corollary 2.15). If (Φ,X) is admissible then $\mathscr{T}_o$ is the set of functions p mapping G into X such that p is ωX-continuous and such that $p(0)$ is in K_o where K_o is defined in 2.18.

PROPOSITION 3.2. (A) <u>If</u> $p \in$ <u>span</u> $\{L_1(G,X) \cap \mathscr{P}\}$ <u>and if</u> $\varphi \in$ <u>span</u>$\{\Phi\}$ <u>then</u> $(\hat{p}(\cdot),\varphi) \in L_1(\hat{G},C)$ <u>and</u> (B) <u>if the Haar measure of</u> G <u>is fixed then the Haar measure of</u> $\hat{G}$ <u>can be so normalized that</u>

$$(3.3) \qquad (p(g),\varphi) = \int_{\hat{G}}(g,\gamma)(\hat{p}(\gamma),\varphi)m(d\gamma)$$

<u>is valid for all</u> $p \in$ <u>span</u>$\{L_1(G,X) \cap \mathscr{P}\}$ <u>and all</u> $\varphi \in$ <u>span</u>$\{\Phi\}$.

Proof: It is evident the results need only hold for $p \in L_1(G,X) \cap \mathscr{P}$, $\varphi \in \Phi$. But this follows from the scalar inversion theorem ([2], p. 22).

A better result is the following.

THEOREM 3.4. <u>Assume</u> Φ <u>is full and countable and</u> (Φ,X) <u>is admissible</u>. (A) <u>If</u> $p \in$ <u>span</u> $\{L_1(G,X) \cap \mathscr{P} \cap \mathscr{T}_o\}$ <u>then</u> $\hat{p} \in L_1(\hat{G},X)$, <u>and</u> (B) <u>if</u> μ <u>is fixed then</u> m <u>can be so normalized that</u>

$$(3.5) \qquad p(g) = \int_{\hat{G}}(g,\gamma)\hat{p}(\gamma)m(d\gamma)$$

<u>for all</u> p <u>in span</u>$\{L_1(G,X) \cap \mathscr{P} \cap \mathscr{T}_o\}$ <u>and all</u> g <u>in</u> G.

Proof: Again we need only prove the results for p in $L_1(G,X) \cap \mathscr{P} \cap \mathscr{T}_o$. For such a p and for φ in Φ we have from 3.3 and 2.14

$$(3.6) \qquad (p(g),\varphi) = \int_{\hat{G}}(g,\gamma)(\hat{p}(\gamma),\varphi)m(d\gamma) = \int_{\hat{G}}(g,\gamma)(\varphi,\nu^{**}(d\gamma))$$

so that for any $E \in \Sigma(\hat{G})$, $\varphi \in \Phi$ $\int_E(\hat{p}(\gamma),\varphi)m(d\gamma) = (\varphi,\nu^{**}(E)) \geq 0$. So, in fact, for any $p \in L_1(G,X) \cap \mathscr{P}$ we have

$$(3.7) \qquad (\hat{p}(\gamma),\varphi) \geq 0, \ \varphi \in \Phi, \ \gamma \in \hat{G}.$$

Now $\infty > (p(0),\varphi_o) = \sum_{i=1}^{\infty} c_i(p(0),\varphi_i) = \sum_{i=1}^{\infty} c_i(\varphi_i,\nu^{**}(\hat{G})) = \sum_{i=1}^{\infty} c_i\int_{\hat{G}}(\hat{p}(\gamma),\varphi_i)m(d\gamma) = \int_{\hat{G}}(\hat{p}(\gamma),\varphi_o)m(d\gamma) \geq \int_{\hat{G}}\|\hat{p}(\gamma)\|m(d\gamma)$ using the monotone convergence theorem, the fact that $\hat{p}(\gamma) \in K_\Phi$ for all $\gamma \in \hat{G}$, and the fact that $\hat{p}$ is continuous so $\|\hat{p}(\cdot)\|$ is measurable. As p is measurable and G is σ-finite, then p is essentially separably valued and so is $\hat{p}$. As $\hat{p}$ is also continuous it is measurable. Hence (A) is established.

Now 3.6 yields $(p(g),\varphi) = (\int_{\hat{G}}(g,\gamma)\hat{p}(\gamma)m(d\gamma),\varphi)$ for any $\varphi \in \Phi$ and almost all $g \in G$. As Φ is full and countable we have $p(g) = \int_{\hat{G}}(g,\gamma)\hat{p}(\gamma)m(d\gamma)$ for almost all g. This proves the theorem.

We give now the sharper theorem which does, however, require $\hat{G}$ to be σ-finite.

THEOREM 3.8. *Assume* Φ *is full and countable and* $\hat{G}$ *is* σ-*finite.* (A) *if* $p \in$ *span* $\{L_1(G,X) \cap \mathscr{P}_d\}$ *then* $\hat{p} \in L_1(\hat{G},X)$, (B) *if* μ *is fixed then* m *can be so normalized that* 3.5 *holds for all* $p \in$ *span*$\{L_1(G,X) \cap \mathscr{P}_d\}$.

Proof: If $p \in L_1(G,X)$ then $\hat{p} \in L_\infty(\hat{G},X)$. If $p \in \mathscr{P}_d$ also and $p(g) = \int_{\hat{G}}(g,\gamma)\nu(d\gamma)$ as in 2.21, then $\nu(E) = \int_E \hat{p}(\gamma)m(d\gamma)$ for $E \in \Sigma(\hat{G})$ such that $m(E) < \infty$. Then $\|\nu\|(E) = \int_E \|\hat{p}(\gamma)\|m(d\gamma)$ for $m(E) < \infty$, or, for any such E, $\int_E \|\hat{p}(\gamma)\|m(d\gamma) \leq \|\nu\|(\hat{G}) < \infty$ as ν has finite variation. Now $\hat{G}$ is σ-finite so if $\{\hat{G}_n\}$ is a sequence in $\Sigma(\hat{G})$ increasing to $\hat{G}$ then $\int_{\hat{G}} \|\hat{p}(\gamma)\|m(d\gamma) = \lim_{n\to\infty} \int_{\hat{G}_n} \|\hat{p}(\gamma)\|m(d\gamma) \leq \|\nu\|(\hat{G}) < \infty$. It follows by the monotone convergence theorem that $\hat{p} \in L_1(\hat{G},X)$. (B) follows readily.

We note that lemma 2.19 and theorem 3.8 give an immediate proof of theorem 3.4 if $\hat{G}$ is σ-finite. Actually theorem 3.4 is the more useful theorem although theorem 3.8 is sharper.

