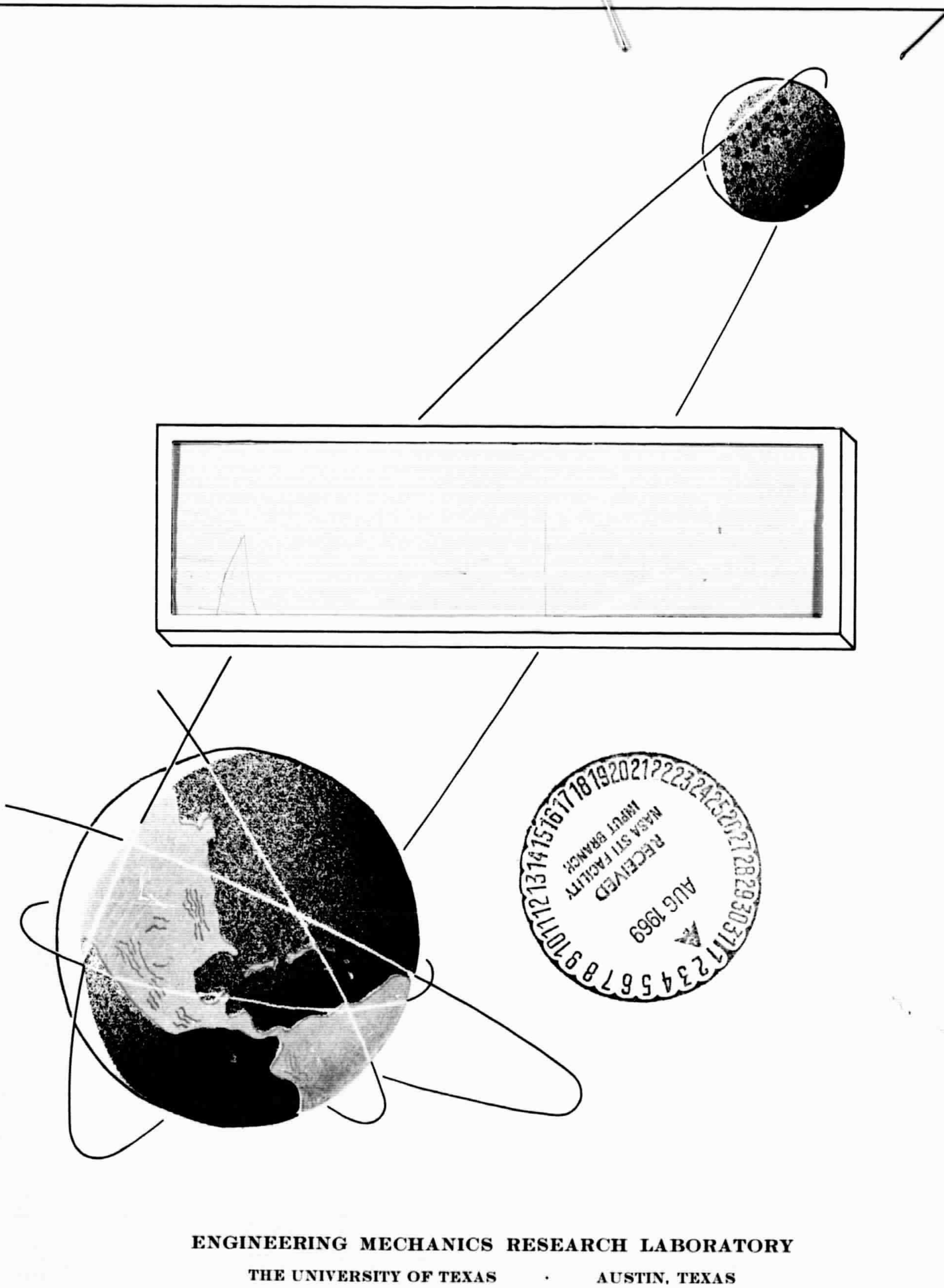

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ON FIXED LINEAR SYSTEMS WITH A GENERALIZED PERFORMANCE CRITERIA

by

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ABSTRACT

This paper considers optimization and sensitivity problems as related to a fixed linear system with a generalized performance criteria. A technique is used which solves for the performance index as a polynomial in the system initial conditions. The method is not restricted to quadratic form loss functions, but applies to any index whose integrand is the product of a time function and a homogeneous polynomial in the states and controls.

I. INTRODUCTION

Considerable research is currently in progress concerning problems that come under the broad headings of optimization and sensitivity studies. These problems, however, are difficult in their complete generality while significant results have been obtained by considering restricted systems, in particular, systems linear in the states and controls. Methods have been found to design optimally linear plants with quadratic loss functions, with some solutions [1,2] assuming the states to be continuously measurable and others [3] optimizing with respect to a set of control release coordinates, the controller having been synthesized by classical means. Controls that are in some sense both optimal and insensitive were generated in [4] by adding a weighted sensitivity function to the performance index. This paper utilizes the results of [3] and [4] and extends them in that the integrand of the performance index is not restricted to a quadratic form but may be a time function multiplying a homogeneous polynomial in the states and controls. The technique is simple in that it consists mainly of a gradient operation and a matrix inversion, but may be tedious, since the matrices involved may be non-numerical in nature.

II. STATEMENT OF THE PROBLEM

The problem may be stated as follows. A plant and controller are given which may be described by the linear state equations

$$\dot{x} = Ax + Bu \qquad (1)$$

$$\dot{u} = Cx + Du \qquad (2)$$

where x and $\dot{x}$ are n-vectors, u and $\dot{u}$ are m-vectors and A, B, C and D are $n \times n$, $n \times m$, $m \times n$ and $m \times m$ time invariant matrices respectively. The optimization problem is to find the control initial condition u(0) so as to minimize a performance measure of the form

$$J = \int_0^\infty f(t)Q(x,u)dt \qquad (3)$$

where f(t) is a Laplace transformable function of time and Q(x,u) is a homogeneous polynomial in x and u. The "optimally-insensitive" solution will be that initial condition u(0) which makes a measure of the form of (3) small yet relatively insensitive to variations of parameters in the matrices of (1) and (2). The importance of the value of the index relative to its sensitivity to parameter variations will be decided by a sensitivity weighting matrix, much as the coefficients of the polynomial Q(x,u) determine the importance of the states vs. controls.

III. PROBLEM SOLUTION

MacFarlane [5] has devised a technique to calculate functionals of the form

$$J = \int_0^\infty f(t)R(x)dt \tag{4}$$

for a system

$$\dot{x} = Fx \tag{5}$$

where f(t) is Laplace transformable, R(x) is a homogeneous polynomial in x, and the system of (5) is such as to insure the existence of the integral of (4). Application to the problem of (1), (2) and (3) is immediate, simply considering the augmented system

$$\dot{y} = \begin{bmatrix} A & \vdots & B \\ \cdots & \vdots & \cdots \\ C & \vdots & D \end{bmatrix} y$$

$$= Hy \tag{6}$$

where y and $\dot{y}$ are m+n vectors and H is an (m+n) x (m+n) matrix. It is assumed that the augmented matrix H assures the existence of the integral of (3) and is stable. This assumption of the existence of the integ-al may be more restrictive than the requirement that H be stable, since the addition of an explicit dependence on time in the integrand may cause divergence of the integral even for an asymptotically stable system. Conversely, certain time-weighting functions may cause the integral to exist even for unstable systems.

Using the results of [5] directly, it may be shown that

$$\int_0^\infty Q(x,u)dt = -P(x(0),u(0)) \tag{7}$$

where P(x, u) satisfies

$$Q(x,u) = \frac{\partial P(x,u)}{\partial x}[Ax + Bu] + \frac{\partial P(x,u)}{\partial u}[Cx + Du],$$

or

$$Q(x,u) = \left(\frac{\partial P}{\partial x}\right)^T Hy, \tag{8}$$

superscript T denoting transpose, and that P(x, u) will be a homogeneous polynomial in x and u of the same degree as Q(x, u). Defining the vector z so that each of its components is of the form of a term of Q(x, u), then for appropriate constant vectors p and q,

$$Q(x,u) = q^T z \tag{9}$$

and

$$P(x,u) = p^T z. \tag{10}$$

Using the identity (8), equating the coefficients of like terms of z yields a matrix M such that

$$q = M^T p. \tag{11}$$

For H a stable matrix, Lyapunov (see [6]) has shown that (8) has a unique solution for P(x, u), and hence the transformation M is non-singular giving

$$p = (M^T)^{-1} q. \tag{12}$$

In this manner, the coefficients of P(x, u) may be determined from those of Q(x, u).