COROLLARY 3.9. *If* p *is given by*

$$p(g) = \int_{\hat{G}}(g,\gamma)\nu(d\gamma) \tag{3.10}$$

where ν *is a weakly regular* Φ-*positive vector measure with finite variation and if* p *is in* $L_1(G,X)$, *then* 3.5 *holds.*

4. The Plancherel Theorem

As usual this theorem is set in a Hilbert space and so we must first develop the necessary structure. Assume now that X is a Banach algebra with continuous involution $x \to x^*$.

DEFINITION 2.14. <u>The triplet</u> (Φ,X,X_o) <u>is strongly admissible if</u> (i) (Φ,X) <u>is admissible,</u> (ii) X_o <u>is a non-trivial subspace of</u> X <u>such that</u> xx^* <u>is in</u> K_o[+] <u>for all</u> x <u>in</u> X, <u>and</u> (iii) <u>there exists</u> $k_o > 0$ <u>such that if</u> $x \in X_o$ <u>then</u>

$$k_o\|xx^*\| \geq \|x\|^2. \tag{4.2}$$

We note that 4.2 is satisfied if X is a C^*-algebra. Now we have

PROPOSITION 4.3. <u>If</u> X <u>is a Banach algebra and if</u> (Φ,X,X_o) <u>is strongly admissible then</u> X_o <u>is a Hilbert space under the norm</u> $\|\cdot\|_o$ <u>where</u> $\|x\|_o^2 = \langle x,x\rangle_o$ <u>and</u> $\langle x,y\rangle_o = (xy^*,\varphi_o)$.

Proof. φ_o is only defined on K and we do not know that if $x,y \in X_o$ then $xy^* \in K$. However we can extend φ_o by setting $(xy^*,\varphi_o) = \sum_{i=1}^{\infty} c_i(xy^*,\varphi_i)$ were $\{c_i\}$, $\{\varphi_i\}$ define φ_o on K. Then $|\langle x,y\rangle_o| = |(xy^*,\varphi_o)| = |\sum_{i=1}^{\infty} c_i(xy^*,\varphi_i)| \leqq \sum_1^{\infty} c_i(xx^*,\varphi_i)^{1/2}(yy^*,\varphi_i)^{1/2}$ where the last inequality follows because φ_i is a positive functional. Hence we can define $\langle x,y\rangle_o$ for $x,y \in X_o$ and $|\langle x,y\rangle_o| \leqq \|x\|_o\|y\|_o$. It follows from 2.18 and 4.2 that $kk_o\|x\|_o^2 \geqq \|x\|^2$ and that $\|\cdot\|_o$ is a norm.

[+] K_o is defined in 2.18.

If $\{x_n\}$ is Cauchy in $\|\cdot\|_o$ then it is Cauchy in $\|\cdot\|$, so $x_n \to x \in X$. As K is closed then $xx^* \in K$. Also $\{x_n\}$ is bounded in $\|\cdot\|_o$ because it is Cauchy, so $\sum_{i=1}^{\infty} c_i(x_n x_n^*, \varphi_i) \leq M$, hence $\sum_{i=1}^{\infty} c_i(xx^*, \varphi_i) \leq M$ or $x \in K_o$. Choose $m(\epsilon)$ such that if $n,m > m(\epsilon)$ then $\|x_n - x_m\|_o < \epsilon$. Then $\sum_{i=1}^{N} c_i([x-x_m][x-x_m]^*, \varphi_i) =$ $\lim_{n \to \infty} \sum_{i=1}^{N} c_i([x_n-x_m][x_n-x_m]^*, \varphi_i) \leq \limsup_{n \to \infty} \sum_{i=1}^{\infty} c_i([x_n-x_m][x_n-x_m]^*, \varphi_i) < \epsilon^2$ so that for $m > m(\epsilon)$ $\|x-x_m\|_o < \epsilon$, or X_o is a Hilbert space.

If X is a Banach algebra and G is σ-finite, then $L_1(G,X)$ is also a Banach algebra ([8]). If X has the involution $x \to x^*$, then we can define an involution on $L_1(G,X)$ as $p \to p^*$ where $p^*(g) = p(-g)^*$.

THEOREM 4.4. <u>If G is σ-finite, X is a Banach algebra with continuous involution, Φ is a full and countable subset of X^* and (Φ, X, X_o) is strongly admissible, then</u> (i) <u>if $\{e_\alpha\}$ is an orthonormal basis for X_o and there exists k_1 such that $|<x, e_\alpha>_o| \leq k_1\|x\|$ for $x \in X_o$ and all α, then the Fourier transform maps $L_1(G,X) \cap L_2(G,X_o)$ onto a dense subset of $L_2(\hat{G}, X_o)$</u>, (ii) <u>for</u> $q,r \in L_1(G,X) \cap L_2(G,X_o)$

$$\int_G q(g)r(g)\mu(dg) = \int_{\hat{G}} \hat{q}(\gamma)\hat{r}(\gamma)m(d\gamma), \tag{4.5}$$

(iii) <u>for</u> $q,r \in L_1(G,X) \cap L_2(G,X_o)$

$$<q,r> = <\hat{q},\hat{r}>, \tag{4.6}$$

where $<q,r> = \int_G <q(g),r(g)>_o \mu(dg)$ and $<\hat{q},\hat{r}> = \int_{\hat{G}} <\hat{q}(\gamma), \hat{r}(\gamma)>_o m(d\gamma)$.

Proof: We shall put $\|q\|_1 = \int_G \|q(g)\| \mu(dg)$ and $\|q\|_2 = \{\int_G \|q(g)\|_o^2 \mu(dg)\}^{1/2}$ for $q \in L_1(G,X) \cap L_2(G,X_o)$. Let $p(g) = (q * q^*)(g)$. As $q \in L_1(G,X)$ so is p with $\|p\|_1 \leq \|q\|_1^2$. It can also be shown that $p \in C_o(G,X_o)^+$ as $q \in L_2(G,X_o)$. Now $p(0) = \int_G q(g)q(g)^* \mu(dg) \in K$ so $(p(0),\varphi_o) = (\int_G q(g)q(g)^* \mu(dg),\varphi_o) = \sum_{i=1}^{\infty} c_i \int_G (q(g)q(g)^*,\varphi_i)\mu(dg) = \int_G (q(g)q(g)^*,\varphi_o)\mu(dg) = \int_G \|q(g)\|_o^2 \mu(dg) = \|q\|_2^2 < \infty$ using the monotone convergence theorem. Hence $p \in L_1(G,X) \cap \mathscr{T}_o$.

Now $C_o(G,X_o) \subset C_o(G,X)$ so $p \in L_\infty(G,X)$. Also $\int_G \int_G \alpha(g)\overline{\alpha(g')}p(g-g')\mu(dg)\mu(dg') = \int_G [\int_G \alpha(g)q(g-g'')\mu(dg)][\int_G \alpha(g') q(g'-g'')\mu(dg')]^* \mu(dg'') = \int_G q'(g)q'(g)^* \mu(dg)$ using the Fubini and Tonelli theorems with $\alpha \in L_1(G,C)$. $q' = \alpha * q \in L_2(G,X_o)$ ([8]) so $q'(g) \in X_o$ a.e. or $q'(g)q'(g)^* \in K_o$ a.e. Hence if $\varphi \in \Phi$ then $(\int_G q'(g)q'(g)^* \mu(dg),\varphi) = \int_G (q'(g),q'(g)^*,\varphi)\mu(dg) \geq 0$ or $p \in \mathscr{P}$.

Consequently theorem 3.4 yields $p(g) = \int_{\hat{G}} (g,\gamma)\hat{p}(\gamma)m(d\gamma)$. Then $\infty > \|q\|_2^2 = <q,q> = \sum_{i=1}^{\infty} c_i(p(0),\varphi_i) = \sum_i c_i \int_{\hat{G}} (\hat{p}(\gamma),\varphi_i)m(d\gamma) = \int_{\hat{G}} (\hat{p}(\gamma),\varphi_o)m(d\gamma) = <\hat{q},\hat{q}>$. We have used the monotone convergence theorem again. Hence the Fourier transform maps into $L_2(\hat{G},X_o)$. By the usual expansion $<q,r> = <\hat{q},\hat{r}>$. This establishes (iii).

+If Y is a Banach space then $C_o(G,Y)$ is the space of continuous functions mapping G into Y, which vanish at infinity if G is only locally compact rather than compact.

Moreover $\int_G q(g)q(g)^*\mu(dg) = p(0) = \int_{\hat{G}}\hat{p}(\gamma)m(d\gamma) = \int_{\hat{G}}\hat{q}(\gamma)\hat{q}(\gamma)^*m(d\gamma)$. Also if x,y are elements of a Banach algebra with involution then

$$(4.7)\qquad 4xy^* = (x+y)(x+y)^* - (x-y)(x-y)^* + i(x+iy)(x+iy)^* - i(x-iy)(x-iy)^*$$

so that (ii) is also proved.

We need only show that $Q = \{\hat{q} \in L_2(\hat{G},X_o): q \text{ in } L_1(G,X) \cap L_2(G,X_o)\}$ is dense in $L_2(\hat{G},X_o)$. As μ is translation invariant so is $L_1(G,X) \cap L_2(G,X_o)$ and hence Q is invariant under multiplication by $(g,\cdot)$ for any $g \in G$. If $r \in L_2(\hat{G},X_o)$ and $< q,r > = 0$ for all $q \in Q$, then $\int_{\hat{G}}(q(\gamma)r(\gamma)^*,\varphi_o)(g,\gamma)m(d\gamma) = 0$ for all $q \in Q$ and $g \in G$. As $(q(\cdot)r(\cdot)^*,\varphi_o) \in L_1(\hat{G},C)$ it follows that $(q(\gamma)r(\gamma)^*,\varphi_o) = 0$ a.e. for every $q \in Q$, or $< q(\gamma), r(\gamma) >_o = 0$ a.e. As $L_1(G,X) \cap L_2(G,X_o)$ is invariant under multiplication by $(\cdot,\gamma)$, $\gamma \in \hat{G}$, then Q is invariant under translation.[+] Hence to every $\gamma_o \in \hat{G}$ there corresponds $q_o \in Q$ such that $q_o(\gamma_o) \neq 0$ so $q_o(\gamma) \neq 0$ in a neighborhood of γ_o as q_o is continuous. If $\{e_\alpha\}$ is the basis of X_o mentioned in the statement of part (i), then $q_o(\cdot) = \sum_\alpha q_\alpha(\cdot)e_\alpha$ so there exists α_o such that $q_{\alpha_o}(\gamma) \neq 0$ in a neighborhood of γ_o. If $q_o(\cdot) = \hat{p}(\cdot)$ then $p = \sum_\alpha p_\alpha e_\alpha$ and as $p \in L_2(G,X_o)$, $p_\alpha \in L_2(G,C)$. By hypothesis $|< x,e_\alpha >_o| \leq k_1\|x\|$ so $p_\alpha \in L_1(G,C)$ and $\hat{p}_\alpha(\gamma) = q_\alpha(\gamma)$. Hence

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[+] By this we mean that f_{γ_o} is in Q for any γ_o in $\hat{G}$ if f is in Q and $f_{\gamma_o}(\gamma) = f(\gamma+\gamma_o)$.

$p_{\alpha_o}(\cdot)e_\alpha \in L_1(G,X) \cap L_2(G,X_o)$ for any α and $p_{\alpha_o}e_\alpha(\cdot) = q_{\alpha_o}(\cdot)e_\alpha \in Q$. Since for each γ in a neighborhood of γ_o $\{q_{\alpha_o}(\gamma)e_\alpha\}_\alpha$ forms a complete set in X_o, and since $0 = \langle q_{\alpha_o}(\gamma)e_\alpha, r(\gamma)\rangle_o$, then $r(\gamma) = 0$ in a neighborhood of γ_o. But γ_o was arbitrary so $r = 0$, or Q is orthogonal only to 0 in $L_2(\hat{G},X_o)$, a Hilbert space. Hence Q is dense in $L_2(\hat{G},X_o)$. This completes the proof.

COROLLARY 4.8. Under the assumptions of the theorem the Fourier transform can be extended in a unique manner to an isometry of $L_2(G,X_o)$ onto $L_2(\hat{G},X_o)$.

Proof: We need only show $L_1(G,X) \cap L_2(G,X_o)$ is dense in $L_2(G,X_o)$. But $C_c(G,X_o)$+ is dense in $L_2(G,X_o)$ ([7]). Hence if $f \in L_2(G,X_o)$ then there exists $\{f_n\}_1^\infty \subset C_c(G,X_o) \cap L_2(G,X_o)$ such that $\|f_n - f\|_2 \to 0$. Then $f_n \in C_c(G,X)$ and f_n is measurable so $f_n \in L_1(G,X)$.

Remark: The equality (4.5) holds for all $q,r \in L_2(G,X_o)$.

5. Examples

Here we give some examples of admissible pairs and strongly admissible triplets.