Since $d(P(x,u))/dt = Q(x,u)$,

$$\frac{d}{dt}(p^T z) = q^T z$$

$$= (M^T p)^T z$$

$$= p^T M z, \tag{13}$$

and (13) is satisfied by

$$\dot{z} = Mz \tag{14}$$

giving

$$z = e^{Mt} z(0). \tag{15}$$

Using (9) and (15) in (3),

$$J = \int_0^\infty f(t) q^T e^{Mt} z(0) dt$$

$$= q^T F(M) z(0) \tag{16}$$

where

$$F(M) = \int_0^\infty f(t) e^{Mt} dt. \tag{17}$$

As is discussed in [5], $F(M)$ is easily evaluated by noting that

$$F(-M) = \int_0^\infty f(t) e^{-Mt} dt,$$

which may be thought of as the matrix equivalent of the Laplace transform. Hence, examples would be

$f(t) = 1$	$F(M) = -M^{-1}$
$f(t) = e^{-at}$	$F(M) = -[M-aI]^{-1}$
$f(t) = t^r$	$F(M) = r![-M^{-1}]^{r+1}$
$f(t) = \sin(at)$	$F(M) = a[M^2+a^2I]^{-1}$

where I denotes the identity matrix.

With the index given in the form of (16), the solution of the system of algebraic equations

$$\frac{\partial J}{\partial u(0)} = 0 \qquad (18)$$

will yield necessary conditions for the existence of a minimum of J.

If information concerning the sensitivity of the index with respect to certain variable parameters is desired, evaluation of J is accomplished as above, except that the matrix to be inverted will now be non-numerical. Thus, the index will be given by (16) as ratios of polynomials in parameters of the matrices of (1) and (2) multiplying control initial conditions. At this point, considerable information concerning the sensitivity of the system is readily available. Specifically, the infinitesimal performance index sensitivity is given by

$$\left.\frac{\partial J}{\partial f}\right|_{f \text{ nominal}} \qquad (19)$$

where f is any one of the parameters. Unfortunately, this information is often of limited value since it only serves as some indication of the effects of a finite parameter change. These finite changes can be investigated, however, with the index given in essentially polynomial form. For the variable f and some maximum permissible index value J max, an inequality solution of (16) will yield constants f_1 and f_2 such that

$$J \leq J \max \tag{20}$$

if

$$f_1 \leq f \leq f_2 \, . \tag{21}$$

Additional information might be desired concerning the effects of variations in state and control initial conditions. This is again simple and, for example, for a single state - single control system with a homogeneous polynomial of order 4 in the loss function, (16) will yield constants $\beta_0, \beta_1, \ldots, \beta_4$ such that

$$\Delta J = \sum_{i=0}^{4} \beta_i [x(0)^i u(0)^{4-i} - x(0)^i_{nominal} u(0)^{4-i}_{nominal}] \tag{22}$$

where

$$\Delta J = J - J_{nominal} \tag{23}$$

Although the above information is doubtless of value, it is basically of an analysis nature yielding little information concerning the design of a more acceptable solution. This latter problem was investigated in [4] by the addition of a sensitivity term to the performance index. Although the controls were restricted to a form

$$u = Kx \tag{24}$$

with a quadratic loss function, the results are here extended to include controls of the form of (2) with loss functions of the form of (3).

It is assumed that the matrices of (1) and (2) contain an ℓ-vector, w, of variable parameters with nominal value w^o. Defining the sensitivity of the performance index with respect to changes in the parameter vector as

$$\frac{\partial I}{\partial w} = \left[\frac{\partial I}{\partial w_1} \; \frac{\partial I}{\partial w_2} \; \cdots \; \frac{\partial I}{\partial w_i}\right]^T \tag{25}$$

a new index is formed

$$I = \int_0^\infty f(t)Q(x,u)dt + \left(\frac{\partial I}{\partial w}\right)^T G \left(\frac{\partial I}{\partial w}\right)$$

$$= J + \left(\frac{\partial I}{\partial w}\right)^T G \left(\frac{\partial I}{\partial w}\right), \tag{26}$$

where G is a positive semi-definite $\ell \times \ell$ weighting matrix and evaluation is made at $w = w^o$. The G matrix determines the importance of the sensitivity of the performance index relative to its actual value. That is, as the norm of G approaches infinity, the problem is concerned completely with sensitivity while if the norm approaches zero, the problem is again a pure optimization.

To find the initial conditions u(0) minimizing I, the integral portion of (26) is evaluated as a function of w and u(0). The gradient operation is performed, evaluation is made at $w = w^o$, and there results I as a function only of the control initial conditions. Hence, the solution of

$$\frac{\partial I}{\partial u(0)} = 0 \tag{27}$$

will yield necessary conditions for a minimum of I.