EXAMPLE 5.1. Let $X = L_1([0,1],C)$ so X is weakly complete, and let Φ consist of elements φ_i such that

+$C_c(G,X_o)$ denotes the set of functions in $C_o(G,X_o)$ having compact support.

(5.2) $$(x,\varphi_i) = \int_0^1 \chi_i(t)x(t)dt \qquad x \in X$$

where $\chi_i(\cdot)$ is the indicator function of one of a countable collection of sets $\{E_i\}$ dense in $\Sigma([0,1])$ under the usual Hausdorff metric. Assume $E_1 = [0,1]$. Then it can be shown ([3], [7]) that Φ is full and that K is the cone of non-negative (a.e.) functions. Let $(x,\varphi_0) = (x,\varphi_1) = \int_0^1 x(s)ds = \|x\|_1$ for $x \in K$. Hence (Φ,X) is admissible and $K_0 = K$.

If p is in $\mathscr{P}$ then $p(0)$ is in $K = K_0$ by propositions 2.8 and 2.9 and by corollary 2.15. So $p \in \mathscr{T}_0$ and the inversion theorem states that if $p \in sp\{L_1(G,L_1([0,1],C)) \cap \mathscr{P}\}$ then $\hat{p} \in L_1(\hat{G},L_1([0,1],C))$ and $p(g) = \int_{\hat{G}}(g,\gamma)\hat{p}(\gamma)m(d\gamma)$.

The author does not know of any non-trivial subspace X_0 which would make (Φ,X,X_0) strongly admissible.

EXAMPLE 5.3. Let $X = H$, a separable Hilbert space with a fixed orthonormal basis $\{e_i\}_1^\infty$. Let H_0 be the set of elements of H with all but a finite number of components zero, with non-zero components being real, rational non-negative, and with norm less than or equal to one. Then $\Phi = H_0$ is full ([3], [7]) and countable and $K_\Phi = \{h \in H: h_i \geq 0\}$.⁺ Let $(h,\varphi_i) = \langle h,e_i \rangle$, $i = 1,2,\ldots$ and $\varphi_0 = \sum_1^\infty \varphi_i$. Then φ_0 maps K into $[0,\infty]$, and for h in K

⁺$h_i = \langle h,e_i \rangle$

$$(h,\varphi_o)^2 = (\Sigma\, h_i)^2 \geq \Sigma\, h_i^2 = \|h\|^2$$

so that (Φ,H) is admissible and $K_o = \{h \in K: \sum_1^\infty h_i < \infty\}$.

H becomes a Banach algebra if we define $hk = \sum_1^\infty h_i k_i e_i$. Let $h^* = \Sigma\, \overline{h}_i e_i$. For h in H hh^* is in K and $(hh^*,\varphi_o) = \sum_i h_i \overline{h}_i = \|h\|^2$. We do not have $k\|hh^*\| \geq \|h\|^2$ for some $k > 0$, but we do have $\|h\|_o = \|h\|$ which is sufficient to show that $X_o = H$. Hence (Φ,H,H) is strongly admissible, and the Plancherel theorem applies. Note that the condition $| < h,e_i > | \leq \|h\|$ also holds.

EXAMPLE 5.4. Let $X = \mathcal{L}(H,H)$, the linear bounded operators mapping the separable Hilbert space H into itself. Let H_o be a countable dense subset of the unit ball in H and let $\Phi = \{\varphi \in X^*: (T,\varphi) = < Th,h >$, $T \in \mathcal{L}(H,H),\ h \in H_o\}$. Let $\{e_i\}$ also be in H_o for some orthonormal basis $\{e_i\}$. Then Φ is full and countable and K_Φ is the cone of positive operators ([3] or [7]). Let $(T,\varphi_o) = \sum_1^\infty < Te_i,e_i >$. So $\varphi_o = \sum_1^\infty \varphi_i$ is the trace, where $(T,\varphi_i) = < Te_i,e_i >$. Then $\varphi_o: K \to [0,\infty]$, $(T,\varphi_o) = \operatorname{tr} T \geq \|T\|$ if T is positive. Hence $(\Phi,\mathcal{L}(H,H))$ is admissible and K_o is the cone of positive operators of finite trace and so a subset of the trace class.

We can see that in one case the condition $p \in \mathcal{T}_o$ is necessary for the inversion theorem to hold. Let G be the circle group so that $\hat{G}$ is countable. Label its elements $\gamma_1, \gamma_2, \ldots$, and let the set function ν be given by

(5.5) $$< \nu(\{\gamma_n\})e_i, e_j > = p_n \delta_{ni} \delta_{nj}, ^{+} \quad i,j,n = 1,2,\ldots$$

where $\infty > M \geqq p_n \geqq 0$. ν can be extended to a countably additive measure of finite semi-variation in the obvious way. Let p be given by

(5.6) $$p(t) = \sum_{n=1}^{\infty} e^{it\gamma_n} \nu(\{\gamma_n\})$$

Then p is in $\mathscr{P}$ (theorem 2.12 (A)) and p is in $L_1(G,X)$ because G is compact and $\|p(t)\| \leqq M$. If $\hat{p}$ is to be in $L_1(\hat{G},X)$ then $\|\nu\|(\hat{G})$ must be finite or $\sum_1^{\infty} p_n = \operatorname{tr} p(0) < \infty$.

Finally let $X_o = \mathscr{N}$, the Hilbert-Schmidt operators ([5a]). Then for T in $\mathscr{N}$, TT^* is in the trace class and is positive so that TT^* is in K_o. Also $\mathscr{L}(H,H)$ is a C^*-algebra so $(\Phi, \mathscr{L}(H,H), \mathscr{N})$ is strongly admissible. A basis for $\mathscr{N}$ is given by $\{T_{ij}\}$ where $< T_{ij}e_k, e_\ell > = \delta_{ik}\delta_{j\ell}$, $k,\ell = 1,2,\ldots$. Then $| < T, T_{ij} >_o| = |< Te_i, e_j >| \leqq \|T\|$, and the condition in (i) of theorem 4.4 also holds.