IV. EXAMPLE PROBLEM

As an example of a non-quadratic time-weighted performance index, consider the system

$$\dot{x} = -x + u$$

$$\dot{u} = -x - u$$

$$x(0) = 1$$

with the index defined by

$$J = \int_0^\infty e^{-t}(x^4 + u^4)dt.$$

Proceeding with the solution,

$$q^T = [\, 1\ 0\ 0\ 0\ 1 \,],$$

$$z = \begin{bmatrix} x^4 \\ x^3 u \\ x^2 u^2 \\ x u^3 \\ u^4 \end{bmatrix},$$

$$M^T = \begin{bmatrix} -4 & -1 & 0 & 0 & 0 \\ 4 & -4 & -2 & 0 & 0 \\ 0 & 3 & -4 & -3 & 0 \\ 0 & 0 & 2 & -4 & -4 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$$

and

$$F(M) = \begin{bmatrix} 0.176 & 0.118 & 0.059 & 0.020 & 0.004 \\ -0.029 & 0.147 & 0.073 & 0.025 & 0.005 \\ 0.098 & -0.049 & 0.141 & 0.049 & 0.010 \\ -0.005 & 0.025 & -0.073 & 0.147 & 0.029 \\ 0.004 & -0.020 & 0.059 & -0.118 & 0.176 \end{bmatrix}.$$

From (16),

$$J = 0.180u(0)^4 - 0.098u(0)^3 + 0.118u(0)^2 + 0.098u(0) + 0.180,$$

and (18) yields a minimum of J at u(0) = -0.267. This gives an index value of J = 0.166, compared with that of J = 0.180 for starting the control at the origin.

Now from the sensitivity viewpoint, the previous problem will be reconsidered including the effects of a variable parameter. That is, given the system

$$\dot{x} = -x + wu$$

$$\dot{u} = -x - u$$

$$x(0) = 1$$

$$w^o = 1$$

with original index

$$J = \int_o^\infty e^{-t}(x^4 + u^4)dt,$$

it is desired to minimize the new index

$$I = J|_{w=1} + 10\left(\frac{\partial J}{\partial w}\right)^2 |_{w=1}.$$

Evaluation of the integral portion of I by (16) yields

$$J = \frac{1}{5(64w^2+500w+625)} [(24w^2+400w+649) + (200w^2+500w+120)u(0)$$

$$+ (48w^3+300w^2+48w+300)u(0)^2+(120w^3-200w-500)u(0)^3$$

$$+ (24w^4+24w^2+400w+625)u(0)^4].$$

Taking the derivative with respect to w and evaluating at $w = 1$, I is given by

$$I = \frac{1}{5(1189)}\,[1073 + 580u(0) + 696u(0)^2 - 580u(0)^3 + 1073u(0)^4]$$
$$+ \left\{\frac{1}{5(1189)^2}\,[1189(448 + 900u(0) + 792u(0)^2 + 130u(0)^3 + 544u(0)^4)\right.$$
$$\left. - 628(1073 + 580u(0) + 696u(0)^2 - 580u(0)^3 + 1073u(0)^4)]\right\}^2 \quad (10).$$

Finally, the minimum of I is found to be 0.180 for an initial condition $u(0) = -0.144$. A graph of I and J versus u(0) is given in Figure 1.

V. CONCLUSIONS AND DISCUSSION

A method has been presented which allows the evaluation of a generalized performance index in terms of initial conditions and system parameters. This evaluation allows sensitivity analysis and the generation of control initial conditions that are either optimal or "optimally-insensitive". While the method is rather complex algebraically, some recent papers have been concerned with the reduction of the complexity of the problem. Specifically, [7] gives a method for evaluating M^{-1} which depends upon finding the eigenvalues and eigenvectors of F, while [8] gives an algorithm for finding M efficiently and finding M^{-1} by inverting matrices of smaller order than M.