6. On a Theorem of Magnus

We use the preceeding theory to deduce a result of Hewitt and Wigner's ([4]). Let $U(\cdot)$ be a continuous n-dimensional unitary representation of G i.e. $U(g+g') = U(g)U(g')$, $U(0) = I$ and U is a continuous mapping of G into $\mathscr{L}(C^n, C^n)$. Then there is a unitary matrix V and characters $\gamma_1, \ldots, \gamma_n$ such that

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$^{+}\delta_{ni}$ is the Kronecker delta.

$$(6.1)\qquad U(g) = V^{-1}\begin{bmatrix}(g,\gamma_1) & & 0\\ & \ddots & \\ 0 & & (g,\gamma_n)\end{bmatrix}V$$

for all g in G ([9]). Hence $U(\cdot)$ is given as a function of n characters. Let A be a symmetric compact neighborhood of 0 in $\hat{G}$ having finite positive measure, and let $E(\gamma_1,\ldots,\gamma_n)$ be the function on $\hat{G}^n$ which equals 1 if $\gamma_j-\gamma_k$ is in A for all j,k, and equals zero otherwise. Let p be in $L_1(G,\mathcal{L}(C^n,C^n))$ and let

$$(6.2)\qquad \hat{p}(U) = \int_G p(g)U(g)\mu(dg)$$

Theorem 6.3. <u>If</u> p <u>is in</u> $L_1(G,\mathcal{L}(C^n,C^n)) \cap L_\infty(G,\mathcal{L}(C^n,C^n))$ <u>and if for any</u> α <u>in</u> $L_1(G,C)$ $\int_G\int_G \alpha(g)\overline{\alpha(g')}p(g-g')\mu(dg)\mu(dg')$ <u>is positive semidefinite, then there is a constant</u> κ, $0 < \kappa < \infty$ <u>such that</u>

$$(6.4)\qquad \kappa\, p(g) = \int_{\hat{G}^n}\hat{p}(U)U(-g)E(\gamma_1,\ldots,\gamma_n)m(d\gamma_1)\ldots m(d\gamma_n).$$

Proof: Using the setting of example 5.4 with $H = C^n$ we have $p \in \mathscr{P}$. Then p is continuous and trace $p(0) < \infty$ so that p is in $\mathscr{T}_0$. Hence $p(g) = \int_{\hat{G}}(g,\gamma)\hat{p}(\gamma)m(d\gamma)$ by theorem 3.4. But $U(g) = \sum_{i=1}^{n}\overline{(g,\gamma_i)}\pi_i$ where the π_i are projections onto mutually orthogonal one-dimensional subspaces ([9]). Then $\hat{p}(U) = \int_G p(g)U(g)\mu(dg) = \sum \hat{p}(\gamma_i)\pi_i$ and $\hat{p}(U)U(-g) = \sum_{i=1}^{n} \hat{p}(\gamma_i)\pi_i \sum_{j=1}^{n} (g,\gamma_j)\pi_j = \sum_{i=1}^{n} \hat{p}(\gamma_i)(g,\gamma_i)\pi_i$. Now let $\kappa = \int_{\hat{G}^{n-1}} E(\gamma_1,\ldots,\gamma_n)m(d\gamma_1)\ldots m(d\gamma_n)$ where we omit integration with respect to γ_i. Then κ is independent of i and of the value of γ_i by the choice of E, and $0 < \kappa \leq m(A)^{n-1} < \infty$. Now we have

$$\int_{\hat{G}^n} \hat{p}(U)U(-g)E(\gamma_1,\ldots,\gamma_n)m(d\gamma_1)\ldots m(d\gamma_n)$$

$$= \sum_{i=1}^{n} \int_{\hat{G}^n} \hat{p}(\gamma_i)(g,\gamma_i)E(\gamma_1,\ldots,\gamma_n)m(d\gamma_1)\ldots m(d\gamma_n)\pi_i$$

$$= \sum_{i=1}^{n} \kappa \int_{\hat{G}} \hat{p}(\gamma_i)(g,\gamma_i)m(d\gamma_i)\pi_i$$

$$= \kappa\ p(g)$$

and the theorem is established.

Note that p could actually be a finite linear combination of functions satisfying the requirements of the theorem. The extension of this theorem to infinite dimensions will be treated elsewhere.

The other result of [4] is

THEOREM 6.5. _If_ p _is in_ $L_1(G,\mathcal{L}(C^n,C^n)) \cap L_2(G,\mathcal{L}(C^n,C^n))$, _then_

(6.6) $$\kappa\ \underline{\text{trace}} \int_G p(g)p(g)^*\mu(dg) =$$

$$\underline{\text{trace}} \int_{\hat{G}^n} \hat{p}(U)\hat{p}(U)^* E(\gamma_1,\ldots,\gamma_n)m(d\gamma_1)\ldots m(d\gamma_n)$$

Proof: The method of proof is similar to the one given above but uses Plancherel's theorem which yields

$$\int_G p(g)p(g)^*\mu(dg) = \int_{\hat{G}} \hat{p}(\gamma)\hat{p}(\gamma)^* m(d\gamma).$$

7. The Maximal Ideals of $L_2(G,\mathcal{N})$

Assume G is a compact abelian group and $\mathcal{N}$ is the space of Hilbert-Schmidt operators in $\mathcal{L}(H,H)$, where H is a separable Hilbert space. We show that the (closed) maximal ideals of $L_2(G,\mathcal{N})$ correspond to $\hat{G}$.

LEMMA 7.1. *If h is a continuous $*$-homomorphism[+] of $L_2(G,\mathcal{N})$ onto $\mathcal{N}$, then $M = \text{kernel}(h)$ is a maximal closed self-adjoint ideal such that $M^\perp$ is isometrically $*$-isomorphic to $\mathcal{N}$.*

Proof: h is continuous so M is a closed 2-sided ideal. M is self adjoint as h is a $*$-homomorphism. $\mathcal{N}$ is a full matrix algebra so it is a simple H^*-algebra. Also $\mathcal{N}$ and $L_2(G,\mathcal{N})/M$ i.e. $M^\perp$ are homeomorphically $*$-isomorphic ([9], p. 181), so $M^\perp$ is a minimal closed ideal and M is a maximal ideal.

Note that if H is infinite dimensional, then M is not regular for if it were there would exist $p \in L_2(G,\mathcal{N})$ such that for all $q \in L_2(G,\mathcal{N})$ $p*q-q \in M$ or $h(p)h(q) = h(q)$. This means $h(p)$ would be an identity in $\mathcal{N}$, but the identity $I \in \mathcal{L}(H,H)$ is not in

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[+]If the isomorphism takes $x \to T$ then $x^* \to T^*$ the adjoint of T.

$\mathcal{N}$. If H is finite dimensional then such a p exists as I is in $\mathcal{N}$ and h is onto. ♠

THEOREM 7.2. <u>To every γ in $\hat{G}$ there corresponds a closed maximal ideal M of $L_2(G,\mathcal{N})$ given by $M = \text{kernel}(h)$ and</u>

(7.3) $$h(p) = \hat{p}(\gamma).$$

Proof. Fix γ in $\hat{G}$ and define h by 7.3. As G is compact then $L_2(G,\mathcal{N}) \subset L_1(G,\mathcal{N})$ and so $\hat{p}$ is given by the usual integral if p is in $L_2(G,\mathcal{N})$. Direct computation shows h is a $*$-homomorphism with norm less than or equal to 1. We need only show h is onto, then the result follows by the preceeding lemma.

Given U in $\mathcal{N}$ let $q(\lambda) = \begin{cases} U & \lambda = \gamma \\ 0 & \lambda \neq \gamma \end{cases}$. Then q is in $L_2(\hat{G},\mathcal{N})$. (N.B. $\hat{G}$ is discrete, $m(\{\gamma\}) = 1$). By the Plancherel theorem there exists p in $L_2(G,\mathcal{N})$ such that $\hat{p} = q$. Then $\hat{p}(\gamma) = U$ or h is onto.

Observe that $M = \ker(h) = \{p\colon \hat{p}(\gamma) = 0\}$ or $M = \{p \in L_2(G,\mathcal{N})\colon \langle p(\cdot)e_j, e_i\rangle = p_{ij}(\cdot) \in M_\gamma,\ i,j = 1\ldots\}$ where M_γ is the ideal in $L_2(G,C)$ corresponding to γ in $\hat{G}$. (N.B. There is a 1-1 correspondence between $\hat{G}$ and the maximal ideals of $L_2(G,C)$ ([1]).) As $\|p\|_2^2 = \int_G \sum_{ij} |p_{ij}(g)|^2 \mu(dg) = \sum_{ij} \|p_{ij}\|_2^2$, we can say that $M = M_\gamma \times M_\gamma \times \ldots$

LEMMA 7.4. <u>Let P_{ij} be the projection of $\mathcal{N}$ onto the ij^{th} basis element. If M is an ideal in $L_2(G,\mathcal{N})$ then $P_{ij}M$ is an ideal in $L_2(G,C)$.</u>

Proof: The basis elements b_{ij} of $\mathcal{N}$ are determined by $< b_{ij}e_r, e_s > = \delta_{is}\delta_{jr}$ where $\{e_r\}$ is a fixed basis of H. Then $P_{ij}p = < p,b_{ij} >_o = \sum_{\ell} < pb^*_{ij}e_\ell, e_\ell > = \sum_{rs} p_{rs}(b_{ij})_{rs} = p_{ij}$. Let α be an element of $L_2(G,C)$ and set $q(\cdot) = \alpha(\cdot)b_{jj}$. Then $p*q$ is in M if p is in M as M is an ideal. Hence

$$P_{ij}(p*q) = \sum_{\ell} p_{i\ell}*q_{\ell j} = \sum_{\ell} p_{i\ell}*\alpha\delta_{\ell j} = p_{ij} * \alpha$$

or $p_{ij} *\alpha$ is in $P_{ij}M$.

Let us write $P_{ij}M = M_{ij}$.

LEMMA 7.5. <u>If M is a closed ideal in $L_2(G,\mathcal{N})$ then the M_{ij} are all identical and closed.</u>

Proof: First we show M_{ij} is closed. Assume $\alpha_n \in M_{ij}$, $\alpha_n \to \alpha$. Then $\alpha_n b_{ij} \to \alpha b_{ij}$. Let $\alpha_n b_{ij} = p_n \in L_2(G,\mathcal{N})$. Then $P_{rs}(p_n) = 0$ for $r \neq i$, $s \neq j$, and $0 \in M_{rs}$, so $P_{rs}(p_n) \in M_{rs}$ for all r,s. Hence p_n is in M. Now $\|p_n-p_m\|_2 = \|\alpha_n-\alpha_m\|_2$ and so p_n is Cauchy, hence converges to an element p of M as M is closed. Also $P_{ij}(p_n) \to P_{ij}(p)$ by continuity of projections. Hence $\alpha = P_{ij}(p) \in M_{ij}$, or M_{ij} is closed.

Now if $p \in M$, $q \in L_2(G,\mathcal{N})$ then $p*q \in M$ and $P_{ij}(p*q) \in M_{ij}$ or $\sum_{\ell} q_{i\ell}*p_{\ell j} \in M_{ij}$. But $q_{i\ell}*p_{\ell j} \in M_{\ell j}$, as $p_{\ell j} \in M_{\ell j}$, so if φ_ϵ in $L_2(G,C)$ is an approximate identity for $L_2(G,C)$, ([1]), and if $q = \varphi_\epsilon(\cdot)b_{ik}$, then the dense subset $\{\varphi_\epsilon * p_{kj}: p \in M, \epsilon = \frac{1}{n}, n = 1,2,\ldots\}$ of M_{kj} is also a subset of M_{ij}. As M_{ij} is closed then $M_{kj} \subset M_{ij}$ for any i,j,k. So $M_{ij} = M_{kj}$ for any i,k,j. Now using $q*p$ we obtain $M_{ij} = M_{ik}$ for any i,k,j, so the M_{ij} are all identical.

THEOREM 7.6. <u>There is a 1-1 correspondence between the closed maximal ideals of $L_2(G,\mathcal{N})$ and $\hat{G}$, i.e. the regular maximal ideals of $L_2(G,C)$ or $L_1(G,C)$. This correspondence is given by</u>

(7.7) $$M_\gamma = \{p \in L_2(G,\mathcal{N}): \hat{p}(\gamma) = 0\}.$$

Proof: By Theorem 7.4 we know every γ corresponds to a closed maximal ideal in $L_2(G,\mathcal{N})$ and 7.7 describes this correspondence. Conversely if $M \in L_2(G,\mathcal{N})$ is closed, maximal then there exists M_o, a closed ideal in $L_2(G,C)$ and $M = \{p \in L_2(G,\mathcal{N}): p_{ij} \in M_o\}$. As M_o is a closed ideal it can be written as

$$M_o = \sum_{i \in I} \oplus N_i = \bigcap_{i \notin I} N_i^\perp$$

where $\{N_i\}_1^\infty$ are the minimal ideals of $L_2(G,C)$ and $N_i \subset M_o$ for

$i \in I$ ([1]) and $N_i^{\perp}$, the orthogonal complement of N_i, is a regular maximal ideal. If M_o is not a regular maximal ideal, then $M_o \subset M_1$, $M_o \neq M_1$ where M_1 is a regular maximal ideal. But then $\hat{\gamma}$ corresponding to M_1 gives rise to a closed maximal ideal $\tilde{M} \in L_2(G,\mathcal{N})$ and $\tilde{M} \supset M$, $\tilde{M} \neq M$. This contradicts the maximality of M. Hence M_o is a regular maximal ideal, and moreover, 7.7 holds. This proves the theorem.