Of course, the major disadvantage of this process is that the control must be designed "a priori". With this in mind, an attempt

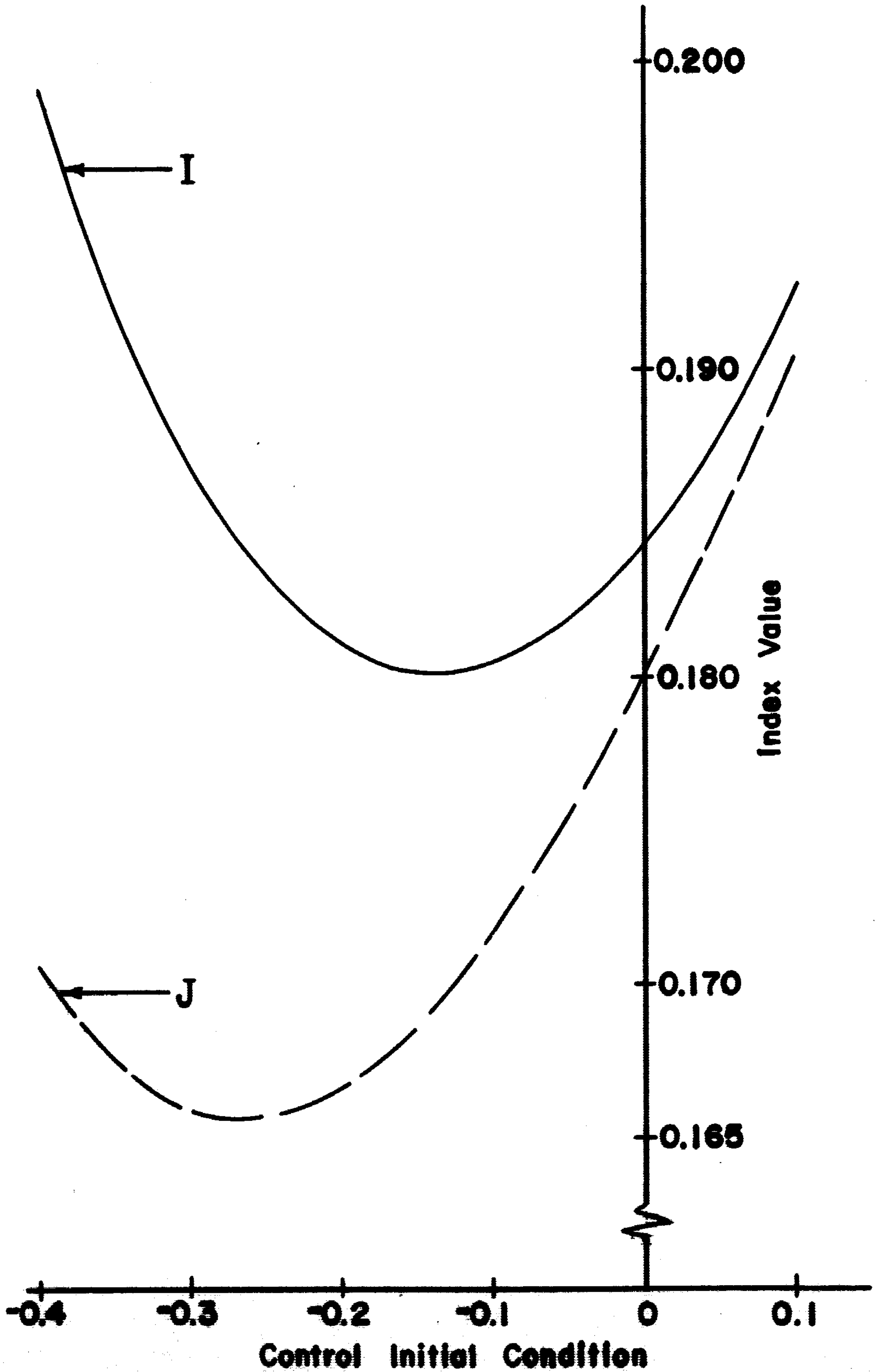

Fig. 1: Performance Indices vs. Control Initial Condition

has been made to leave as free parameters the elements of the C and D matrices of (2), evaluate the index as a function of these parameters, and then optimize with respect to them. That is, an attempt was made to find a control optimal over the space of all controls of the form of (2). A difficulty was immediately encountered, however, as is evidenced by the following scalar example. Given a system

$$\dot{x} = ax + bu \tag{28}$$

$$\dot{u} = cx + du \tag{29}$$

and a performance index

$$J = \int_0^\infty (x^2 + u^2)dt, \tag{30}$$

the globally optimal control is known by standard techniques to be

$$u = kx \tag{31}$$

with control initial conditions

$$u(0) - kx(0). \tag{32}$$

This control is obviously an element of the restricted control space since

$$\begin{aligned} \dot{u} &= k\dot{x} \\ &= kax + kbu. \end{aligned} \tag{33}$$

Consequently, the values ka, kb and kx(0) would be found for c, d and u(0) by the process presented herein. Examination of the augmented system, however, reveals that the augmented matrix is singular, an obviously unacceptable solution.

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