We note that if H is infinite dimensional then $\mathcal{N}$ is, and none of the closed maximal ideals are regular, whereas if H is finite dimensional all are. We also note that if H has dimension $n < \infty$ then by an argument similar to the one in [1], page 161, the closed ideals of $L_1(G,\mathcal{L}(H,H)) = L_1(G,\mathcal{N})$ correspond in a one to one fashion to the closed ideals of $L_2(G,\mathcal{N})$ and so the maximal ideals of $L_1(G,\mathcal{L}(H,H))$ can be studied through the transform on $\hat{G}$. Unfortunately we cannot prove this for non-compact groups.

8. Convolution Equations for Operators

The above theory can be used to solve operator integral equations much as in the scalar case. Let G be a locally compact, abelian, σ-finite group, H be a separable Hilbert space, and $\mathcal{L}(H,H)$, $\mathcal{N}$ be as before.

PROPOSITION 8.1. If $q \in L_2(G,\mathcal{N})$, $p \in L_1(G,\mathcal{L}(H,H))$ then $p*q \in L_2(G,\mathcal{N})$ and $\|p*q\|_2 \leq \|p\|_1\|q\|_2$.

Proof: This is straightforward and will be omitted. See also [7].

Consider now

$$q(g) = \int_G p(g-g')q(g')\mu(dg') + r(g) \tag{8.2}$$

or equivalently

$$q = p*q + r \tag{8.3}$$

where $p \in L_1(G,\mathcal{L}(H,H))$, $r \in L_2(G,\mathcal{N})$. We are looking for solutions q of 8.3 in $L_2(G,\mathcal{N})$.

THEOREM 8.4. <u>If</u> r <u>is in</u> $L_2(G,\mathcal{N})$, p <u>is in</u> $L_1(G,\mathcal{L}(H,H))$ <u>and if</u> $\sup_{\gamma\in\hat{G}} \|\hat{p}(\gamma)\| < 1$ <u>then</u> 8.3 <u>has a solution in</u> $L_2(G,\mathcal{N})$.

Note that $\|p\|_1 \geq \|\hat{p}(\gamma)\|$, $\gamma \in \hat{G}$.

Proof: Consider $I-\hat{p}(\gamma)$. As $\|\hat{p}(\gamma)\| < 1$ we know that $(I-\hat{p}(\gamma))^{-1}$ exists for each $\gamma \in \hat{G}$ and $\|(I-\hat{p}(\gamma))^{-1}\| \leq (1-\|\hat{p}(\gamma)\|)^{-1}$. It follows $(I-\hat{p}(\cdot))^{-1} \in L_\infty(\hat{G},\mathcal{L}(H,H))$ and so $\|(I-\hat{p}(\cdot))^{-1}\hat{r}(\cdot)\|_2 \leq \|(I-\hat{p}(\cdot))^{-1}\|_\infty\|\hat{r}(\cdot)\|_2$. Hence there exists $q \in L_2(G,\mathcal{N})$ such that $\hat{q}(\cdot) = (I-\hat{p}(\cdot))^{-1}\hat{r}(\cdot)$ by the Plancherel theorem. Let $w(g) = (p*q)(g)$ so $w \in L_2(G,\mathcal{N})$ by proposition 8.1. It can be shown by an approximation argument that $\widehat{p*q}(\gamma) = \hat{p}(\gamma)\hat{q}(\gamma)$. Then $\widehat{r+w} = \hat{r}+\hat{w} = \hat{r}+\hat{p}\hat{q} = (I+\hat{p}(\cdot)(I-\hat{p}(\cdot))^{-1})\hat{r} = (I-\hat{p})^{-1}\hat{r} = \hat{q}$. Hence q satisfies 8.3.

COROLLARY 8.5. <u>The above solution is unique in</u> $L_2(G,\mathcal{N})$.

Proof: If q_o is any other solution of $(8.\hat{3})$ in $L_2(G,\mathcal{N})$ then $\hat{q}_o = \hat{r}+\hat{p}\hat{q}_o$ so $\hat{q}_o = (I-\hat{p})^{-1}\hat{r} = \hat{q}$ or $q_o = q$.

We wish to extend the above theorem to cases where $\|p\|_1 \geq 1$. This can be done by utilizing some results due to Falb and Freedman ([8]). Let W be the set of all continuous linear operators Z mapping $L_2(G,\mathcal{N})$ into itself such that there is a uniformly continuous function $z(\cdot)$ mapping $\hat{G}$ into $\mathcal{L}(\mathcal{N},\mathcal{N})$ with $\widehat{Zp}(\gamma) = z(\gamma)\hat{p}(\gamma)$ for all γ in $\hat{G}$, all p in $L_2(G,\mathcal{N})$. We use the norm

(8.6)
$$\|Z\|_W = \sup_{\gamma\in\hat{G}} \|z(\gamma)\|_{\mathcal{L}(\mathcal{N},\mathcal{N})}$$

where $\|x\|^2_{\mathcal{N}} = \sum_i \|xe_i\|^2$ for x in $\mathcal{N}$. For p in $L_2(G,\mathcal{N})$ $p(g) = \hat{\hat{p}}(-g)$ for almost all g in G. Also $\mathcal{N}$ is a B-algebra so W is a B-algebra by the same proof as in [8].

Let B be given by

(8.7)
$$B = \{T \in \mathcal{L}(L_2(G,H), L_2(G,H)):\ Tx(g) = \int_G p(g-g')x(g')dg' + \lambda x(g) \text{ for some } p \in L_1(G,\mathcal{L}(H,H)) \text{ and } \lambda \in C\}.$$

We see that under the norm $\|\cdot\|_B$, given by $\|T\|_B = \|p\|_1+|\lambda|$, B becomes a Banach space, in fact a B-algebra isometrically isomorphic to $L_1(G,\mathcal{L}(H,H)) \oplus C$. Also if $T = (p,\lambda)$ then $\hat{T}(\gamma) = \hat{p}(\gamma)+\lambda I$.

We shall now identify B with $\tilde{B}$, a B-algebra of linear operators of $L_2(G,\mathcal{N})$ into itself. For h in H, p in $L_2(G,\mathcal{N})$, g in G, and T in B let $\tilde{T}$ be defined by

$$(8.8) \qquad (\tilde{T}p)(g)h = T(p(\cdot)h)(g),$$

so if $T = (q,\lambda)$ then $\tilde{T}p = q*p+\lambda p$ and $\|\tilde{T}\| = \|T\|_B$. Hence $\tilde{B}$ and B are isometrically isomorphic (in the algebra sense). As $\tilde{T}p = \hat{q}\hat{p} + \lambda\hat{p}$ by proposition 8.1, we have $\tilde{B} \subset W$, although the norms are different.

Let $\mathcal{M}$ be the maximal ideal space of $L_1(G,C) \oplus C$ (or just $L_1(G,C)$ if G is discrete), so we can put $\mathcal{M} \cong \hat{G} \cup \{\infty\}$, the one point compactification of $\hat{G}$. Then define $\sigma(\hat{T}(\gamma)) = \{\lambda: \hat{T}(\gamma)-\lambda I$ does not have an inverse in $\mathcal{L}(H,H)\}$. Also $\Sigma_{\tilde{B}}(\tilde{T}) = \{\lambda: \tilde{T}-\lambda$ does not have an inverse in $\tilde{B}\}$, $\Sigma_W(Z) = \{\lambda: Z-\lambda$ does not have an inverse in $W\}$ and $\Sigma_B(T) = \{\lambda: T-\lambda$ does not have an inverse in $B\}$. Evidently $\Sigma_B(T) = \Sigma_{\tilde{B}}(\tilde{T})$. As B, W and $\tilde{B}$ have identities then $T-\lambda$, $Z-\lambda$ and $\tilde{T}-\lambda$ are defined for λ in C.

DEFINITION 8.9. Let T be in B and let $\{e_i\}$ be an orthonormal basis of H. Let $H_n = \text{span}\{e_1,\ldots,e_n\}$ and let E_n be the projection of H onto H_n. Then $T_n = E_nTE_n$ is in B and T is approximable if $\hat{T}_n(\gamma)$ converges to $\hat{T}(\gamma)$ uniformly on $\hat{G} \cup \{\infty\}$.

PROPOSITION 8.10. _T in B is approximable if and only if each_ $\hat{T}(\gamma)$ _is a completely continuous element of_ $\mathcal{L}(H,H)$ _for each_ γ _in_ $\hat{G} \cup \{\infty\}$, _and the map_ $\gamma \to \hat{T}(\gamma)$ _is continuous on_ $\hat{G} \cup \{\infty\}$.

Proof: See [8].

Now we have

THEOREM 8.11. _If T in B is approximable, then_ $\Sigma_W(\tilde{T}) \subset \bigcup_{\gamma \in \hat{G} \cup \{\infty\}} \sigma(\hat{T}(\gamma)) \subset \Sigma_B(T) = \Sigma_{\tilde{B}}(\tilde{T})$.

Proof: The proof is the same as that given in [8] for $L_2(G,H)$ rather than $L_2(G,\mathcal{N})$. We need only note if $x \in \mathcal{N}$, $A \in \mathcal{L}(H,H)$ then $\|Ax\|_{\mathcal{N}} \leq \|A\|\|x\|_{\mathcal{N}}$ so $A \in \mathcal{L}(\mathcal{N},\mathcal{N})$ and in fact $\|A\|_{\mathcal{L}(H,H)} = \|A\|_{\mathcal{L}(\mathcal{N},\mathcal{N})}$ so that $\sup_{\gamma \in \hat{G}} \|\hat{T}(\gamma)\|_{\mathcal{L}(H,H)} = \sup_{\gamma \in \hat{G}} \|\hat{T}(\gamma)\|_{\mathcal{L}(\mathcal{N},\mathcal{N})}$. For more details see [7] and [8].

We say p in $L_1(G,\mathcal{L}(H,H))$ is approximable if the corresponding element $(p,0)$ in B is.

THEOREM 8.12. _Let p in_ $L_1(G,\mathcal{L}(H,H))$ _be approximable, let r be in_ $L_2(G,\mathcal{N})$, _and let_ $1 \notin D$, _a domain containing_ $\bigcup_{\gamma \in \hat{G}} \sigma(\hat{p}(\gamma))$ _in its interior. Then 8.3 has a unique solution in_ $L_2(G,\mathcal{N})$.

REMARK. We note first that if $p \in L_1(G,\mathcal{L}(H,H))$ then $\hat{p}$ is in $C_0(\hat{G},\mathcal{L}(H,H))$ so $\hat{p}(\infty) = 0$. Hence p is approximable if and only if $\hat{p}(\gamma)$ is a completely continuous element of $\mathcal{L}(H,H)$ for every $\gamma \in \hat{G}$.

Proof: $\bigcup_{\gamma\in\hat{G}\cup\{\infty\}} \sigma(\hat{p}(\gamma)) = \bigcup_{\gamma\in\hat{G}} \sigma(\hat{p}(\gamma)) \cup \{0\}$ so we can extend D to D' such that $1 \notin D'$ and D' contains $\bigcup_{\gamma\in\hat{G}\cup\{\infty\}} \sigma(\hat{p}(\gamma))$ in its interior. Now we can define $F(p) \in W$ where $F(t) = (1-t)^{-1}$ is analytic on D', a domain containing $\Sigma_W(p)$. If Δ is the identity in W then $F(p) = (\Delta-p)^{-1} \in W$ and $F(\hat{p}(\gamma)) = (I-\hat{p}(\gamma))^{-1}$, $\gamma \in \hat{G}$, ([9], page 203). If Γ is a simple closed rectifiable curve enclosing $\bigcup_{\gamma\in\hat{G}\cup\{\infty\}} \sigma(\hat{p}(\gamma))$ in D' then we have for $x \in L_2(G,\mathcal{N})$

$$(8.13)\qquad \widehat{F(p)x}(\gamma) = \frac{1}{2\pi i} \widehat{\int_\Gamma F(t)(t\Delta-p)^{-1}dt\, x}\,(\gamma)$$

$$= \frac{1}{2\pi i} \int_\Gamma F(t)\widehat{(t\Delta-p)^{-1}x}(\gamma)dt$$

$$= \frac{1}{2\pi i} \int_\Gamma F(t)(t-\hat{p}(\gamma))^{-1}dt\; \hat{x}(\gamma)$$

$$= F(\hat{p}(\gamma))\hat{x}(\gamma).$$

Hence if $r \in L_2(G,\mathcal{N})$ and $q = F(p)r \in L_2(G,\mathcal{N})$ then $\hat{q}(r) = F(\hat{p}(r))\hat{r}(\gamma) = (I-\hat{p}(\gamma))^{-1}\hat{r}(\gamma)$. Consequently $r+p*q = \hat{r}+\hat{p}\hat{q} = \hat{r}+\hat{p}(I-\hat{p})^{-1}\hat{r} = (I-\hat{p})^{-1}\hat{r} = \hat{q}$ so by the Plancherel theorem q is a solution of 8.3.

Uniqueness can be proved by the method of corollary 8.5.

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